

On Prime Numbers Between kn And $(k+1)n$

Wing K. Yu

Abstracts

In this paper along with the previous studies on analyzing the binomial coefficients, we will complete the proof of a theorem. The theorem states that for two positive integers n and k , if $n \geq (k - 1)$, then there always exists at least a prime number p such that $kn < p \leq (k + 1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$.

Keywords

Bertrand's Postulate - Chebyshev's Theorem, Legendre's Conjecture, Oppermann's Conjecture

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1. Introduction

The Bertrand-Chebyshev's theorem States that for any positive integer n , there always exists a prime number p such that $n < p \leq 2n$. Pafnuty Chebyshev proved this in 1850 [1]. In 2006, M. El Bachraoui [2] extended the theorem and proved that for any positive integer n , there exists a prime number p such that $2n < p \leq 3n$. In 2011, Andy Loo [3] proved that when $n \geq 2$, there are prime numbers in the interval $(3n, 4n)$. In 2013, Vladimir Shevelev et al. [4] proved that when the integer $k \leq 100,000,000$, only $k = 1,2,3,5,9,14$, for all $n \geq 1$, the interval $(kn, (k + 1)n)$ contains prime numbers. This raises the question: for all $k \geq 1$, under what conditions does the interval $(kn, (k + 1)n)$ contain prime numbers? Previously, the author partially answered this question in the paper [5] by analyzing the binomial coefficients $\binom{\lambda n}{n}$ where λ is a positive integer. In that paper, the author proved that when $n \geq (\lambda - 2) \geq 25$, i.e., when $n \geq (\lambda - 2)$ and $\lambda \geq 27$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. In this article, we will use the same method to complete the entire work on this problem. We will prove that when $n \geq (\lambda - 2)$ and $\lambda \geq 3$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. Then, by converting λ to k , we prove that for two positive integers n and k , when $n \geq (k - 1)$, there is always at least a prime number p such that $kn < p \leq (k + 1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$.

We will use the same definition and concepts from [5] in this section and in section 2. In section 3, we will prove that for λ from 3 to 26, when $n \geq (\lambda - 2)$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. In section 4, we will convert λ to k , and complete this article.

Definition: $\Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\}$ denotes the prime number factorization operator of the integer expression $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \geq p > b$. In this operator, p is a prime number, a and b are real numbers, and $\lambda n \geq a \geq p > b \geq 1$.

It has some properties:

$$\text{It is always true that } \Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\} \geq 1 \tag{1.1}$$

If there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$, then $\Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\} = 1$, or vice versa, if $\Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\} = 1$, then there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. (1.2)

If there is at least one prime number in $\binom{\lambda n}{n}$ in the range of $a \geq p > b$, then $\Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\} > 1$, or vice versa, if $\Gamma_{a \geq p > b} \left\{ \binom{\lambda n}{n} \right\} > 1$, then there is at least one prime number in $\binom{\lambda n}{n}$ within the range of $a \geq p > b$. (1.3)

Let $v_p(n)$ be the p -adic valuation of n , the exponent of the highest power of p that divides n . We define $R(p)$ by the inequalities $p^{R(p)} \leq \lambda n < p^{R(p)+1}$, and determine the p -adic valuation of $\binom{\lambda n}{n}$.

$v_p\left(\binom{\lambda n}{n}\right) = v_p((\lambda n)!) - v_p(((\lambda - 1)n)!) - v_p(n!) = \sum_{i=1}^{R(p)} \left(\left\lfloor \frac{\lambda n}{p^i} \right\rfloor - \left\lfloor \frac{(\lambda - 1)n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right) \leq R(p)$
because for any real numbers a and b , the expression of $\lfloor a + b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \leq p^{R(p)} \leq \lambda n$ **(1.4)**

If $\lambda n \geq p > \lfloor \sqrt{\lambda n} \rfloor$, then $0 \leq v_p\left(\binom{\lambda n}{n}\right) \leq R(p) \leq 1$. **(1.5)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n . Among the first six consecutive natural numbers are three prime numbers 2, 3, and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1 \pmod{6}$

and $p \equiv 5 \pmod{6}$. Thus, $\pi(n) \leq \left\lfloor \frac{n}{3} \right\rfloor + 2 \leq \frac{n}{3} + 2$. **(1.6)**

From the prime number decomposition, when $n > \lfloor \sqrt{\lambda n} \rfloor$,

$$\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \cdot \Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\}$$

$$\text{When } n \leq \lfloor \sqrt{\lambda n} \rfloor, \binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\}$$

$$\text{Thus, } \binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \cdot \Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\}$$

$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} = \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

If $n \geq \lambda - 2$, and there is a prime number p in $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\}$, then

$$p \geq n + 1 = \sqrt{(n + 2)n + 1} > \sqrt{\lambda n}. \text{ From (1.5), } 0 \leq v_p \left(\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) \leq R(p) \leq 1.$$

Thus, if $n \geq \lambda - 2$, every prime number in $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\}$ has a power of 0 or 1. **(1.7)**

Referring to (1.5), $\Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \leq \prod_{n \geq p} p$. It has been proven [6] that for $n \geq 3$, $\prod_{n \geq p} p < 2^{2n-3}$.

When $n = 2$, $\prod_{n \geq p} p = 2^{2n-3}$, then for $n \geq 2$, $\Gamma_{n \geq p > \lfloor \sqrt{\lambda n} \rfloor} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \leq \prod_{n \geq p} p \leq 2^{2n-3}$.

Referring to (1.4) and (1.6), $\Gamma_{\lfloor \sqrt{\lambda n} \rfloor \geq p} \left\{ \frac{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \right\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$.

Thus, for $\lambda \geq 3$ and $n \geq 2$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$ **(1.8)**

2. Lemmas

Lemma 1: For $\lambda \geq 3$ and $n \geq 2$, $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4} \right) \cdot \left(\frac{\lambda}{\lambda - 1} \right)^{\lambda - 1} \right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}} = f_8(n, \lambda)$ **(2.1)**

Proof:

Let a real number $x \geq 3$, and $f_1(x) = \frac{2(2x-1)}{x-1}$;

then, $f_1'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus, $f_1(x)$ is a strictly decreasing function for $x \geq 3$.

Since $f_1(3) = 5$, and $\lim_{x \rightarrow \infty} f_1(x) = 4$, for $x \geq 3$, we have $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$.

