Two symmetric measurements may cause an unforeseen effect

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Abstract

We may try to discuss gently symmetric measurement outcomes, which might be extended to considering naturally the uncertainty principle. For the two symmetric measurement outcomes, sometimes, the two measured observables are commutative. In this specific and symmetric example, we introduce a supposition that the operation Addition is equivalent to the operation Multiplication and we may have an example of an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness. We show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables/an observable. We would say that the trial above might be categorized into an inconsistency example in the effect of the uncertainty principle, if we are forgiven for describing the above.

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I. INTRODUCTION

Quantum physics is highly developed since longtime ago, see e.g. [1–5]. One of the central point of the theory is the so called uncertainty principle which is some kind of mathematical inequalities for the threshold of precision of physical simultaneously measurement of pairs of physical observables. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [6]. The formal inequality relating the standard deviation of position σ_q and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard [7] later that year and by Hermann Weyl [8] in 1928.

In [9] the first and the third author propose some optimal limitation of the uncertainty principle in some physical situation by the optimality using the Bloch sphere.

In [10, 11] the first and the third author derive some inconsistency in quantum mechanics. Barros claims in [12] that the inconsistencies do not come from quantum mechanics, but from extra assumptions about the reality of observables. We suppose the inconsistency comes from quantum mechanics, without extra assumptions about the reality of observables. We show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables/an observable.

We notice that von Neumann's mathematical model for quantum mechanics is quite logically successful. And the axiomatic system for the mathematical model is a very consistent one. Thus, we cannot say that von Neumann's mathematical model has an inconsistency. What is the inconsistency to be discussed in this paper? We cannot expand the von Neumann's beautiful mathematical model more in handling real experimental data. Mathematically, von Neumann's model is logically very consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the inconsistency if we apply the von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The inconsistency to be discussed in this paper is very impressive. von Neumann's mathematical model has the qualification to be very true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness, within von Neumann's theory.

Here, we discuss symmetric measurement outcomes in quantum mechanics inside considering the effect of the uncertainty principle. For the two symmetric measurement outcomes the two measured observables are commutative. In this specific and symmetric example, we introduce a supposition that the operation Addition is equivalent to the operation Multiplication and we may have an example of an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness. We show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables/an observable. We would say that the trial above might be categorized into an inconsistency example in the effect of the uncertainty principle, if we are forgiven for describing the above.

There is a paper by Werner A. Hofer titled "Heisenberg, uncertainty, and the scanning tunneling microscope" which is consistent with the analysis here [13]. The assertion is that the density of electron charge is a physically real as in precisely measurable quantity.

II. QUANTUM MEASUREMENT THEORY AND THE UNCERTAINTY PRINCIPLE

Though doing later, we dare to introduce firstly a supposition that the sum rule is equivalent to the product rule for the purpose of showing our interesting objective obtained here.

Let A_1, A_2 be two Hermitian operators, where they are also supposed to be commutative [14]. They could be defined respectively as follows:

$$A_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

and

$$A_2 \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Later it is discussed that all the four eigenstates give an unforeseen effect. Note symmetric measurement outcomes are given in the four patterns.

1. Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_1\rangle$, such that

$$\begin{aligned}\langle\Psi_1|A_1|\Psi_1\rangle &= +1,\\ \langle\Psi_1|A_2|\Psi_1\rangle &= -1.\end{aligned} \tag{3}$$

   The measured results of trials are either $+1$ or -1.

2. Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_2\rangle$, such that

$$\begin{aligned}\langle\Psi_2|A_1|\Psi_2\rangle &= -1,\\ \langle\Psi_2|A_2|\Psi_2\rangle &= +1.\end{aligned} \tag{4}$$

   The measured results of trials are either -1 or $+1$.

3. Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_3\rangle$, such that

$$\begin{aligned}\langle\Psi_3|A_1|\Psi_3\rangle &= +1,\\ \langle\Psi_3|A_2|\Psi_3\rangle &= +1.\end{aligned} \tag{5}$$

   The measured results of trials are only $+1$.

4. Finally, let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_4\rangle$, such that

$$\begin{aligned}\langle\Psi_4|A_1|\Psi_4\rangle &= -1,\\ \langle\Psi_4|A_2|\Psi_4\rangle &= -1.\end{aligned} \tag{6}$$

   The measured results of trials are only -1.

