Proving correctness of Collatz conjecture

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Abstract

This proves Collatz conjecture is correct using binary representation.

Each transition of generated numbers is determined in advance based on Collatz conjecture formula. But its path of transition is not so simple regarding to each initial value repeating increasing and decreasing.

Behavior of global flow to termination value 1 is investigated using concept of probability.

1. Introduction

Collatz problem has repeating of operations. At first the meaning of the operations is investigated using binary representation. Then whether its behavior is according to Collatz conjecture or not is inspected.

2. Binary representation of number

Positive integers can be represented as following sample using binary representation.

bin11001

unit of digit: bit

possible number for a bit: 0 or 1

digit number: Most right digit is 1, ascending number is assigned from right to left .

3. Operations

Procedure of collatz problem has following operations.

It starts with positive odd number integer.

- It is multiplied by 3, then increased by 1. (1)
- If result is even number, it is divided by 2 until it become odd number. (2)

Repetition of above operations makes terminal number 1 even if it is started from any positive odd number integer.

Broken down operations of above operations is as follows.

- Number n is calculated using formula (2n+n)+1, then result becomes n. (3)
- If result n is even number, it is divided by 2 until it becomes odd number. Then result becomes n (4)

These are repeated until n becomes 1.

More broken down calculation are as follows

- $n_1 = 2n$ (5)
- $n_2 = n + n_1$ (6)
- $n_3 = n_2 + 1$ (7)
- If n_3 is even number, m times division $n_4 = \frac{n_3}{2^m}$ is done until n_4 becomes odd. (8)

4. Behavior and meanings of each calculation

About calculation (5) (6) (7) (8), related behavior is investigated using binary representation.

About (5) $n_1 = 2n$,

n_1 is the number which is shifted n to left by one digit on binary representation.

Sample 1 n=bin10111、n_1=bin101110

About (6) $n_2 = n + n_1$

This addition is n and n_1 which is shifted n to left by one digit. n_2 is always odd number.

Sample 2

n			1	1	1
$+n_1$		1	1	1	0
$= n_2$	1	0	1	0	1

About (7) $n_3 = n_2 + 1$,

Adding 1 to odd number n_2 makes n_3. n_3 is always even number. This also makes carry over to digit 2.

About (8) $n_4 = n_3/2$,

n_3 is shifted one digit to right. If this can be done m times until n_4 becomes odd number, *multiplier for making odd number* (MMON) is $\frac{1}{2^m}$.

Sample 3 n_3=bin10110 → n_4=bin1011 MMON=$\frac{1}{2^1}$

5. Characteristics and recognition of the operations

Operation (1) which is broken down to (5) (6) (7) can be recognized as following function also 'next random number' generator using 'Linear congruential generators'. Therefore from bit 2 to i-1 of next input x is randomized.

$y = f(x) = 3x + 1$=x+2x+1 x: positive odd integer, y: positive even integer (9)

Generated y could have *number of most right continuous value 0 digits* (NMRZD) as sample 4. This is related with NMON.

sample 4　101011000, NMRZD=3, NMON=$\frac{1}{2^3}$

Considering NMRZD, characteristics of y and function (9) is investigated.

6.　Characteristics of generated y value

Characteristics of y output of the function (9) are reviewed here.

If arbitrary positive odd integer x has i digits, calculation is done as follows.

digit no.	i+2	i+1	i	i-1				1
x			1	・	・	・	・	1
+2x+1		1	・	・	・	・	1	1
=y	・	・	・	・	・	・	・	0

fig.1

This processing and y has following characteristics.

- digit 1 of x=digit 2 of 2x=1
  digit 1 of 2x =digit 1 of y=0
  digit i of x=digit (i+1) of 2x=1
  digit j of x=digit (j+1) of 2x=0 or 1　$2 \leq j \leq i-1$

-Number of digit for y is i+1 or i+2.

On above, if carry over from digit i-1 does not exist and digit i-1 of x is not 1, y has following structure from digit i+1 to 1. This case y has i+1 digits.

y=bin 11・・・0

In other cases(carry over from digit i-1 exists and/or digit i-1 of x is 1), y has following structure from digit i+2 to 1.　This case y has i+2 digits.

y=bin 1・・・・0

-In the case, number of digits i of x is even number and most right i-1 digits of y are all value 0, value of digit i for y is always 1.

Reason of this is that digits structure of x should be as fig.2 in order that most right i-1 digits of y are all 0. And in this case, digit i of y is 1 never be 0. Therefore maximum NMRZD is i-1.

digit no.	i+2	i+1	i	i-1	8	7	6	5	4	3	2	1
x			1	1	0	1	0	1	0	1	0	1
+2x+1		1	1	0	1	0	1	0	1	0	1	1
=y	1	0	1	0	0	0	0	0	0	0	0	0

fig.2

In the case, number of digits i of x is odd number and most right i-1 digits of y are all value 0, value of digit i and i+1 for y are always 0.

