Ergodic theory on a circle

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Abstract

We consider the motion on particles on a circle. Application on initial value problems with nonlinear boundary conditions will be done. .

1 Introduction

We begin with the Riemann mapping theorem which asserts that if D is a simply connected domain $\mathbb{C}$ whose boundary contains at least two points, there is a conformal mapping ψ of the open unit disk Δ onto D. We can map 0 to any fixed point of D, and also specify the argument of $\psi'(0)$, and then the Riemann mapping is unique (see, [2].

The paper based with the iterations $f_n(z)$ $(f_2(z) = f(f(z)), f_3(z) = f(f_2(z))))$ etc. of various classes of regular functions along with lines of the theory initiated by Fatou and Julia for entire or rational $f(z)$. H. Rodströ showed that a theory of the Fatou – Julia type exist precisely for the ration and entire cases considered by them. Now such function have applications in physics of Mandelbrot fractals (see, for example, [12, 16]).

The main objects of the theory is the non-empty perfect $J(f)$ of points where sequence $f_n(z)$ is not normal (in the sense of Montel). Now $J(f)$ is called the Jukia set. Hence, $C(\mathfrak{F})$ is the complement of J, so that $C(\mathfrak{F})$ splits into components (domains of normality). The situation is similar to the one–dimensional theory of the unimodal maps with one extremum, where an analog of a Julia set is formed by the so-called separator of the map, which contains the pre-images of repelling fixed points (see, [22]).

As in $1D$ case, we must know the fixed points of $f(z)$ and the limit subsequences of $f_n(z)$. Components of $C(\mathfrak{J}$ contain attractive fixed points with $f(\alpha) = \alpha$ and $|f'(\alpha)| < 1$. We call such domains the 'immediate domains of attraction'. It can be proved that an entire function of order $< \frac{1}{2}$ has no unbounded domain of attraction for of its fixed points. Thus, such type of function describe localized complex structures in $\mathbb{C}$ in physics (see, for example, [12, 16]).

Further, estimations for the growth of functions with large infinite domains of attraction (particularly, including half–planes) are obtained. It is shown that if an attractive fixpoint of order 1 has bounded domain of attraction, then the boundary D contains repulsive – fix points of every order. As noted by P. Hhattacharyya [34]:'There are accessible boundary points – in general boundary may be very wild' [1969]. Some results about completely invariant domains of normality are obtained. These are components of $C(\mathfrak{F})$ invariant under

$z \to f(z)$ and $z \to f^{-1}(z)$. These domains usually form bounded clusters. There are boundaries between clusters which form Jordan curve. In some cases, such boundary curve in $\mathbb{C}$ - space may be even non-differentiable. But in normal domains on $C(\mathfrak{F})$ dynamics is simple.

Often the half-plane can be transformed to a circle by a homeomorphism, so that we can study the dynamics of transformation of the circle to itself. The number of components $C(\mathfrak{F})$ is $0, 1$ or ∞ in the entire case. If there is a completely invariant domain it is conjectured that there no other domains and some results in this direction. The situation is very simple because we have two attractive fixpoints 0 and $+\infty$, and one repelling fixed points.

Harmonic measure arise in a natural way for Julia sets of polynomial. If $P(z)$ is a polynomial, we denote by ∞ its domain of attraction to infinity. The Julia set of P is then the boundary of F_∞. It was demonstrated by Brolin [29, 11] that harmonic measure on F_∞ is balanced (has constant Jacobian under mapping $P(p)(...P)z)$. Carleson and Jones studed numerically the thermodynamical pressure β (which characterize the spectrum of the dynamical system) for domains of attraction to infinity for quadratic polynomials $f(z) = z^2 + c$. Non-rigorous estimation is $\beta = 0.24$ for $c = -0.560 + 0.6640$ [Smirnov].

Revolutionized study of topology in 2 and 3 dimensions, showing interplay between analysis, topology, and geometry. The central new idea is that a very large class of closed 3-manifolds should carry a hyperbolic structure - be the quotient of hyperbolic space by a discrete group of isometries, or equivalently, carry a metric of constant negative curvature. Although this is a natural analogue of the situation for 2 -manifolds, where such a result is given by Riemann's uniformisation theorem, it is much less plausible - even counter-intuitive - in the 3-dimensional situation (see, [**?**, 34]).

The iteration dynamics could be used in the study of complex dynamical systems (in particular, condensed matter physics). This lead to a interest the classical work of Gaston Julia, Pierre Fatou and Paul Montel on complex analytic dynamics. The work of Benoit Mandelbrot on 'fractal geometry' and the realization of the importance of fractal geometry in physics (see, [12]) leads to famous Mandelbrot's works which involve earlier mathematical Cantor's and Hausdorff's works.

Here, we shall state a few important results without proofs and show how these abstract results may be applied, for example, to the motion of electron along equatorial orbits (see, [11]. Recall that important physical fittings work on equatorial orbits as noted by Victor Maslov. The type of dynamics we are interested in initially has been considered by Back [12] for the 'determinislic' Langevin problem as the velocity of a particle, though other interpretations are possible as well.

2 The physical meaning of the Mandelbrot set

There is a theorem of Douady and Hubburd assuring that the Mandelbrot set is connected. There is also the Bӧttcher – Fatou lemma.

Assume that

$$f(z) = z^k + a_{k+1} z^{k+1} + ... \tag{1}$$

with $k \geq 2$ is anaclitic near 0. Define

$$\phi_n(z) = (f^n)(z)^{1/k^n} = z + a_1 z^2 + ... \tag{2}$$

At a neighbourhood U of $z = 0$ the homeomorphism

$$\phi(z) = \lim_{n\to\infty} \phi_n(z) : U \to B_r(0), \tag{3}$$

where $B_r(0)$ is a disk centered at 0, we have locally

$$\phi \circ f \circ \phi^{-1}(z) = z^k \tag{4}$$

and $\phi(0) = 0$ and $\phi'(0) = 1$.

The function

$$h(z) := \log\left(\frac{bf(z)^{1/k}}{z}\right) \tag{5}$$

with the chosen root

$$f(z)^{1/k} = z + O(z^2) \tag{6}$$

3 Postulation of the classical problem

Below it will be used an example, which has been considered in [12]. It is most convenient to interpret the dynamical variable as the 'velocity' of in classical mechanics. For example, let $v(t) := (u_1(t), u_2(t))$ be the velocity (particular, charged) of particle.

Let

$$z(t) = u_1(t) + iu_2(t) \tag{7}$$

be the complex variable. Then the expression $|z(x,t,h)|^2$ is vied as probability density in quantum mechanics. The measure depends on a small parameter h (or on a large parameter ω.

Consider the following dynamics [12]. At points t_n the particle gets a kick of strength $c = a_1 + ia_2$, where a_1 is the kick strength in x_1 - direction, a_2 that in x_2 - direction. Then we can consider the velocity $v_n^- = (u_{1,n}^-), u_{2,n}^-$ and $v_n^- = (u_{n,1}^+, u_{n,2}^+$ before and after the kick. As a result, we have

$$u_{n,1}^+ = u_{n,1}^- + a_1 \quad \text{and} \quad u_{n,2}^+ = u_{n,2}^- + a_2 \tag{8}$$

that is equivalently

$$z_n^+ = z_n^- + c \tag{9}$$

where index $(\pm)$ labels values before $(-)$ and after $(+)$ the kick.

Next, we assume that there is independent magnetic field in x_3 - direction, which is constant with respect to space. It means that at points t_n the field change its value, but on interval $t_n < t < t_{n+1}$ it is constant, i.e., $B := (0, 0, B_n)$. Assume that there is a forth $A := A(v)$ which ia acting on the particle in v - direction, so that $\dot{v} = A(v)$. The example is a linear damping force $A(v) := -v$ or $A(v) := -v + v_3$, which is acting and arising from a well- known double - well potential. These are typical potential.

Consider the following dynamics [12]. At points where the particle gets a kick of strength $c = a_1 + ia_2$, where a_1 is the kick strength in x_1 - direction, a_2 that in x_2 - direction. Then we can consider the velocity $v_n^- = (u_{1,n}^-, u_{2,n}^-)$ and $v_n^+ = (u_{n,1}^+, u_{n,2}^+$ before and after the kick. As a result, we have that

$$u_{n,1}^+ = u_{n,1}^- + a_1 \quad \text{and} \quad u_{n,2}^+ = u_{n,2}^- + a_2 \tag{10}$$

that is equivalently

$$z_n^+ = z_n^- + c \tag{11}$$

where index $\pm$ labels values before $(-)$ and after $(+)$ the kink.

Next, we assume that there is independent magnetic field in x_3 - direction, which is constant with respect to space. It means that at t_n field changes its value, but on interval $t_n < t < t_{n+1}$ it is constant, i.e. $B = (0, 0, B_n)$. Further, we assume that there is a force $A := A(v)$ which is acting, but also the Lorentz force $F = qv \times B$ (labels of vectors we omitted). The from [12] it follows that

$$\tau_n = f(v_n^+, \varphi_n^+) \tag{12}$$

where $\tau_n = t_{n+1} - t_n$, and

$$B_n = h(v_n^+, \varphi_n^+, \tau_n). \tag{13}$$

Let $g(v_0, t)$ be a solution of the initial value problem

$$\dot{g} = A(g), \quad g(v_0, 0) = v_0. \tag{14}$$

Then integration of equations of motion gives that (between two successive kicks) the velocity of the particle obey the following decreet dynamics [12]:

$$v_{n+1}^- = g(\tau_n, v_n^+), \tag{15}$$

$$\varphi_{n+1}^- = \varphi_n^- + \omega_n \tau_n \tag{16}$$

where v_{n+1}^- and φ_{n+1}^- denote the modulo value and the angle of the velocity before the next kick.

We see that this system produce the discrete dynamical system which may be solved step by step by iterations of initial data. We can say that the system is produced by 'boundary conditions' that are formed by the continuous analogies at points of kicks between particles. Iedeed, if the decreet system may be prolonged on the continuous case then we can use Beck's method. Now instead of evolution equation (102) we use the linear ShrÖdinger equation. Then the ShrÖdinger equation will be reduced by Maslov's method [11] to the canonical system of the transport equation for the amplitude and to the Hamilton -Jacobi equation for the phase (see, [26]). Next, using functional boundary conditions the such boundary problem with additional initial conditions can be reduced to a system of difference equations with continuous time.

The careful method of reduction may be find in [16, 22]. The method is similar as in [12]. Indeed, in [12] it is shown that (independently on v) the angle φ rotates with $\omega_n = -\frac{qB_n}{m}$, where we put $q=-1$ and mass $m=1$. It means that $\omega_n = B_n$. Further, putting τ_n and B_n from (100),(235) into equations (103),(104) we arrive at [12]:

$$v_{n+1}^{-} = g(f(v_n^+, \varphi_n^+), v_n^+), \tag{17}$$

$$\varphi_{n+1}^{-} = \varphi_n^+ h(v_n^+, \varphi_n^+, \tau_n) f(v_n^+, \varphi_n^+). \tag{18}$$

Substituting (100) into (18) from (17),(18) it follows that

$$v_{n+1}^{-} = g(f(v_n^+, \varphi_n^+), v_n^+), \tag{19}$$

$$\varphi_{n+1}^{-} = \varphi_n^+ h(v_n^+, \varphi_n^+, \tau_n) f(v_n^+, \varphi_n^+). \tag{20}$$

We see that this system produce decreet dynamical system which can be solved step by step by iteration of initial data.

Since at time t_{n+1} a kick of strength c is acting the change z_n^- to z_n^+ according to

$$z_n^+ = z_n^- + c \tag{21}$$

gives

$$v_{n+1}^+ e^{i\varphi_{n+1}^+} = v_{n+1}^- e^{i\varphi_{n+1}^-} + c. \tag{22}$$

As a result, we obtain non-trivial case of coupled analytic relation

$$v_{n+1}^+ e^{i\varphi_{n+1}^+} = g(f(v_n^+, \varphi_n^+), v_n^+) \times e^{i(\varphi_n^+ + h(v_n^+, \varphi_n^+, f(v_n^+, \varphi_n^+)))} + c. \tag{23}$$

Now we consider time difference τ_n between kicks that are large if the velocity is small and small if the velocity is large, and that do not depend on the angle φ_n. A possible example is

$$\tau_n = f(v_n^+, \varphi_n^+) = \frac{1}{2} \ln\left(1 + \frac{1}{(v_n^+)^2}\right). \tag{24}$$

Next, if the force is produced by a potential $V(v) = \frac{1}{2}v^2 - \frac{1}{4}v^4$ then the solution of the initial value problem is

$$g(t, v_0) = \frac{v_0}{(v_0^2 + (1 - v_0^2)e^{2t})^{1/2}}. \tag{25}$$

Putting (242) into (25) one obtain that

$$g(\tau_n, v_n^+) = (v_n^+)^2. \tag{26}$$

Further, substituting (242)(that is now non-dependent on φ_n^+ and τ_n) into (242) we obtain that

$$v_{n+1}^+ e^{i\varphi_{n+1}^+} = g(f(v_n^+, \varphi_n^+), v_n^+) \times e^{i(\varphi_n^+ + h(v_n^+, \varphi_n^+, f(v_n^+, \varphi_n^+)))} + c. \tag{27}$$

4 Rotation on charged particle on the circle

Let f be a circle map. Instead of work up with f itself we shall use a lifting on f. A continuous map $F : R \to R$ is called a lifting of f if $e \circ F = f \circ e$, where $e(u) = e(2\pi i u)$ is the natural projection from R to S^1. If F is a lifting of f then $F \in m$ is also a lifting of f for each $m \in z$ and F^n is a lifting of f^n. There is an integer d such that

$$F(u+1) = F(u) + d, \tag{28}$$

for each $n \in R$. The number d is calleв the degree of f and is denoted by $def(f)$. We can show that $def(f^n) = def(f)^n$. We say that a point $u \in R$ is periodic $(mod\, 1)$ of period q for F if $F^q(u) - u \in Z$ but $F^i(u) - u \ni Z$ for $j = 1, 2, ..., q-1$. Clearly, u is a periodic $(mod\, 1)$ point of F of period q if and only if $e(u)$ is a periodic point of f of period q.

Let F be a lifting of a circle map f. We shall denote by $Per\,(f)$ the set of periods of all periodic $(mod\, 1)$ points of F. Clearly, $Per\,(f) = Per\,(F)$. Let f be a circle of degree 1 and let F be a lifting of f. For $u \in R$ we define its F - rotation number as

$$\limsup_{n\to\infty} \frac{F^n(u) - u}{n} \tag{29}$$

and denote it by $\varrho_F(u)$. Since f has degree 1, we have that

$$\varrho_F(u) = \varrho_F(u) + m \tag{30}$$

for all $m \in Z$. If f is a periodic point $(mod\, 1)$ of period q of F then

$$\varrho_F(u) = \frac{F^q(u) - u}{q} \in Q. \tag{31}$$

The set

$$\{\varrho_F(u) : u \in R\} = \{\varrho_F(u) : u ß[0,1]\} \tag{32}$$

is denoted by F_F. In [?] is proved that F_F is a closed interval (perhaps, degenarated to a point) of R. Thus, L_F will be called the rotation interval of F. The rotation interval of a lifting of a circle map of degree 1 capture a lot of its dynamical properties and plays a fundamental role in there study.