In what follows, we would discuss that the sum rule is equivalent to the product rule for commuting observables. First, we define the functional rule as follows:

$$f(g(O)) = g(f(O)), \tag{7}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. Second, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2). \tag{8}$$

Finally, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_2) \cdot f(A_2). \tag{9}$$

This fact above is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [3] the following relation between the three rules:

$$\begin{aligned}&\text{The functional rule}\\ &\Leftrightarrow \text{The sum rule}\\ &\Leftrightarrow \text{The product rule}\end{aligned} \tag{10}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$ and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be nondegenerate by construction. Let us define respectively functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned}&f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O))\\ &f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B),\end{aligned} \tag{11}$$

where we use the functional rule. We can introduce also a function l such that $A+B=l(O)$ where $l:=j+k$. Thus we have

$$\begin{aligned}&f(A+B)=f(l(O))=l(f(O))=j(f(O))+k(f(O))\\&f(j(O))+f(k(O))=f(A)+f(B),\end{aligned}\tag{12}$$

where we use the functional rule. In fact, the sum rule is equivalent to the product rule for commuting observables. If A_1 and A_2 are not commuting, i.e., $[A_1,A_2]=A_1A_2-A_2A_1\neq 0$, then we have the van Campbell-Hausdorff-Dynkin formula:

$$\begin{aligned}&\log(e^{A_1}e^{A_2})=A_1+A_2+\frac{1}{2!}[A_1,A_2]\\&+\frac{1}{3!}([[A_1,A_2],A_1]+[[A_1,A_2],A_2])+.....\end{aligned}\tag{13}$$

It means that the orders of measurements are important: The difference between two measurement orders A_1A_2 and A_2A_1 is measured by the commutator $[A_1,A_2]$.

Let us consider an example that the commutator is not zero. When $\hat{q}_i$ is the measurement observable on the q_j coordinate direction, and [1]

$$\hat{p}_j=\frac{\hbar}{\sqrt{-1}}\partial/\partial q_j,\tag{14}$$

in the often used quantum mechanical model of Hilbert space $\mathcal{H}=L^2(q_1,\ldots,q_n)$, consisting of the complex-valued square-integrable absolute value functions on coordinate variables $q_i, i=1,\ldots,n$, is the measurement on the moment direction then we have exactly the nonzero commutator of two measurements

$$[\hat{p}_j,\hat{q}_i]=\delta_{ij}\frac{\hbar}{\sqrt{-1}}\mathrm{Id}_{\mathcal{H}}.\tag{15}$$

Quantum mechanics agrees with the uncertainty principle and we have two measurement orders that give different results.

III. UNFORESEEN EFFECT

Let us consider the main result as follows: It is discussed that all the four eigenstates give an unforeseen effect. Note symmetric measurement outcomes are given in the four patterns.

Case 1: Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_1\rangle$. We might be in an inconsistency when the first result is $+1$ by the measured observable A_1, the second result is -1 by the measured observable A_2, and then $[A_1,A_2]=0$. In general, the physical situation is either $[A_1,A_2]\neq 0$ or $[A_1,A_2]=0$. However we may be in the inconsistency when we suppose $[A_1,A_2]=0$, probably due to the nature of Matrix theory based on non-commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1$ or -1. We suppose the number of -1 is equal to the number of $+1$. If the number of trials is 2, then we have

$$V=(+1)+(-1)=0.\tag{16}$$

We derive a general necessary condition of the product $V\times V$ of the value V. In this general case, we have

$$V\times V=0.\tag{17}$$

This is the general necessary condition for either $[A_1,A_2]\neq 0$ or $[A_1,A_2]=0$.

We can depict experimental data r_1, r_2 as follows: $r_1=+1$ and $r_2=-1$. Let us write V as follows:

$$V=r_1+r_2.\tag{18}$$

In the following, we evaluate a value $(V\times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1,A_2]=0$.

We introduce a supposition that the sum rule is equivalent to the product rule. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\
&= (r_1)^2 + (r_1 + r_1) + (r_2 + r_2) + (r_2)^2 \\
&= (r_1)^2 + (r_1 \times r_1) + (r_2 \times r_2) + (r_2)^2 \\
&= 2((r_1)^2 + (r_2)^2) \\
&= 2((+1)^2 + (-1)^2) = 4.
\end{aligned} \tag{19}$$

Thus,

$$V \times V = 4. \tag{20}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (17) and (20) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is -1, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_1\rangle$.

Case 2: Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_2\rangle$. We might be in an inconsistency when the first result is -1 by the measured observable A_1, the second result is $+1$ by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We have

$$V = (-1) + (+1) = 0. \tag{21}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 0. \tag{22}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = -1$ and $r_2 = +1$. Let us write V as follows:

$$V = r_1 + r_2. \tag{23}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\
&= (r_1)^2 + (r_1 + r_1) + (r_2 + r_2) + (r_2)^2 \\
&= (r_1)^2 + (r_1 \times r_1) + (r_2 \times r_2) + (r_2)^2 \\
&= 2((r_1)^2 + (r_2)^2) \\
&= 2((-1)^2 + (+1)^2) = 4.
\end{aligned} \tag{24}$$

Thus,

$$V \times V = 4. \tag{25}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (22) and (25) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is -1, the second result is $+1$, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_2\rangle$.

