$\pi/(2+s_n+c_n)$

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Abstract
We give some formulas for $\frac{\pi}{(2+s_n+c_n)}$, $n=1,2,3,\ldots$

Introduction

Recall that

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right)$$

$$s_n=2\sin\left(\frac{\pi}{2^{n+1}}\right)=\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n-radicals}\quad ,n=1,2,3,\ldots$$

$$c_n=2\cos\left(\frac{\pi}{2^{n+1}}\right)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n-radicals}\quad ,n=1,2,3,\ldots$$

The Gauss hypergeometric function is defined by

$$F(a,b,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\,z^n\quad ,|z|<1$$

where

$$(a)_n=a(a+1)(a+2)\ldots(a+n-1)\ ,n=1,2,3,\ldots,\text{ and }(a)_0=1$$

In this note we give some formulas for $\frac{\pi}{(2+s_n+c_n)}$, $n=1,2,3,\ldots$.

Formulas

Entry 1. For $n = 1,2,3, \ldots$, we have

$$\begin{aligned}\frac{\pi}{2+s_n+c_n} = {} & \frac{2^{n-1}}{2^n+1}F(1,1+2^{-n},2+2^{-n},-1) \\ & + \frac{2^{n-1}}{2^n+1}F\left(1,\frac{1}{2}+2^{-n-1},\frac{3}{2}+2^{-n-1},-1\right) \\ & - \frac{2^{n-1}}{2^{n+1}+1}F(1,1+2^{-n-1},2+2^{-n-1},-1) \\ & - 2^{n-1}F(1,-2^{-n},1-2^{-n},-1) + \frac{2^{n-1}}{2^n-1}F\left(1,\frac{1}{2}-2^{-n-1},\frac{3}{2}-2^{-n-1},-1\right) \\ & + 2^{n-1}F(1,-2^{-n-1},1-2^{-n-1},-1)\end{aligned}$$

Entry 2. For $a > 1$ and $n = 1,2,3, \ldots$, we have

$$\begin{aligned}\frac{\pi}{2+s_n+c_n} = {} & \frac{a^{1+2^{-n}}}{2(1+2^{-n})(1+a)}F\left(1,1,2+2^{-n},\frac{a}{1+a}\right) \\ & + \frac{a^{1+2^{-n}}}{2(1+2^{-n})(1+a^2)}F\left(1,1,\frac{3+2^{-n}}{2},\frac{a^2}{1+a^2}\right) \\ & - \frac{a^{2+2^{-n}}}{2(2+2^{-n})(1+a^2)}F\left(1,1,\frac{4+2^{-n}}{2},\frac{a^2}{1+a^2}\right) \\ & + \sum_{k=0}^{\infty}\left(\frac{a^{-4k-2+2^{-n}}}{4k+2-2^{-n}} - \frac{a^{-4k-3+2^{-n}}}{4k+3-2^{-n}}\right)\end{aligned}$$

Entry 3. For $a > 1$ and $n = 1,2,3, \ldots$, we have

$$\begin{aligned}& \frac{\pi}{2+s_n+c_n} \\ & = \frac{2\,a^{1+2^{-n}}}{(1+a^2)(2+a)}\sum_{k=0}^{\infty}\left(\frac{a}{2+a}\right)^k \sum_{m=0}^{k}\binom{k}{m}\frac{(-2)^m}{m+1+2^{-n}}\ F\left(1,1,\frac{m+3+2^{-n}}{2},\frac{a^2}{1+a^2}\right) \\ & + \sum_{k=0}^{\infty}\left(\frac{a^{-4k-2+2^{-n}}}{4k+2-2^{-n}} - \frac{a^{-4k-3+2^{-n}}}{4k+3-2^{-n}}\right)\end{aligned}$$

Entry 4. For $a > 1$ and $n = 1,2,3, \ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = \frac{2\,a^{1+2^{-n}}}{(2+a)} \sum_{k=0}^{\infty} \left(\frac{a}{2+a}\right)^k \sum_{m=0}^{k} \binom{k}{m} \frac{(-2)^m \left(\sqrt{1+a^2}\right)^{-k-1-2^{-n}}}{m+1+2^{-n}}\; G(m,n,a)$$
$$+ \sum_{k=0}^{\infty} \left(\frac{a^{-4k-2+2^{-n}}}{4k+2-2^{-n}} - \frac{a^{-4k-3+2^{-n}}}{4k+3-2^{-n}} \right)$$

where

$$G(m,n,a) = F\left(\frac{m+1+2^{-n}}{2}, \frac{m+1+2^{-n}}{2}, \frac{m+3+2^{-n}}{2}, \frac{a^2}{1+a^2} \right)$$