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(e^{x \cdot \ln \frac{x}{x-1}}\right)' = e^{x \cdot \ln \frac{x}{x-1}} \cdot \left(x \cdot \ln \frac{x}{x-1}\right)'$

$$f_2'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} + x \cdot \left(\ln \frac{x}{x-1}\right)'\right) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} + x \cdot \frac{x-1}{x} \cdot \frac{x-1-x}{(x-1)^2}\right)$$

$$f_2'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} - \frac{1}{x-1}\right) \tag{2.1.1}$$

$$\ln(2.1.1), \text{ for } x \geq 3, \frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \dots$$

Using the formula: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$, we have

$$\ln \frac{x}{x-1} = \ln \frac{1}{1+\frac{-1}{x}} = -\ln\left(1+\frac{-1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \dots$$

Thus, for $x \geq 3$, $\ln \frac{x}{x-1} - \frac{1}{x-1} < 0$.

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x \geq 3$, $f_2'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} - \frac{1}{x-1}\right) < 0$.

Thus $f_2(x)$ is a strictly decreasing function for $x \geq 3$. Since $f_2(3) = 3.375$ and

$$\lim_{x \rightarrow \infty} f_2(x) = e \approx 2.718, \text{ for } x \geq 3, 3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e \tag{2.1.2}$$

Since for $x \geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$$f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$$

$$\text{When } x = \lambda \geq 3, \text{ we have } \frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^\lambda \tag{2.1.3}$$

$$\text{When } \lambda \geq 3 \text{ and } n = 2, \binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1) \tag{2.1.4}$$

$$\frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n - \lambda + 1}} = \frac{\lambda^{2\lambda - \lambda + 1}}{2(\lambda-1)^{2(\lambda-1) - \lambda + 1}} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^\lambda \tag{2.1.5}$$

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \geq 3$, referring to (2.1.4) and (2.1.5), when $\frac{\lambda(\lambda-1)}{2}$

multiplies both sides of (2.1.3), we have

$$\left(\frac{\lambda(\lambda-1)}{2}\right) \left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n} > \left(\frac{\lambda(\lambda-1)}{2}\right) \left(\frac{\lambda}{\lambda-1}\right)^\lambda = \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n - \lambda + 1}}$$

$$\text{Thus, } \binom{\lambda n}{n} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n - \lambda + 1}} \text{ when } \lambda \geq 3 \text{ and } n = 2. \tag{2.1.6}$$

By induction on n , when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n - \lambda + 1}}$ is true for n , then for $n + 1$,

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n + \lambda}{n+1} = \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)(\lambda n + 1)}{(\lambda n + \lambda - n - 1)(\lambda n + \lambda - n - 2) \cdots (\lambda n - n + 1)(n + 1)} \cdot \binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)(\lambda n + 1)}{(\lambda n + \lambda - n - 1)(\lambda n + \lambda - n - 2) \cdots (\lambda n - n + 1)(n + 1)} \cdot \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)}{(\lambda n + \lambda - n - 1)(\lambda n + \lambda - n - 2) \cdots (\lambda n - n + 1)} \cdot \frac{\lambda n + 1}{n} \cdot \frac{1}{(n + 1)} \cdot \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}}$$

Notice $\frac{\lambda n + 1}{n} > \lambda$, and $\frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)}{(\lambda n + \lambda - n - 1)(\lambda n + \lambda - n - 2) \cdots (\lambda n - n + 1)} > \left(\frac{\lambda}{\lambda - 1}\right)^{(\lambda - 1)}$

because $\frac{\lambda n + \lambda}{\lambda n + \lambda - n - 1} = \frac{\lambda}{\lambda - 1}$; $\frac{\lambda n + \lambda - 1}{\lambda n + \lambda - n - 2} > \frac{\lambda}{\lambda - 1}$; \cdots ; $\frac{\lambda n + 2}{\lambda n - n + 1} > \frac{\lambda}{\lambda - 1}$. Thus,

$$\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda - 1}}{(\lambda - 1)^{(\lambda - 1)}} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n + 1)} \cdot \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}} = \frac{\lambda^{\lambda(n+1) - \lambda + 1}}{(n + 1)(\lambda - 1)^{(\lambda - 1)(n + 1) - \lambda + 1}} \quad (2.1.7)$$

From (2.1.6) and (2.1.7), we have for $\lambda \geq 3$ and $n \geq 2$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}}$

Referring to (1.8), for $\lambda \geq 3$ and $n \geq 2$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot 2^{2n - 3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$.

Then, $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot 2^{2n - 3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}}$.

Since when $\lambda \geq 3$ and $n \geq 2$, then $2^{2n - 3} > 0$ and $(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} > 0$.

$$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} \cdot 2^{2n - 3} \cdot n(\lambda - 1)^{(\lambda - 1)n - \lambda + 1}} = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda - 1}\right)^{\lambda - 1} \right)^{(n - 1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}}$$

Thus, Lemma 1 is proven.

Lemma 2: When $n \geq (\lambda - 2) \geq 9$, $f_3(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda - 1}{4}\right) \cdot e \right)^{(n - 1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}}$ is an increasing function with

respect to the product of λn and with respect to n . (2.2)

Proof:

Referring to (2.1.2), when $\lambda \geq 3$, $\left(\frac{\lambda}{\lambda - 1}\right)^\lambda \geq e$. From (2.1), when $\lambda \geq 3$ and $n \geq 2$,

$$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > f_8(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda - 1}{4}\right) \cdot \left(\frac{\lambda}{\lambda - 1}\right)^\lambda \right)^{(n - 1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}} \geq \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda - 1}{4}\right) \cdot e \right)^{(n - 1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}} = f_3(n, \lambda) > 0 \quad (2.2.1)$$

Let $x \geq 2$ and $y \geq 4$ both be real numbers. When $x = y - 2$,

$$f_3(x, y) = \frac{2(x + 2)^2 \cdot \left(\left(\frac{x + 1}{4}\right) \cdot e \right)^{(x - 1)}}{(x + 2) \cdot x^{\frac{\sqrt{x \cdot (x + 2)}}{3} + 3}} > f_4(x) = \frac{2(x + 2)^2 \cdot \left(\left(\frac{x + 1}{4}\right) \cdot e \right)^{(x - 1)}}{(x + 2) \cdot x^{\frac{x + 1}{3} + 3}} > 0 \quad (2.2.2)$$

$$f_4'(x) = f_4(x) \cdot \left(\frac{2}{x+2} + \ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln((x+2) \cdot x) - \frac{10}{3x} - \frac{8}{3(x+2)} \right) = f_4(x) \cdot f_5(x)$$

$$\text{where } f_5(x) = \frac{2}{x+2} + \ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln((x+2) \cdot x) - \frac{10}{3x} - \frac{8}{3(x+2)}$$

$$f_5'(x) = \frac{4x+6}{(x+1)^2 \cdot (x+2)^2} + \frac{x^2+2x-2}{3x(x+1)(x+2)} + \frac{10}{3x^2} + \frac{8}{3(x+2)^2} > 0 \text{ when } x \geq 2.$$

Thus, $f_5(x)$ is a strictly increasing function for $x \geq 2$.