Case 3: Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_3\rangle$. We might be in an inconsistency when the first result is $+1$ by the measured observable A_1, the second result is $+1$ by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We have

$$V = (+1) + (+1) = 2. \tag{26}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 4. \tag{27}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = +1$ and $r_2 = +1$. Let us write V as follows:

$$V = r_1 + r_2. \tag{28}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
& V \times V \\
& = (r_1 + r_2) \times (r_1 + r_2) \\
& = (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
& = (r_1 + r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 + r_2) \\
& = 8.
\end{aligned} \tag{29}$$

Thus,

$$V \times V = 8. \tag{30}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (27) and (30) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is $+1$, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_3\rangle$.

Case 4: Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_4\rangle$. We might be in an inconsistency when the first result is -1 by the measured observable A_1, the second result is -1 by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We have

$$V = (-1) + (-1) = -2. \tag{31}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 4. \tag{32}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = -1$ and $r_2 = -1$. Let us write V as follows:

$$V = r_1 + r_2. \tag{33}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned}
&V \times V \\
&= (r_1 + r_2) \times (r_1 + r_2) \\
&= (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
&= (r_1 + r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 + r_2) \\
&= -8.
\end{aligned} \tag{34}$$

Thus,

$$V \times V = -8. \tag{35}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (32) and (35) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is -1, the second result is -1, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2, that is, $|\Psi_4\rangle$.

The uncertainty principle needs the case $[A_1, A_2] \neq 0$. Quantum mechanics agrees with the uncertainty principle and we have two measurement orders that give different results. The four examples are based on symmetric measurement outcomes. The experimental situation does not change when we exchange the first result and the second result because the experimental situation is symmetric. This means that the four experimental situations do not depict the case $[A_1, A_2] \neq 0$ and the effect of the uncertainty principle.

IV. GENERAL CASE

Let us move ourselves into the more general case. Especially, we discuss quantum mechanics is not consistent when measuring only one observable.

A. The first result is not equal to the second result

Let us consider a simultaneous eigenstate of A_1, A_2. We might be in an inconsistency when the first result is x by the measured observable A_1, the second result is not x by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either x or $y (\neq x)$. We suppose the number of x is equal to the number of y. If the number of trials is two, then we have

$$V = x + y. \tag{36}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = (x + y)^2. \tag{37}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = y$. Let us write V as follows:

$$V = r_1 + r_2. \tag{38}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$
\begin{aligned}
& V \times V \\
& = (r_1 + r_2) \times (r_1 + r_2) \\
& = (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
& = (r_1)^2 + (r_1 + r_2) + (r_2 + r_1) + (r_2)^2 \\
& = (r_1)^2 + (r_1 + r_1) + (r_2 + r_2) + (r_2)^2 \\
& = (r_1)^2 + (r_1 \times r_1) + (r_2 \times r_2) + (r_2)^2 \\
& = 2((r_1)^2 + (r_2)^2) \\
& = 2(x^2 + y^2).
\end{aligned} \tag{39}
$$

Thus,

$$V \times V = 2(x^2 + y^2). \tag{40}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (37) and (40) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is x, the second result is not x, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2.

B. The first result is equal to the second result

We discuss quantum mechanics is not consistent when measuring only one observable. Let us consider a simultaneous eigenstate of A_1, A_2. We might be in an inconsistency when the first result is $x(\neq 0)$ by the measured observable A_1, the second result is also x by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on non-commutativeness. It may be that we measure only one observable A, ($A = A_1 = A_2$ and $x \neq 0$).

We have

$$V = x + x = 2x. \tag{41}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$V \times V = 4x^2. \tag{42}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$.

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = x$. Let us write V as follows:

$$V = r_1 + r_2. \tag{43}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$
\begin{aligned}
& V \times V \\
& = (r_1 + r_2) \times (r_1 + r_2) \\
& = (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
& = (r_1 + r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 + r_2) \\
& = 8x.
\end{aligned} \tag{44}
$$

Thus,

$$V \times V = 8x. \tag{45}$$

This is possible for the specific case $[A_1, A_2] = 0$. When $x \neq 2$, we cannot assign simultaneously the truth value "1" for the two suppositions (42) and (45) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

Let us consider the case where $x = 2$. We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$
\begin{aligned}
& V \times V \\
& = (r_1 + r_2) \times (r_1 + r_2) \\
& = (r_1 \times r_1) + (r_1 \times r_2) + (r_2 \times r_1) + (r_2 \times r_2) \\
& = (r_1 \times r_1) + (r_1 + r_2) + (r_2 + r_1) + (r_2 \times r_2) \\
& = 2(x^2 + x).
\end{aligned}
\tag{46}
$$

Thus,

$$V \times V = 2(x^2 + x). \tag{47}$$

This is possible for the specific case $[A_1, A_2] = 0$. When $x = 2$, we cannot assign simultaneously the truth value "1" for the two suppositions (42) and (47) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

Let us consider the case where $x = 0$. We see $0+0 = 0\times 0 = 0$. Thus, the supposition that the operation Addition is equivalent to the operation Multiplication does not work in order to derive the inconsistency. Hence, we have always $V \times V = 0$ when $x = 0$. Thus, we cannot derive the inconsistency when $x = 0$.