4.1 Main properties of the rotation interval

From [13] it follows that if f be a circle map and if F be a lifting, then one of the following properties hold: (a) f has a horseshoe. (b) There exist $q \in N, p \in Z$ and $I = [n, n+1] \subset R$ such that $(F^q - p)(I) \subset I$. (c) $Per(f) = \phi$.

Now we use the following result [13]. Let f be a continuous map of the interval $I = [a.a+1]$ into itself such that with $d = -1, 0, 1$. Assume that ϖ - limit set $\varpi_u(f)$ is a simple set for any $u \in I$ and that $a \in \varpi_u(f)$ and $a + 1 \in \varpi_u(f)$ for each $u, y \in I$. Then the following statement is hold. (a) If $\varpi_u(f) \neq \varpi_y(f)$ then $\varpi_u(f)$ and $\varpi_y(f)$are fixed points of f. (b) If $\varpi_y(f) = \varpi_u(f)$ then $\varpi_u(f)$ is a periodic orbit of period 2.

4.1.1 Case 1

If $d = 0$, the we shall prove that $(a, a+1) \ni \varpi(f)$, and the above statement hold. Indeed, from ([13], Theorem A) it follows that for each interval map f satisfies one and only one of the following conditions: (a') f has a horseshoe. (b') The chain recurrent set of f is the union of all simple sets of f.

5 Remark 1

The same is true for a frequency of the motion of the charged particle with double potential for the Beck problem [12]. ■

To extend Theorem A to circle maps we have to reformulate the above notation in this context. We shall represent the circle S^1 as the set $\{z \in \mathbb{C} : |z| = 1\}$. Any continuous map which form S^1 into itself will be called a circle map. (For example, a map $z \to z^2$ forms a circle map. But it is generally not true for $z \to z^2 + c$, where $c \neq 0$). We note that the notation of a simple set and of horseshoe extends naturally to circle maps by simply replacing closed intervals by closed arcs of the circle (that is, subsets of S^1) which are homeomorphic to closed intervals of the real line.

We note also that if an interval map f has a horseshoe I_1, I_2 then we always have that, for each $i, j = 1, 2$ there exist a closed interval $I_j^i \subset I_i$ such that $f^n(I_j^i) = I_j$. However, this is not (see, [13]). In this sense, there is difference between the circle and interval.

6 The theory for a circle

Let $f : X \to X$ be a map from X into itself and $u \in X$, where X is a topological space. If $u \in X$ we shall denote by $\varpi_(f)$ the ϖ - limit set of u which is defined to be the set of all accumulation points of $\{f^n(u) : u \geq 0\}$. We also use the notation $\varpi(f)$ to denote $\bigcup_{u \in X} \varpi_(f)$.

Then as follows from [13]: (a") f has a horseshoe. (b") There exist $n > 0$such that $\varpi_u(f^n)$ is a simple set for each $u \in S^1$. (c") $Per(f) = \phi$.

An interval map having a horseshoe was called turbulent by Block (see, [?, ?]). A similar notion was called an L - schema. If $I_1, I_2 \subset S^1$ is a horseshoe of a circle map g it may be happen that $g^n(I_1) = S_1$ in such a way that $g^n_(Int(I_1))$ is injective on I_1 and $g^n(a) = g^n(b) \in Int(I_2)$ where a and b denote the two endpoints of I_1. Then clearly does not hold for each $i, j = 1, 2$ there exist a closed arc $(\check{I}_j^i) \subset arc(\check{I}_2)$ such that $g^n arc(\check{I}_j^i) = arc(\check{I}_j)$.

6.1 Difference between circle and interval maps

We note that above conditions type (a) that f has a horseshoe are transformed in the space of circle maps with help of lifting F. A criterion for positive topological entropy. On another hand, conditions of type (b) (that is, there exist $n > 0$ such that $\varpi_u(f^n)$ is a simple set for each $u \in S^1$) in the case of interval map, is equivalent of condition of type (b) of Theorem A (see, also, Theorem 2 of cite7). that is, the chain recurrent set of f is the union of all simple sets of f). Hence, only for $n = 1$ these conditions for interval map and circle map, respectively, are equivalent. However, for circle maps it is, generally, not. It is connecters with geometry on circle and interval. In this case, the topological picture of the chain recurrent set can be described in detail (see, [13]).

7 Topology of a circle map for charged particle in magnetic field on a plane

For the Beck's problem [12, 16], there is the following dynamics:

$$v_{n+1}^{-} = g(\tau_n, v_n^{+}, \varphi_{n+1}^{-} = \varphi_{n+1}^{-} + \omega_n \tau_n), \tag{33}$$

where g is a given function, τ_n and ω_n ate known parameters. If we consider a problem in $\mathbb{C}$ then $z = |z|e^{i\varphi}$, where $v = |z|$.

7.1 The separating case

Define $d_n = \omega_n \tau_n$. Assume that $d_n := \{-1, 0, 1\}$. Let f be f be a continuous map of the interval $I = [n, n+1]$ onto itself. Then

$$f(n+1) = f(n) + d. \tag{34}$$

We say that there are the simple interval maps if this set either consists of a unique periodic orbit or does not contain any periodic orbit.

Recall the set of chain recurrent points of f is denoted by $\mathcal{C}R(f)$ and is defined to be of all $u \in X$ for $f : X \to X$ such that for each $\varepsilon > 0$ there exists $\}(u_i)_{i=0}^{n}$ such that

$$|f(u_i) - f(u_{i+1})| < \epsilon, \quad i = 0, 1, ..., n-1. \tag{35}$$

7.2 f has a horseshoe

If $deg(f) \ni (-1, 0, 1)$ then clearly f has a horseshoe. Assume now that $deg(f) = 0$.Then

$$F(n+1) = F)n), \quad u \in R. \tag{36}$$

Hence, $F(R) = F([0, 1]) = [a, b]$. If $b < a + 1$ then we set $q = 1, p = 0$ and $I = [a, a + 1]$. And there exists $q \in N, p \in Z$ and $I = [n, n + 1] \subset R$ such that

$$(F^q - p)(I) \subset I. \tag{37}$$

Further, assume that $b > a+1$. Let $c \in R$ be such that $F(c) = a$. Clearly, $F(c+1) = q$ and there exist $d \in (c, c+1)$ such that $F(c) = a$, $F(c+1) = q$ and there exist $d \in (c, c+1)$ such that $F(d) = b$.

The main moment is the following. Set $I_1 = e([c, d])$ and $I_2 = e([d, c + 1])$. Clearly, I_1 and I_2 are arcs of S^1. Now we obtain the horseshoe as above.

8 An application to Beck's problem

Remind the above formulation in more convenient form for to an application to the physical Beck problem [12]. Let $e : R \to S^1$ be the 'exponential map'. If we view D_1 as a unit disk in $\mathbb{C}$ then on the boundary S^1 we obtain

$$e(t) = e^{i2\pi t} \tag{38}$$

We view also S^1 as $(x, y) \in R^2$ such that $x^2 + y^2 = 1$. Then

$$e(t) = (\cos 2\pi t, \sin 2\pi t). \tag{39}$$

Moreover,

$$e(t_1) = e(t_2) \Leftrightarrow t_1 - t_2 \in Z. \tag{40}$$

Define $e_0 : I \to S^1$ the restriction of e on I. Since the trinogonometrc functions are periodic of period 1, we have

$$e(t + C) = e(t), \quad \text{where} \quad C \in Z. \tag{41}$$

Now consider the Path-Lifting Lemma: Let $g : [0, 1] \to S^1$ be a continuous map, and let $x \in R$ such that $e(x) = g(0)$. Then there is a unique continuous map $\tilde{g} : [0, 1] \to R$ such that

$$e(\tilde{g}(t)) = g(t), \quad t \in [0, 1], \tag{42}$$

where $\tilde{g}(0) = x$.

A map $\tilde{g}$ is called a lift of g. For the 'initial conditions' $\tilde{g}(0) = x$, we get the unique lift of g at x.

8.1 The degree of a circle map

Example 1. Let us consider the linear expending map $E_2 : S^1 \to S^1$ (noninvertible) map, so that

$$E_2(x) = 2x \pmod 1, \tag{43}$$

(see, Figure 1 and Figure 2).

The number of periodic points

$$P_n(f) = \{\text{fixed points of} f^n\}. \tag{44}$$

Thus, number of fixed point $P_n(E_2) = 2^n - 1$ and periodic points of E_2 are dense in S^1 ■.

We remember that $S^1 = R/Z$ and there is a projection $\pi : R \to S^1$ that

$$x \to [x]. \tag{45}$$

then

$$\pi \circ F = f \circ \pi, \tag{46}$$

where F is called a lift of f, and $f : S^1 \to S^1$ is continuous, $F : R^1 \to R^1$ is continuous. F is unique up to integer translation.

It is important that $F(x+1) - F(x)$ is an integer independent on F and x together. If f is a homeomorphism, then degree $|deg(f)| = 1$. Further, $F(x+1)$ is also a lift of F. Since

$$\pi(F(x+1)) = f(\pi(x+1)) = f(\pi(x+1)) = f(\pi(x)), \tag{47}$$

$F(x+1) - F(x)$ is an integer independent on x.

If F and G are lifts, then

$$F(x+1) - F(x) - (G(x+1) - G(x)) = k - k = 0. \tag{48}$$

Next, if $deg(f) = 0$, then $F(x+1) = F(x)$ for all $x \in R$, where F is not monotone and f is not monotone. If $|deg(f)| > 1$, then $|F(x+1) - F(x)| > 1$. There is $y \in (x, x+1)$ such that $|F(y) - F(x)| = 1$ and f is not monotone.

Example 2. For linear expending map for each integer $m \neq 1$

$$E_m(x) = mx \quad (mod\,1), \tag{49}$$

periodic points are $P_n(E_m) = |m^n - 1|$ and periodic points of E_m are dense on S^1.

9 Example.Donady's Rabbit

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The Julia set of the map $z \to z^2 + c$, where c satisfies $c^3 + 2c^2 + c + 1$ and $\Im(c) > 0$ has been obtained by Blanchard (see, [?], Figure 2.1), where $c \approx -0.12256117 + 0.74486177i$. This Julia set is connected, and the Fatou set consists of infinitely many, simply connected domains.

10 Reduction to real and complex planes

Recall that if $c = 0$ then relation (60) in complex space $\mathcal{C}$ can be reduced to the system of discrete difference equations in R^2, so that

$$e^{i\varphi^+_{n+1}} = e^{i(\varphi^+_n + h(v^+_n, \varphi^+_n, f(v^+_n, \varphi^+_n)))}, \tag{50}$$

$$v^+_{n+1} = g(f(v^+_n, \varphi^+_n), v^+_n), \tag{51}$$

i.e.,

$$\varphi^+_{n+1} = \varphi^+_n + h(v^+_n, \varphi^+_n, f(v^+_n, \varphi^+_n))), \tag{52}$$

From the above assumptions it follows that the system can be reduced to

$$v^+_{n+1} = (v^+_n)^2, \tag{53}$$

i.e.,

$$\varphi^+_{n+1} = \varphi^+_n + \omega_n \frac{1}{2} \ln \left(1 + \frac{1}{(v^+_n)^2}\right), \tag{54}$$

10.1 Dynamic of charged particle on a circle

If $v^+_n = \infty$, then we have a circle S^1 as the quotient space of the real line by the group of transformations by integers $S^1 := R/Z$, and we consider the circular ordering S^1. Let $\pi : R \leftrightarrow S^1$ be the quotient map. In S^1, we consider the metric and orientation which are induced by the metric and orientation of the real line π. We are interested by the dynamics of the homeomorphism $f : S^1 \to S^1$. The simplest homeomorphism is rotation: those are the orientation which presents isometries on the circle. If y_1 is a periodic point of period n of a rotation f then any other point y is also periodic point on the same period. If a rotation f does not have a periodic point then the orbit $O_f(y) = \{f^n(y), n \in Z\}$ is dense on the circle.

It means that the transition from $v^+_n < \infty$ to $v^+_n = \infty$ (or from $B_n = 0$ to $B_n \neq 0$) changes catastrophically behavior of trajectories of the considered dynamical system. The same is true when the kicks includes in the Beck problem.

We must discuss a result of Poincaré: (1) If $f : S^1 \to S^1$ is a homeomorphism with a periodic point then any orbit is asymptotic to a periodic orbit and (if f preserves orientation) any two periodic orbits have the same period. (2) If f does not have a periodic point then there exist a rotation $g : S^1 \to S^1$ such that any orbit of f has the same order as any orbit of g. The map $h : O_f(y) \to O_f(y')$ defined by $h(f^n(y)) = g^n(y')), n \in Z$ is monotone.

From this fact it follows (since any orbit of g is dense on the on the circle) that h extends continuously to a monotone map $h : S^1 \to S^1$ which satisfies to the equation $h \circ f = g \circ h$. We say that h is a semi - conjugacy between f and g. From the above equation it follows that $h \circ f^n = g^n \circ h$, that is, h sends orbits of f into orbits on g. In general h is not a conjugacy because the inverse image of some point may be an interval.

In this situation there is a very simple observation of the dynamics of an invertible continuous map of an interval. Let I be a closed interval and $f : I \to I$ be a continuous injective map, and f is orientation preserving (i.e., monotone increasing). Then any orbit of f is asymptotic t a fixed point. If f is orientation reversing (i.e., monotone decreasing), then the iterate $f^2 = f \circ f$ is monotone increasing. We get that any orbit of f ia asymptotic to either the fixed point of f or to a oeriodic orbit of period 2.

11 Circle homeomorphisms

Define $y^+_n = \frac{1}{(v^+_n)^2}$ (below indexes for $\frac{1}{(v^+_n)^2} = y^+_n$ will be omitted) and use the series

$$\ln(1 + y) = \int_0^y \frac{ds}{1+s} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{n+1} \tag{55}$$

has radius of convergence to 1 so that y converges absolutely for $|y| < 1$ (i.e., for $(v^+_n)^2 > 1$). For $y = 1$ (i.e., for $(v^+_n)^2 = 1$) we obtain the series $\sum_{n=0}^{\infty} \frac{(-1)^n y^{n+1}}{n+1}$.

Thus, the problem is reduced for the system

$$v^+_{n+1} = (v^+_n)^2, \tag{56}$$

$$\varphi_{n+1}^{+} = \varphi_n^{+} + \frac{\omega_n}{2}\int_0^{y_n^{+}} \frac{ds}{1+s}. \tag{57}$$

But we confined itself only the case $n = 0$. Then the last system may be written as

$$v_{n+1}^{+} = (v_n^{+})^2, \tag{58}$$

$$\varphi_{n+1}^{+} = \varphi_n^{+} + \frac{\omega_n}{2}\frac{1}{v_n^{+})^2}, \tag{59}$$

where

$$\omega_n = -\frac{qB_n}{m}. \tag{60}$$

Here, the particle is with charge $q = 1$ and mass $m = 1$, which means that $\omega_n == B_n$. Here also it is assumed by Beck [12] that there is a time - dependent magnetic field in x_3 - direction, which is constant with respect to space. So that at time points t_n the field changes its value, for $t \in (t_n, t_{n+1})$ it has constant value $B = (0, 0, B_n)$ (where vectors here omitted). Thus, between kicks, not only the force $A(v)$, but also the Lorentz force $F = qv \times B$ is acting on the particle, with $v = (u(t), w(t), 0)$.