When $x = 9$, $f_5(x) = \frac{2}{9+2} + \ln\left(\frac{9+1}{4}\right) + \frac{4}{3} - \frac{2}{9+1} - \frac{1}{3} \ln(9) - \frac{1}{3} \ln(9+2) - \frac{10}{27} - \frac{8}{33} > 0$. Thus, for $x \geq 9$, $f_5(x) > 0$. Then, $f_4'(x) = f_4(x) \cdot f_5(x) > 0$.

Thus, $f_4(x)$ is a strictly increasing function for $x \geq 9$.

From **(2.2.2)**, when $x = y - 2$, $f_3(x, y) > f_4(x) > 0$. Thus, when $x = (y - 2) \geq 9$ and $xy \geq 99$, then $f_3(x, y)$ is an increasing function respect to the product of xy . **(2.2.3)**

$$\frac{\partial f_3(x, y)}{\partial x} = f_3(x, y) \cdot \left(\ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot \ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x} \right) = f_3(x, y) \cdot f_6(x, y) \quad \text{(2.2.4)}$$

$$\text{where } f_6(x, y) = \ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot \ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}$$

$$\text{When } x = y - 2, \text{ then } f_6(x, y) = f_7(x) = \ln\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot (\ln(x+2) + \ln(x) + 2) - \frac{3}{x}$$

$$\text{When } x \geq 2, f_7'(x) = \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\frac{1}{x+2} + \frac{1}{x} \right) + \frac{\ln(x+2)+\ln(x)+2}{6x\sqrt{x}(x+2)} + \frac{3}{x^2}$$

$$\begin{aligned} f_7'(x) &= \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \frac{x+x+2}{x(x+2)} + \frac{\ln(x+2)+\ln(x)+2}{6x\sqrt{x}(x+2)} + \frac{3}{x^2} \\ &= \frac{1}{x+1} - \frac{1}{3\sqrt{x}} \cdot \frac{x+1}{x\sqrt{x+2}} + \frac{2}{6x\sqrt{x}(x+2)} + \frac{\ln(x+2)+\ln(x)}{6x\sqrt{x}(x+2)} + \frac{3}{x^2} \\ &= \frac{1}{x+1} - \frac{x}{3x\sqrt{x}(x+2)} + \frac{\ln(x+2)+\ln(x)}{6x\sqrt{x}(x+2)} + \frac{3}{x^2} \end{aligned} \quad \text{(2.2.5)}$$

When $x \geq 2$, then $3\sqrt{x(x+2)} + (x+1) > 0$

$$\left(3\sqrt{x(x+2)} + (x+1) \right) \cdot \left(3\sqrt{x(x+2)} - (x+1) \right) = \left(3\sqrt{x(x+2)} \right)^2 - (x+1)^2 = 8x^2 + 16x - 1 > 0$$

Thus, $\left(3\sqrt{x(x+2)} + (x+1) \right) \cdot \left(3\sqrt{x(x+2)} - (x+1) \right) > 0$

$$3\sqrt{x(x+2)} - (x+1) > 0$$

$$3\sqrt{x(x+2)} > (x+1) \text{ then } \frac{1}{x+1} > \frac{1}{3\sqrt{x(x+2)}}$$

When $x \geq 2$, $\left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}} \right) > 0$, and from **(2.2.5)**, $\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x}(x+2)} + \frac{3}{x^2} > 0$.

$$\text{Then } f_7'(x) = \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}} \right) + \frac{\ln(x+2)+\ln(x)}{6x\sqrt{x}(x+2)} + \frac{3}{x^2} > 0.$$

Thus, when $x \geq 2$, $f_7(x)$ is a strictly increasing function.

When $x = (y - 2) \geq 2$, since $f_6(x, y) = f_7(x)$, $f_6(x, y)$ is an increasing function respect to xy .

$$\text{When } x = (y - 2) = 9, f_6(x, y) = \ln\left(\frac{11-1}{4}\right) + 1 - \frac{\sqrt{11}}{6\sqrt{9}} \cdot \ln(99) - \frac{\sqrt{11}}{3\sqrt{9}} - \frac{3}{9} > 0.$$

When $x \geq (y-2) \geq 2$, $\frac{\partial f_6(x,y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{3}{x^2} > 0$.

Thus, when $x \geq (y-2) \geq 9$, $f_6(x, y) > 0$, and it is an increasing function with respect to x and to the product of xy , then, from **(2.2.4)**, $\frac{\partial f_3(x,y)}{\partial x} = f_3(x, y) \cdot f_6(x, y) > 0$.

Thus, when $x \geq (y-2) \geq 9$, $f_3(x, y)$ is an increasing function with respect to x . **(2.2.6)**

Referring to **(2.2.3)** and **(2.2.6)**, when $x \geq (y-2) \geq 9$, then $xy \geq 99$, $f_3(x, y)$ is an increasing function with respect to the product of xy and with respect to x .

Let $x = n$ and $y = \lambda$. Then when $n \geq (\lambda-2) \geq 9$, $f_3(n, \lambda)$ is an increasing function with respect to the product of λn and with respect to n .

Thus, **Lemma 2** is proven.