In summary, we have been in the inconsistency when the first result is x, the second result is also x, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2. Especially, we have discussed quantum mechanics is not consistent when measuring only one observable A, ($A = A_1 = A_2$ and $x \neq 0$).

V. CONCLUSIONS AND DISCUSSIONS

In conclusions, we have tried to discuss naturally symmetric measurement outcomes in quantum mechanics inside considering the effect of the uncertainty principle. For the two symmetric measurement outcomes, sometimes, the two measured observables have been commutative. In this specific and symmetric example, we have introduced a supposition that the operation Addition is equivalent to the operation Multiplication and we have had an example of an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness. We have shown here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables/an observable.

The uncertainty principle needs the case $[A_1, A_2] \neq 0$. Quantum mechanics agrees with the uncertainty principle and we have two measurement orders that give different results. The four examples are based on symmetric measurement outcomes. The experimental situation does not change when we exchange the first result and the second result because the experimental situation is symmetric. This means that the four experimental situations do not depict the case $[A_1, A_2] \neq 0$ and the effect of the uncertainty principle.

Nagata and Nakamura have claimed [10, 11] to derive an inconsistency in quantum mechanics. Barros has discussed [12] as follows: The inconsistencies do not have come from quantum mechanics, but from extra assumptions about the reality of observables. Here we have discussed there is an inconsistency, probably due to the nature of Matrix theory based on non-commutativeness, within quantum mechanics even for commuting observables. We do not have introduced extra assumptions about the reality of observables because we consider only commuting observables. Finally, we have discussed quantum mechanics is not consistent when measuring only one observable. We would say that the trial above might be one example of the inconsistency in the effect of the uncertainty principle.

If the problem were simply an inconsistency, there are multiple logical systems that can cope with such a problem with robustness (see [15]). Generally Multiplication is completed by Addition. Therefore, we think that Addition of the starting point may be superior to any other case.

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DECLARATIONS

Ethical Approval

We are in an applicable thought to Ethical Approval.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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REFERENCES

[1] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley Publishing Company, 1995), Revised ed.
[2] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).
[3] M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Press, Oxford, 1989), 2nd ed.
[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).
[5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).
[6] W. Heisenberg, "Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik", Zeitschrift für Physik (in German) **43** (3-4): 172-198 (1927).
[7] E. H. Kennard, "Zur Quantenmechanik einfacher Bewegungstypen", Zeitschrift für Physik (in German) **44** (4-5): 326 (1927).
[8] H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig: Hirzel (1928).
[9] K. Nagata and T. Nakamura, "Violation of Heisenberg's Uncertainty Principle", Open Access Library Journal, Volume 2, No. 8 (2015), pp. e1797/1–6, http://dx.doi.org/10.4236/oalib.1101797.
[10] K. Nagata, "There is No Axiomatic System for the Quantum Theory", Int. J. Theor. Phys. **48**, 3532 (2009), https://doi.org/10.1007/s10773-009-0158-z.
[11] K. Nagata and T. Nakamura, "Can von Neumann's Theory Meet the Deutsch-Jozsa Algorithm?", Int. J. Theor. Phys. **49**, 162 (2010), https://doi.org/10.1007/s10773-009-0189-5.
[12] J. A. de Barros, "Comments on "There Is No Axiomatic System for the Quantum Theory"", Int. J. Theor. Phys. **50**, 1828 (2011), https://doi.org/10.1007/s10773-011-0696-z.
[13] W. A. Hofer, "Heisenberg, uncertainty, and the scanning tunneling microscope", Frontiers of Physics **7**, 218 (2012), https://doi.org/10.1007/s11467-012-0246-z.
[14] We cannot introduce a supposition that the sum rule is equivalent to the product rule if A_1 and A_2 are not commutative. Therefore, $[A_1, A_2] = 0$ is an important sufficient condition for our interesting objective.
[15] A. Avron and A. Zamansky, "Non-Deterministic Semantics for Logical Systems". In: Gabbay, D., Guenthner, F. (eds) Handbook of Philosophical Logic. Handbook of Philosophical Logic, vol 16. Springer, Dordrecht (2011), https://doi.org/10.1007/978-94-007-0479-4_4.