12 A simple model produce fractal?

13 Example

As an example, it will be studied the limit case (60, 59). Initially, let us consider the real - valued family of quadratic maps

$$\varphi; x \to x^2 + \alpha, \tag{61}$$

where $\alpha \in R$. We begin with $c = 0$. If $-2 \leq\leq 1/4$ then $\varphi(\hat{I}) \subseteq \hat{I}$, where $\hat{I} = [-\beta_0, \beta_0]$, and

$$\beta_{0,1} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha}, \tag{62}$$

where β_0 is the repelling fixed point, and β_1 is attractive fixed point. Then interval $I :=$ $(-\beta_0 + \varepsilon, \beta_0 - \varepsilon)$ for small $0 < \varepsilon < \alpha + \beta_0$ is invariant and, moreover, $\bar{\varphi I} \subset I$.

14 Quantum initial value boundary problem

In this paper it will be considered complex solutions of real analytic equations with functional two point boundary conditions and with special initial conditions of the WKB type. It turns out that for the such problem it is possible to apply the known Maslov technique of reduction of $PDR's$ of quantum mechanics to a so called canonical system which contains to connected $PDE's$ that are approximating the general system with accuracy $O(h^2)$, where h is a small positive parameter. A first goal is to fined an asymptotic of solutions as $h \to 0$. This problem has been solved by Maslov in special classes of initial conditions on each finite time internal $[0, t(h)$, where $t = t(h)$.

It turns out that for the such problem it is possible to apply the known Maslov's technique of reduction of $PDE's$ of quantum mechanics to the so called canonical system which contains (with accuracy $O(h^2)$) both the Hamilton - Jacobi and transport equations. The Maslov method allow to reduce the quantum problem to the solutions of equations of classical mechanics and hence to find the quasi-classical asymptotic as $h \to 0$ for initial data of the WKB type.

We prolong this method on initial boundary value problem on the example of the Shrödinher linear equation with non-linear boundary conditions (particular, periodic conditions). As known, the Shrödinher equation without potential with Hamiltonian $H(p,q) = \frac{p^2}{2}$ in the first approximation on h can be reduced to the Hamilton - Jacobi equation

$$S_t + \frac{1}{2} S_x^2 = 0 \tag{63}$$

and to the transport equation

$$\varphi_t + S_x \varphi_x + \frac{1}{2} S_{xx} = 0, \tag{64}$$

where φ is a phase and S is an amplitude. If we confined itself only by the linear phase

$$S(x,t) = \lambda_1 x + \lambda_2 x \tag{65}$$

then equations (63), (67) become

$$S_t + \frac{1}{2} S_x^2 = 0 \tag{66}$$

and to the transport equation

$$\varphi_t + S_x \varphi_x + \frac{1}{2} S_{xx} = 0, \tag{67}$$

Let us consider for the first equation the periodic boundary conditions

$$e^{iS(0,t)/h} = e^{iS(l,t)/h} \tag{68}$$

and the initial condition

$$S(x,0) = \lambda x. \tag{69}$$

Boundary conditions describes the motion of a particle on circle S^1.

A solution of equation (67) with an initial condition (69) is $S(x,t,\lambda) = \lambda \left(x - \frac{1}{2}\lambda\right)$. Here, $\lambda = p$, where p is an impulse of the particle. Then from the periodic boundary conditions on the phase, which is independent on the amplitude, we arrive at

$$\frac{p}{l} = \frac{\pi}{2} \pm 2\pi k, \quad k = o, 1, ... \tag{70}$$

15 The boundary conditions on circles

It is known that any quantum equation can be reduced in the WKB approximation (with a given accuracy $O(h^2)$, where $h > 0$ is a small parameter) to a system of two $PDE's$ equations – the transport diffusion equation for amplitude and the Hamilton - Jacobi equation for phase(see, [11]). Such equations will be called by the canonical system.

We consider only the linear Shrödinger equation without potential on interval $(0, l)$ with functional nonlinear boundary conditions of the form:

$$\psi(0, t) = \Phi(\psi(l, t)), \tag{71}$$

where $\psi \in \mathcal{C}$ and $\Phi : \mathcal{C} \to \mathcal{C}$ is an analytic function.

Then (as it will be shown below) the linear Shr$\ddot{o}$ equation of quantum mechanics (without potential) with nonlinear boundary conditions (71) admits the reduction to the corresponding canonical system of classical mechanic (with accuracy $O(h^2)$), which have the form:

$$\varphi(0, t) = \Phi_1(\varphi(l, t), S(l, t)), \tag{72}$$

$$S(0, t) = \Phi_2(\varphi(l, t), S(l, t)), \tag{73}$$

where φ is the amplitude and S is the phase of the wave function ψ that is found in the WKB approximation in the linear approximation on small positive parameter h.

15.1 Discussion on structural stability of the boundary conditions

It will be shown that the quantum boundary problem is to study of the analytic dynamical system $R_a : z \to \Phi(z)$ on the Riemann sphere $\bar{S}$ to $R_a \to \Phi(z)$, where a is depending on a parameter a. The following happens: before perturbation the round (unit) circle C is invariant under iterations of R is expending on S so that $|\dot{R}(z)| > 1$. The map R has dense orbits, and is even ergodic on S with respect to linear measure.

This situation strongly remind one of Poincar$\acute{r}$'s original perturbations (see, [15]). For example, for the map $R : z \leftrightarrow z^2 + az$ for small a. The Poincar$\acute{r}$ limit set then changes from a round circle to a non-differentiable Jordan circle. Fatou and Julia (see, [15]) where well aware to the analogy with Poincar$\acute{e}'s$ work. Sullivan continue this analogy and it has been considered the modern theory of quasiconformal mapping into the dynamical theory of iteration of complex analytical mapping. But we can not study here this very exotic and complex situation when the map $R_a : \mathcal{C} \to \mathcal{C}$ is quasiconformal deformations for the simplest map $z \to z^2 + az$ on the entire sphere for $0 < |a| < 1$, since in this case we have deal with non-differentiable homeomorphisms h.

Bellow it will be considered only more simpler and clear differential dynamics with differentiable homeomorphisms h conjugating one system R_1 to another R_2 so that $h \circ R_1 = r_2 \circ h$. Such h is Lipschitz. An important point is that nearby complex analytic dynamical system tend to be conjugate using homeomorphisms which are Lipschitz and quasi-conformal. Such situations has been studied, for example,

16 The Shr$\ddot{o}$dinger equation in the dimensionless form

We begin with the clear method of reduction Since in literature there are no of the clear description of reduction of the Shr$\ddot{o}$edinger equation to the dimensionless form:

$$-i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}. \tag{74}$$

Here, $\psi:=\psi(x,t):R^2\to\mathbb{C}$, where $\mathbb{C}$ is a complex space, $\hbar$ is the Planck constant. Let us divide the two parts of equation on a value mv^2, where m is the mass of particles, v is their velocity and $p=mv$ is an impulse. Introduce scaling time $\bar{t}=t/\tau$ and relaxation time τ, and consider a dimensionless constant $h=\frac{\lambda}{v\tau}$. Then

$$-ih\frac{\partial\psi}{\partial\bar{t}}=\frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2}, \tag{75}$$

17 Postulation of the problem

Let us consider equation (74) with the two-point boundary conditions

$$\psi(0,t)=\Phi(\psi(l,t)),\ t>0, \tag{76}$$

an initial condition are

$$\psi(x,0)=\psi_0(x)),\ 0<x<l. \tag{77}$$

Here, Φ is the real function such that $\Phi:\mathbb{C}\to\mathbb{C}$, where $\mathbb{C}$ is a complex space.

It will be proved that solutions of initial value boundary problem (74), (77), (77) can be approximated the function $\psi=\varphi e^{iS/h}$, where $h\to 0$, and S is a real function in (x,t,p) – space, where $p=\frac{\partial S}{\partial x}$.

18 Postulation of boundary conditions

We consider the boundary conditions

$$z_{|x=0}=\Phi(z)_{|x=l}, \tag{78}$$

where $z\in\mathcal{C}$, $\Phi\in\mathcal{C}$, and Φ is an analytic function from D on D, so D is the closed disk (particular, a circle). The map Φ may be polynomial of order n or the rational function.

Here, we confined itself by a case when $\Phi:S^1\to S^1$, where S^1 is a circle, and assume that Φ is the conform mapping, so the boundary conditions become

$$\psi=\varphi e^{iS}_{|x=0}=\Phi(\varphi e^{iS})_{|x=l}=\Phi_1(\varphi)e^{i\Phi_2(S)})_{|x=l}, \tag{79}$$

so that now we have uncoupled boundary conditions.

18.1 The boundary conditions

Let us consider for the Srödinger equation instead of the general boundary function $\Phi(z)$ (see, (77) a polynomial $w(z)$ of degree n which maps $z-$ plane conformably upon n – sheeted Riemann surface w, exempt at branch points of w which defined by the vanishing of $\frac{dw}{dz}$ where angle fail to be preserved. The unit circle in the z – plane is mapped upon a curve C, so that the shape is determined exempt perhaps of a translation and rotation, and magnification when the positions of the roots of $w(z)$ is known.

We assume that

$$w(z) = cz^k(z - z_{k+1})(z - z_{k+2})...(z - z_n), \tag{80}$$

where $z_{k+1}, ..., z_n$ are the roots of the polynomial other than 0. Then it is evident that the change in the argument of w as z varies from z' to z'' is equal to the sum of the angles which are generated at the roots of w by vectors joining the latter to z, each root being counted with its proper multiplicity (see, [25].

The ratio between the length of an element of arc in the w plane and the corresponding element in the z plane, or the distortion of the map, is dw/dz. The angle between the two elements (or the twist) is $\arg dw/dz$. If we know dw/dz then we can to draw the perturbations at each point, and hence the way which describes how the twist varies when z moves from z' to z''.

If roots of the polynomial are on the same side of a straight line then the line is transformed into a curve which turns continuously about the origin 0. If roots of the derived function dw/dz places at the same side of a straight line, then the line is transformed about the origin 0 into a curve with curvature that is either positive or negative.

Let us determine in the $z-$ plane a region R_n such that the polynomial has roots, which are placed within or on the boundary of R_n. Then the unit circle is transformed upon a simple region. By observing values of a for which the function

$$w(z) = (a - z)^n \tag{81}$$

fails to have this property, we can fins the boundary of the region R_n.

Thus we study asymptotic solutions of the Srödinger equation as $h \to 0$. The Caushy problem for equation (74) with an initial condition

$$\psi(x, 0) = \varphi(x, 0)e^{iS(x,0)/h}, \tag{82}$$

where $\varphi(x, 0)$ and $S(x, 0)$ has been solved by Maslov [?] on each finite interval $0 < t \leq t_0$, where $t_0 = t(h_0)$.

Asymptotic of this type is known in quantum mechanics as the WKB – method. Further, substitution WKB – function into (74) for Hamiltonian $H(p, q) = \frac{p_2}{2}$, we obtain the exact equation:

$$\left(S_t + \frac{1}{2}S_x^2\right)\varphi + (-ih)\left(S_x\varphi_x + \varphi_t + \frac{1}{2}S_{xx}\varphi\right) + \frac{(ih)^2}{2}\varphi_{xx} = 0. \tag{83}$$

Next in order that $\psi(x, t)$ be an asymptotic solution of (74) modulo $)(h^2)$, it is sufficient that $S(x, t)$ be a solution for the Hamilton – Jacobi equation

$$S_t + \frac{1}{2}S_x^2 = 0, \tag{84}$$

and $\varphi(x,t)$ satisfy to the transport equation

$$\varphi_t + S_x\varphi_x + \frac{1}{2}S_{xx}\varphi = 0. \tag{85}$$

18.2 An initial problem for the Hamilton - Jacobi equation

We begin with an initial problem for the Hamilton - Jacobi equation

$$\frac{\partial S}{\partial q} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = 0, \tag{86}$$

with an initial data $S(0,t) = \zeta q$, where ζ is a parameter.

Here, $H(q,p,t) = \frac{1}{2}p^2$ and the corresponding hamiltonian system has the form:

$$\dot{q} = p, \quad \dot{q} = 0 \tag{87}$$

and we arrive at $q = \tilde{q}(q_0,p_0,t)$ and $p = \tilde{p}(q_0,p_0,t) = p_0$, where (p_0,q_0) are initial values of impulse and coordinate.

For $S_{|t=0} = \xi q$ initial data have the form $q_{|t=0} = q_0$, $p_{|t=0} = \xi$, and the trajectories are $q = \tilde{q}(q_0,t) = \xi t + q_0$ and $p = \tilde{p}(q_0,t) = \xi$. The Jacobian is $J = \frac{\partial \tilde{q}(q_0,t)}{\partial q_0} = 1$ and a solution $S(q,t)$ exist for any $t \in [0,+\infty)$. Since

$$q_0 = q - \xi t \quad \text{and} \quad p\frac{\partial H}{\partial p} - H = \frac{1}{2}p^2, \tag{88}$$

From mechanics it is known that solution $S(x,t)$ can be written as

$$S(x,t) = S_0(x_0) + \int_0^t (pdx - Hdt)_{|x_0=x_0(x,t)} \tag{89}$$

Here, $x := x(x(p,t)$ is a solution of the equation $p = \frac{\partial S}{\partial x}$, where p can be considered as the additional coordinate in (x,p,t) – space. Then on characteristics $\frac{dx(p,t)}{dt} = p$ we have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{dx}{dt} = -H(p) + p\frac{dx}{dt}. \tag{90}$$

The Hamiltonian is $H = \frac{p^2}{2}$. Hence, integratinf (234) from a point $(0,t_0)$ to a point $(0,t_1)$ we obtain

$$S(x(t),t) = S(x(t_0),t_0) - \frac{p_2}{2}(t-t_0) + p(x(t) - x(t+0)). \tag{91}$$

Integrating from the boundary $x(t_0) = 0$ to the boundary $x(t_1 = t_0 + l/p) = l$, from (95) it follows that

$$S(l, t_0 + l/p) = S(0,t_0) - \frac{p^2}{2}\frac{l}{p} + pl = S(0,t_0) + \frac{pl}{2}. \tag{92}$$

19 Boundary conditions for the Shrödinger equation

Now we consider the boundary conditions for equation (74) of the form

$$\psi(0,t) = \Phi(\psi(l,t)), \tag{93}$$

where $\Phi : \mathcal{C} \to \mathcal{C}$ is an analytic mapping. As example, we think that Φ is the real function $z \to z^2$ on the Riemenn sphere C so that $z \to z^2 + az$. But this example is very complex, and we assume that it is possible to use the circle C of radius φ, so that there is the WKB – approximation $\psi = \varphi e^{iS/h}$.

