The Generating Function Technique and Algebraic Ordinary Differential Equations

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Abstract

In the past, theorems have shown implementing a (former) power series method to derive solutions to algebraic ordinary differential equations, or AODEs. First, this paper will give a quick synopsis of these "bottom-up" approaches while further elaborating on a recent theorem that established the (modified) generating function technique, or [m]GFT, as a powerful method for solving differentials equations. Instead of building a (formal) power series, the latter method uses a predefined set of Laurent series comprised of product ring-based generating functions to produce an analytic solution. Next, this study will utilize the [m]GFT to create several analytic solutions to a few example AODEs. Ultimately, one will find [m]GFT may serve as a powerful "top-down" method for solving linear and nonlinear AODEs.

1.) Introduction

AODEs use differential algebra to define differential equations with only one independent variable [1]. Numbers theory and computer-based algebra are extensively utilized and supported by this field. Finally, AODEs have many formulations, such as differential Galois theory and modules (i.e., *M*, *D*, etc.).

Most methods for solving AODEs involve a "bottom-up" approach regarding a formal power series. In other words, an individual tries to establish an analytic solution by finding a pattern within the leading coefficient of a solitary power series [2]. If the value turns out to be a combinatorial number, then the power series becomes formal [3]. This process often involves enacting many iterations, hence making the "bottom-up" means of acquiring an analytic solution very time-consuming [2,3].

This article will consider a new "top-down" approach for finding solutions to AODEs. It is well-known that (formal) power series can form a new analytic function [4]. For instance, GFT, which incorporates a set of Laurent series of product ring-based formal power series or generating functions, can be used to discover analytic solutions to both linear and nonlinear partial differential equations [5]. This method might be the pinnacle of power series methods to develop new functions; hence, [m]GFT is viewed as a “top-down” means for solving differential equations, such as AODEs, since it uses preformed generating functions to find analytic solutions.

There are several sections in this paper. Section two will have a more thorough discourse on methods and theorems which implement a "bottom-up" basis for deriving analytic solutions to AODEs. On the other hand, the theorem explaining why [m]GFT, an effective “top-down” instrument in solving nonlinear partial differential equations, will be further elucidated and expanded upon to show it as a method for finding solutions AODEs in section three. Section four will show the application of [m]GFT on a few examples of AODEs. Finally, the conclusion, or section five, will gleam a terse description and beneficial facets of [m]GFT.

2.) (Formal) Power Series and their solutions to AODEs

Consider the following power series, also known as a Taylor series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If x_0, or the center of the series, was equal to null or zero, then the above expression is considered a Maclaurin series. Some coefficients of a power series incorporated a division by factorial n. It was the main purgative for an individual to define the value of this coefficient for each serial term [6]. Also, the Cauchy-Hadamard theorem stated that power series converged at specific values [7,8]: when a series was deemed convergent, it formed an analytic function [9].

There were many algorithms and methods for establishing power series solutions to various differential equations via iterative or “bottom-up” means. For instance, N. Thieu Vo and associates devised an algorithm that helped an individual iteratively access each coefficient of a prospective power series solution to an AODE [10]. Building an analytic solution via single power series is generally time-consuming, but others in the field are finding ways to accelerate the process [11].

3.) The modified Generating Function Technique Revisited

The central theorem that established the GFT as a method for solving (nonlinear) PDEs claimed that a Laurent series of formal power series derived analytic solutions to many (nonlinear) differential equations [5]. The theorem suggested that formal power series within the set of Laurent series are polynomial rings. Upon applying the polynomial rings within a differential equation, an individual would form a free ideal ring whose generators were necessary to form algebraic equations. Setting these algebraic equations to zero, then solving for as many coefficients and constants as possible would allow the individual to establish analytic solutions to many differential equations.

Definition 3.1. *The predefined set of Laurent series of formal power series or generating functions served as the general solution to an AODE of interest and was a symmetric (Lie) algebra.*

The general solution y of [m]GFT was defined as follows:

$$y(x) = \sum_{i=1}^{2}\sum_{j=-p_s}^{p_s}\left(a(i,j)\left(\sum_{k=0}^{\infty}2S_k(0)^i\left(\prod_l f_l(x)\right)^k\right)^j + b(i,j)\left(\sum_{k=0}^{\infty}2C_k(0)^i\left(\prod_l f_l(x)\right)^k\right)^j\right),$$

where f_l is the l-th auxiliary function, $S_k$and $C_k$were the square root of the Fibonacci and Chebyshev U combinatorial numbers about zero, respectively. Note:

$$S_k(0) = \sin\left(\frac{\pi k}{2}\right),$$

and

$$C_k(0) = \cos\left(\frac{\pi k}{2}\right).$$

The other coefficients accompanying each formal power series were $a(i,j)$ and $b(i,j)$. The formal power series used in the general solution were "complete" polynomial rings.

