Non-Relativistic Quantum Mechanical Motion in a Uniformly Accelerated Frame

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Abstract

In this M.Sc. Dissertation we have developed a formalism to obtain the solution of Schrödinger equation in a non-inertial frame. The frame is moving with an acceleration. The classical part of the formulation has been developed following Landau and Lifshitz [1] and obtained the single particle Hamiltonian. With the standard form of canonical quantization rule, we have setup the Schrödinger equation in non-inertial frame. See also the references [2–4] for some beautiful pedagogical discussion on non-inertial frame of reference. The physical situation of our study is basically the solution obtained by Fowler and Nordheim for the emission of electrons from cold metal surface under the action of strong electric field [7] known as cold field emission of electrons. However in the present formulation, it is the gravity, simulated by the acceleration acts on mass of the particle and causes emission. Hence we have got some flavor of Hawking radiation [8] and Unruh effect [9] within the limited scope of our non-relativistic approach.

1. INTRODUCTION

It is well known that a reference frame at rest or moving with uniform velocity is an inertial frame. Whereas frame undergoing an accelerated motion with respect to some inertial frame is known as a non-inertial frame. When a frame is rotating with uniform or non-uniform angular velocity relative to an inertial frame is also a non-inertial frame [10–12] (see also [13, 14]). As a preamble, let me give a brief outline to obtain the single particle Lagrangian following Landau and Lifshitz [1]. This part we believe to be needed for the sake of completeness. Let us first consider an inertial frame K_0. The Lagrangian of the particle is given by

$$L_0 = \frac{1}{2}mv_0^2 - U(\vec{r}) \tag{1}$$

Hence using the non-relativistic form of Euler-Lagrange formulation one can obtain the equation of motion of the particle. Next we consider a non-inertial frame K' moving with a time varying rectilinear velocity $\vec{V}(t)$ relative to the inertial frame K_0. The velocity of the particle, say the electron in these two frames are related by the Galilean transformation $\vec{v}_0 = \vec{v}' + \vec{V}(t)$. Hence the single particle Lagrangian in this non-inertial frame is given by

$$L' = \frac{1}{2}mv'^2 - m\vec{W}.\vec{r} - U(\vec{r}) \tag{2}$$

where $\vec{W} = d\vec{V}/dt$ is the acceleration of the frame. Hence one can obtain the equation of motion of the particle. Let us now consider another frame K, whose origin coincides with that of K' but rotates relative to K' with an angular velocity Ω. The velocity v' of the particle in K' frame is related to the velocity v in the K frame by the relation $\vec{v}' = \vec{v} + \vec{\Omega} \times \vec{r}$. Then the Lagrangian of a particle in a non-inertial frame having rotational motion and also non-uniform rectilinear motion is given by

$$L = \frac{1}{2}mv^2 + m\vec{v}.(\vec{\Omega} \times \vec{r}) + \frac{1}{2}m(\vec{\Omega} \times \vec{r})^2 - m\vec{W}.\vec{r} - U(\vec{r}) \tag{3}$$

Then using the standard relation, the Hamiltonian is given by

$$H = \vec{p}.\vec{v}(\vec{p}) - L, \tag{4}$$

Hence we have

$$H(\vec{r}, \vec{p}) = \frac{p^2}{2m} + U(\vec{r}) + m\vec{W}.\vec{r} - \frac{1}{2}(\vec{\Omega} \times \vec{r}).(\vec{\omega} \times \vec{r}) \tag{5}$$

