# A PROOF OF THE KAKEYA MAXIMAL FUNCTION CONJECTURE FROM A SPECIAL CASE

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Abstract. First in this paper we will prove the Kakeya maximal function conjecture in a special case when tube intersections behave like line intersections. This paper highlights how different tube intersections can be than line intersections. However, we show that the general case can be deducted from the linelike case.

## 1. Introduction

A line $l_i$ is defined as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and} \quad t \in \mathbf{R} \quad \text{s.t} \quad y = a + xt\}$$

We define the $\delta$-tubes as $\delta$ neighbourhoods of lines:

$$T_i^\delta := \{x \in \mathbf{R}^n | |x - y| \leq \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant $C_n$ depending only on $n$ such that $A \leq C_n B$. We say that tubes are $\delta$-separated if their angles are $\delta$-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define $a$ as it's center of mass. Define the Kakeya maximal function as
$f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T_\omega^\delta(a) \cap B(0,1)} \int_{T_\omega^\delta(a) \cap B(0,1)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension $n$. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon} ||f||_p, \tag{1.1}$$

for all $\epsilon > 0$ and some $n \leq p \leq \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$|| \sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)} ||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon} N^{1/p'} \delta^{(n-1)/p'}, \tag{1.2}$$

for all $\epsilon > 0$, and for any set of $N \leq \delta^{1-n}$ $\delta$-separated of $\delta$-tubes. See for example [3] or [2]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [3, 2]. Moreover it's enough to consider the case where the $\delta$-separated set is maximal. We will prove that

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**Theorem 1.1.** *Let there be a $N \lesssim \delta^{1-n}$ $\delta$-separated $\delta$-tubes. Assume that the intersection of each pair of different tubes contains only maximally one intersection of given dyadic order. Then we have*

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)} \leq C_n(\log(\frac{1}{\delta}N\delta^{n-1})^{(n-1)/n}.$$

It is a fact that the intersection of each pair of different lines contains only maximally one point. So this paper emphasis the difference between line and tube intersections and it can be said that we first prove the Kakeya maximal function conjecture in a linelike case. However, we have the general case also.

**Corollary 1.2.** Let there be a $N \lesssim \delta^{1-n}$ $\delta$-separated $\delta$-tubes. Then we have

$$||\sum_{\omega\in\Omega} 1_{B(0,1)}1_{T_\omega(a_\omega)}||_{n/(n-1)} \leq C_n(\log(\frac{1}{\delta}N\delta^{n-1})^{(n-1)/n}.$$

One of our results is the following: a generalization of a lemma of Corbóda.

**Lemma 1.3.** *[A generalization of a lemma of Corbóda] For $\delta$-separated tube intersections of order $2^k > 1$ it holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1}2^{-k/(n-1)}.$$

It‘s not hard to check that the above bound is essentially tight.

## 2. Previously known results

We will use the following bound for the pairwise intersections of $\delta$-tubes:

**Lemma 2.1** (Corbòda). *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,1)$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [2].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its $\delta$-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta$-separated subset of $\Omega$.

**Lemma 2.2.** *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [2].

## 3. A proof of the generalization of the lemma of Corbóda

Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i}(x)1_{B(0,1)}(x) \leq 2^{k+1}\}.$$

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \leq n-1$, and let‘s suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between $T_\omega$ and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 1.3 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \leq |T_\omega \cap T_{\omega'} \cap E_{2^k}| \leq |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1+\beta/(n-1)} \leq \delta^{n-1} 2^{-k/(n-1)}. \tag{3.1}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we $\delta$ - separate the cap via lemma 2.2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube $T_\omega$ in the intersection there exists a tube $T_{\omega'}$, such that the angle between $T_\omega$ and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.3.

## 4. The proof of the linelike case

We define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i} 1_{B(0,1)} \leq 2^{k+1}\}.$$

We have for $k > 0$ that

$$E_{2^k} = \bigcup_{i=1}^{M} \bigcap_{j=1}^{2^k} T_{ij}.$$

We assume the special case that

$$E_{2^k} \cap T_l \cap T_m \subset \bigcap_{j=1}^{2^k} T_{ij}, \tag{4.1}$$

for $l \neq m$. We then say that the intersection $T_l \cap T_m$ is linelike, because the above holds for tubes replaced by lines. However it's easy to construct examples of situations where (4.1) does not hold. Now, via standard dyadic decomposition

$$\sum_k (2^k)^{n/(n-1)} |E_{2^k}| \sim || \sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)} ||_{n/(n-1)}^{n/(n-1)}$$

It suffices to proof that

$$|E_{2^k}| \lesssim 2^{-kn/(n-1)} N \delta^{n-1}. \tag{4.2}$$

We use Fubini to deduct

$$\begin{aligned} (2^k)^3 |E_{2^k}| &\sim \int_{E_{2^k}} (\sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i})^3 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \int 1_{B(0,1)} 1_{T_i} 1_{T_j} 1_{T_l} \\ &\sim \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} |B(0,1) \cap T_i \cap T_j \cap T_l \cap E_{2^k}| \end{aligned} \tag{4.3}$$