Lemma 3: When $n \geq 24$ and $26 \geq \lambda \geq 3$, $f_8(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$ is a strictly increasing function respect to n . **(2.3)**

Proof:

Referring to **(2.2.1)**, when $\lambda \geq 3$ and $n \geq 2$, $f_8(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \geq f_3(n, \lambda) > 0$

Let a real number $x = n \geq 2$. When $\lambda \geq 3$, then $f_8(x, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}} > 0$ **(2.3.1)**

When λ is an integer constant in the range of $10 \geq \lambda \geq 3$,

$$\frac{\partial f_8(x, \lambda)}{\partial x} = 2\lambda^2 \cdot \frac{\frac{\partial}{\partial x} \left(\left(\left(\frac{\lambda}{4} \right) \cdot \left(\frac{\lambda}{\lambda-1} \right)^{\lambda-1} \right)^{(x-1)} \right) \cdot \left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3} \right) - \left(\left(\frac{\lambda}{4} \right) \cdot \left(\frac{\lambda}{\lambda-1} \right)^{\lambda-1} \right)^{(x-1)} \cdot \frac{\partial}{\partial x} \left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3} \right)}{\left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3} \right)^2}$$

$$\frac{\partial f_8(x, \lambda)}{\partial x} = f_8(x, \lambda) \cdot \left(\ln \left(\frac{\lambda}{4} \right) + \ln \left(\frac{\lambda}{\lambda-1} \right)^{\lambda-1} - \frac{\sqrt{\lambda} (\ln(x) + \ln(\lambda) + 2)}{6\sqrt{x}} - \frac{3}{x} \right) = f_8(x, \lambda) \cdot f_9(x, \lambda) \text{ where}$$

$$f_9(x, \lambda) = \ln \left(\frac{\lambda}{4} \right) + \ln \left(\frac{\lambda}{\lambda-1} \right)^{\lambda-1} - \frac{\sqrt{\lambda} (\ln(x) + \ln(\lambda) + 2)}{6\sqrt{x}} - \frac{3}{x}$$

$\frac{\partial f_9(x, \lambda)}{\partial x} = \frac{\sqrt{\lambda} \ln(\lambda) + \sqrt{\lambda} \ln(x)}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 1$ and $\lambda > 2$; then, $f_9(x, \lambda)$ is a strictly increasing function respect to x .

When $x = 24$ and $10 \geq \lambda \geq 3$, $f_9(x, \lambda) > 0$. The calculations are below.

$$\text{When } \lambda = 3, f_9(x, \lambda) = \ln \left(\frac{3}{4} \right) + \ln \left(\frac{3}{3-1} \right)^{3-1} - \frac{\sqrt{3} (\ln(24) + \ln(3) + 2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.0283 > 0.$$

$$\text{When } \lambda = 4, f_9(x, \lambda) = \ln\left(\frac{4}{4}\right) + \ln\left(\frac{4}{4-1}\right)^{4-1} - \frac{\sqrt{4}(\ln(24)+\ln(4)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.2814 > 0.$$

$$\text{When } \lambda = 5, f_9(x, \lambda) = \ln\left(\frac{5}{4}\right) + \ln\left(\frac{5}{5-1}\right)^{5-1} - \frac{\sqrt{5}(\ln(24)+\ln(5)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.4744 > 0.$$

$$\text{When } \lambda = 6, f_9(x, \lambda) = \ln\left(\frac{6}{4}\right) + \ln\left(\frac{6}{6-1}\right)^{6-1} - \frac{\sqrt{6}(\ln(24)+\ln(6)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.6113 > 0.$$

$$\text{When } \lambda = 7, f_9(x, \lambda) = \ln\left(\frac{7}{4}\right) + \ln\left(\frac{7}{7-1}\right)^{7-1} - \frac{\sqrt{7}(\ln(24)+\ln(7)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.7183 > 0.$$

$$\text{When } \lambda = 8, f_9(x, \lambda) = \ln\left(\frac{8}{4}\right) + \ln\left(\frac{8}{8-1}\right)^{8-1} - \frac{\sqrt{8}(\ln(24)+\ln(8)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.8044 > 0.$$

$$\text{When } \lambda = 9, f_9(x, \lambda) = \ln\left(\frac{9}{4}\right) + \ln\left(\frac{9}{9-1}\right)^{9-1} - \frac{\sqrt{9}(\ln(24)+\ln(9)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.8755 > 0.$$

$$\text{When } \lambda = 10, f_9(x, \lambda) = \ln\left(\frac{10}{4}\right) + \ln\left(\frac{10}{10-1}\right)^{10-1} - \frac{\sqrt{10}(\ln(24)+\ln(10)+2)}{6\sqrt{24}} - \frac{3}{24} \approx 0.9347 > 0.$$

From **(2.3.1)**, when $x \geq 2$ and $\lambda \geq 3$, $f_8(x, \lambda) > 0$.

Since $\frac{\partial f_8(x, \lambda)}{\partial x} = f_8(x, \lambda) \cdot f_9(x, \lambda)$, when $x = 24$ and $10 \geq \lambda \geq 3$, $\frac{\partial f_8(x, \lambda)}{\partial x} > 0$.

Thus, when $x \geq 24$ and $10 \geq \lambda \geq 3$, $f_8(x, \lambda)$ is a strictly increasing function respect to x .

Let $x = n \geq 24$ and $10 \geq \lambda \geq 3$, $f_8(n, \lambda)$ is a strictly increasing function respect to n . **(2.3.2)**

From **(2.2.1)**, when $\lambda \geq 3$ and $n \geq 2$, $f_8(n, \lambda) \geq f_3(n, \lambda) > 0$. Referring to **(2.2)**, when $\lambda \geq 11$ and $n \geq (\lambda - 2)$, $f_3(n, \lambda)$ is an increasing function with respect to the product of λn and with respect to n .

When $n \geq 24$ and $26 \geq \lambda \geq 11$, since $n \geq (\lambda - 2)$ and $f_8(n, \lambda) \geq f_3(n, \lambda)$, $f_8(n, \lambda)$ is an increasing function with respect to the product of λn and with respect to n . **(2.3.3)**

From **(2.3.2)** and **(2.3.3)**, when $n \geq 24$ and $26 \geq \lambda \geq 3$, $f_8(n, \lambda)$ is an increasing function with respect to n .

Thus, **Lemma 3** is proven.

Lemma 4: When $n \geq n_0 \geq 24$ and $26 \geq \lambda \geq 3$, if $f_8(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$, then

there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(2.4)**

Proof:

From **(2.3)**, when $n \geq n_0 \geq 24$ and $26 \geq \lambda \geq 3$, $f_8(n, \lambda)$ is an increasing function respect to n .

Let integers $m \geq n \geq n_0 \geq 24$.

