ON FINITE GROUPS AND GALOIS THEORY

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ABSTRACT. We comment on Artin's reformulation of Galois Theory incorporating MacLane's non-abelian extensions theory, and Eilenberg's Category Theory ideology.[1]

CONTENTS

1. INTRODUCTION

Abstract groups, axiomatically, belong to two kingdoms: commutative (Abelian) $Z-modules$ on one hand, obedient under the action of some non-commutative transformation group (on the other hand). Of course, for classification purposes for instance, one of the major breakthroughs of 20th century [1], the abstract approach is mandatory.

Understanding groups require the Lie-connection with its generators, the so called Lie Algebra and Lie Groups Theory.

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For the more complex kingdom of non-commutative groups, there are three subdivisions of Lie type, according to the type of coefficients [1]:

Classical Groups	*Chevalley Groups*	*Sporadic Groups*
$R = Q_\infty, C, H$	$GL(n, F_p)$	$``S(n, F_1)''$

In this article we focus on Galois Theory relating algebraic number fields and groups of permutations, "upgrading" Artin's classical presentation by considering short exact sequences of group extensions with possibly non-abelian kernel, in order to address the problem of splitting of primes, in addition to the classical emphasis on Galois subgroups-subfields correspondence.

We will also try to separate the group theoretical aspects, belonging to the arithmetic-geometric aspects of number systems, for example dissecting the familiar number system $(Z/nZ, +, \cdot)$ into $(Z/nZ, +)$ as a "discrete space of vectors" with a geometry given by its Abelian automorphisms $(Z/nZ^\times, \cdot)$, since it pedagogically corresponds via *linearization* (e.g. group ring construction), to the cyclotomic numbers and their Galois groups. By Weber Theorem this is enough to make quite important points for Galois Theory, corresponding to the finite subgroups of $SO(2)$.

We will hint only to the non-commutative case, where the 3D non-commutative finite groups enter the picture, as for example the historically famous Kleinian investigation of the icosians and icosahedral group in connection with solvability of the 5-th degree polynomial equation.

At this stage it is exiting to foresee entering the realm of binary point groups as Weyl groups and root systems for the exceptional Lie algebras E_6, E_7, E_8, and beyond F, G, H, with their associated sporadic groups, hoping for gaining some unified viewpoint to relate all the above three subdivisions.

Indeed, as known from ADE-classification, one side of the "big picture" refers to Klein singularities, i.e. orbifolds $C^n/Gamma$ of finite subgroups of $SU(n)$ [3], witch correspond to integral lattices in *Quaternionic Number Theory*, the same lattices leading to the exceptional Lie algebras and sporadic groups. The connection with cyclotomic theory seems to be deeper, reflected in the ubiquitous Gauss and Eisenstein integral quadratic extensions, as precisely the simply laced case of ADE Dynkin Diagrams.

The main results of the paper include: 1) a reformulation of $USTs$ of prime decompositions in terms of group extensions, via Jordan-Holder Theorem; 2) applications

[1] The hint towards Tate's field with one element F^1, as a unifying principle for including say permutations as representations on par with linear representations, remains to be justified later on.

of (1) to Galois Theory, incorporating galois covering maps into what we will call Galois-Riemann Theory (see [4]).

2. On Finite Groups of Lie Type

Regarding the above diagram, more importantly is the distinction, at second glance, between "characteristic 0" Lie groups, noting that the real numbers $R = Q_\infty$ are the *analytic completion* [2] (or Cauchy sequences, Dedekind cuts), and the "finite characteristic", which is the "infinitesimal level" for the completion to the p-adic numbers Q_p.

Since the p-adic integers Z_p are deformations of the formal series with finite fields as coefficients $F_p[[x]]$, part of a generalization Lie Theory by Tate, Liubin etc., the above comparison of "groups of Lie type" is of course, partially "unfair". For a Deformation Theory presentation of p-adic numbers, as "bases spaces", and their relation with field extensions, as "vector bundles", see [6].