Reason of this is that digits structure of x should be as fig.3 in order that most right i-1 digits of y are all 0. And in this case, digit i and i+1 of y are 0 never be 1. Therefore as a result maximum NMRZD is i+1.

bit no.	i+2	i+1	i	i-1	7	6	5	4	3	2	1
x			1	0	1	0	1	0	1	0	1
+2x+1		1	0	1	0	1	0	1	0	1	1
=y	1	0	0	0	0	0	0	0	0	0	0

fig.3

7. Digits structure analysis of y

Considering above characteristics, calculation is done for each NMRZD.

Case1;

digit 2 of y=1、digit 1 of y=0, as following sample, NMRZD =1

bin1・・・・・10

Meaning of this digits structure: This can be divided by 2, that is, this can be shifted one digit to right. MMON is1/2.

Number of cases for x:

Number of cases which is satisfied this condition: 2^{i-3}

(most left digit i=1, most right digit 1=1, digit 2= 1 ,

Other i-3 digits are able to have value 0 or 1.)

Number of all possible cases: 2^{i-2}

(most left digit i=1, most right digit 1=1,

other i-2 digits are able to have value 0 or 1.)

Therefore

possibility rate of this case; $\frac{\text{nuber of cases which is satisfied this case}}{\text{number of all possible cases}} = \frac{2^{i-3}}{2^{i-2}}$ =1/2

Case2;

digit 3 of y=1、digit 2 of y=0, digit 1 of y=0, as following sample, NMRZD =2

bin1・・・・100

Same way as Case1,

MMON =$\frac{1}{2^2}$

possibility rate of this case= $\frac{1}{2^2}$

Case3;

In general (including Case1, Case2), when $1 \le j \le i-2$,

digit (j+1) of y=1, digits from 1 to j of y=0, NMRZD =j (10)

MMON =$\frac{1}{2^j}$

possibility rate of this case= $\frac{1}{2^j}$ (11)

Case4;

digits from 1 to (i-1) of y=0

possibility rate of this case= $\frac{1}{2^{i-2}}$ (12)

As fig.2, fig.3

If digit number i of x is even, digit i of y is 1. Therefore

NMRZD is i-1. (13)

MMON=$\frac{1}{2^{i-1}}$

If digit number i of x is odd, digit i and digit i+1 of y is 0. Therefore

Actual NMRZD is i+1. (14)

MMON=$\frac{1}{2^{i+1}}$

8. Verification of the analysis for digits structure of y

Total possibility rate S is

S=$\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^{i-2}}+\frac{1}{2^{i-2}}$=1.

S=1 indicates above analysis lists up all cases regarding to NMRZD.

9. Expected NMRZD

Base on (10)(11)(12)(13)(14) and the fact that number of digits for x can be even or odd equally, expected NMRZD is calculated on following formula.

($NMRZD_j$ is NMRZD when NMRZD=j.)

Expected NMRZD=$\frac{1}{2}$(expected NMRZD for even i + expected NMRZD for odd i)

$$=\frac{1}{2}(\sum_{j=1}^{i-2}(NMRZD_j \times \text{possibility rate for } NMRZD_j)$$

$$+NMRZD_{i-1} \times \text{possibility rate for } NMRZD_{i-1})$$

$$+\frac{1}{2}(\sum_{j=1}^{i-2}(NMRZD_j \times \text{possibility rate for } NMRZD_j)$$

$$+NMRZD_{i+1} \times \text{possibility rate for } NMRZD_{i+1})$$

$$=\sum_{j=1}^{i-2}(NMRZD_j \times \text{possibility rate for } NMRZD_j)$$

$$+\frac{1}{2}(NMRZD_{i-1} \times \text{possibility rate for } NMRZD_{i-1})$$

$$+\frac{1}{2}(NMRZD_{i+1} \times \text{possibility rate for } NMRZD_{i+1})$$

Therefore

Expected NMRZD=$\frac{1}{2^1}+\frac{2}{2^2}+\cdot\cdot\cdot+\frac{i-2}{2^{i-2}}+\frac{1}{2}(\frac{i-1}{2^{i-2}}+\frac{i+1}{2^{i-2}})=(2-\frac{i}{2^{i-2}})+\frac{i}{2^{i-2}}=2$

This calculation result does not depend on i, it means expected NMRZD of output from operation (1) is 2 for all positive odd integers.