Entry 5. For $n = 1,2,3, \ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = 2^{-2+2^{-n}} \sum_{k=0}^{\infty} \frac{F(-k-2^{-n}, k+2-2^{-n}, k+3-2^{-n}, 1/2)}{k+2-2^{-n}}$$
$$+ 2^{-1-2^{-n}} \sum_{k=0}^{\infty} \frac{F(-k-1+2^{-n}, k+1+2^{-n}, k+2+2^{-n}, 1/2)}{k+1+2^{-n}}$$

Entry 6. For $n = 1,2,3, \ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = 2^{-2+2^{-n}} \sum_{k=0}^{\infty} \frac{1}{k+2-2^{-n}} \sum_{m=0}^{k} \frac{2^{-m}(-k+m-2^{-n})_m}{m!}$$
$$+ 2^{-1-2^{-n}} \sum_{k=0}^{\infty} \frac{1}{k+1+2^{-n}} \sum_{m=0}^{k} \frac{2^{-m}(-k+m-1+2^{-n})_m}{m!}$$

Entry 7. For $n = 1,2,3, \ldots$, we have

$$\frac{\pi}{2+s_n+c_n} =$$
$$\sum_{k=0}^{\infty} 2^{-k-3} \sum_{k=0}^{\infty} \binom{2k-m+2}{m} m! \left(\frac{1}{(k-m+2-2^{-n})_{m+1}} + \frac{1}{(k-m+1+2^{-n})_{m+1}} \right)$$

Entry 8. For $n = 1,2,3,\ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m F(k+2,k+m+1+2^{-n},k+m+2+2^{-n},-1)}{k+m+1+2^{-n}} + \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m F(k+2,k+m+2-2^{-n},k+m+3-2^{-n},-1)}{k+m+2-2^{-n}}$$

Entry 9. For $n = 1,2,3,\ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = 2^{-2+2^{-n}}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{-k}}{k+2-2^{-n}}\left(\frac{1+(-1)^{[k/2]}}{2}\right) + \frac{2^{2^{-n}}}{5}\sum_{k=0}^{\infty}2^{-k}\sum_{m=0}^{k}\binom{k}{m}\frac{(-2)^m}{m+1+2^{-n}}F\left(1,1,\frac{m+3+2^{-n}}{2},\frac{4}{5}\right)$$

$$\frac{\pi}{2+s_n+c_n} = 2^{-2+2^{-n}}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{-k}}{k+2-2^{-n}}\left(\frac{1+(-1)^{[k/2]}}{2}\right) + \frac{2^{1+2^{-n}}}{9}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k\sum_{m=0}^{k}\binom{k}{m}\frac{(-2)^m}{2m+1+2^{-n}}F\left(1,1,2m+2+2^{-n},\frac{2}{3}\right)$$

Entry 10. For $n = 1,2,3,\ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = 2^{-2+2^{-n}}\sum_{k=0}^{\infty}\frac{2^{-k}}{k+2-2^{-n}}\sum_{m=0}^{k}\frac{(-2^{-n})_{k-m}}{(k-m)!}c_m + 2^{-1-2^{-n}}\sum_{k=0}^{\infty}\frac{2^{-k}}{k+1+2^{-n}}\sum_{m=0}^{k}\frac{(-1+2^{-n})_{k-m}}{(k-m)!}c_m$$

where

$$c_m = 2c_{m-1} - 2c_{m-2}\ ,\ c_0 = 1, c_1 = 2$$

$$c_{4m+3} = 0\ ,\ m = 0,1,2,3,\ldots$$

$$c_m = \sum_{r=0}^{[m/2]}(-1)^r\binom{m-r}{r}2^{m-r}\ ,\ m = 0,1,2,3,\ldots$$

$$c_m = \left(\frac{1+i}{2}\right)\left((1-i)^m - i\,(1+i)^m\right)\ ,\ m = 0,1,2,3,\ldots;\ i = \sqrt{-1}$$