(Algebraic)
Extensions

$$\begin{array}{ccccccccc}
\bar{F}_p & & & & & & & & \\
\dots & & & & & & & & \\
\uparrow & & & & & & & & \\
F_{p^2} & \longleftarrow & Z/p[\xi] & \longleftarrow & Z/p^2[\xi] & \twoheadleftarrow & \dots & & Q_q = Z_p[\xi] \\
\uparrow & & \uparrow & & & & & & \uparrow \\
F_p & \twoheadleftarrow & Z/pZ & \twoheadleftarrow & Z/p^2 & \twoheadleftarrow & \dots & \twoheadleftarrow & Z_p
\end{array}$$

(Algebraic)
Deformations

Further considerations will be provided in the conclusion and in a separate article.

In what follows we will focus on Galois Theory, especially the "2D"-Abelian case of cyclotomic extensions and subfields.

3. Galois Theory and Group Extensions

We analyse normal subfield extensions $k \to K \to L$, and correspondingly the Galois group extension $H- > G- > G/H$, where the normal subgroup H corresponds to K.

The point of view adopted is that of *relative homotopy theory*, studying the relative homotopy groups $\pi_n(X;Y)$, rather then the "unspecified" $D = \{*\}$ homotopy groups $\pi_n(X)$. In the case of Galois Theory (Galois covers), $n = 1$ and one considers the extension L/k relative to the extension K/k.

[2] in the "wrong direction" of the carrying over 2-cocyle, and the "natural" completion in the opposite direction

Remark 3.1. From a category Theory point of view, we will consider on the *Sets* category side, where groups live, short exact sequences $0 \to N \to E \to G \to 0$, as previously announced. Fields "live" on the "vector spaces" side, where we get through functors like the free vector space associated to a basis, with corresponding forgetful functor as adjoint, or group ring and group of units as a pair of adjoint functors. This will be kept lightly in the background, not required as a background for the reader.

3.1. **From Arithmetic to Algebraic via Linearization.** The modular *arithmetic* takes place in the ring $(Z/nZ, +, \cdot)$. The *Klein geometry* interpretation considers the multiplicative structure $G = (Z/nZ^\times, \cdot)$ acting by automorphisms on the Abelian group $A = (Z/nZ, +)$, as a discrete space of vectors [3]

The additive and multiplicative characters map the "arithmetic site" to the "algebraic site" of complex numbers, with its 2D-geometry "on top":

$$e : A \times A \to U(n) \to C^\times, \quad e_t(k) = exp(2\pi itk/n)$$

$$\sigma : G \to Gal(Q(\zeta_n)/Q), \quad \sigma_t(\zeta) = \zeta^t.$$

Via taking linear combinations we get the cyclotomic ring of integers $Z[A]$ with $Z-mod$ lattice basis the n-th roots of unity $1, \zeta ... \zeta^{n-1}$, and the corresponding group ring of the Galois group of automorphisms $Q[Gal]$.

The fields ("number systems" point of view) are just the corresponding fraction fields, usually emphasized in traditional texts.

3.2. **On the structure of groups.** The structure of groups can be analyzed by considering the Hasse diagram of its subgroups (POSet Category), together with the Jordan-Holder Theorem which provides the composition series describing how the group is obtained via extensions from its factors (Towers of group extensions as paths subject to a "homotopy" structure, via isomorphisms: the "uniqueness" in JHT).

4. Applications to the Decomposition of Primes

The main point is that the decomposition of primes in field extensions (or rather in the corresponding extensions of lattices, i.e. in the category $Z - Mod$ of Z-modules), corresponds to *divisions of groups relative to a subgroup*, which in the case of Galois theory is normal, hence corresponding to a group extension.

The details are left to the reader.

5. Conclusions

In this preliminary research program, the connection between finite group theory and Galois theory is considered. A few ideas and facts are stated, yet to be developed in the future.

[3]A set with a coordinate system: a number of Hamiltonian paths (rank of the group) vs. a toric discret manifold, via Chinese Remainder Theorem.

References

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