Since our intention is to study the non-relativistic quantum mechanical motion of the particle in a non-inertial frame undergoing accelerated motion with respect to some inertial frame, in the Hamiltonian the rotation part is discarded. In a separate work we shall report the effect of rotation on spinor field. Neglecting the rotational motion of the frame the Hamiltonian of the particle reduces to

$$H(\vec{r},\vec{p}) = \frac{p^2}{2m} + U(\vec{r}) + m\vec{W}.\vec{r} \tag{6}$$

where $U(\vec{r})$ is some background potential. Before we go into detail study of quantum mechanical motion of the particle in an accelerate frame, let us first discuss the effect of acceleration of such non-inertial frame on macroscopic classical objects. It is our common experience that when we are inside a vehicle or a train or inside an aircraft, then at the time of acceleration of the carrier we are pushed backward. The reverse is true when the motion is retarded. Same is true for any solid object on a non-inertial frame undergoing an accelerated or retarded motion. Now the origin of any kind of push or pull on massive objects must be gravity. The source (if not a human source or a mechanical source) may not be visible or exist in reality. It is called a fictitious source. Therefore whenever there is either acceleration or retardation a force in the opposite direction will act. We will see later that this concept is alsoapplicable for the quantum mechanical systems, i.e., in the subatomic world. This is one of the definitions of Principle of Equivalence. More precisely the definition is that when a non-inertial frame undergoing accelerated / retarded motion it is equivalent to the presence of gravity in the rest frame. See also the references [15, 16] for some interesting discussion on equivalence principle published in this journal. However, the reverse is not in general true [11] (see also [15]). The magnitude of acceleration / retardation is exactly equal to the strength of gravity. Of course in Newtonian mechanics the definition of principle of equivalence is that the inertial and the gravitational masses are exactly same. To express it mathematically, let us write down the equation of motion of a macroscopic object in an accelerated frame using Hamilton's equation:

$$\frac{d\vec{p}}{dt} = -\frac{dH}{d\vec{r}}$$

Hence the equation of motion is given by

$$\frac{d\vec{p}}{dt} = -m\vec{W} \tag{7}$$

Since in reality a force is acting on a mass in the direction opposite to $\vec{W}$, we can rewrite the above equation of motion in presence of gravity in the rest frame. The form of the equation of motion is given by

$$\frac{d\vec{p}}{dt} = m\vec{g} \tag{8}$$

where $\vec{g}$ and $\vec{W}$ are mutually in the opposite direction, but $|\,\vec{g}\,|=|\,\vec{W}\,|$, i.e., the strength of gravity and the magnitude of the acceleration are exactly equal. Hence the Newtonian form of equivalence principle follows automatically.

In the next section we shall investigate the quantum mechanical motion in some uniformly ccelerated frame and finally give the conclusion. To the best of our knowledge this type of formalism has not been reported earlier.

2. QUANTUM MECHANICAL MOTION

In quantum mechanical scenario, H, $\vec{p}$, $\vec{W}$ and $U(r)$ are treated as operators. Then we have

$$H = -\frac{\hbar^2}{2m}\nabla^2 + m\vec{W}.\vec{r} + U(\vec{r}) \tag{9}$$

See [17] for some beautiful discussion on equivalence principle and quantum mechanecs. Let us now consider a two body quantum mechanical system, e.g., either deuteron or hydrogen atom or hydrogen molecule or hydrogen like atoms etc. Then we can recast the above Hamiltonian into individual coordinates in the following form (indicated by the indices 1 and 2 for the two components):

$$H = H_1 + H_2 = -\frac{\hbar^2}{2m_1}\nabla_1^2 + m_1\vec{W}.\vec{r}_1 - \frac{\hbar^2}{2m_2}\nabla_2^2 + m_2\vec{W}.\vec{r}_2 + U(|\,\vec{r}_1 - \vec{r}_2\,|) \tag{10}$$

Here $U(|\,\vec{r}_1 - \vec{r}_2\,|)$ is the two body potential and we have neglected the presence of any background potential. Now instead of individual coordinates, let us express the above Hamiltonian in terms of relative and centre of mass coordinates, which are given by

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \ \text{ and } \ \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \ \text{ respectively} \tag{11}$$