Entry 11. For $n = 1,2,3, \ldots$,and $i = \sqrt{-1}$ we have

$$\frac{\pi}{2+s_n+c_n} = \sum_{k=0}^{\infty} 2^{-k-2}\left(\frac{k!}{(1+2^{-n})_{k+1}} F_1\left(k+1,1,1,k+2+2^{-n},\frac{1+i}{2},\frac{1-i}{2}\right) + \frac{k!}{(2-2^{-n})_{k+1}} F_1\left(k+1,1,1,k+3-2^{-n},\frac{1+i}{2},\frac{1-i}{2}\right)\right)$$

$$\frac{\pi}{2+s_n+c_n} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1-i}{2}\right)^k\left(\frac{k!}{(1+2^{-n})_{k+1}} F_1\left(k+1,1,1,k+2+2^{-n},\frac{1+i}{2},\frac{1}{2}\right) + \frac{k!}{(2-2^{-n})_{k+1}} F_1\left(k+1,1,1,k+3-2^{-n},\frac{1+i}{2},\frac{1}{2}\right)\right)$$

$$\frac{\pi}{2+s_n+c_n} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1+i}{2}\right)^k\left(\frac{k!}{(1+2^{-n})_{k+1}} F_1\left(k+1,1,1,k+2+2^{-n},\frac{1}{2},\frac{1-i}{2}\right) + \frac{k!}{(2-2^{-n})_{k+1}} F_1\left(k+1,1,1,k+3-2^{-n},\frac{1}{2},\frac{1-i}{2}\right)\right)$$

$$\frac{\pi}{2+s_n+c_n} = \sum_{k=0}^{\infty} 2^{-k-2}\sum_{m=0}^{k} 2^m\left(\frac{k!}{(1+2^{-n})_{k+1}} F_1\left(k+1,m+1,m+1,k+2+2^{-n},\frac{i}{\sqrt{2}},-\frac{i}{\sqrt{2}}\right) + \frac{k!}{(2-2^{-n})_{k+1}} F_1\left(k+1,m+1,m+1,k+3-2^{-n},\frac{i}{\sqrt{2}},-\frac{i}{\sqrt{2}}\right)\right)$$

where F_1 is the Appell hypergeometric function

$$F_1(a,b,c,d,x,y) = \sum_{m,k=0}^{\infty}\frac{(a)_{m+k}(b)_m(c)_k}{(d)_{m+k}m!\,k!}x^m y^k \quad, max(|x|,|y|) < 1$$

Entry 12. For $n = 1,2,3, \ldots$, we have

$$\frac{\pi}{2+s_n+c_n} = \frac{1}{4}\left(\psi\left(\frac{2+2^{-n}}{4}\right) - \psi\left(\frac{1+2^{-n}}{4}\right) + \psi\left(\frac{3-2^{-n}}{4}\right) - \psi\left(\frac{2-2^{-n}}{4}\right)\right)$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, $z \neq 0,-1,-2, \ldots$, is the Psi function.

Entry 13. For $n = 1,2,3, ...$, we have

$$\frac{\pi}{2+s_n+c_n} = \frac{4}{9}\sum_{k=0}^{\infty} 3^{-k}\sum_{m=0}^{k}\sum_{r=0}^{k-m}(-2)^r\binom{k-m}{r}\sum_{s=0}^{m}(-2)^s\binom{m}{s}\left(\frac{1}{r+2s+1+2^{-n}} + \frac{1}{r+2s+2-2^{-n}}\right)$$

Entry 14. For $n = 1,2,3, ...$, we have

$$\frac{\pi}{2+s_n+c_n} = 2^n\int_0^w\left(\left(-\frac{1}{3}-\frac{2}{3}\sqrt{\frac{3-2x}{x}}\cos\left(\frac{2\pi}{3}+\frac{\theta(x)}{3}\right)\right)^{2^{-n}} - \left(-\frac{1}{3}-\frac{2}{3}\sqrt{\frac{3-2x}{x}}\cos\left(\frac{4\pi}{3}+\frac{\theta(x)}{3}\right)\right)^{2^{-n}}\right)dx$$

where

$$\theta(x) = \cos^{-1}\left(\frac{9+20x}{6-4x}\sqrt{\frac{x}{3-2x}}\right)$$

$$w = \frac{s^2}{1+s+s^2+s^3}$$

$$s = \frac{1}{3}\left(27-3\sqrt{78}\right)^{1/3} + 3^{-2/3}\left(9+\sqrt{78}\right)^{1/3}$$

References

1. Andrews, G.E., Number Theory. Dover, New York, 1994.
2. Arndt, J., and Haenel, C., π unleashed. Springer-Verlag, 2001.
3. Jolley, L.B.W., Summation of Series. 2nd ed., Dover, 1961.
4. Kalman, D., Six ways to sum a series. College Math. J., 24, 1993.
5. Koepf, W., Hypergeometric Summation. Advances Lectures in Mathematics, Vieweg, 1998.
6. Olver, F.W.J., et al., NIST Handbook of Mathematical Functions, Cambridge University Press , 2010.