Then from (93) it follows that

$$\varphi(0,t)e^{iS(0,t)/h} = \Phi\left(\varphi(l,t)e^{iS(l,t)/h}\right). \tag{94}$$

Further, it will be considered only analytic dynamical system which admits the decomposition, so that

$$\Phi(\varphi e^{iS/h}) = \Phi_1(\varphi)e^{i\Phi_2(S/h)}. \tag{95}$$

Then from (93) and (95) we see that the functional boundary conditions an be separated, so that

$$\varphi(0,t)e^{iS(0,t)/h}) = \Phi_1(\varphi(l,t))e^{i\Phi_2(S(l,t)/h)}. \tag{96}$$

From (96) it follows that for the transport equation the boundary conditions are

$$\varphi(0,t) = \Phi_1(\varphi(l,t)), \tag{97}$$

and for the Hamilton – Jacobi equation the boundary conditions are

$$e^{iS(0,t)/h}) = e^{i\Phi_2(S(l,t)/h)}. \tag{98}$$

From (98) it follows that

$$S(0,t)) = \Phi_2(S(l,t)) = \Phi_2(S(0,t-l/p)+pl/2). \tag{99}$$

Substituting (99) into (98) we obtain

$$e^{iS(0,t)/h}) = e^{i\Phi_2(S(0,t-l/p)/h+pl/2h)}. \tag{100}$$

Particularly, if $\Phi_2 := Id$, where Id is the identical map, from (100) it follows that

$$e^{iS(0,t)/h}) = e^{i(S(0,t-l/p)/h+pl/2h)}. \tag{101}$$

This boundary conditions on the phase is satisfied if and only if $e^{\frac{pl}{2h}} = 1$, that is

$$\frac{pl}{2} = \pi h\left(\frac{1}{2}+k\right), \quad k = 0,1,... \tag{102}$$

19.1 Bohr quantization condition

Let we have the one – parameter family of closed curves $\Lambda^1(E)$, where $E \subset R^1$, in the phase space $R_p \times R_x$. Then the Bohr quantization condition is

$$\int_{\Lambda^1(E)} = \pi h \left(\frac{1}{2} + k\right) \tag{103}$$

gives quasi-classical energetic levels of the problem to which responde the curves $\Lambda^1(E)$. From (103) and (102) it follows that for our concrete problem we have

$$\frac{pl}{2} = \int_{\Lambda^1(E)} pdx = \pi h \left(\frac{1}{2} + k\right). \tag{104}$$

20 Interpretation of the boundary conditions

A set $W \in R^n$ is called a star–shaped region or star convex set if there is $x_0 \in S$ such that for each $x]inS$ the segment $x_0, x] \in S$. In our case, $W := (x_1, x_2)$, where $x_1 = \varphi$ and $x_2 =$S are amplitudes and phases of the wave function, respectively. Recall that φ and S are solutions of the canonical system to which in WKB approximation is reduced the Srödinger equation. Thus, we consider the problem for $n = 2$.

For example, if $A \in R^n$ then the set

$$B := \{at \,|\, a \in A, t \in [0, 1]\} \tag{105}$$

which is produced by connecting all points from A to the origin 0 is a star region. Assume that there are polar coordinates $(\varphi, S) \in R^2$, where φ is an amplitude and $S \in [0, 2\pi)$ (with accuracy to shift) is the phase, so that $\varphi \in B$. Next, we assume that the function

$$\Phi := \varphi e^{iS} \to \Phi_1(\varphi) e^{i\Phi_2(S)} \tag{106}$$

is star convex. For instance, annulus is not a star domain, but ones may be prolonged to the star domain if we include the origin 0. Thus, we assume that the map Φ in the boundary conditions for the quantum problem may be considefred as the mao $\Phi : R^2 \to R^2$, which is constructed by a transformation (106).

It is very important that 'For conformal transformations the non-linearity problem is similar to the non-linearity problem in dimension-one' [15]. The mail questions as to whether the images of an initial neighborhood return infinitely, often to intersect the initial neighborhood or whether the iterated images wander off to accumulate elsewhere. This is the basic questions of topo logical dynamics.

For these conformal dynamical systems the linear distortion simplifies to the problem of determining the scalar multiples or conformal factors. It is a case so called abelian computations as for dimension 1. And we shall see common features of 1-dimensional systems and conformal systems in higher dimensions.

There is a rich supply of dynamical examples when all the derivatives encountered are scalar multiples of orthogonal transformations (i.e. conformal transformations). For example, these include all differentiable examples in a space of 1 real dimension, and all complex analytic examples on a 1 complex dimensional manifold.

When we use terms with the linear part of the distortion of an iterated composition (i.e. the so called tangent map), then there arise a difficult problem of non-linearity, the there are unbounded deviation of the iterated compositions at a neighborhood from any linear approximation.

We confined itself only by conformal transformations when the non-linearity problem is similar to the non-linearity problem of dimension 1. In this case, a natural measure of non-linearity of a transformation f is

$$L(f) = \ln f''. \tag{107}$$

Here,

$$L(f \circ g) = (L(f) \circ g \circ g' + L(g), \tag{108}$$

where $(\circ)$ is the composition of the corresponding mapping.

Note that in dimension 1 a natural measure of non - projectively of a transformation on f is the well - known Swartzian derivative

$$L(f \circ g) = (L(f) \circ g \circ g' + L(g), \tag{109}$$

Next, to understand distortion lemmas of C^2 Denjoy theory, we remined some well - known results. Let

$$g_n = f_n \circ f_{n-1} \circ ... \circ f_1(f \circ g) = (L(f) \circ g \circ g' + L(g), \tag{110}$$

be the non-linear geometric distribution in an iterated composition. Let f' a numerical composition which denotes a numerical measure of distribution that multiplies under composition.

Then linear distortion for conformal maps admits the estimation

$$(x_1, y_1)(g_i(x_0), g_i(y_0)), \tag{111}$$

where x_0 and y_0 are two initial points,

$$grad\,(\lg f_1)) \leq N \tag{112}$$

for each of generated transformationf_i and that

$$\sum_{i=0}^{n} \rho(x_i, y_i) \leq L \tag{113}$$

and

$$(x_1, y_1) = (g_i(x_0), g_i(y_0). \tag{114}$$

From these estimations it can be shown that

$$|\lg g_n'(x_0)/g_n'(y_0)| \leq NL \tag{115}$$

is bounded independently of n by LN. This is a main coroll for one - dimensional transformation. It means that $g_n \in \mathcal{C}$ implies that $g_{n=1} \in \mathcal{C}$, where $\mathcal{C}$ be a collection of compositions (or words).

20.1 1

From this main statement it can be proved the very important result: $\{x|\sum_{g_n\in\mathcal{C}} g'_n(x)<\infty\}$ is an open set (see, [15], Lemma 2).

21 Applications to the quantum boundary conditions

We begin for a case which has been proved by Denjoy: (i) C^2 diffeomorphism of the circle without a periodic point has only dense orbits (and thus by Poincare is topologically conjugate to a rotation).

Next, there is the following statement (Rosenberg and Sullivan): (ii) A complex analytic homeomorphism of a neighborhood of an invariant rectifiable closed curve in with no periodic points on the curve has only dense orbits on the curve.

From (ii) it follows that in principle we can solve the quatre problem for general analytic boundary conditions with the mapping $u \to \Phi(u)$, where $u \in$ which is idiomorphic to the conform mapping. Of coarse, it is possible only in very special cases (see, [?]). The case (i) can be also applied for the amplitude of the wave function.

To conclude, we considered the main statement: (iii) (Sacksteder): In a C2-codimension one foliation of a compact manifold by simply connected non-compact leaves all leaves must be topologically dense. The statements (i)–(iii) represent main results of topological dynamics. As noted by Sullivan [15]: ' There are also direct corollaries in these three cases relative to the natural one-dimensional 'measure and measurable dynamics.

22 Example 1. A Siegel disk

A Siegel disk is a stable regions which is cyclic and on which the appropriate power of f is analytically conjugate to a rotation of the standard unit disk. A situation that describes the boundary problem of quantum mechanics. Siegel [1942] proved these there occur near a non-hyperbolic periodic point if $1/R$ - argument of the derivative is far from the rationals. Far from the rationals means $|\theta - p/q| >> c/q^\nu$ for some $c > 0, v > 0$, and all p/q reduced fractions. The Siegel disk is a good candidate for the quantum boundary conditions.

23 Example 2. Transitive boundary conditions

Look for invariant sets. That is $S \subset X$ such that $f(S) = S$, where $f := \coprod \Phi_2$ in the decoupled boundary conditions in the Shrödinger equation for the phase. Decompose X into invariant sets $S_1, ..., S_k$ and wandering components (i.e. points that under a forward orbit approach one of the invariant sets and under backward iterates approach another set). Then a system (X, f) is transitive if there exists some $x \in X$ such that $\mathcal{O}^+(x)$ is dense in X, where $\mathcal{O}^+(x) = \{f^n(x)\,|\,n \in N\}$. This implies that there is a decomposition of the space into one invariant set and wandering components. In other words, the f is transitive if and only if for any open set U and V in there exist $n \in N$ such that $f^n(U) \bigcup V \neq$.

24 Behaviour of phase of wave function for the quantum problem

It is obvious that the evolution with time of phase of wave function repeat properties of a $\mathcal{C}^2$ - diffeomorphism Φ_2 for the separated boundary conditions on the circle. This diffeomorphism has an invariant Cantor set such that the complementary intervals have finite total length, and for the Cantor set these intervals wandering. The deviation of n - fold composition at points on interval are commensurable.

The main property of such diffeomorphism is that the total sum of derivatives along one orbit is comparable to the total sum of derivatives along one orbit to the total length of intervals and so is finite. It has been proved by Denjoy (see, [15]). In fact from the proof for an interval I about $x_0|(f^n)'y|$ is bounded by a constant times, and so it tends uniformly to zero. By recurrence at x_0 some $f_n(x_0)$ lies close to x_0. Thus we can assume $f_n(I) \subset I$ and we have a periodic point. This contradicts the assumption, proving f is topologically transitive.

25 Boundary conditions homeomorphic the circle

We recall the definition of external rays: For any compact and full subset $\mathcal{K} \subset \mathbb{C}$ consisting on more than a single point, there is a unique conformal isomorphism $\Phi : \mathbb{C} - \mathcal{K} \to \mathbb{C} - \mathcal{K}$ fixing ∞, normalising so as to have positive real derivatives at ∞. Inverse images of radial line in $\mathbb{C} - \bar{D}$ are called external rays, and an external ray at some angle θ is said to land if the limit $\lim_{r \to 1} \Phi^{-1}\left(re^{i\theta}\right)$ exist.

The impression of this external ray is the set of all limit points of $\Phi^{-1}\left(r'e^{i\theta'}\right)$ for $r' \to 1$ and $\theta' \to \theta$. rays on the forvard orbit of the ray pair. As noted in [5] and [?], we will denote external rays of the Multibrot set by parameter rays in order to distinguish them from dynamic rays of Julia sets. All the parameter rays at rational angles are known to land (see, Milnor [?]). Array pair is a collection of two external rays which land at a common point. A dynamics ray pair is characteristic if it separates the critical value from the critical point and from the other rays on the forward orbit of the ray pair. The landing point of a periodic or preperiodic dynamic ray pair is always on a repelling or preperiodic orbit. If a preperiodic dynamic is characteristic, then its landing is necessary on a repelling orbit.

From [5] and [?] (see, also, [31], Theorem 2.3) it follows Theorem about Correspondence of Ray Pairs: For every $d \geq 2$ and every unicritical polynomial $z \to z^d + c$ there are bijections:

• between the ray pairs in parameter space at periodic angles, separating 0 and c, and the characteristic periodic pairs in the dynamic plane of c landing at repelling orbit; and

• between the ray pairs in parameter space at preperiodic angles, separating 0 and c, and the characteristic periodic pairs in the dynamic plane of c.

• this bijection of ray pairs preserves external angles.

It is assumed that the separating ray pairs do not go through the point c (the critical value or the parameter). The critical value is never on a periodic ray pair because, depending on whether the corresponding Julia set is connected or not. It would either be periodic and thus super attracting and could not be the landing point of the dynamic rays.

All the rational rays landing at Misiurewicz cut the complex plane into as many open parts as that are rays. Note that Misiurewicz originally investigated maps in which all critical points are non-recurrent. That is, there is a neighbourhood of every critical point that is not visited. A parameter $\mathbb{C}$ be a Misiurewicz point $M_{k,n}$ if

$$f_c^k(z_c) = f_c^{(k+n)}(z_c) \tag{116}$$

and

$$f_c^{(k-1)}(z_c) = f_c^{(k+n-1)}(z_c) \tag{117}$$

for all $c \in M_{k,n}$, where k, n are positive integers. So for a quadratic polynomial a unique critical point is pre-periodic.

26 Polynomials of degree d

For any $P \in P_d$ define the real valued function h_p on $\mathcal{K}$ by

$$h_p(z) = \lim_{n\to\infty} \frac{\lg_+ |P^n(z)|}{d^n}, \tag{118}$$

where $\lg_+$ is the supremum of lg and 0.

Remind that key difference from the close connection between holomorphic and harmonic functions in the plane is the following: A real – valued function on $\Omega \subset \mathbb{R}^2$ is harmonic if and only if it is locally the real part of a holomorphic function. If a complex function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic, then there are Cauchy– Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{119}$$

or, equivalently, the Wintinger derivative is zero, i.e.,

$$f_{\bar{z}} = 0. \tag{120}$$

For example, the function $f(z) = |z|^2$ is complex differentiable at exactly one point ($z_0 = 0$). It is not holomorphic because there is no open set around 0 on which f is complex differentiable.

26.1 Mandelbrot and Multibrot Sets

We discuss polynomials of the form $z \to z^d + c$ with $z \in \mathbb{C}$ and $c \in \mathbb{C}$ of degree $d \geq 2$. We consider, up to normalization, only those polynomials which have a single critical point. We call these polynomials unicritical or unsingular. We define the Multibrot set of degree d as the correctness locus of these families (see, [31]).

Remind that the Julia set of $f(z) = z^2 + c$ is the set of all that $z \in \mathbb{C}$ where the behavior of iterates is 'chaotic'. The Fatou or set is the set of $z \in \mathbb{C}$ where iterations are 'normally'. The unit disk is thus the locus of chaotic behaviour, where $|z > 1|$.

When in the WKB - method the boundary functions $\varphi x, te^{iS}$, where S is real, we consider only separated boundary conditions

$$\varphi(0,t) = \Phi_1(\varphi(l,t)) \tag{121}$$

$$S(0,t) = \Phi_2(S(0,t)), \tag{122}$$

then the quantum problem is reduced to the canonical system of the two couple equations with uncoupled boundary conditions.

27 The classification of conformal boundary conditions

Here we classify boundary conditions for the Shrödinger equation with nonlinear boundary conditions that are produced in the problem by some nonlinear mapping $\Phi : \mathcal{C} \to \mathcal{C}$. Now let $f : \mathcal{C} \to \mathcal{C}$ be a rational map with iterates $f^n = f \circ f \circ ... \circ f$. We will assume that there are the Julia set J and the Fatou set Ω according to the dynamics of f. The Julia set is a closed, perfect set, defined as • the smallest closed set with $f^{-1}(J) = J$ and $|J| > 2$ • the set of accumulation points of any orbit $\Gamma(x) \subset \mathcal{C}$ • the closer of the set of repelling fixed points of $\gamma \in \Gamma$ • the set of points near which Γ does not form a normal family.