When $26 \geq \lambda \geq 3$, if $f_8(n_0, \lambda) > 1$, then $f_8(n, \lambda) > 1$, and thus, $f_8(m, \lambda) > 1$; then from **(2.1)**,

$$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > f_8(n, \lambda) > 1, \text{ and } \Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > f_8(m, \lambda) > 1. \quad \text{(2.4.1)}$$

Note that $m \geq n \geq n_0 \geq 24 \geq \lambda - 2$ since $26 \geq \lambda \geq 3$.

From **(1.7)**, when $n \geq \lambda - 2$, every prime number in $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\}$ has a power of 0 or 1;

when $m \geq \lambda - 2$, every prime number in $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\}$ has a power of 0 or 1. **(2.4.2)**

$$\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} =$$

$$= \Gamma_{\lambda m \geq p > (\lambda-1)m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \cdot \prod_{i=1}^{i=\lambda-2} \left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \cdot \Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right)$$

$$\ln \prod_{i=1}^{\lambda-2} \left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right),$$
 for every distinct prime number p in these ranges, the numerator $(\lambda m)!$ has the product of $p \cdot 2p \cdot 3p \cdots ip = (i)! \cdot p^i$. The denominator $((\lambda-1)m)!$ also has the same product of $(i)! \cdot p^i$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to **(1.2)**, $\prod_{i=1}^{\lambda-2} \left(\Gamma_{\frac{(\lambda-1)m}{i} \geq p > \frac{\lambda m}{i+1}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right) = 1$.

Thus, $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} = \Gamma_{\lambda m \geq p > (\lambda-1)m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \cdot \prod_{i=1}^{i=\lambda-2} \left(\Gamma_{\frac{\lambda m}{i+1} \geq p > \frac{(\lambda-1)m}{i+1}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right)$

$$\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} = \prod_{i=1}^{i=\lambda-1} \left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right). \quad (2.4.3)$$

$\prod_{i=1}^{i=\lambda-1} \left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right)$ is the product of $(\lambda-1)$ sectors from $i=1$ to $i=(\lambda-1)$. Each of these sectors is the prime number factorization of the product of the consecutive integers between $\frac{(\lambda-1)m}{i}$ and $\frac{\lambda m}{i}$.

Referring to **(2.4.3)**, when $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > 1$, then $\prod_{i=1}^{i=\lambda-1} \left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right) > 1$.

Referring to **(1.1)**, $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \geq 1$. Thus, when $m \geq n \geq \lambda-2$, at least one of the sectors in $\prod_{i=1}^{i=\lambda-1} \left(\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} \right) > 1$.

Let $\Gamma_{\frac{\lambda m}{i} \geq p > \frac{(\lambda-1)m}{i}} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > 1$ be such a sector and let $m = ni$ where $(\lambda-1) \geq i \geq 1$ from **(2.4.3)**. Thus, when $m = ni \geq n \geq \lambda-2$,

$$\Gamma_{\frac{\lambda ni}{i} \geq p > \frac{(\lambda-1)ni}{i}} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\} > 1. \quad (2.4.4)$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{(\lambda ni) \cdot (\lambda ni - 1) \cdots (\lambda ni - i) \cdots (\lambda ni - 2i) \cdots (\lambda ni - (n-1)i) \cdots (\lambda ni - ni + 1) \cdot ((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{i \cdot (\lambda n) \cdot (\lambda ni - 1) \cdots i \cdot (\lambda n - 1) \cdots i \cdot (\lambda n - 2) \cdots i \cdot (\lambda n - n + 1) \cdots (\lambda ni - ni + 1) \cdot ((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains all the factors of (λn) , $(\lambda n - 1)$, $(\lambda n - 2)$, ..., $(\lambda n - n + 1)$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up all the consecutive integers in the range of $\lambda n \geq p > (\lambda-1)n$ in

$$\frac{(\lambda n)!}{((\lambda-1)n)!}. \text{ Thus, } \frac{(\lambda ni)!}{((\lambda-1)ni)!} \text{ contains } \frac{(\lambda n)!}{((\lambda-1)n)!}.$$

Referring to the definition, all prime numbers in $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the ranges of $\lambda ni \geq p > \lambda n$ and

$(\lambda-1)n > p$ do not contribute to $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\}$, nor does i for $(\lambda-1) \geq i \geq 1$. Only

the prime numbers in the prime factorization of $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the range of $\lambda n \geq p > (\lambda-1)n$

present in $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\}$. Since $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive

integers in this range, $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\}$.

Referring to **(2.4.4)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda ni)!}{((\lambda-1)ni)!} \right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > 1$. **(2.4.5)**

Referring to **(2.4.1)**, **(2.4.3)**, **(2.4.4)**, and **(2.4.5)**, when $26 \geq \lambda \geq 3$ and $n \geq n_0 \geq 24$,

if $f_8(n, \lambda) > 1$, then $f_8(m, \lambda) > 1$ and $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > 1$;

when $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > 1$.

Thus, when $26 \geq \lambda \geq 3$ and $n \geq n_0 \geq 24$, if $f_8(n, \lambda) > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > 1$,

referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$.

Thus, **Lemma 4** is proven.

Lemma 5: When $n \geq (\lambda-2) \geq 25$, there exists at least a prime number p such that

$(\lambda-1)n < p \leq \lambda n$. **(2.5)**

Proof:

Referring to **(2.2)**, when $n \geq (\lambda-2) \geq 9$, $f_3(n, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right) \cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$ is an increasing function

with respect to the product of λn and with respect to n .

Let integers $n_1 = 25$, and $\lambda_1 = 27$. Then $f_3(n_1, \lambda_1) = \frac{2 \cdot 27^2 \cdot \left(\left(\frac{27-1}{4}\right) \cdot e\right)^{(25-1)}}{(25 \cdot 27)^{\frac{\sqrt{675}}{3}+3}} \approx \frac{1.2495E+33}{9.7843E+32} > 1$.

When $m = n = n_1 = (\lambda-2) = (\lambda_1-2) = 25$, $f_3(m, \lambda) = f_3(n, \lambda) = f_3(n_1, \lambda_1) > 1$.

From **(2.2)**, when $m = n = (\lambda-2) \geq 25$, $f_3(n, \lambda) > 1$ and $f_3(m, \lambda) > 1$ since $f_3(n, \lambda)$ is an

increasing function with respect to the product of λn . When $m \geq n \geq (\lambda-2) \geq 25$, then

$f_3(n, \lambda) > 1$ and $f_3(m, \lambda) > 1$ since $f_3(n, \lambda)$ is also an increasing function with respect to n .