Then the two body Schrödinger equation is given by

$$\begin{aligned} &H\psi(\vec{r},\vec{R}) = E\psi(\vec{r},\vec{R}) \ \text{ or} \\ &\left[-\frac{\hbar^2}{2M}\nabla_R^2 + M\vec{W}.\vec{R} - \frac{\hbar^2}{2\mu}\nabla_r^2 + U(r)\right]\psi(\vec{r},\vec{R}) = E\psi(\vec{r},\vec{R}) \ \text{ or} \\ &(H_r + H_R)\psi(\vec{r},\vec{R}) = E\psi(\vec{r},\vec{R}) \end{aligned} \tag{12}$$

where $M = m_1+m_2$, the total mass concentrated at the centre of mass and $\mu = m_1m_2/(m_1+m_2)$, the reduced mass. Now with the separation of variables $\psi(\vec{r},\vec{R}) = \xi(\vec{r})\phi(\vec{R})$, we have

$$\left[-\frac{\hbar^2}{2M}\nabla_R^2 + M\vec{W}.\vec{R}\right]\phi(\vec{R}) = E_{cm}\phi(\vec{(}R) \tag{13}$$

the Schrödinger equation in the centre of mass coordinate, and

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + U(r)\right]\xi(\vec{r}) = E_{rel}\xi(\vec{r}) \tag{14}$$

the Schrödinger equation in the relative coordinate. Obviously the relative part is independent of the acceleration of the non-inertial frame. It is just the conventional form of Schrödinger equation in relative coordinate. Therefore at this point we may conclude that the internal activities of any two body quantum system are independent of the nature of the frame of reference. On the other hand, the center of mass part depends on the acceleration of the non-inertial frame. Which indicates that the centre of mass motion will be affected by the acceleration of the frame. Here we have not considered any back ground field. If θ is the angle between $\vec{W}$ and $\vec{r}$, then $M\vec{W}.\vec{r} = \pm MWr \mid \cos\theta \mid$, where + sign is for the first and the fourth quadrants, whereas $-$ sign is for the second and the third quadrants, except for $\theta = \pi/2$ and $3\pi/2$. We will see later that along these two directions, the solutions are just the free particle outgoing spherical waves. In the first and fourth quadrants, $\vec{r}$ is measured along the direction of acceleration $\vec{W}$. We call it as the forward hemisphere. Whereas for second and third quadrants, $\vec{r}$ is measured along the opposite direction of $\vec{W}$. We can call this direction as backward hemisphere. The problem is of course symmetric in azimuthal coordinate. Now redefining $R' = (MW \mid \cos\theta \mid)^{1/3}R \longrightarrow R$ and $E_{cm}^* = E_{cm}/(MW \mid \cos\theta \mid)^{2/3} \longrightarrow E$, eqn.(13) reduces to

$$\left[-\frac{\hbar^2}{2M}\nabla_R^2 \pm R\right]\phi(\vec{R}) = E\phi(\vec{R}) \tag{15}$$

Further redefining

$$R \longrightarrow \left(\frac{2M}{\hbar^2}\right)^{1/3} R \;\; \text{and} \;\; E \longrightarrow \left(\frac{2M}{\hbar^2}\right)^{1/3} E$$

the above differential equation may be rewritten in the following form

$$\left[-\nabla_R^2 \pm R\right]\phi(\vec{R}) = E\phi(\vec{R}) \tag{16}$$

Now for the sake of simplicity we assume spherical symmetry, the angular momentum $\vec{L} = 0$ and substituting $\phi(R) = u(R)/R$, the above differential equation reduces to

$$\left[-\frac{d^2}{dR^2} \pm R\right]u(R) = Eu(R) \tag{17}$$

Now instead of two body if we consider a single body, then obviously there is nothing called relative motion. As a consequence the equation representing the relative motion (eqn.(14)) does not exist. Whereas the centre of mass coordinate R here is nothing but the single particle coordinate which coincides with the centre of mass coordinate. We further assuming that $U(r)$, which is now $U(R)$ is some constant background potential. Then in the present situation the equation representing the centre of mass motion (eqn.(13)) is the equation for the single particle motion. Since our intention was to show that all kinds of activities which are determined by the relative motion are independent of the characteristic of the non-inertial frame of reference, we started with a two body quantum system. Now we shall use eqn.(13) satisfied by a single body and try to extract some interesting physics with the limited scope of non-relativistic approach.