For us it is convenient to consider the Julia set J as the closer of the set of repelling periodic points of f because the dynamics of a single conformal endomorphism can exhibit similar structure as the dynamics of 1 - dimensional map on circle S^1 or an interval I. Then the Julia set has the similar structure as the so called separator of one - dimensional diffeomorphism which is defined as

$$D = \bigcup_{n \geq 0} f^{-n}(\bar{P}_-), \tag{123}$$

where $\bar{P}_-$) is the closer of repelling fixed points. It is a closed set of zero measure nowhere dence on I. It ca be finite (in particular, empty), countable or uncountable.

28 Iterations

The filled–in Julia set K_f is the polynomial like mapping is defines

$$K_f = \{z \in U' | f^{on}(z) \in U' \quad \text{for} \quad all\, n\}. \tag{124}$$

When $z \in K_f$ then we say that a point z in U' does not escape under iterations by f. Let P be cubic polynomial with critical points ω_1 and ω_2 such that $h_p(\omega_2) < h_p(\omega_1)$.

Here,

$$h_p(z) = \lim_{n \to \infty} \frac{1}{3^n} \ln_+ |P_{0n}(z)|. \tag{125}$$

is called by the potential. Then we can determine the filled–in Julia set as

$$K_p = \{z \in \mathcal{C} | h_p(z) = 0\}. \tag{126}$$

For any $z_0 \in \mathcal{C}$ let

$$U_p(z_0) = \{z \in \mathcal{C} | h_p(z) > h_p(z_0) = 0\} \tag{127}$$

so that

$$U_p(z_0) = \mathcal{C} - K_p \quad \text{if} \quad z_0 \in K_p. \tag{128}$$

Assume that the polynomial $P(z)$ has a critical point ω_1 which escape to infinity, that is $h_p(\omega_1) > 0$, and another critical point ω_2 escaping to infinity more slowly than ω_1, i.e. $h_p(\omega_2) < h_p(\omega_1)$. Note that $h_p(\omega_2) = \mathcal{C} - K_p$ if ω_2 does not escape. Then the boundary $\partial h_p(\omega_2)$ given by transcendental equation if $h_p(\omega_2) > 0$, and ones is given by fractal if $h_p(\omega_2) = 0$. They are fairly complected, depending on a complex number $\xi \in \mathcal{C} - \tilde{D}$, a real number $h < \ln |\xi|$ and many (sometimes infinitely) combinatorial date.

Note that polynomial P of degree d can be analytically conjugated to $z \to z^d$ at a neighboured of ∞. There is the conjugating map φ_p and polar coordinate at a neighbourhood ∞.

29 Symmetries of the Mandelbrot set

We define Mandelbrot set of degree d as a connectedness locus of these families, that is

$$\mathfrak{M}_d = \{c \in \mathbb{C} : \text{the Julia set of} \quad z \to z^2 + c \quad \text{is connected}\}. \tag{129}$$

Then $\mathfrak{M}_2$ is the familiar Mandelbrot set. It is known that all the Mandelbrot sets are connected and symmetric with respect to the real axes, and they have $(d-1)$ - fold rotation symmetries (see, [?]).

In [25], Alexander, Giblin and Newton describe the symmetry groups of certain Jukia set. As they call them, generalized Mandelbrot and Mandelbar sets. They give proves that these 'fractal' are invariant under certain symmetry groups and conjecture that other symmetry. All of these 'fractal' come from interacting polynomials of the type

$$z \to z^d + c \quad \text{or} \quad z \to \bar{z}^d + c \tag{130}$$

on the $\mathbb{C}$. For each $d \geq 2$, there are one holomorphic and one untiholomorphic family of polynomials.

If $|z| > max\,(|c|, 2^{1/(d-1)})$, then iterations toward infinity. The complex number 0 is the only critical point in $\mathbb{C}$. This is a unique point at which this map is not locally injective, i.e., where their derivatives vanish. This illustrate many important principle in complexe dynamics for a holomorphic map which is closely related to the orbits of the critical point.

By the symmetry of a subset we have a rigid motion of this subset into itself, orientation – preserving or not. Symmetries are translation, rotations, line reflections, and composition thereof. All such set to be compact. They therefor have unique (smallest) circumscribed circles which have to remain invariant under the symmetries. All rotations have a common center. We see that each of sets have a rotational symmetry around the origin (except $\mathfrak{M}_2$). Hence all their symmetries leave the origin fixed.

29.1 Symmetries of Julia sets

The filled - in Julia sets are closed sets. So their topological boundaries (the actual Julia sets) have exactly the same symmetries and it is suffices to investigate the symmetries of the filled Julia set. Two of them are shown in ([?], Figure 1).

If $c = 0$, then filled - in Julia set is the filled unit disk $|z \geq 1|$. For $c \neq 0$, the rotational symmetry of every filled - in Julia set $K_{d,c}$ or $K^*_{d,c}$ around the origin is exactly d - fold, so their symmetry groups are either $\mathbb{C}_d$ or D_d. In the holomorphic case $\mathfrak{K}_{d,c}$, there are reflection symmetries if and only if c^{d-1} is real. In the antiholomorphic case, reflection symmetries exist if and only if c^{d+1} is real. In both cases, if axes of reflection exist, there are of than of original angles. On of them is line trough 0 and c.

Further we consider the problem with accuracy $O(h^2)$, where $O(h^2) \to 0)$ as $t \to +\infty$, and we find solutions in the form

$$\psi(x,t,h) = e^{iS(x,t)/h}\varphi(x,t,h). \tag{131}$$

Here, $S(x,t)$ and $\varphi(x,t,h)$ are real phase and amplitude, respectively. Below, where it will not cause misunderstandings, h will be omitted.

Assume that $C^0([0,l] \times [0,+\infty)$ is the space of bounded continuous functions and C^2 is the space of twice differentiable functions with the norm $||f||_{C^2} = \sum_{k=0}^{2} \sup ||f^k||$, where $||f^0||$ is the norm in $C^0([0,l] \times [0,+\infty$. The function $\psi \in C^2$ belongs to if its real and imaginary parts belong $C^2([0,l] \times [0,+\infty$. Then in C^2 - norm there is the convergence

$$||S(x,t)||_{C^2} \Rightarrow ||\Phi_1[p_1(t-x/p)||_{C^2}, \tag{132}$$

where $p_1(\zeta)$ is $2^N l/p$ is some periodic piecewise constant function with finite number Γ of points of discontinuities on the period. Further

$$||\varphi(x,t)||_{C^2} \Rightarrow ||\Phi_2[p_1(t-x/p)||_{C^2}, \tag{133}$$

where $p_1(\zeta)$ is $2^N l/p$. By definition, $\Gamma = \varrho^{-1}(D)$, where $D = \bigcup_{n\geq 0} G^{-n}(A^{\pm})$, $A^{\pm}$ is a set of saddle points of codimension one and $\varrho(\zeta) = (S_0(\zeta), \varphi_0(\zeta))$ is an initial curve in R^2, which is determined by initial data of the boundary problem, and N is least common multiple of the map $G : (S,\varphi) \to (\Phi_1(S,\varphi), \Phi_2(S,\varphi))$ [22].

30 Method of reduction of problem to system of integro-difference equations

In these section, it will be shown that the boundary problem can be reduced to a system of transport equations for amplitudes $|\psi(x,t)$ and to a system of the Hamilton-Jacobi equations for the phase $S(x,t)$ with some boundary conditions. These equations may be coupled or uncoupled. We confined itself only a case when asymptotically (for large times) these equations are decomposed on two independent boundary problems for the amplitudes and the phases, respectively.

Indeed, substituting (131) into equation (75), we obtain that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi + (-ih)\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi \triangle S\right) + \frac{(-ih)^2}{2}\triangle\varphi = 0. \tag{134}$$

Next, we find solutions with accuracy $O(h^2)$ so that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi + (-ih)\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi \triangle S\right) = 0. \tag{135}$$

As a result, we obtain the classical Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 = 0 \tag{136}$$

$$\varphi^2_{x=0} = \Phi_2(S,\varphi)_{x=l}, \tag{137}$$

$$S_{x=0} = \Phi_1(S,\varphi)_{x=l}. \tag{138}$$

Let, for simplicity, $\Phi_2 := \Phi_2(\varphi)$ and $\Phi_1 := \Phi_1(S)$. Then we have two-point boundary conditions

$$S(0,t) = \Phi_1[S(l,t)]. \tag{139}$$

Similarly, we have the transport equation

$$\frac{\partial \varphi}{\partial t} + p\frac{\partial \varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\varphi = 0 \tag{140}$$

with the boundary conditions

$$\varphi(0,t) = \Phi_2[\varphi(l,t)]. \tag{141}$$

Here, the maps Φ_1, $\Phi_2 \in C^1(I \to I)$ are assumed structural stable , where I is an open closed interval. Note that the structural stable maps form an open dense subset (see, [22], p.233).

30.1 Reduction of the problem to integro-difference equations of the Volterra type

In order to solve these equations, we use the method of characteristics. Initially, we consider the Hamilton system of ODE with hamiltonian $H(x,p) = \frac{1}{2}p^2$ as

$$\dot{x} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial x} \tag{142}$$

with the initial conditions

$$x(0) = x_0, \quad p(0) = \frac{\partial S}{\partial x}(x_0) = p. \tag{143}$$

For a constant p, the function $x := x(p,t)$ is a solution of equation

$$p - \frac{\partial S(x,t)}{\partial x} = 0, \tag{144}$$

where p can be considered as additional coordinate in (x,p,t) - space. Then on characteristics $dx(p,t)=p$ we have the equation:

$$\frac{dS(x(t,p),t)}{dt}=\frac{\partial S(x(t,p),t)}{\partial t}+\frac{\partial S(x(t,p),t)}{\partial x}\frac{dx(t,p)}{dt}=-H(p)+p\frac{dx(t,p)}{dt}. \tag{145}$$

Next, by integration along characteristics $dx/dt=p$ with help of boundary conditions the problem can be reduced to the system of integro-difference equations:

$$S(x,t)=\Phi_1[S(x,t-x/p)]+\frac{p}{2}l+\frac{p}{2}(l-x), \tag{146}$$

$$\varphi(x,t)=\varphi(0,t-x/p)+\int_{t-x/p}^{t}\frac{\partial^2 S}{\partial x^2}[(p(s-t)+x,s]\varphi[(p(s-t)+x,s)]ds= \tag{147}$$

$$\Phi_2[\varphi(l,t-x/p)]+\int_{t-l/p}^{t-x/p}\frac{\partial^2 S}{\partial x^2}[(p(s-t+l/p)+x,s]\varphi[(p(s-t+l/p)+x,s)]ds] \tag{148}$$

$$+\int_{t-x/p}^{t}\frac{\partial^2 S}{\partial x^2}[(p(s-t)+x,s]\varphi[(p(s-t)+x,s)]ds. \tag{149}$$

Indeed, the first equation follows from the relation

$$S(x,t)=S(0,t-x/p)+\frac{p}{2}l=\Phi_1[S(l,t-x/p)]+\frac{p}{2}l=\Phi_1[S(x,t-l/p)]+\frac{p}{2}(l-x)+\frac{p}{2}l \tag{150}$$

which can be obtained from (145) with help of the boundary conditions for the phase.

Further, we define an initial function $S(x,0)=S(0,t-x/p)+\frac{p}{2}l$, where $S(0,t-x/p)$ is determined on interval $-l/p<t-x/p<0$ from initial data of the boundary problem. Define the map

$$\Phi_{1,\mu}=\Phi_1[S(l,t-x/p)]+\frac{p}{2}l+\frac{p}{2}(l-x), \tag{151}$$

where, $\mu:=\mu(l,x,p)$ and $\mu(l,x,p)=\frac{p}{2}l+\frac{p}{2}(l-x)$. As the above, we assume that the map $\Phi_{1,\mu}\in C^2(I,I)$ is structural stable. As a result, we obtain the difference equation

$$S(x,t)=\Phi_{1,\mu}[S(x,t-x/p)], \tag{152}$$

depending on x and $\mu\in R$ as parameters. Then solutions of the DE can be find, step by step, with help of iterations of the initial function $S_0(x,t)$, which is given on interval $-l/p<t-x/p<0$. Indeed, if $\Phi_{1,\mu}\in C^2(I,I)$, then, as noted the above, the map is structural stable. If a set $Fix\,(\Phi_{1,\mu})$ of fixed point is finite then there is a set of initial functions $S_0(t)$ on $[-l/p,0)$ such that solutions of the difference equation are asymptotic $2^{N_1}l/v$ - periodic piecewise constant functions with finite points of discontinuities on a period, as shown in [22], where, N_1 is least common multiple of periods of attractive circles of the map $(\Phi_{1,\mu})$. Further, from (153) it follows that the phase depends on $\zeta=t-x/p$. As a result, a limit function is $p_{1,\mu}(\zeta)$ as $\zeta\to+\infty$.