Note: the general solution y with a Frobenius adjustment[18] is:

$$y(x) = x^r\sum_{i=1}^{2}\sum_{j=-p_s}^{p_s}\left(a(i,j)\left(\sum_{k=0}^{\infty}2S_k(0)^i\left(\prod_l f_l(x)\right)^k\right)^j + b(i,j)\left(\sum_{k=0}^{\infty}2C_k(0)^i\left(\prod_l f_l(x)\right)^k\right)^j\right).$$

Definition 3.2. *The auxiliary function f_l, in the primary expression defining the general solution y, was a polynomial ring based upon the dependent variable, or intermediate, x. The dependent variable x was linearized, exponentiated, or hypergeometric transformed.*

Definition 3.3. *The multiplication of auxiliary functions f_l established a polynomial product ring.*

For this paper's ppurpose, we defined the auxiliary function f_l as a linearized, exponentiated, or hypergeometric transformed Laurent polynomial ring l, or:

$$L_l(x) = \alpha_l(0) + x^2\alpha_l(2) + \frac{\alpha_l(-2)}{x^2} + x\alpha_l(1) + \frac{\alpha_l(-1)}{x},$$

or

$$L_l(x) = \alpha_l(0) + x^{3/2}\alpha_l(3) + \frac{\alpha_l(-3)}{x^{3/2}} + x^2\alpha_l(4) + \frac{\alpha_l(-4)}{x^2} + x\alpha_l(2) + \sqrt{x}\alpha_l(1) + \frac{\alpha_l(-1)}{\sqrt{x}} + \frac{\alpha_l(-2)}{x},$$

where α_l was simply a coefficient/constant within an unnormalized Alexander knot polynomial. This algebraic entity added to the topology of the space of future analytic solutions to an AODE that was derived using [m]GFT [12,13]. (Note: the former auxiliary function L_l was used in the rest of this section and for deriving solutions to the example AODEs given in the next section of this study since it possessed a significantly lower computational cost than its counterpart.) To establish the auxiliary function f_l, the Laurent series was either linearized ($l = 1$), exponentiated ($l = 2$), or the hypergeometric $transformed$ ($l = 3$).

Since one considered the composition of formal power series as a set of polynomial rings, (s)he could claim each completed polynomial ring was a symmetric algebra [14].

Lemma and definition 3.4. *Plugging the set of Laurent series described above into an AODE of interest would form another set of symmetric algebras, referred to as Hopf algebras.*

Hopf algebras were known to be skewed polynomial rings called noncommutative principal ideal rings, or R_{PI} [15]. In AODEs, the generator of these structures became apparent when an individual times the common denominator/multiplicative inverse of the greatest common (zero) divisor, or $(R_{\mathrm{GCD}})^{-1}$, of post transformed equation with the transformed equation to yield a particular product ring, called a principal integral/ideal domain, or R_{PID} [16]. In other words, the numerator of a principal ideal ring was a principal ideal domain.

$$R_{\mathrm{PI}} = R_{\mathrm{PID}} R_{\mathrm{GCD}}$$

$$R_{\mathrm{PI}}(R_{\mathrm{GCD}})^{-1} = R_{\mathrm{PID}} R_{\mathrm{GCD}}(R_{\mathrm{GCD}})^{-1}$$

$$R_{\mathrm{PI}}(R_{\mathrm{GCD}})^{-1} = R_{\mathrm{PID}}$$

A principal ideal domain was also known as a free ideal ring, or *fir* [17]. For example, the linearized, exponentiated, and hypergeometric transformed variable x may serve as generators, or $\langle x, e^{L_2(x)}, {}_1F_1(\beta_1; \beta_2; L_3(x))\rangle$, for the RPID produced from some AODEs even with variable coefficients and/or inhomogeneous components. (Note: the general solution may only use part of generator set, like $\left\langle x^p, e^{L_2(x)}\right\rangle$.) The generator is used to derive more algebraic equations or objects from the product ring of the differential equation and its common denominator/multiplicative inverse of zero divisor [5]. These algebraic objects are likely symmetric.

Corollary 3.5. *By setting the algebraic equations derived from the product ring discussed above to null or zero, an individual should be able to define at least one set of coefficients/constants properly. After plugging in these coefficients/constants into the set of Laurent series described in definition 3.1, the last item may become an analytic solution to the AODE of interest.*

One exploited the algebraic equations for the values of the coefficients/constants a, b, α_l, β_1, and β_2 whenever possible after setting all the algebraic equators equal to null or zero. Using computer mathematics software, like Mathematica®, individuals derived sets of known coefficients/constants that yielded analytic solutions to AODEs.