As has already been mentioned that for $\theta = \pi/2$ or $\theta = 3\pi/2$, the effect of acceleration on the particle motion vanishes and the equation reduces to

$$\left[\frac{d^2}{dR^2} + E\right] u(R) = 0 \tag{18}$$

The solution is well known, which is an outgoing spherical wave, given by

$$\phi(R) = \frac{u(R)}{R} = C\frac{\exp(-iER)}{R}$$

where C is the normalization constant. This is not the gravity induced emission. It could be some kind of spontaneous emission, if any.

Let us now consider the differential equation given by eqn.(17) in the backward hemisphere and re-defining $R \longrightarrow R + E$, we have

$$\left[\frac{d^2}{dR^2} + R\right] u(R) = 0 \tag{19}$$

To get an analytical solution we use the transformation

$$u(R) = R^n u'(R)$$

where n is an unknown constant, not necessarily an integer. Now discarding the prime symbol. we have

$$R^2\frac{d^2u}{dR^2} + 2nR\frac{du}{dR} + [n(n-1) + R^3]u(R) = 0 \tag{20}$$

To identify it with some known differential equation, let us put $R = \beta z^{2/3}$, with β as another constant and z is the new variable. Then we have

$$z^2\frac{d^2u}{dz^2} + \left(n + \frac{1}{4}\right)\frac{4}{3}z\frac{du}{dz} + \frac{4}{9}[n(n-1) + \beta^2 z^2]u(z) = 0 \tag{21}$$

Putting $n = 1/2$ and $\beta = 3/2$. the above equation can be written as

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - \frac{1}{9}\right) u(z) = 0 \tag{22}$$

Comparing with the standard form of differential equation for Ordinary Bessel function

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - n^2\right) u(z) = 0 \tag{23}$$

whose solution is $J_n(z)$, the solution for eqn.(22) is $J_{1/3}(z)$. Since we expect oscillatory solution at a large distance from the source of emission, i.e., in the asymptotic region, we use the function $H^{(2)}_{1/3}(z)$, the Hankel function of second kind of order 1/3 instead of ordinary Bessel function. This is exactly the solution obtained by Fowler and Nordheim for cold field emission of electrons from metal surface under the action of strong electric field [7]. However, in the present situation, the electric field has been replace by strong gravitational field. This gravitational field is acting on mass and causes emission. This is the gravity induced emission. It is happening in a non-inertial frame undergoing accelerated motion. The action of gravity is along the opposite direction (here it is the backward hemisphere). It has already been shown for the classical Newtonian case as well. The source of gravity is again fictitious like the classical case. This emission process under the action of gravity may be called as the non-relativistic version of Hawking radiation. In the relativistic picture, the creation of particle anti-particle pairs and radiation occurs at the event horizon of black holes. Of course the mechanism of particle production or the creation of radiation can not be explained in the non-relativistic approach. The Dirac vacuum, which is the source of particles, anti-particles or radiation, does not exist in the case of Schrödinger equation. There is no non-relativistic counter part of Schwinger mechanism [18]. Therefore the solution is giving only a flavor of these quantum field theoretic phenomena. One can also get the flavor of Unruh effect. Which is one of the strange field theoretic phenomena, in which an accelerated observer while traveling through Dirac vacuum will observe a thermal spectrum of particle excitations. One explanation is the energy transfer from the accelerated frame to the vacuum when the non-inertial frame interacts quantum mechanically with the vacuum. This is also equivalent to the gravitational force acting on the particles or anti-particles or radiation because of the accelerated motion of the frame. However, in the non-relativistic scenario one can not distinguish a particle from its anti-particle counter part. In such approach, the constituents of Dirac vacuum, which are sleeping quietly are excited and pushed out by strong gravitational field produced by some fictitious source. The direction of force is along the opposite direct of the acceleration of the non-inertial frame. This is nothing but the Unruh effect and also equivalent to Hawking radiation. Of course this is just the qualitative explanation.