30.2 Reduction of the problem to transport equations

The second equation follows from the integration of transport equation (140) so that along characteristics we have the ODE

$$\frac{d\varphi}{dt} = -\frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi. \tag{153}$$

Integration of this equation from a point (x,t) to a point $(x, t-l/p)$ (see, Fig.3) with help of the boundary conditions for amplitude leads to equation (167). On the other hand, this equation has a solution

$$\frac{d\varphi}{dt} = -\frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi \quad \text{at} \quad dx/dt = p. \tag{154}$$

After multiplying on φ this equation can be written as

$$\varphi(x(t),t) = \varphi(x(t_0),t_0)e^{-\int_{t_0}^{t}\frac{\partial^2 S}{\partial x^2}[p(s-t)+x,s]\varphi[p(s-t)+x,s]ds}. \tag{155}$$

30.3 Determination of phase in the transport equation from the Hamilton-Jacobi boundary problem

Now we can find the phase $S(x,t)$ in this equation. Indeed,

$$S(x,t) = S(0, t-x/p) + \frac{1}{2}px. \tag{156}$$

Further, using the boundary conditions for the phase, from (170 we obtain that

$$S(l,t) = S(l,t) + \frac{p}{2}l = \Phi_1[S(l,t-l/p)] + \frac{p}{2}l. \tag{157}$$

This equation has asymptotic solution $S(l,t) \Rightarrow p_1(l,t)$. Then from (170) we obtain that $S(x,t) \Rightarrow \Phi_1[p_1(l,t-x/p)] + \frac{1}{2}px$. Then

$$p^2\frac{\partial^2 S}{\partial x^2} = \Phi_1''[S(l,t-x/p)][S'(l,t-x/p)]^2 + Phi_1'[S(l,t-x/p)][S''(l,t-x/p)] \tag{158}$$

where we used the relation

$$\frac{\partial^2 S}{\partial x^2}[p(s-t)+x,s] = S''(0,t-x/p) = \Phi_1''[S(l,t-x/p)][S'(l,t-x/p)]^2 + \Phi_1'[S(l,t-x/p)][S''(l,t-x/p)] \tag{159}$$

where $S(l,t-x/p)$ tends to a $2^{N_1}(t-x/p)$ periodic function $p_1(l,t-x/p) \in A_1^+$, where A_1^+ is a set of attractive points of the map Φ_1. (Here, the index μ is omitted in the map $Phi_{1,\mu}$). Then we obtain the equation

$$\varphi(x(t),t)=\varphi(x(t_0),t_0)e^{-S''(0,t-x/p)\int_{t_0}^{t}\varphi[p(s-t)+x,s]ds}. \tag{160}$$

1.1hf"

We assume that $S(l,t) = A_1^+ + \varepsilon S(\tilde{l},t)$, where $\varepsilon > 0$ is a small parameter. Then lumbarization of difference equation (170) at each point $a \in A_1^+$ leads to the equation

$$S(\tilde{l},t)=\Phi_1'(a)S(l,t\tilde{-}l/p) \tag{161}$$

where $|\Phi_1'(a)| < 1$. Solutions of this equations are $S(l,t) = e^{kt}$, where $k = \frac{p}{l}\ln|\Phi_1'(a)|$. Equation (161) has exponentially decreasing explicit solutions, which tends to zero as $t \to +\infty$. Hence, at a neighbourhood of the point $a \in A_1^+ = 0$ equation (160) can be written as

$$\varphi(x(t),t)=\varphi(x(t_0),t_0)\exp -k^2e^{k(t-x/p)}\int_{t_0}^{t}\varphi[p(s-t)+x,s]ds. \tag{162}$$

Since $k < 0$, we can us the approximation $e^z \approx 1 - z$ and rewrite equation (**??**) in the form

$$\varphi(x(t),t)=\varphi(x(t_0),t_0)(1+k^2e^{k(t-x/p)}\int_{t_0}^{t}\varphi[p(s-t)+x,s]ds). \tag{163}$$

Then, in class of functions $\varphi \in C^2$, the last term in equation (163) can be neglected, because

$$|k^2e^{k(t-x/p)}\int_{t_0}^{t}\varphi[p(s-t)+x,s]ds|\le k^2e^{k(t-x/p)}(t-t_0)M, \tag{164}$$

where $M = \sup|\varphi(x,t)|$ at $(x,t) \in [0,l] \times R^+$, and $k^2e^{k(t-x/p)}(t-t_0) \to 0$ as $t \to +\infty$. It means that the function $\varphi(x(t),t) = \varphi(x(t_0),t_0)$ asymptotically. As a result, we have

$$\varphi(l,t)=\varphi(0,t-l/p)=\Phi_2[\varphi(l,t-l/p)]. \tag{165}$$

This equation has asymptotically $2^{N_1}l/p$ - periodic piecewise constant solutions $p_2(t) \in A_2^+$, where A_2^+ is a set of attractive points of the map $\Phi_2 \in C^2(I,I)$.

This statement can be proved as follows. Integrating the transport equation, we obtain the integro-difference equation

$$\varphi(l,t)=\Phi_2[\varphi(l,t-l/p)]+\frac{1}{2}\int_{t-l/p}^{t}\frac{\partial^2 S}{\partial x^2}[(p(t-s)+l,s]\varphi[(p(t-s)+l,s]ds. \tag{166}$$

Then

$$\varphi(x,t) = \varphi(0, t - x/p) + \int_{t-x/p}^{t} \frac{\partial^2 S}{\partial x^2}[(p(s-t)+x, s]\varphi[(p(s-t)+x, s)]ds = \tag{167}$$

$$\Phi_2[\varphi(l, t - x/p)] + \int_{t-l/p}^{t-x/p} \frac{\partial^2 S}{\partial x^2}[(p(s-t+l/p)+x, s]\varphi[(p(s-t+l/p)+x, s)]ds]$$

$$+ \int_{t-x/p}^{t} \frac{\partial^2 S}{\partial x^2}[(p(s-t)+x, s]\varphi[(p(s-t)+x, s)]ds.$$

Let $S \in P_1^+$, $\varphi \in P_2^+$, where P_1^+, P_2^+ are sets of attractive fixed points. Let $S = P_1^+ + \varepsilon\tilde{S}$ and $\varphi = P_1^+ \varepsilon\tilde{\varphi}$. Substituting these functions into equation (178), we obtain

$$\tilde{\varphi}(l,t) = \Phi_2'(P_2^+)\varphi(l, t \,\tilde{-}\, l/p) + \frac{P_2^+}{2}\int_{t-l/p}^{t} \frac{\partial^2 \tilde{S}}{\partial x^2}[(p(t-s)+l, s]ds. \tag{168}$$

Further, from the relation

$$S(x,t) = S(0, t - x/p) + \frac{p}{2}x. \tag{169}$$

it follows that

$$\frac{\partial S}{\partial x}(x,t) = \frac{1}{p}S'(0, t - x/p) + \frac{p}{2}, \tag{170}$$

$$\frac{\partial^2 S}{\partial x^2}(x,t) = \frac{1}{p^2}S''(0, t - x/p). \tag{171}$$

Then from (172) we arrive at

$$\tilde{S}](x,t) = \Phi_2'[\tilde{S}(x, t - l/p)]\tilde{S}'(x, t - l/p). \tag{172}$$

Now, we assume that $\tilde{S} = S + \varepsilon\hat{S}$. Then for $\hat{S}$ we have the linear difference equation

$$\tilde{S}(x,t) = \lambda_1\tilde{S}'(x, t - l/p) \tag{173}$$

where $\lambda_1 = \Phi_1'(P_1^+)$. Since $\tilde{S}(x,t) = \tilde{S}(0, t - x/p) + \frac{2}{p}x$, we obtain that a solution of the equation is $\tilde{S}'(x,t) = \tilde{S}'(0, t - x/p) + \frac{2}{p}$. Hence

$$\tilde{S}'(0, t - x/p) = \lambda_1\tilde{S}'(0, t - x/p - l/p). \tag{174}$$

Let $\zeta = t - x/p)$. Then from (174) we get

$$\tilde{S}'(0, t - x/p) = \lambda_1\tilde{S}'(0, t - x/p - l/p). \tag{175}$$

A solution of the equation is $S'(\zeta) = e^{k\zeta}$, where $k = \frac{2}{p}\ln\lambda_1|$, where $|\lambda_1| < 1$. Further

$$\frac{\partial^2 \tilde{S}}{\partial x^2}[(p(t-s)+l, s] = \frac{1}{p^2} S''(0, t - x/p), \tag{176}$$

where $x(s,t) := p(t-s)+l$. Then $t - x/p = s - l/p$. As a result, equation (178) can be reduced to the equation

$$\tilde{\varphi}(l,t) = \Phi_2'(P_2^+)\varphi(l, t \tilde{-} l/p) + \frac{P_2^+}{2}\frac{1}{p^2} e^{k(t-l/p)}(1 - e^{kl/p}. \tag{177}$$

Now we introduce a variable $y(t) = e^{kt}$. Then $y(t) = e^{kl/p} y(t - l/p)$. As a result, we obtain the system of two difference equations

$$\tilde{\varphi}(l,t) = \lambda_2 \tilde{\varphi}(l, t - l/p) + \frac{P_2^+}{2}\frac{1}{p^2}(e^{-kl/p} - 1) y(t - l/p), \tag{178}$$

$$y(t) = e^{kl/p} y(t - l/p), \tag{179}$$

where $\lambda_2 = \Phi_2'(P_2^+)$. Eigenvalues of this system are $\chi_1 = \lambda_2$, $\chi_2 = e^{kl/p}e^{kl/p}$, where $|\chi_1| < 1$, $0 < \chi_2 < 1$. Hence, the fixed point $(\tilde{\varphi}, y) = (0,0)$ attracts all trajectories of the dynamical system. It means that solutions of the boundary problem are asymptotically stable. The stability is proved only for a special class of initial functions $\mathfrak{H} = \{\tilde{\varphi}(t) \in O_\delta(P_2^+),\ y(t) \in O_\delta(P_1^+, t \in [-l/p, 0)\})$, where $O_\delta(a)$ is a small neighbourhood of a point a.

30.4 Separatrix and saddle type fixed points of codimensional 1

If there are repelling fixed points P_2^-, then in R^2 there is a fixed point $P^\pm$ of saddle type. In this case, there is a separatrix, which divides a plain on the two regions U_1^+, U_1^+ such that trajectories from U_1^+ are attracted by a fixed point A_1^+. The trajectories from U_2^+ are attracted by a fixed point A_2^+. This is a typical property of the hyperbolic dynamical system (see, Fig.1).

Thus, the attractor of the boundary problem contains asymptotic $2^{N_1} l/p$, $2^{N_2} l/p$ - periodic piecewise constant functions $p_1(t)$, $p_2(t)$ for phases and amplitudes, respectively, if initial data S_0, $\varphi_0 \in \mathfrak{O}_\sigma(P_1^+, P_2^+$, where σ is small. If S_0, $\varphi_0 \in \mathfrak{O}_\sigma(P_1^-, P_2^+$ then the fixed point $a^\pm$ is a saddle type with unstable manifold $W^u a^\pm$ of codimension 1. If initial curve $S_0(t)$, $\varphi_0(t)$ intersect the manifold $W^u a^\pm$ at a point $t_0 \in [-p/l, 0$ transversally then the point determines a set $\mathfrak{d}$ of points of 'discontinuities' on the period of the limit functions $p_1(t)$, $p_2(t)$.

31 General functional boundary conditions

If we consider the boundary conditions of the form

$$\varphi_{x=0} = \Phi_2(S,\varphi)_{x=l}, \quad S_{x=0} = \Phi_1(S,\varphi)_{x=l} \tag{180}$$

then the problem can be reduced the integro-difference equations

$$S(l,t) = \Phi_1[S(l,t-l/p),\varphi(l,t-l/p). \tag{181}$$

$$\varphi(l,t) = \Phi_2[S(l,t-l/p),\varphi(l,t-l/p) - \frac{1}{2}\int_{t-l/p}^{t} \frac{\partial^2 S}{\partial x^2}[(p(s-t)+l,s]\varphi[(p(s-t)+l,s]ds. \tag{182}$$

Since

$$S(x,t) = S(0,t-x/p) + \frac{p}{2}x, \tag{183}$$

from (183) and parametrization $x(s,t) := p(s-t)+l$ we obtain that

$$\frac{\partial^2 S}{\partial x^2}(x(s,t),t) = \frac{\partial^2 S}{\partial x^2}(0,t-x(s,t)/p)) = \frac{1}{p^2}S''(0,t-l/p). \tag{184}$$

Then relation (182) can be rewritten as

$$\varphi(l,t) = \Phi_2[S(l,t-l/p),\varphi(l,t-l/p)] - \frac{1}{2p^2}S''(0,t-l/p)\int_{t-l/p}^{t} \varphi[(p(s-t)+l,s]ds. \tag{185}$$

Next, from (185) it follows that

$$\begin{gathered}\varphi'(l,t) = \Phi_2'[S(l,t-l/p),\varphi(l,t-l/p)]\varphi'(l,t-l/p) - \\ \frac{1}{2p^2}S'''(0,t-l/p)\int_{t-l/p}^{t}\varphi[(p(s-t)+l,s]ds + S''(0,t-l/p)[\varphi(l,t)-\varphi(0,t)]\end{gathered} \tag{186}$$

where $(')$ is the derivative along the direction $dx(t)/dt = p$.

Further, we assume that $S(x,t)$ is a map which is homeomorphic to the quadratic map. Then $S'''(0,t-l/p) \equiv 0$, (187) we obtain the relation

$$\begin{gathered}\varphi'(l,t) = \Phi_2'[S(l,t-l/p),\varphi(l,t-l/p)]\varphi'(l,t-l/p) - \\ \frac{1}{2p^2}S''(0,t-l/p)[\varphi(0,t)-\varphi(0,t-l/p)].\end{gathered} \tag{187}$$

Now, using the functional boundary conditions $\varphi(0,t) = \Phi_2[S(l,t),\varphi(l,t)]$, we obtain

$$y'(l,t) = S''(0,t-l/p)y'(l,t), \tag{188}$$

where

$$y(l,t) = \varphi(l,t) = \Phi_2[S(l,t-l/p), \varphi(l,t-l/p)]. \tag{189}$$

Equation (190) has the solution

$$y(l,t) = y(l,t_0)e^{S''(0,t-l/p)}(t-t_0). \tag{190}$$

Next, we assume that $S''(0,t-l/p) \leq 0$ for each $t \in [-l/p, 0)$. Then $y(l,t) \to 0$ as $t \to \infty$, $y(l,t) = y(l,t_0)$ if the phase is linear function. Then asymptotically the problem is reduced to the coupled difference equations

$$S(l,t) = \Phi_1[S(l,t-l/p), \varphi(l,t-l/p)] + \frac{pl}{2}. \tag{191}$$

$$\varphi(l,t) = \Phi_2[S(l,t-l/p), \varphi(l,t-l/p)] \tag{192}$$

Rigorous proof can be found in book [22]) for difference equations of the form

$$\frac{d}{dt}[\omega(u(t), \omega(u(t+1))] = F[\omega(u(t), \omega(u(t+1))], \tag{193}$$

which are integrable. Indeed, the operator of (194) can be decomposed on the product of a difference operator $\mathfrak{F}[y(t)] := \omega(u(t), \omega(u(t+1)$ and the differential operator $\mathfrak{D}[F] := y(t) - F[y(t)]$. It means that each equation of the form (194) can be reduced to a family of non-autonomic difference equations

$$\omega(u(t), \omega(u(t+1))] = v(t, \lambda), \tag{194}$$

where $v(t,\lambda)$ is a solution of equation $v' - F(v) = 0$, depending on a parameter $\lambda = v(0)$ (see, [22], p.163).

Asymptotic behaviour of difference equations (191),(192) is known [30, 33]. If the map $G : R^2 \to R^2$, which is produced by these equations, has a finite number of fixed points A then functions $S(l,t)$, $\varphi(l,t)$ tend to an asymptotic $2^N l/p$ periodic piecewise constant functions $p_1(t) \in A$ and $p_2(t) \in A$ for almost all points $t \in (-l/p, \infty)$, excluding finite or infinite points of discontinuities (see, Fig.2). We know that it is possible if the map G is hyperbolic. That is a spectrum of a differential $T(G)$ has no real points with values equal in modulus 1. If also stable manifolds $W^s(A^+)$ intersect unstable manifolds $W^u(A^-)$ and $W^u(A^\pm)$ transversally and an initial curve $(S_0(-l/p,0), \varphi(-l/p,0))$ intersect an unstable manifold $W^u(A^\pm)$ of codimension 1 then G is structural stable and hyperbolic. Here, A^+ is a set of attractive points of G and A^- is a set of repelling points, and $A^\pm$ is a set of saddle type points of codimension 1.

It is known that a structure of attractor of dynamical can be described by a structure non-wandering points of dynamical system. Let us define a set of non-wandering points

$\Omega(G) = Per(G) = Fix(G^N)$, where $Per(G)$ is a set of periodic points, $Fix(G)$ is a set of fixed points and N is least common multiple of attractive circle of the map G. Let $W = \bigcup_{a\in\Omega(G)} W^s(a)$, where $W^s(a) = \{u \in W :\ \lim_{m\to+\infty} G^{mN}(u) = a\}$ is a stable manifold of a fixed point a of G. Particulary, for any point $u \in W$ there is the finite limit

$$\lim_{j\to+\infty} G^{Nj}(u) := G^*(u). \tag{195}$$

Let $u(t) = (S(t), \varphi(t)) \in C^0(R^+ \to R^2)$.