It was important to note that when the AODE contained variable coefficients and inhomogeneous components, either an exponential or trigonometric function, an individual limited the auxiliary function f_l as an exponentiated expression of the independent variable x, or let $l = 2$.

Mathematica® was used to derive solutions to the following AODEs. Thus, one can follow most of the work in this paper by examining the Mathematica® spreadsheet.

4.) Examples

This study section will consider and solve three examples AODEs found within the Kamke set [18]. The first equation, ODE No. 29, in the paper is as follows:

$$F(x, y, y') = y'(x) - xy(x)^2 - 3xy(x) = 0.$$

By setting the domain of p_s of the predefined set of Laurent series, a composition of formal power series between [0,2], then plugging it into the above equation, one obtains a significant expression involving the variable x. Next, (s)he times the common denominator/multiplicative inverse of the greatest common (zero) divisor of the expression with the original equation/principal ideal ring to derive a product ring, or free ideal ring. The coefficients associated with the generator $\langle x, e^{x(\alpha_2(1)+\alpha_2(2)x)}, F_1(\beta_1; \beta_2; \alpha_3(0) + x(\alpha_3(1) + x\alpha_3(2)))\rangle$ produces one hundred fifty-nine algebraic equations. After setting these algebraic equations to null or zero, one derives two hundred seventy-nine sets of coefficients/constants, thus possible analytic solutions. The expression below is an example of an analytic solution derived for AODE given above:

$$y(x) = -\frac{3\alpha_1(0)e^{\alpha_2(0)+\frac{3x^2}{2}}\,{}_1F_1(\beta_1; \beta_2; \alpha_3(0))}{\alpha_1(0)e^{\alpha_2(0)+\frac{3x^2}{2}}\,{}_1F_1(\beta_1; \beta_2; \alpha_3(0)) + i}$$

Another AODE from the Kamke set is ODE No. 1115:

$$F(x, y, y', y'') = xy''(x) - (3x - 2)y'(x) + (3 - 2x)y(x) = 0.$$

By setting the domain of p_s predefined set of Laurent series consisting of formal power series also between [-1,0], then plugging it into the above equation, one obtains a significant expression involving the variable x. Again, (s)he times the common denominator/multiplicative inverse of the greatest common (zero) divisor of the expression and the equation itself to derive a product ring, or free ideal ring—the coefficients associated with the generator $\left\langle x, e^{\frac{\alpha_2(-1)+x(\alpha_2(0)+\alpha_2(1)x)}{x}},\ {}_1F_1\left(\beta_1;\beta_2;\frac{\alpha_3(-1)}{x}+\alpha_3(0)+x\alpha_3(1)\right)\right\rangle$ yield thirty-six algebraic equations. After solving the coefficients/constants, one derived twenty-seven sets of possible solutions. The following is an example of an analytic solution derived by [m]GFT for the above AODE:

$$y(x)=a(1,-1)\ {}_1F_1\left(1-\frac{6}{\sqrt{17}};2;\sqrt{17}x\right)e^{\alpha_2(0)-\frac{1}{2}(\sqrt{17}-3)x}.$$

The final AODE from the Kamke set is ODE No. 1700:

$$F(x,y,y',y'')=y(x)y''(x)-y'(x)^2+1=0.$$

By setting the domain of p_s predefined set of Laurent series comprised of formal power series again between [-1,0], then plugging it into the above equation, one produces a significant expression involving the variable x. Again, (s)he multiplies the common denominator/multiplicative inverse of the greatest common (zero) divisor of the expression and the equation itself to derive a product ring, or free ideal ring. The coefficients associated with the generator $\langle x, e^{\alpha_2(1)x},\ {}_1F_1(\beta_1;\beta_2;x\alpha_3(1))\rangle$ establish one hundred thirty-seven algebraic equations. These sets of coefficients/constants generate thirty-one different results. An example of an analytic solution derived by [m]GFT for the above AODE is as follows:

$$y(x)=\frac{(a(1,-1)+a(2,-1))e^{\frac{x}{\sqrt{a(2,-1)^2-a(1,-1)^2}}}}{2\alpha_1(0)}+\frac{1}{2}\alpha_1(0)(a(1,-1)-a(2,-1))e^{-\frac{x}{\sqrt{a(2,-1)^2-a(1,-1)^2}}}.$$

5.) Conclusion

Unlike many "bottom-up" methods that build analytic solutions from solitary formal power series, [m]GFT uses a predefined set of Laurent series of generating functions comprised of linearized, exponentiated, and hypergeometric transformed polynomial product rings to establish a solution to differential equations. Once applied to differential equations, like AODEs, it also establishes Hopf symmetric algebras. Finally, [m]GFT is not only a powerful tool for solving (nonlinear) partial differential equations, but it possesses great potential as a means for finding analytic solutions to many linear and nonlinear differential equations, unlike other methods.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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