Now the solution of eqn.(17) in the forward hemisphere can also be obtained following the same mathematical technique. The solution is given by $J_{1/3}(iz)$. This is related to the Modified Bessel function of first kind with argument z, i.e., $I_{1/3}^{(1)}(z)$. Since it diverges asymptotically, i.e., as $z \longrightarrow \infty$, it is therefore an unphysical solution. In reality we will never see the emission of particles or radiation along the direction of acceleration under the action of gravity. This is a consequence of the principle of equivalence. This is also true in the classical Newtonian picture.

3. CONCLUSION

In our investigation it has been noticed that the relative motion of two body quantum system will not be affected by the nature of reference frame. For a specific example, the electric quadrupole moment of deuteron, which is of course quite small, will not change whether it is in inertial frame or in non-inertial frame. However. because of motion of the frame, there will be Doppler shift of frequencies when observed from a rest frame. Of course the later is purely classical in nature and the cause is essentially because of the relative motion between source and the observer.

We have noticed that both in classical scenario for macroscopic objects and quantum picture for the sub-atomic world, a force due to gravity will act on the objects. This is consistent with the principle of equivalence. Hence we have got some flavor of Unruh effect and Hawking radiation and in consistent with the field theoretic formulation. we found that these two effects are identical physical phenomena.

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Appendix

For the sake of completeness we are giving the mathematical along with graphical representation of the Hankel function of second kind of order 1/3 in complex argument space. Generally, the Hankel function of second kind of order ν is represented as:

$$H_\nu^{(2)}(z) \equiv J_\nu(z) - iY_\nu$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν and $Y_\nu(z)$ is the Bessel function of the second kind of order ν. Figure (1) shows the real part of the Hankel function of second kind of order 1/3 in a complex plane. We adopted the numerical method to obtain the the plots.

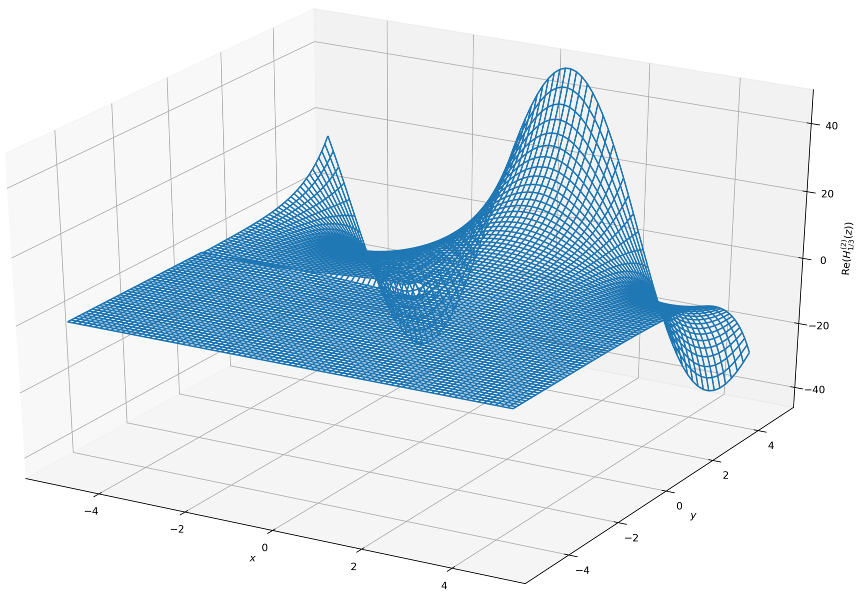

Figure 1: Real part of the Hankel function of second kind of order 1/3, $Re\left(H_{1/3}^{(2)}(z)\right)$ has been plotted in complex plane with three dimensional view

Figure (2) shows the imaginary part of the Hankel function of second kind of order 1/3 in a complex argument space. The asymptotic nature of the Hankel function of second kind of order 1/3 suggests that this may be useful for am oscillatory solution at a large distance from the source of emission.

