Computation of multiple binomial Series based on geometric series

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Abstract: This paper presents addition of multiple binomial series based on geometric series. In general, a finite multiple summations of a geometric series are called binomial series. Addition of multiple binomial series is a sum and summation of multiple binomial series.

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I. Multiple summations of a geometric series

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\cdots\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i$$

When substituting r = 1 in the above binomial series, it becomes double summation of a geometric series,

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n} x^{i_2} = \sum_{i_2=0}^{n} x^{i_2} + \sum_{i_2=1}^{n} x^{i_2} + \sum_{i_2=2}^{n} x^{i_2} + \cdots + \sum_{i_2=n}^{n} x^{i_2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n.$$

$$1 + 2x + 3x^2 + \cdots + (n+1)x^n = \sum_{i=0}^{n}(i+1)x^i = \sum_{i=0}^{n} V_i^1 x^i.$$

When substituting r = 2, it becomes triple summation of a geometric series,

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i_2=0}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=2}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \cdots + \sum_{i_2=n}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i=0}^{n} V_i^2 x^i.$$

Similarly, if the above process continues up to r times, the r^{th} equation becomes as follows:

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\cdots\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r}.$$

If substituting r = 0, the series becomes the actual geometric series,

$$\sum_{i=0}^{n} V_i^0 x^i = \sum_{i_1=0}^{n} x^{i_1} = 1 + x + x^2 + x^3 + \cdots + x^n \left(\because\ V_0^0 = V_i^0 = v_0^i = 1, i \in N\right).$$

II. Sum and summations of multiple binomial series [Annamalai, 2018]

$$\sum_{i=0}^{n} V_i^{p+1} x^i = \sum_{i=0}^{n} V_i^p x^i + \sum_{i=1}^{n} V_{i-1}^p x^i + \sum_{i=2}^{n} V_{i-2}^p x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^p x^i + \sum_{i=k}^{n} V_{i-k}^p x^i ,$$

where $V_k^p = \prod_{i=1}^{p} \frac{(k+i)}{p!} = \frac{(k+1)(k+2)(k+3)\cdots(k+p)}{p!}$ & V_k^p is binomial coefficient.

For Example,

$$\sum_{i=0}^{5} V_i^2 x^i = \sum_{i=0}^{5} V_i^1 x^i + \sum_{i=1}^{5} V_{i-1}^1 x^i + \sum_{i=2}^{5} V_{i-2}^1 x^i + \sum_{i=3}^{5} V_{i-3}^1 x^i + \sum_{i=4}^{5} V_{i-4}^1 x^i + \sum_{i=5}^{5} V_{i-5}^1 x^i$$

$$\sum_{i=0}^{5} V_i^2 x^i = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$$

$$\sum_{i=0}^{5} V_i^1 x^i + \sum_{i=1}^{5} V_{i-1}^1 x^i + \sum_{i=2}^{5} V_{i-2}^1 x^i + \sum_{i=3}^{5} V_{i-3}^1 x^i + \sum_{i=4}^{5} V_{i-4}^1 x^i + \sum_{i=5}^{5} V_{i-5}^1 x^i$$

$$= (1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^{5)} + (x + 2x^2 + 3x^3 + 4x^4 + 5x^5)$$

$$+(x^2 + 2x^3 + 3x^4 + 4x^5) + (x^3 + 2x^4 + 3x^5) + (x^4 + 2x^5) + x^5$$

$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$$

Here, both sides are equal.

We can prove the binomial identity for *p*=1, 2, 3, . . .

Hence, the sum and summations are proved.

III. Reference

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