Referring to **(2.2.1)**, when $m \geq n \geq (\lambda-2) \geq 25$, $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > f_3(n, \lambda) \geq f_3(n_1, \lambda_1) > 1$;

and $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > f_3(m, \lambda) \geq f_3(n, \lambda) > 1$. **(2.5.1)**

Referring to **(2.4.2)**, when $m \geq n \geq (\lambda-2) \geq 25$, every prime number in $\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\}$ and in

$\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\}$ has a power of 0 or 1. **(2.5.2)**

Referring to **(2.5.1)**, **(2.4.3)**, **(2.4.4)**, and **(2.4.5)**, when $m \geq n \geq (\lambda-2) \geq 25$,

$\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > 1$, and $\Gamma_{\lambda m \geq p > m} \left\{ \frac{(\lambda m)!}{((\lambda-1)m)!} \right\} > 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{(\lambda n)!}{((\lambda-1)n)!} \right\} > 1$;

referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **Lemma 5** is proven. It was proven in [5, pp 1324-1329] with more details.

3. A Prime Number Between $(\lambda-1)n$ and λn when $26 \geq \lambda \geq 3$ and $n \geq \lambda-2$

Proposition 1: For $\lambda = 3$, when $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.1)**

Proof:

Referring to **(2.3)** for $\lambda = 3$, when $n \geq n_0 \geq 24$, $f_8(n, \lambda)$ is a strictly increasing function on n .

Let $n_0 = 83$. When $\lambda = 3$ and $n \geq n_0 = 83$,

$$f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\binom{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1} \right)^{\lambda-1} \right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} = \frac{18 \cdot (1.6875)^{(83-1)}}{(249)^{\frac{\sqrt{249}+3}{3}}} \approx \frac{7.7493E+19}{6.2000E+19} > 1.$$

Thus, for $\lambda = 3$, when $n \geq 83$, $n > \lambda - 2$ and $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$; then, referring to **(2.4)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

For $\lambda = 3$, when $82 \geq n \geq 1 = \lambda - 2$, **Table 1** shows that there exists a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **(3.1)** is proven.

Table 1. For $82 \geq n \geq 1$ and $\lambda = 3$, there is a prime number p such that $2n < p \leq 3n$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p	3	5	7	11	13	17	19	19	23	23	29	29	31	31
n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
p	37	37	41	41	43	43	47	47	53	53	59	59	61	61
n	29	30	31	32	33	34	35	36	37	38	39	40	41	42
p	67	67	71	71	73	73	79	79	83	83	89	89	97	97
n	43	44	45	46	47	48	49	50	51	52	53	54	55	56
p	101	101	103	103	107	107	109	109	113	113	127	127	131	131
n	57	58	59	60	61	62	63	64	65	66	67	68	69	70
p	139	139	149	149	151	151	157	157	163	163	167	167	173	173
n	71	72	73	74	75	76	77	78	79	80	81	82		
p	179	179	181	181	191	191	193	193	197	197	119	199		

Proposition 2: For $5 \geq \lambda \geq 4$, when $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.2)**

Proof:

From **(2.3)**, for $5 \geq \lambda \geq 4$, when $n \geq n_0 \geq 24$, $f_8(n, \lambda)$ is a strictly increasing function on n .

Let $n_0 = 40$. When $n \geq n_0 = 40$ and $5 \geq \lambda \geq 4$, $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$. The calculations are below.

$$\text{When } \lambda = 4, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{32 \cdot (2.3703)^{(40-1)}}{(160)^{\frac{\sqrt{160}}{3}+3}} \approx \frac{1.3258E+16}{8.0503E+15} > 1.$$

$$\text{When } \lambda = 5, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{50 \cdot (3.0518)^{(40-1)}}{(200)^{\frac{\sqrt{200}}{3}+3}} \approx \frac{7.8968E+18}{5.6253E+17} > 1.$$

Thus, for $5 \geq \lambda \geq 4$, when $n \geq 40$, $n > \lambda - 2$ and $f_8(n, \lambda) > 1$; then, referring to **(2.4)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

For $5 \geq \lambda \geq 4$, when $39 \geq n \geq \lambda - 2$, **Table 2** shows that there exists a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **(3.2)** is proven.

Table 2. For $5 \geq \lambda \geq 4$ and $39 \geq n \geq \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
3n	6	9	12	15	18	21	24	27	30	33	36	39	42
p	7	11	13	17	19	23	29	31	31	37	37	41	43
4n	8	12	16	20	24	28	32	36	40	44	48	52	56
p		13	17	23	29	31	37	41	43	47	53	59	61
5n		15	20	25	30	35	40	45	50	55	60	65	70
n	15	16	17	18	19	20	21	22	23	24	25	26	27
3n	45	48	51	54	57	60	63	66	69	72	75	78	81
p	47	53	59	59	61	61	67	67	71	73	79	79	83
4n	60	64	68	72	76	80	84	88	92	96	100	104	108
p	67	71	73	79	79	83	89	97	101	103	107	109	113
5n	75	80	85	90	95	100	105	110	115	120	125	130	135
n	28	29	30	31	32	33	34	35	36	37	38	39	
3n	84	87	90	93	96	99	102	105	108	111	114	117	
p	89	89	97	97	101	101	103	107	109	113	127	131	
4n	112	116	120	124	128	132	136	140	144	148	152	156	
p	113	127	131	137	139	149	151	157	163	173	179	191	
5n	140	145	150	155	160	165	170	175	180	185	190	195	

Proposition 3: For $11 \geq \lambda \geq 6$, when $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.3)**

Proof:

Referring to **(2.3)** for $11 \geq \lambda \geq 6$, when $n \geq n_0 \geq 24$, $f_8(n, \lambda)$ is a strictly increasing function on n .

