A PROOF OF THE LINE LIKE KAKEYA MAXIMAL FUNCTION CONJECTURE

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Abstract. In this paper we will prove the Kakeya maximal function conjecture in a special case when tube intersections behave like points. We achieve this by showing there exist large essentially disjoint tube-subsets.

Contents

1. Introduction

A line l_i is defined as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and} \quad t \in \mathbf{R} \quad \text{s.t} \quad y = a + xt\}$$

We define the δ-tubes as δ-neighborhoods of lines:

$$T_i^\delta := \{x \in \mathbf{R}^n | |x - y| \leq \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant C_n depending only on n such that $A \leq C_n B$. We say that tubes are δ-separated if their angles are δ-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define a as it‘s center of mass. Define the Kakeya maximal function as
$f^*_\delta : S^{n-1} \to \mathbb{R}$ via

$$f^*_\delta(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T^\delta_\omega(a) \cap B(0,1)} \int_{T^\delta_\omega(a) \cap B(0,1)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f^*_\delta||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon} ||f||_p, \tag{1.1}$$

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for all $\epsilon > 0$ and some $n \leq p \leq \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon} N^{1/p'}\delta^{(n-1)/p'}, \tag{1.2}$$

for all $\epsilon > 0$, and for any set of $N \leq \delta^{1-n}$ δ-separated of δ-tubes. See for example [2] or [1]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [1, 2]. Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i}(x)1_{B(0,1)}(x) \leq 2^{k+1}\}.$$

We will prove the following theorem.

Theorem 1.1. *Let Ω be a set of $N \lesssim \delta^{1-n}$ δ-separated δ-tubes. Moreover assume that for all $k > 0$, $l \neq m$, it holds that*

$$T_l \cap T_m \cap E_{2^k} = \bigcap_{j=1}^{2^k} T_{jlm}.$$

Here T_l and T_m are any δ-separated tubes with respect to Ω. Then we have

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)} \leq C_n(\log(\frac{1}{\delta})^{(n-1)/n}(N\delta^{n-1})^{(n-1)/n}.$$

It is a fact that the intersection of each pair of different lines contains only one point and our conditions hold with tubes replaced by lines. But the author of this paper emphasis the difference between line and tube intersections and it can be said that we prove the Kakeya maximal function conjecture only in a line like case. One of our results is the following: a generalization of a lemma of Corbóda.

Lemma 1.2. *[A generalization of a lemma of Corbóda] For δ-separated tube intersections of order $2^k > 1$ it holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1}2^{-k/(n-1)}.$$

It's not hard to check that the above bound is essentially tight.

2. Previously known results

We will use the following bound for the pairwise intersections of δ-tubes:

Lemma 2.1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,1)$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [1].

For any (spherical) cap $\Omega \subset S^{n-1}$, $|\Omega| \gtrsim \delta^{n-1}$, $\delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 2.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [1].

3. A proof of the generalization of the lemma of Corbóda

Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i}(x) 1_{B(0,1)}(x) \leq 2^{k+1}\}.$$

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \leq n-1$, and let‘s suppose that tube T_j intersecting $T_i \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between T_j and T_i is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 1.2 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \leq |T_i \cap T_j \cap E_{2^k}| \leq |T_i \cap T_j| \lesssim \delta^{n-1+\beta/(n-1)} \leq \delta^{n-1} 2^{-k/(n-1)}. \tag{3.1}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_i \cap T_j$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2.2 we get that the cap can contain at most $\sim 2^k$ tube-directions. However, the cap contains at least 2^k tube directions. Thus, for any tube T_i in the intersection there exists a tube T_j, such that the angle between T_i and T_j is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.2.

4. The proof of the line like case

We defined

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i}(x) 1_{B(0,1)}(x) \leq 2^{k+1}\}.$$

We have for $k > 0$ that

$$E_{2^k} = \bigcup_{i=1}^{M} \bigcap_{j=1}^{\sim 2^k} T_{ij}.$$

The number M is just the number of distinct intersections of given order. The cases $k < 2$ are trivial for our purposes and we omit them. We assume the special case that

$$E_{2^k} \cap T_l \cap T_m \subset \bigcap_{j=1}^{\sim 2^k} T_{ij}, \tag{4.1}$$

for $l \neq m$. We then say that the intersection $T_l \cap T_m$ is point like, because the above holds for tubes replaced by lines.

Example 4.1. The following is a basic example of a case when our asumptions are not fulfilled. Let us assume that we have a standard hairbrush on $\mathbf{R}^3$. We have an unit length handle with $\sim 2^k$-tubes (orthogonal with respect to the handle) intersecting the handle. The intersections should be δ-spaced on the handle. So there could be δ^{-1}-intersections on the handle. So if we have almost parallel δ-tube intersecting the handle, then we see that (4.1) is not fulfilled.