Next, we define a set of initial functions

$$\mathfrak{H} = \{h(t) \in C^0([-l/p, 0), R^2) :\ h(0) = G[h(0)]\}. \tag{196}$$

Then for each function $h(t) \in \mathfrak{H}$ there is periodic piecewise constant function $p^*[h(\cdot)] :\ R^+ \to \Omega(G)$ such that

$$p^*[h(t)] = G^i[G^*(h(t-i))] = G^*[G^i(h(t-il/p))], \tag{197}$$

where $t \in [i, i + l/p)$, $i = 0, 1, ...$ $p^*[h(t)]$ is constant if and only if $h(t) \in \mathfrak{H}'$, where

$$\mathfrak{H}' = \bigcup_{a\in Fix(G)} \mathfrak{H}_a, \tag{198}$$

where

$$\mathfrak{H}_a = \{h(t) \in \mathfrak{H} :\ h(t') \in W^s(a),\ t' \in [-l/p, 0)\}. \tag{199}$$

Here, $\mathfrak{H}_a \neq \phi$, where ϕ is empty set if and only if $a \in Fix(G)\}$.

Then each solution $u(t$ of a system of difference equations with initial functions $u(t)_{[-l/p,0)} \in \mathfrak{H}_a$ tends to a constant a if $t \to +\infty$. Each solution $u(t)$ with initial functions $u(t)_{[-l/p,0)} \in \mathfrak{H}/\mathfrak{H}'$ is asymptotic periodic piecewise constant function so that

$$\lim_{j\to+\infty} ||u(t' + Nj) - p^*[h(t')]||_{R^2} = 0, \tag{200}$$

where $t' \in R^+$.

Further, for any $\varepsilon > 0$ and any solution $u(t$ such that $u(t)_{[-l/p,0)} \in \mathfrak{H}_a$ we have the limit:

$$\lim_{j\to+\infty} \sup ||u(t + Nj) - p^*[u(t')]||_{R^2} = 0. \tag{201}$$

Note that

$$\lim_{j\to+\infty} \sup ||u(t + Nj) - p^*[u(t')]||_{R^2} \neq 0. \tag{202}$$

Thus, asymptotic solutions have the form

$$\psi(x,t) = e^{ip_1(t-x/p)/h} p_2(t - x/p)) + O(h^2), \tag{203}$$

where $O(h^2) \to 0$ as $h^2 \to 0$. For application, such type solutions describe distributions of order parameter for the linear Ginzburg-Landau equation with nonlinear boundary conditions (see,[?] p.270).

32 Boundary problem for quantum equation of general type

In this section, we consider the Shr$\ddot{o}$dinger type equation with a symbol-polyoma:

$$-ih\frac{\partial y}{\partial t} + P_n\left(-ih\frac{\partial y}{\partial x}\right) = 0. \tag{204}$$

Here

$$P_n(P) = \sum_{j=0}^{n} a_j p^j, \tag{205}$$

where $a_j \in R$ and $p^j \in C$, S is the complex space, $h > 0$ is the small parameter.

Solutions will be find in the form

$$y(x,t) = \exp\frac{i}{h} S(x,t)\varphi(x,t). \tag{206}$$

Then, substituting (206) into equation (204), we obtain the equation

$$\exp\frac{i}{h}S(x,t)\left[\left(\frac{\partial S}{\partial x} - ih\frac{\partial \varphi}{\partial t}\right) + P_n\left(\frac{\partial S}{\partial t} - ih\frac{\partial \varphi}{\partial x}\right)\right] = 0. \tag{207}$$

Coefficients of (207) can be fined from the function

$$F(h) = (\lambda_2 - ihE') + P_n(\lambda_1 - ihp') \tag{208}$$

by the formula

$$F(h) = \sum_{k=0}^{n} \frac{h^k}{k!}\frac{d^k}{dh^k}F(h). \tag{209}$$

Then calculations lead to the equation

$$\left(\frac{\partial S}{\partial t} + P_n\left(\frac{\partial S}{\partial x}\right)\right)\varphi - ih\left(\frac{\partial \varphi}{\partial t} + \frac{\partial P_n}{\partial p}\left(\frac{\partial S}{\partial x}\right)\frac{\partial \varphi}{\partial x}\right) + \sum_{k=2}^{n}\frac{-ih^k}{k!}\frac{\partial P_n}{\partial p}\frac{\partial^k \varphi}{\partial x^k} = 0. \tag{210}$$

As a result, we get the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + P_n\left(\frac{\partial S}{\partial x}\right) = 0, \tag{211}$$

for phases S, and the transport equation

$$\frac{\partial \varphi}{\partial t} + \frac{\partial P_n}{\partial p}\left(\frac{\partial S}{\partial x}\right)\frac{\partial \varphi}{\partial x} = 0, \tag{212}$$

for the amplitudes φ.

For equation (211) with Hamiltonian $H(p) = P_n(p)$ we consider the Hamilton system of ODE

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = \frac{\partial H}{\partial x} \tag{213}$$

with initial conditions

$$x(0) = x_0, \quad p(0) = \frac{\partial S_0(x)}{\partial x}_{|x=x_0}. \tag{214}$$

Let $x(x_0, t)$, $p(x_0, t)$ are solutions of the Hamilton system. Then equations $x = x(x_0, t)$, $p = p(x_0, t)$ define a manifold $\mathfrak{L}$ in space (x, p, t). If from $x = x(x_0, t)$ it follows that $x_0 = x_0(x, t)$ then the projection of $\mathfrak{L}$ on (x, t) - space is a solution of the Hamilton-Jacobi equation in (x, t) - space.

To solve the Hamilton-Jacobi equation, we use a method of characteristics, noting that along the characteristics $dx(t)/dt = \dot{H}_p H$ solutions of the Hamilton-Jacobi equation can be written as

$$\frac{dS(x(t), t)}{dt} = \frac{\partial S}{\partial t} + pdx = -H(p) + pdx. \tag{215}$$

Remark that there is another interpretation of relation (215). Indeed, let us consider a set of points (x, t), which is given by relations (214), where $x_0 \in R$. Then we can find a solution $S(x, t)$ on surface L so that the function $x(p, t)$ is a solution of equation

$$p - \frac{\partial S(x, t)}{\partial x} = 0 \tag{216}$$

because if on $\mathfrak{L}$ take place the equality $\frac{\partial S(x,t)}{\partial x} = p$ then the solution $x = x(p, t)$ is the transition function from coordinates (p, t) to coordinates (x, t) on the surface $\mathfrak{L}$.

We recall that the function of an action $S(q, t)$ is the integral

$$S_{q_0, t_0}(q, t) = \int_{\varpi} L dt \tag{217}$$

along the extremal ϖ, connecting points q_0, t_0 and q, t, where $L = p\dot{q} - H$, is the Lagrangian of a dynamical system. Thus, the Lagrangian is the Legendre transformation of the hamiltonian H (see, [?], p.210).

Next, we consider the Hamilton system in a space $R^1_x \bigoplus R_{1p} \bigoplus R^1_t$. Let $x(x_0, t)$, $p(x_0, t)$ be a solution of the Hamilton system. Then equations

$$x = x(x_0, t), \quad p = p(x_0, t) \tag{218}$$

determines a manifold $\mathfrak{L}$ of dimension $n = 2$ in the $R^1_x \bigoplus R_{1p} \bigoplus R^1_t$ with the boundary $\{t = 0\}$. At this manifold the form

$$-H(x, p, t)dt + pdx = \Omega \tag{219}$$

is closed. Remained that a form is closed if $d\Omega = 0$. Additionally, if the manifold $\mathfrak{L}$ is connected then Ω is exact. Then $\Omega = d\Lambda$ where Λ is a differential form.

Next, interpreting the Hamiltonian system as a vector field (dynamical system) at a symplectic manifold $\mathfrak{M}$, we can consider solutions of the Hamiltonian system as integral trajectories of a vector field. There is the important question of the solvability of the equation $\Omega = d\Lambda$, where we assume that Λ is connected with respect to the topological structure of the manifold $\mathfrak{M}$. Then a solution of the equation

$$dS' = \Omega_{\mathfrak{L}'}, \quad S'_{t=0} = S_0(x) \tag{220}$$

exists. Indeed, let U be an open neighbourhood in the space $R^1_x \bigoplus R^1_t$ of the set $\{t = 0\}$, where U is a projection of a manifold $\mathfrak{L}$ such that equation $x = x(x_0, t)$ is solvable with respect to x_0, that is $x_0 = x_0(x, t)$.

32.1 Reduction to canonical system

From the above observation it follows thet the function

$$S(x, t) = S'(x_0 = x_0(x, t), t) = (\pi_x^{-1})^* S'(x_0, t), \tag{221}$$

is a solution in U of the initial problem. Here, $\pi_x : \mathfrak{L} \to R^1_x \bigoplus R^1_t$ is a projection (see, [20], p.25).

In this case, from (210) it follows that the origin initial problem for equations of quantum mechanics is reduced (with accuracy $O(h^2)$) to a system of the Hamilton-Jacobi and transport equations:

$$\frac{\partial S}{\partial t} + P_n\left(\frac{\partial S}{\partial x}\right) = 0, \tag{222}$$

$$\frac{\partial \varphi}{\partial t} + \frac{dP_n}{dp}\left(\frac{\partial S}{\partial x}\right)\frac{\partial \varphi}{\partial x} = 0. \tag{223}$$

These equations are called by the canonical system.

Here, the phase S on manifold L satisfies to the differential form

$$d\hat{S} = -Hdt + pdx \tag{224}$$

where the Hamiltonian $H(p) := P_n(p)$. Then from (224) we arrive at

$$S(x(t), t) = S_0(x(t_0), t_0) - P_n(p)(t - t_0) + p[x(t) - x(t_0)]. \tag{225}$$

Now we consider the functional boundary conditions

$$S(0, t) = \Phi_1[S(l, t)], \quad t > 0, \tag{226}$$

where $\Phi_1 : I \to I$ is a function. Smooth initial conditions are

$$S(x, 0) = S_0(x), \quad 0 < x < l. \tag{227}$$

In applications, such boundary conditions describe the changing between input $S(0, t)$ and output phases in the electronic device the multiplier of the phase N and the amplifier J of a frequency of the input signal at $x = l$. Additionally, there is the resistor R. The device is designed so that we can change independently the phase and amplitude. The map Φ_1^{-1} describes coupling between input and output phases at $x = l$ and $x = 0$, respectively. Similarly, we can organise connection between input and output amplitudes at $x = l$ and $x = 0$. It is possible a case, when such coupling exists for input (output) phases and amplitude together. In the first case, the problem is reduced to the two independent difference equations for phases and amplitudes, separably. In the second case, the problem is reduced to the two coupled difference equation. As a result, the problem is reduce to the research of asymptotic behaviour of trajectories of $1D$ decouple dynamical systems or $2D$ couple systems [**?**, **?**].

32.2 Asymptotic behaviour of difference equations

From the above sections it follows that solutions of boundary problem (223), (229) can be find by the method of characteristic, which together with boundary conditions leads to an initial problem for difference equations with delay arguments. Indeed, from relation (229) we arrive at

$$S(l, t) = S(0, t - l/\dot{P}_n(p)) - P_n(p)(l/\dot{P}_n(p)) + pl = \Phi_1[S(l, t - l/P_n(p)) + \mu, \tag{228}$$

where $\mu = -P_n(p)(l/\dot{P}_n(p)) + pl$ is a parameter of the problem. We rewrite this difference equation as

$$S(l, t) = \Phi_1[S(l, t - l/P_n(p)) + \mu). \tag{229}$$

Solutions of this equation can be find step by step if we know the initial function $S_0(l, t)$ on the interval $t \in [-l/p, 0)$.

We assume that a map Φ_1 is structural stable. It is known that if $\Phi_1 \in C^2$ then such maps are dense in C^2 - metric. Partially, we consider a hyperbolic map. In $1D$ case, a set of periodic points of the map Φ_1 is the set $Per\,\Phi_1 = P^+ \bigcup P^-$, where a set of attractive fixed points P^+ is always finite. The set P^- of repelling fixed points is finite, countable or countable. The main role in topology of trajectories plays the so-called separator D, which is defined by the formula

$$D = \bigcup_{n \geq 0} h^{-n}(\bar{P}^-)z, \tag{230}$$

where $\bar{P}^-$ is closer of P^-. The separator D may be finite if P^- is finite or infinite (countable or uncountable). We call corresponding limit solutions by distributions of relaxation, pre-turbulent or turbulent type, respectively.

32.3 Transport equation

In this section we consider the transport equation (223). Denote $\frac{dP_n}{dp} := \dot{P}_n(p)$. Solutions will be find as $\varphi(x,t) := u(t - x/dotP_n(p))$. The boundary conditions of the problem are

$$\varphi(0,t) = \Phi_2[\varphi(l,t)], \quad t > 0. \tag{231}$$

The initial conditions are

$$\varphi(x,0) = \varphi_0(x), \quad 0 < x < l. \tag{232}$$

The problem can be reduced to the difference equation

$$\varphi(l,t) = \Phi_2[\varphi(l, t - l/P_n(p))]. \tag{233}$$

Indeed, if $\Phi_2 : I \to I$ hyperbolic then asymptotic solutions of (233) represent piecewise constant $2^N l/P_n(p)$ - periodic function $p_2(l,t)$, where $p_2(l,t) \in P_2^+$. Here, P_2^+ is a set of attractive fixed points of Φ_2. A solution can be written as

$$y(x,t) = e^{\frac{i}{h} p_1(\zeta)} p_1(\zeta), \quad \zeta = t - l/\dot{P}_n(p). \tag{234}$$

Here $p_1(\zeta) \in P_1^+$, $p_2(\zeta) \in P_2^+$, where P_1^+, P_2^+ are fixed attractive points of the maps Φ_1, Φ_2. Set of points of discontinuities (see, Fig.1) is determined by separators D_1, D_2 of Φ_1, Φ_2.

Let us initially postulate the simplest conditions

$$y(0,t) = e^{\frac{i}{h} S(0,t)} \varphi(0,t), \quad y(l,t) = e^{\frac{i}{h} S(0,t)} \varphi(0,t) \tag{235}$$

with additional asymptotic

$$y(0,t) \Rightarrow e^{\frac{i}{h} p_1(0,t)} p_2(0,t)), \quad y(l,t) \Rightarrow e^{\frac{i}{h} p_1(l,t)} p_2(l,t)) \tag{236}$$

as $t \to \infty$ for almost all points $t \in R^+$.

On the other hand, it can be considered the boundary conditions

$$y(0,t) = \Phi[y(l,t)] \tag{237}$$

where $\Phi : I \to I$.