Now, for each three different tubes $T_i, T_j$ and $T_l$ there are only $\sim 2^k$ tubes that $B(0,1) \cap T_i \cap, \ldots, T_{2^k} \cap E_{2^k} \neq \emptyset$. Moreover,

(4.4)

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N}|B(0,1)\cap T_i\cap T_j\cap T_l\cap E_{2^k}|$$

$$\lesssim N + C\sum_{i=1}^{N}\sum_{j=1}^{N}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}|$$

$$+ \sum_{i=1,i\neq j,i\neq l}^{N}\ \sum_{j=1,j\neq i,l\neq j}^{N}\ \sum_{l=1,l\neq i,l\neq j}^{\sim 2^k}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}\cap T_l|$$

$$\lesssim \delta^{n-1}N + 2^k\delta^{n-1}N + \sum_{i=1,i\neq j,i\neq l}^{N}\ \sum_{j=1,j\neq i,l\neq j}^{N}\ \sum_{l=1,l\neq i,l\neq j}^{\sim 2^k}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}\cap T_l|$$

In the above from Fubini

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}| \sim (2^k)^2|E_{2^k}| \lesssim 2^k\delta^{n-1}N,$$

where we used that

$$\sum_k 2^k|E_{2^k}| \sim ||\sum_{i=1}^{N}1_{T_i}||_1 = \sum_{i=1}^{N}|T_i| \sim \delta^{n-1}N.$$

Next we can sum $T_j$ away and obtain

$$\begin{aligned}
&= \sum_{i=1,i\neq j,i\neq l}^{N}\ \sum_{j=1,j\neq i,l\neq j}^{N}\ \sum_{l=1,l\neq i,l\neq j}^{\sim 2^k}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}\cap T_l| \\
&\lesssim \sum_{i=1,i\neq l}^{N}\ \sum_{l=1,l\neq i}^{\sim 2^k}2^k|B(0,1)\cap T_i\cap E_{2^k}\cap T_l|
\end{aligned} \tag{4.5}$$

This "summing away" is based on Fubini:

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{\sim 2^k}|B(0,1)\cap T_i\cap T_j\cap E_{2^k}\cap T_l| \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{\sim 2^k}\int_{B(0,1)\cap T_i\cap T_l\cap E_{2^k}}1_{T_j} \\
&= \sum_{i=1}^{N}\sum_{l=1}^{\sim 2^k}\int_{B(0,1)\cap T_i\cap T_l\cap E_{2^k}}\sum_{j=1}^{N}1_{T_j} \\
&\lesssim \sum_{i=1}^{N}\sum_{l=1}^{\sim 2^k}\int_{B(0,1)\cap T_i\cap T_l\cap E_{2^k}}2^k \\
&= \sum_{i=1}^{N}\sum_{l=1}^{\sim 2^k}2^k|B(0,1)\cap T_i\cap E_{2^k}\cap T_l|
\end{aligned}$$

Now, it follows from the lemma 1.3 that we have

$$|B(0,1)\cap T_j\cap T_l\cap E_{2^k}|\lesssim 2^{-k/(n-1)}\delta^{n-1}, \tag{4.6}$$

for $i\neq l$. Thus, the claim (4.2), follows from the equations (4.3), (4.4), (4.5) and (4.6).

## 5. The proof the general case

We divide each $\delta$-tube to $L$ paraller $\delta'$-tubes. So we have

$$|E_{2^k}|\sim|\{x\in\mathbf{R}^n|2^k\leq\sum_{i=1}^N\sum_{j=1}^L 1_{T_{ij}^{\delta'}}1_{B(0,1)}\leq 2^{k+1}\}|.$$

Now, we define

$$E'_{j2^k}:=\{x\in\mathbf{R}^n|2^k\leq\sum_{i=1}^N 1_{T_ij^{\delta'}}1_{B(0,1)}\leq 2^{k+1}\}.$$

Thus,

$$\sum_{j=1}^L|E'_{j2^k}|\sim|E_{2^k}|.$$

We make $\delta'$ so small that we have linelike intersections, in other words

$$E'_{2^k}\cap T_l^{\delta'}\cap T_m^{\delta'}\subset\bigcap_{j=1}^{2^k}T_{ij}^{\delta'}.$$

So we have

$$|E'_{2^k}|\lesssim 2^{-kn/(n-1)}N\delta'^{(n-1)}$$

via previous theorem 1.1. And we have

$$|E_{2^k}|\sim\sum_{j=1}^L|E'_{j2^k}|\lesssim 2^{-kn/(n-1)}NL\delta'^{(n-1)}\sim 2^{-kn/(n-1)}N\delta^{n-1},$$

whuch proves the corollary 1.2.

## References

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