Via standard dyadic decomposition

$$\sum_k (2^k)^{n/(n-1)} |E_{2^k}| \sim || \sum_{i=1}^N 1_{B(0,1)} 1_{T_i} ||_{n/(n-1)}^{n/(n-1)}$$

So it suffices to proof that

$$|E_{2^k}| \lesssim 2^{-kn/(n-1)} N \delta^{n-1}. \tag{4.2}$$

We use Fubini to deduct

$$(2^k)^2 |E_{2^k}| \sim \int_{E_{2^k}} (\sum_{i=1}^N 1_{T_i})^2 = \sum_{i=1}^N \sum_{j=1}^N \int 1_{T_i} 1_{T_j}$$
$$\sim \sum_{i=1}^N \sum_{j=1}^N |T_i \cap T_j \cap E_{2^k}|.$$

Next, we show a very useful technique to reduce to essentially disjoint sets. By essential disjointness we mean that the possible orders of intersections are very low e.g bounded my a small constant.

First, we sum the other index away and rearrange so that each intersection is summed only $\sim (2^k)^2$-times:

$$\sum_{i=1, i \neq j}^N \sum_{j=1}^N |T_i \cap T_j \cap E_{2^k}|$$
$$\sim 2^k \sum_{i=1}^N |T_i \cap E_{2^k}| \sim 2^k \sum_{i=1, i \neq j}^N \sum_{j=1}^{M_i} |T_i \cap T_j \cap E_{2^k}|.$$
$$\sim (2^k)^2 \sum_{i=1, i \neq j}^{N/2^k} \sum_{j=1}^{M_i} |T_i \cap T_j \cap E_{2^k}|$$

So in the above the summands over j are disjoint for each i. We see that as many as $\sim N/2^k$ of the sets $E_{2^k} \cap T_i$ are essentially disjoint: meaning that the order of intersections is very low. We sum over each intersection once and obtain

$$\begin{aligned} &\sum_{i=1, i \neq j}^{N/2^k} \sum_{j=1}^{M_i} |T_i \cap T_j \cap E_{2^k}| \\ &\sim \sum_{i=1}^{N/2^k} |E_{2^k} \cap T_i| \\ &\sim |E_{2^k}|. \end{aligned} \tag{4.3}$$

In the last sum above each intersection is still summed over once. This means that the sets $T_i \cap E_{2^k}$ are essentially disjoint in the above sum $\sum_{i=1}^{N/2^k} |E_{2^k} \cap T_i|$.

We define

$$E'_{2^k} := E_{2^k} \cap \bigcup_{i=1}^{N/2^k} T_i.$$

On the other hand essentially $E_{2^k} \subset \bigcup_{i=1}^{N/2^k} T_i$, by (4.3). Thus,

$$|E'_{2^k}| \sim |E_{2^k}|.$$

The above reduction is the reduction to essentially disjoint tubes!

Remark 4.2. It's well worth noticing that any N tubes contain $\sim N/2^k$ essentially disjoint tubes, if there is a bound 2^k for the order of intersections!

We wan't to show that

$$|E'_{2^k}| \lesssim 2^{-kn/(n-1)}\delta^{n-1}N.$$

When we make our reduction to the essentially disjoint tubes for each T_i we choose some $N/2^k$ tubes which to spare.

Remark 4.3. An interesting fact is that if we have

$$\sum_{i=1}^{N/2^k} |T_i \cap B(0,1) \cap E_{2^k}| \sim \sum_{i=1}^{N/2^k} |T_i \cap B(0, 2^{k/(n-1)}) \cap E_{2^k}|.$$

Now we would have $2^{k/(n-1)}$-long $\delta-$tubes T'_i and T'_j , but the size or the position of the intersections does not change.

Now, for each T_i we can choose the tubes "dividing" T_j disjoint and essentially parallel to T_i. For each T_i the tubes T_j can be parallel but still δ-separated with respect to T_i. It's clear that we need only bounded many these kind of tubes to cover T_i. We get from from the generalized lemma of Córdoba 1.2 and from the fact that the intersection is point like, that the intersection is essentially contained in a $2^{-k/(n-1)}$-long δ-tube. So we get from the point like condition (4.1) and from the generalized lemma of Córdoba 1.2 that

$$\begin{aligned}
2^{kn/(n-1)}|E_{2^k}| &\sim 2^{kn/(n-1)} \sum_{i=1}^{N/2^k} |T_i \cap B(0,1) \cap E_{2^k}| \\
&\sim 2^{kn/(n-1)} \sum_{i=1}^{N/2^k} \sum_{j=1, i\neq j}^{\sim 1} |T_i \cap T_j \cap E_{2^k}| \\
&\lesssim 2^{kn/(n-1)} \sum_{i=1}^{N/2^k} \sum_{j=1, j\neq i}^{\sim 1} |T_i \cap T_j \cap B(x_j, 2^{-k/(n-1)})| \\
&\lesssim 2^{kn/(n-1)} N/2^k |T_i \cap T_j \cap B(x_j, 2^{-k/(n-1)})| \\
&\lesssim N 2^{-k} 2^{kn/(n-1)} \delta^{n-1} 2^{-k/(n-1)} \\
&\lesssim N\delta^{n-1}.
\end{aligned}$$

Thus, we are done proving the theorem (1.1).

References

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[2] T. Tao, *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999)

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