Next, from

$$S(l,t) = S(0, t - l/\dot{P}_n(p)) + \mu, \quad \varphi(l,t) = \varphi(0, t - l/\dot{P}_n(p)) \tag{238}$$

it follows that (237) can be written as

$$e^{\frac{i}{h}S(0,t)}\varphi(0,t) = \Phi\left[e^{\frac{i}{h}(S(0,t-l/\dot{P}_n(p))+\mu)}\varphi(0,t-l/\dot{P}_n(p))\right]. \tag{239}$$

For simplicity, we assume that quantization equations $e^{i\frac{\mu}{h}} = 1$ are satisfied, and define $z(t) = e^{\frac{i}{h}S(0,t)}\varphi(0,t)$. Then equation (240) can be written as

$$y(t) = \Phi[y(t-\Delta)]. \tag{240}$$

32.4 Example 1

Indeed, consider a case $G : (Re\, y, Im\, y) \to (2\tan Re\, y, Re\, y + \frac{1}{2}Im\, y)$. Then the system produce a map $\Phi : R^2 \to (-2,2) \times R$. Note that there is a set Π such that $G : \bar{\Pi} \to \Pi$. Here, $G = (-a,a) \times (-b,b)$ and $a \in (\alpha, 2)$, where $\alpha > 0$ is solution of equation $2\tan Re\, y = Re\, y$ and $b > 2a$.

Then a set of non-wandering points is $\Omega(G) = \{(-\alpha,-2\alpha),(0,0),(-\alpha,-2\alpha)\}$, where $\alpha \approx 1.8$ [30]. Attractive fixed points of the map G are $A^+ = \{(-\alpha,-2\alpha),(\alpha,2\alpha)\}$, and a saddle point is $A^\pm = \{((0,0)\}$. Eigenvalues $(2,1/2)$ correspond to eigenvectors are $(3,2)$, $(0,1)$. The vectors are tangential concern separatrices $W^u(0,0)$, $W^s(0,0)$, respectively. Similarly, differentials $T(\Phi)(\alpha,2\alpha)$, $T(G)(-\alpha,-2\alpha)$ have eigenvalues $(2/\cosh^2\alpha, 1/2)$ with eigenvectors $\left(1, \left(\frac{2}{2/\cosh^2\alpha} - \frac{1}{2}\right)^{-1}\right)$ and $(0,1)$. Stable manifolds are $W^s(\alpha,2\alpha) = \{(Re\, y, Im\, y) \in G : Re\, y \in (0,a), Im\, y) \in (-b,b)\}$ and $W^s(-\alpha,-2\alpha) = \{(Re\, y, Im\, y) \in G : Re\, y \in (-b,b)\}$.

Let $h(t)$ be such that $h(0) \in W^s(-\alpha,-2\alpha)$, $h(t_1), h(t_2) \in W^s((0,0))$, where $t_1 < t_2$ and $t_1, t_2 \in (-\Delta, 0)$. The set of points of discontinuities $\Gamma(h(t)) = \{t_1,t_2\}$. As a result, solutions of the system tend to a periodic piecewise constant function, as shown on Figure 4. A limit function $\mathfrak{P}(h(t)) = (\pm\alpha, \pm 2\alpha)$ as $t \in [0,t_1) \bigcup [t_2, \Delta)$. Note that at points t_1, t_2 a limit function has only one 'jump' on a period.

33 First approximation

33.1 First approximation

In the first approximation, a solution of the transport equation can be represented as

$$\varphi(x,t) = \varphi_0(x,t) + h\varphi_1(x,t). \tag{241}$$

Then (with accuracy $O(h^3)$) we have the equation

$$\frac{\partial\varphi_1}{\partial t}+\dot{P}_n(p)\frac{\partial\varphi_1}{\partial x}=-\frac{1}{2}\ddot{P}_n(p)\frac{\partial^2\phi_0}{\partial x^2}, \tag{242}$$

which (on characteristics) can be written as ODE

$$\frac{d\varphi_1}{dt}=-\frac{1}{2}\ddot{P}_n(p)\frac{\partial^2\phi_0}{\partial x^2} \tag{243}$$

where $\phi_0(x,t)$ is known, and

$$\phi_0(x,t):=\phi_0(t-x/\dot{P}_n(p)). \tag{244}$$

Thus, we have ODE (243). Solutions of this equation are determined along $dx(t)/dt=\dot{P}_n(p)$. Indeed, integration of (243) from (l,t) to $(0,t-\Delta)$ leads to the integro-difference equation of the Volterra type

$$\varphi_1(l,t)=\varphi_1(0,t-\Delta)+\frac{1}{2}\ddot{P}_n(p)\int_{t-\Delta}^{t}\frac{\partial^2\phi_0}{\partial x^2}[\dot{P}_n(p)(s-t+\Delta),s]ds. \tag{245}$$

Since $\phi_0[x(s),s)=\phi_0(s-x(s)/\dot{P}_n(p)$, from (247) we arrive at

$$\varphi_1(l,t)=\varphi_1(0,t-\Delta)-\frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)}\phi_0''(t-\Delta). \tag{246}$$

From boundary conditions $\varphi(0,t)=\Phi[\varphi(l,t)]$ it follows that equation (246) can be written as

$$\varphi_1(l,t)=\Phi[\varphi_1(l,t-\Delta)]-\frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)}\phi_0''(t-\Delta). \tag{247}$$

33.2 Another method of reduction

The same result can be obtained from representation of equation (247) as

$$\frac{\partial\phi_0}{\partial t}=-\frac{1}{2}\ddot{P}_n(p)\frac{\partial^2\phi_0}{\partial x^2}(\zeta,t), \tag{248}$$

where $\zeta=t-x/\dot{P}_n(p))$. Indeed, the similar integration on characteristic leads to the equation

$$\varphi_1(t,t)=\varphi_1(t-\Delta,t-\Delta)-\frac{1}{2}\ddot{P}_n(p)\int_{t-\Delta}^{t}\frac{\partial^2\phi_0}{\partial x^2}(\zeta(s))ds. \tag{249}$$

But $\zeta(s)=s-x(s)/\dot{P}_n(p)$ alon the line $\dot{x(s)}=\dot{P}_n(p)$. Then

$$d\zeta=ds-\frac{\dot{x(s)}}{\dot{P}_n(p)}ds=0. \tag{250}$$

Hence, $\zeta=\zeta^*$ is a constant, and it is easy to check that $\zeta^*=t-\Delta$. As a result, from (249) it follows the equation

$$\varphi_1(t,t)=\varphi_1(t-\Delta,t-\Delta)-\frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)}\phi_0''(t-\Delta). \tag{251}$$

Equations (246),(251) are identical equations, but in different coordinates (x,t), (ζ,t), respectively. Indeed, if we denote $z_1(t) = \varphi_1(t,t)$ then we obtain the equations

$$z_1(t) = z_1(t-\Delta) - \frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)} z_2(t-\Delta) \tag{252}$$

where $z_2(t-\Delta) = \phi_0''(t-\Delta)$.

33.3 Some estimations

In this section, it will be shown that in the above integro-difference equation (IDE) the integral is decreeing for smooth probe functions then IDE tends to a corresponding difference equation. It means that asymptotically solutions of IDE tend to solutions of the difference equations and ones have the same asymptotic. It is possible for a special asymptotic behaviour of the phase. At first, we must prove that the function $\phi_0''(t) \Rightarrow 0$ as e^{-kt} for almost all $t \in R^+$, where $k > 0$. Indeed, let us consider the difference equation

$$u(t+\Delta) = f[u(t)],\ t > 0, \tag{253}$$

where $f \in C^2(I, I)$ is a given function. Then

$$u'(t+\Delta) = f'[u(t)]u'(t) \tag{254}$$

and if $|f'[u]| < 1$ for $u \in I$, then $u'(t) \Rightarrow 0$ as $t \to \infty$. That is it is possible if initial data lie at a some neighborhood of an attractive point of the map which is produced by the origin boundary problem. If P^+ contains one point, the statement is evident. But if we have more then one points, there exist repelling fixed points, which correspond to large derivatives of solutions at a neighborhood of the repelling point. This problem can be solved by the method which has been developed by Sharkovsky [22].

To begin the prove, let us consider the difference equation

Further

$$u''(t+\Delta) = f''[u(t)][u'(t)]^2 + f'[u(t)]u''(t). \tag{255}$$

Hence,

$$u''(t+\Delta) = f''[u(t)][u'(t)]^2 + f'[u(t)]u''(t) \tag{256}$$

and

$$|u''(t+\Delta)| \leq \nu|u''(t)| + O(1/t), \tag{257}$$

where $0 < \nu < 1$. Next, $|u''(t)| \Rightarrow 0$ as $t \to \infty$. Points of discontinuities Γ of the limit solutions are produced by pre-images $U = f^-n[u^-]$, $n = 1, 2, 3, ...$ of repelling fixed points $u^- \in A^-$ of the map f, where $|f'[u^-]| > 1$.

Next, from (257) it follows that $|\phi_0''(t-\Delta)| \Rightarrow 0$ as $t \to \infty$. From (257) it follows that we can use the asymptotic approximation

$$u''(t+\Delta) = f'[P^+]u''(t) \tag{258}$$

for each $u \in \mathfrak{O}_\varepsilon(P^+)$, where P^+ is a set of attractive fixed points of the map f and $\mathfrak{O}_\varepsilon(P^+)$ is a ε - neighborhood of the set P^+. This is a consequence of structural stability the map f.

The solution of equation (258) is $u''(t) = e^{kt}$, where $k = \ln f'(P^+)/\Delta$. Then we can insert the value k into equation (251). As a result, we obtain

$$z_1(t) = \Phi[z_1(t-\Delta)] - \frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)} e^{k(t-\Delta)} \tag{259}$$

where $k < 0$. Next, we define $z_2(t-\Delta) = e^{k(t-\Delta)}$. Then $z_2(t) = e^{k\Delta} z_2(t-\Delta)$. Hence, equation (259) can be written as the system of two difference equations:

$$z_1(t) = \Phi[z_1(t-\Delta)] - az_2(t-\Delta), \tag{260}$$

$$z_2(t) = e^{k\Delta} z_2(t-\Delta) \tag{261}$$

where $a = \frac{\Delta}{2}\frac{\ddot{P}_n(p)}{\dot{P}_n^2(p)}$. These equations produce the $2D$ - map $\hat{\Phi} : (z_1, z_2) \to (\Phi(z_1) - az_2, bz_2)$, where $b = e^{k\Delta}$. Since $k < 0$, all trajectories at plane (z_1, z_2) tend to a line z_1 as $t \to +\infty$ so that $z_2 \to 0$. It means that asymptotically we have deal with $1D$ - dimensional map which is described by iterations of the difference equation:

$$z_1(t) = \Phi[z_1(t-\Delta)]. \tag{262}$$

Fixed points of these map can be find from equations $\Phi(z_1) - az_2 = z_1$ and $bz_2 = z_2$. Hence, the points are $\mathfrak{A} = (Fix\,\Phi,\, 0)$. If $Fix\,\Phi = A^+$ then a point $A = (A^+, 0)$ is attractive. It means that eigenvalues of the Jacobi matrix $T\,\Phi$ are $k_1 = \Phi'(P^+)$ and $k_2 = b$, where $0 < b < 1$. Next, if $Fix\,\Phi = A^\pm$ then a point $A = (A^\pm, 0)$ is saddle type.

34 Example A

Let us consider the two parameter family of Henon maps which represent quadratic diffeomorphisms at a plane that can be written as

$$(x, y) \to (y, y^2 - \alpha - \beta x), \tag{263}$$

with constant Jacobian determinant γ. A set of (α, β) for which there is an attracting periodic orbit is quite shallow - shaped configuration (see, [?]). Such region corresponds to an attracting orbit 5. If $|\beta|$ is small then the dynamics of the $2D$ – map is similar to the the dynamics of the $1D$ – map $y \to y^2 - \alpha$.

The Henon map can be approximated by a linear map exapt at points near $y = 0$, then the dynamics may beconsidered as a composition of two quadratic maps. It means that now a Henon map is the product of the two quadratic maps of the form $\Phi : (x, y) \to (x^2 - \alpha, y)$ or otherwise. If $-\alpha \le \frac{1}{4}$, then the map has the two fixed points (β_0, β_0) and (β_1, β_1) on R^2, where

$$\beta_{0,1} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha}. \tag{264}$$

If $\Phi(x, y) \in R^2 \ I \times I$ then an orbit of the map Φ is bounded. Hence the set K_R ca be described as the real part of the 'filled Julia set' for each components of the map Φ.

Thus, for $|\beta| \ll 1$, asymptotic orbits for the Henon map can be described as the Decart of the $2D$ Julia set $J := J(x) \times J(y)$, where $J(x) \approx J(y)$ and both are 'filled Julia sets'. Note that there is prolongation of the Milnor one-dimensional classification of the two - dimensional case. Indeed, we will say that Φ belongs to the trivial class $R_0 \times R_0$ in $K_R(\Phi)$, where $K_R(\Phi)$ contains at most a single point $(\beta_1, \beta_1) \in R^2$.

If Φ does not belong to the trivial class then there are at least two distinct fixed points (β_0, β_0) and (β_1, β_1). Let $\Pi := I \times I$ be a smallest closed invariant interval that contains $K_R(\Phi)$. Thus every trajectory of the dynamical system which starts outside of Π must escape to ∞, but the boundary $\partial\Pi$ must have bounded orbits in R^2. Indeed, since we have deal with the degenerated Henon map, which is decomposed on the two independent quadratic maps, with the two fixed points (β_0, β_0) and (β_1, β_1).

If $-\alpha > \frac{1}{4}$ then each component of the decoupled Henon map Φ tend to a fixed point $B_0 = (\beta_0, \beta_0)$, where B_0 is an attractive fixed point on the diagonal $x = y$. For $-\alpha > \frac{1}{4}$, points of each orbits of the map $\Phi : R^2 \to R^2$ tend to $+\infty$ by iterations $\Phi^n(x, y)$ as $n \to \infty$. Thus a set of parameters $\alpha < -\frac{1}{4}$ determines the class R_0 if $|\beta| = 0$. Next, we determine a family of the Henon maps $\Phi(\alpha, \beta)$ for small β and show that a new class $R_0(\alpha, \beta)$ is diffeomorphic to the class $R_0(\alpha, 0)$ (compare with ([**?**], figure 2).

To make it we assume that the map Φ is transitive. It means that there are intervals U_1 and U_2 about two critical points, so that a first return map from the interval $U = U_1 \bigcup U_2$ to itself is defined and smooth, interchanging these two components. It means that $f^{\circ p}(U_1) \subset U_2$ and $f^{\circ p}(U_2) \subset U_1)$ for some $p \geq 0$ and $q \geq 1$. For example, a universal model for this behaviour take place for 'biquadratic map', that is, for the composition of the two quadratic maps.

In the above conditions, we assume that there are neighborhoods U_1 and U_2 such that the fixed return map carries both U_1 and U_2 into U_2. Thus the orbit of U_1 is 'captured' by the periodic orbit of U_2. There are also the disjoint periodic orbits. It means that again there are disjoint neighborhoods U_1 and U_2, but now the first return map carries each U_1 into itself, so that $f^{\circ p}(U_1) \subset U_1)$ and $f^{\circ q}(U_2) \subset U_2)$.

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