Let $n_0 = 28$. When $n \geq n_0 = 28$ and $11 \geq \lambda \geq 6$, $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$. The calculations are below.

$$\text{When } \lambda = 6, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{72 \cdot (3.7325)^{(28-1)}}{(168)^{\frac{\sqrt{168}+3}{3}}} \approx \frac{2.0014E+17}{1.9515E+16} > 1.$$

$$\text{When } \lambda = 7, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{98 \cdot (4.4128)^{(28-1)}}{(196)^{\frac{\sqrt{196}+3}{3}}} \approx \frac{2.5033E+19}{3.7501E+17} > 1.$$

$$\text{When } \lambda = 8, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{128 \cdot (5.0930)^{(28-1)}}{(224)^{\frac{\sqrt{224}+3}{3}}} \approx \frac{1.5686E+21}{5.9689E+18} > 1.$$

$$\text{When } \lambda = 9, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{162 \cdot (5.7730)^{(28-1)}}{(252)^{\frac{\sqrt{252}+3}{3}}} \approx \frac{5.8527E+22}{8.1510E+19} > 1.$$

$$\text{When } \lambda = 10, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{200 \cdot (6.4529)^{(28-1)}}{(280)^{\frac{\sqrt{280}+3}{3}}} \approx \frac{1.4602E+24}{9.7946E+20} > 1.$$

$$\text{When } \lambda = 11, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}+3}{3}}} \approx \frac{242 \cdot (7.1328)^{(28-1)}}{(308)^{\frac{\sqrt{308}+3}{3}}} \approx \frac{2.6414E+25}{1.0560E+22} > 1.$$

Thus, for $11 \geq \lambda \geq 6$, when $n \geq 28$, $n > \lambda - 2$ and $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$; then, referring to (2.4), there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

For $11 \geq \lambda \geq 6$, when $27 \geq n \geq \lambda - 2$, Table 3 shows that there exists a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, (3.3) is proven.

Table 3. For $11 \geq \lambda \geq 6$ and $27 \geq n \geq \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

	n	4	5	6	7	8	9	10	11	12	13	14	15
	$5n$	20	25	30	35	40	45	50	55	60	65	70	75
$\lambda=6$	p	23	29	31	37	41	47	53	59	61	67	71	79
	$6n$	24	30	36	42	48	54	60	66	72	78	84	90
$\lambda=7$	p		31	37	47	53	59	61	67	79	83	89	97
	$7n$		35	42	49	56	63	70	77	84	91	98	105
$\lambda=8$	p			47	53	59	67	79	83	89	97	101	109
	$8n$			48	56	64	72	80	88	96	104	112	120
$\lambda=9$	p				59	67	79	83	89	97	107	113	127
	$9n$				63	72	81	90	99	108	117	126	135
$\lambda=10$	p					79	83	97	107	113	127	137	139
	$10n$					80	90	100	110	120	130	140	150
$\lambda=11$	p						97	107	113	127	137	149	157
	$11n$						99	110	121	132	143	154	165

	<i>n</i>	16	17	18	19	20	21	22	23	24	25	26	27
	$5n$	80	85	90	95	100	105	110	115	120	125	130	135
$\lambda=6$	<i>p</i>	83	89	97	101	103	107	113	127	131	137	139	149
	$6n$	96	102	108	114	120	126	132	138	144	150	156	162
$\lambda=7$	<i>p</i>	101	107	113	127	131	137	139	149	151	157	163	167
	$7n$	112	119	126	133	140	147	154	161	168	175	182	189
$\lambda=8$	<i>p</i>	113	127	131	137	149	149	151	163	173	179	191	199
	$8n$	128	136	144	152	160	168	176	184	192	200	208	216
$\lambda=9$	<i>p</i>	139	137	149	163	173	179	191	199	211	223	227	229
	$9n$	144	153	162	171	180	189	198	207	216	225	234	243
$\lambda=10$	<i>p</i>	157	163	173	179	191	199	211	223	227	229	239	251
	$10n$	160	170	180	190	200	210	220	230	240	250	260	270
$\lambda=11$	<i>p</i>	163	179	191	199	211	223	227	239	251	257	269	281
	$11n$	176	187	198	209	220	231	242	253	264	275	286	297

Proposition 4: For $26 \geq \lambda \geq 12$, when $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.4)**

Proof:

Referring to **(2.3)** for $26 \geq \lambda \geq 12$, when $n \geq n_0 \geq 24$, $f_8(n, \lambda)$ is a strictly increasing function on n . Let $n_0 = 25$. When $n \geq n_0 = 25$ and $\lambda = 12$, then

$$f_8(n, \lambda) \geq f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{288 \cdot (7.8126)^{(25-1)}}{(300)^{\frac{\sqrt{300}}{3}+3}} \approx \frac{7.7000E+23}{5.4078E+21} > 1.$$

For $n \geq n_0 = 25$ and $26 \geq \lambda \geq 13$, then $f_8(n, \lambda) \geq f_8(n_0, \lambda)$. It can be seen from **Table 4** that the values of $f_8(n_0, \lambda)$ as well as $f_8(n, \lambda)$ are all greater than 1.

(Detailed calculations are in **Appendix**.)

Table 4. When $n \geq n_0 = 25$ and $26 \geq \lambda \geq 13$, then $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$.

λ	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$f_8(n_0)$	156	157	145	125	102	78.7	58.5	41.9	29.1	19.7	13.0	8.35	5.29	3.29

Thus, for $26 \geq \lambda \geq 12$, when $n \geq 25 > \lambda - 2$, $f_8(n, \lambda) \geq f_8(n_0, \lambda) > 1$; then, referring to **(2.4)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

For $13 \geq \lambda \geq 12$ and $24 \geq n \geq \lambda - 2$, **Table 5** shows that there exists a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

For $26 \geq \lambda \geq 14$ and $24 \geq n \geq \lambda - 2$, **Table 6** shows that there exists a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, **(3.4)** is proven.

Combining **(3.1)**, **(3.2)**, **(3.3)**, and **(3.4)**, when $26 \geq \lambda \geq 3$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **(3.5)**

Table 5. For $13 \geq \lambda \geq 12$ and $24 \geq n \geq \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

	n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	$11n$	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264
$\lambda=12$	p	113	127	137	149	157	167	179	191	199	211	223	241	251	269	277
	$12n$	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288
$\lambda=13$	p		137	149	157	179	191	199	211	223	241	251	269	277	283	293
	$13n$		143	156	169	182	195	208	221	234	247	260	273	286	299	312