Therefore after operation (1) is executed, y can be divided by 2^2(MMON $=\frac{1}{2^2}$) as an average during operation (2). Then next n becomes

$$\frac{3n+1}{2^2} \tag{15}$$

Generally for integer n>1, following relation can be satisfied.

$$\frac{3n+1}{2^2} < n \tag{16}$$

10. Examination

On (15) (16), repetition of operation (1) (2) decreases n as an average. Finally, n reaches to 1 which is minimum positive integer.

Expected value is expected one statistically, but it is sure to be realized after sufficient many trials.

In detail, 3n+1 increase n, and dividing by 2 is only decreasing factor.

This means operation (1) 3n+1 increase value of n and/or increase number of digits of n, and operation (2) $\frac{n}{2}$ decrease number of digits when NMRZD >0.

So (15) (16) means the combination of operation (1) (2) actually decrease n as an average. Through this process at one step before final, n becomes 1×2^m. Finally it is divided by 2^m and reaches to minimum number of digit 1 and minimum positive value 1.
This should prove Collatz conjecture is correct.
But this could have exception because there is possibility of looping. It prevent from doing enough true iterations of (1) (2), therefore conjectured target cannot be reached.

11. Possibility of looping
Following loop as a sample is investigated.

$$n_{1,} n_{2,} n_{3,} \cdots n_{m-1,} n_{m,} \qquad n_m = n_1 \tag{17}$$

There are two required conditions in order to have looping.

A. On (15), relation $n_1 and\, n_m$ is $n_1 > n_m$ in an average, but relation $n_1 \cong n_m$ should be required in order to be $n_m = n_1$.
B. On addition to A, digits structure should be same exactly in order to be $n_m = n_1$.

About A, on (15), m times iteration makes n_m in an average as follows from n_1.

$$n_m \cong (\frac{3n_1+1}{2^2})^m \cong \left(\frac{3}{2^2}\right)^m n_1 = (\frac{3}{4})^m n_1 \qquad (n \gg 1)$$

This decrease n, therefore increasing factor is required to accomplish (17).
Increase factor is only for NMON=$\frac{1}{2}$ case on (11). In such case, next n is $\frac{3n+1}{2}$.
When existence rate of this factor is about 0.4 to total iteration m, $n_1 \cong n_m$ is realized on following calculation.

$$n_m \cong \left(\frac{3n_1+1}{2^2}\right)^l \left(\frac{3n_1+1}{2}\right)^{m-l} \cong \left(\frac{3}{2^2}\right)^l \left(\frac{3}{2}\right)^{m-l} n_1 \cong \left(\frac{3}{2^2}\right)^{0.6m} \left(\frac{3}{2}\right)^{0.4m} n_1 \cong n_1 \tag{18}$$

Because on (11) occurrence possibility of $\frac{3n+1}{2}$ is $\frac{1}{2}$, possibility for $n_1 \cong n_m$ is $\left(\frac{1}{2}\right)^{0.4m}$

This depends on total iterations.
For example,

$\left(\frac{1}{2}\right)^{0.4m} = \left(\frac{1}{2}\right)^4 = 0.06$ when total iteration=10

$\left(\frac{1}{2}\right)^{0.4m} = \left(\frac{1}{2}\right)^{40} = \frac{9}{10^{13}}$ when total iteration=100

About B, if number of digits is i or i digit of x=1, other digits can have value 0 or 1 for x to have value around 2^{i-1}. Actually in this case i-2 digits (except i digit of x=1 and 1 digit of x=1) can have value 0 or 1. Therefore possibility to have same digit structure in order to $n_1 = n_m$ is

$$\left(\frac{1}{2}\right)^{i-2} \quad (19)$$

This depends on i. If i is large, possibility is small. For example, if i=35(x value around billion), possibility is around $\frac{1}{10^{10}}$.

From above samples, longer span looping has smaller possibility to occur.
Larger number which has large digit number i have smaller possibility to occur.
Actually value 1 will enter looping if continued. It is span 1, i=1 loop.

12. Conclusion

On (15), Collatz conjecture is basically correct.
But there is possibility to have looping which prevent from reaching to target. Therefore it is systematically incorrect. Or we can say ' Collatz conjecture is correct if there is no looping'.
However possibility of looping is very small also no looping case is discovered yet in spite of many trials up to large initial number case. This shows Collatz conjecture is actually or statistically correct till now maybe also in future.
It should be better if the degree of correctness would be shown by more accurate non-existence possibility of looping.

13. Consideration

In order to manipulate behavior of many objects system, it is sometimes difficult to do calculation for each object. Examples of such case are,

-It is difficult to get each object's position, initial value etc. Ex. molecular in space
-Conditional calculation is required for individual objects. Ex. Condition is 'if it becomes even number.' as Collatz problem case.

In such case, statistical method could be useful. Ex. Statistical mechanics

For the statistical method, there could be anything which prevents from doing enough times effective trial as this looping case.
In such case, we could do only providing probability of such anything, as a solution.