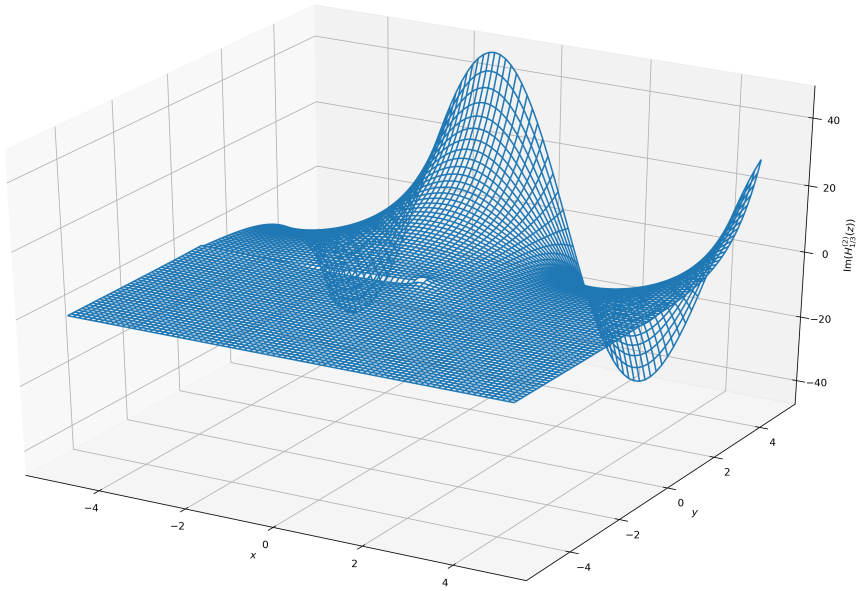

Figure 2: Imaginary part of the Hankel function of second kind of order 1/3, $Imag\left(H^{(2)}_{1/3}(z)\right)$ has been plotted in complex plane with three dimensional view

Here, Figure (3) represents the absolute value of the Hankel function of second kind of order 1/3 in a complex argument space. Where, $\left|\left(H^{(2)}_{1/3}(z)\right)\right| = \sqrt{\left(J^{(2)}_{1/3}(z)\right)^2 + \left(Y^{(2)}_{1/3}(z)\right)^2}$

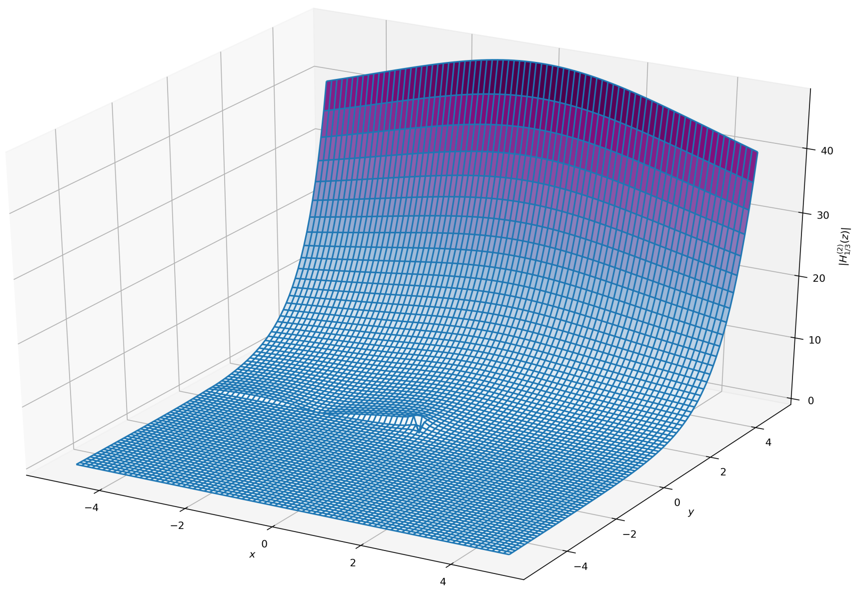

Figure 3: Absolute value of the Hankel function of second kind of order 1/3, $\left|\left(H^{(2)}_{1/3}(z)\right)\right|$ has been plotted in complex plane with three dimensional view