Table 6. For $26 \geq \lambda \geq 14$ and $24 \geq n \geq \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

	n	12	13	14	15	16	17	18	19	20	21	22	23	24
	$13n$	156	169	182	195	208	221	234	247	260	273	286	299	312
$\lambda=14$	p	157	179	191	199	211	223	241	251	269	277	293	307	317
	$14n$	168	182	196	210	224	238	252	266	280	294	308	322	336
$\lambda=15$	p		191	199	211	227	241	269	277	293	307	317	331	347
	$15n$		195	210	225	240	255	270	285	300	315	330	345	360
$\lambda=16$	p			211	227	241	269	277	293	307	317	331	347	373
	$16n$			224	240	256	272	288	304	320	336	352	368	384
$\lambda=17$	p				241	269	277	293	307	331	347	373	389	401
	$17n$				255	272	289	306	323	340	357	374	391	408
$\lambda=18$	p					277	293	307	331	347	373	389	401	419
	$18n$					288	306	324	342	360	378	396	414	432
$\lambda=19$	p						307	331	347	373	389	401	419	449
	$19n$						323	342	361	380	399	418	437	456
$\lambda=20$	p							347	373	389	401	419	449	461
	$20n$							360	380	400	420	440	460	480
$\lambda=21$	p								389	419	433	461	467	491
	$21n$								399	420	441	462	483	504
$\lambda=22$	p									433	461	467	491	521
	$22n$									440	462	484	506	528
$\lambda=23$	p										467	491	521	541
	$23n$										483	506	529	552
$\lambda=24$	p											521	541	563
	$24n$											528	552	576
$\lambda=25$	p												563	587
	$25n$												575	600
$\lambda=26$	p													613
	$26n$													624

4. A Prime Number Between kn and $(k + 1)n$ when $n \geq (k - 1)$

Proposition 5: For two positive integers n and k , when $n \geq (k - 1)$, there exists at least a prime number p such that $kn < p \leq (k + 1)n$. (4.1)

Proof:

Referring to (2.5), when $n \geq (\lambda - 2) \geq 25$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. This statement is the same as that when $\lambda \geq 27$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Referring to (3.5), when $26 \geq \lambda \geq 3$ and $n \geq \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, when $\lambda \geq 3$ and $n \geq \lambda - 2$, there is at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Let integer $k = (\lambda - 1)$, then for $n \geq 1$ and $k \geq 2$, when $n \geq (k - 1)$, there exists at least a prime number p such that $kn < p \leq (k + 1)n$. (4.2)

The Bertrand-Chebyshev's theorem points out that for $n \geq 1$ and $k = 1$, there exists at least a prime number p such that $kn < p \leq (k + 1)n$. (4.3)

From (4.2) and (4.3), we can conclude that for $n \geq 1$ and $k \geq 1$, when $n \geq (k - 1)$, there exists at least a prime number p such that $kn < p \leq (k + 1)n$. Thus, **Proposition 5** becomes a theorem, **Theorem 4.1**, and Bertrand-Chebyshev's theorem is a special case of this theorem.

5. References

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6. Appendix

Calculation of $f_8(n_0, \lambda)$ when $n_0 = 25$ and $26 \geq \lambda \geq 13$.

$$\text{When } \lambda = 13, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{338 \cdot (8.4924)^{(25-1)}}{(325)^{\frac{\sqrt{325}}{3}+3}} \approx \frac{6.6934E+24}{4.2676E+22} \approx 156 > 1.$$

$$\text{When } \lambda = 14, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{392 \cdot (9.1721)^{(25-1)}}{(350)^{\frac{\sqrt{350}}{3}+3}} \approx \frac{4.9265E+25}{3.1424E+23} \approx 157 > 1.$$

$$\text{When } \lambda = 15, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{450 \cdot (9.8518)^{(25-1)}}{(375)^{\frac{\sqrt{375}}{3}+3}} \approx \frac{3.1448E+26}{2.1746E+24} \approx 145 > 1.$$

$$\text{When } \lambda = 16, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{512 \cdot (10.5315)^{(25-1)}}{(400)^{\frac{\sqrt{400}}{3}+3}} \approx \frac{1.7743E+27}{1.4234E+25} \approx 125 > 1.$$

$$\text{When } \lambda = 17, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{578 \cdot (11.2112)^{(25-1)}}{(425)^{\frac{\sqrt{425}}{3}+3}} \approx \frac{8.9862E+27}{8.8497E+25} \approx 102 > 1.$$

$$\text{When } \lambda = 18, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{648 \cdot (11.8909)^{(25-1)}}{(450)^{\frac{\sqrt{450}}{3}+3}} \approx \frac{4.1374E+28}{5.2574E+26} \approx 78.7 > 1.$$

$$\text{When } \lambda = 19, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{722 \cdot (12.5705)^{(25-1)}}{(475)^{\frac{\sqrt{475}}{3}+3}} \approx \frac{1.7498E+29}{2.9904E+27} \approx 58.5 > 1.$$

$$\text{When } \lambda = 20, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{800 \cdot (13.2502)^{(25-1)}}{(500)^{\frac{\sqrt{500}}{3}+3}} \approx \frac{6.8616E+29}{1.6366E+27} \approx 41.9 > 1.$$

$$\text{When } \lambda = 21, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{882 \cdot (13.9298)^{(25-1)}}{(525)^{\frac{\sqrt{525}}{3}+3}} \approx \frac{2.5127E+30}{8.6291E+28} \approx 29.1 > 1.$$

$$\text{When } \lambda = 22, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{968 \cdot (14.6094)^{(25-1)}}{(550)^{\frac{\sqrt{550}}{3}+3}} \approx \frac{8.6507E+30}{4.4015E+29} \approx 19.7 > 1.$$

$$\text{When } \lambda = 23, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{1058 \cdot (15.2891)^{(25-1)}}{(575)^{\frac{\sqrt{575}}{3}+3}} \approx \frac{2.8161E+31}{2.1742E+30} \approx 13.0 > 1.$$

$$\text{When } \lambda = 24, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{1152 \cdot (15.9687)^{(25-1)}}{(600)^{\frac{\sqrt{600}}{3}+3}} \approx \frac{8.7081E+31}{1.0425E+31} \approx 8.35 > 1.$$

$$\text{When } \lambda = 25, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{1250 \cdot (16.6483)^{(25-1)}}{(625)^{\frac{\sqrt{625}}{3}+3}} \approx \frac{2.5691E+32}{4.8599E+31} \approx 5.29 > 1.$$

$$\text{When } \lambda = 26, f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}^{(n_0-1)}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{1352 \cdot (17.3279)^{(25-1)}}{(650)^{\frac{\sqrt{650}}{3}+3}} \approx \frac{7.2594E+32}{2.2080E+32} \approx 3.29 > 1.$$