Gravidynamics of an Affine Connection on a Minkowski Background

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Abstract

In this paper, a post-Riemannian formalism is constructed based on a minimalistic set of modifications and suggested as the framework for a classical alternative to General Relativity (GR) which, notably, can be formulated in Minkowski spacetime. Following the purely geometrical exposition, arguments are advanced for the transport of matter and radiation, a Lagrangean quadratic in the gravitational field strengths is considered, and several of the resulting properties are analyzed in brief. Simple models are then set up to explore the astrophysical and cosmological reach of the proposed ideas, including their potential (and so far tentative) agreement with the 'classical tests' of GR. Some arguments are also presented towards quantization within the proposed formalism, and a few other issues are discussed.

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Men can construct a science with very few instruments, or with very plain instruments; but no one on earth could construct a science with unreliable instruments. A man might work out the whole of mathematics with a handful of pebbles, but not with a handful of clay which was always falling apart into new fragments, and falling together into new combinations. A man might measure heaven and earth with a reed, but not with a growing reed. [1]

1 Introduction

The ongoing search for a fully-developed quantum theory of gravity, probably the most renowned open problem in theoretical physics, has by now become a staple to a broad audience of experts and nonexperts alike, as it resisted over a century of attempts at quantizing it [2]; however, in spite of its prominence, it also seems to have overshadowed a number of other issues of a purely classical nature, as illustrated by a fairly recent (meta)list published by Coley [3], which includes over seventy (!) open problems rooted in plain General Relativity (GR). The author comments on the situation thus: "GR problems have typically been under-represented in lists of problems in mathematical physics [...], perhaps due to their advanced technical nature"; yet at the same time, some of the problems listed include examples such as "Show that a solution of the linearised (about Minkowski space) Einstein equations is close to a (non-flat) exact solution" and "Prove rigorously the existence of a limit in which solutions of the Einstein equations reduce to Newtonian spacetimes" (RB19 and RB21 respec.), which are surprisingly basic considering the level of maturity of the field.

Aside from unsolved technical problems, however, we can still point to known (and often well-established) features of GR such as the convoluted treatment of singularities, and causal and spinor structures [4], the possibility of closed timelike curves [5], and the lack of a true local conservation law [6] - features that strike one more like bugs. It is not my point to deny that these (perceived) bugs can be handled within the current state of the art - indeed, there is extensive literature concerned with each of these issues; rather, it is to suggest that the conceptual and technical problems that we're faced with even at the classical level are due to the idiosyncractic mathematical formulation of GR that has dominated the field since its inception in 1915 - an overly formal one that tends to alienate a larger audience of physicists, but which, in light of modern developments in our understanding of differential geometry, might be exchanged with a simpler, more intuitive apparatus without incurring any loss in any of the fundamental structures or ideas of physical import.

It may be briefly pointed, from a historical perspective, that the conventional view of the physical nature of the metric seems to have taken shape ca. 1907-1912 - i.e., between the publication of the famous special-relativistic review that introduced the equivalence principle, and the fateful reencounter of Einstein and Grossman in Zürich; in particular, Born's work on the rigid body seems to have had a major theoretical influence in the development of GR, both directly and indirectly [7, 8]. After General Relativity was established, it enjoyed empirical prestige via its prediction of several effects such as the bending of light by massive bodies - in fact, so much so that by the mid-1970s, alternatives to the theory weren't taken very seriously except as possible PPN foils to GR [9], or something of the sort. At the same time, however, a countercurrent of ideas inspired by the then-nascent gauge formalism gave rise to a particular family of theories under the umbrella metric-affine (gauge) gravity (MAG), which is not so much an *alternative* to GR as an *augmentation* thereof - in which not only the metric, but also the linear connection and the coframe (or alternatively the soldering) are taken as dynamical variables [10, 11, 12].

Although we recognize that the MAG programme introduced important insights as to the nature of the gravitational field, it is still plagued by difficulties perhaps best expressed by Mielke [12]: "With reference to the proper foundation of a gauge theory of gravity, however, there is no absolute agreement among the members of the scientific community. It is the incorporation of a *dynamical geometry* as realized by Einstein via the pseudo-Riemannian metric that seems to prevent a direct transfer of the Yang–Mills gauge program". Here we see a sharp conflict: our best theory for all the non-gravitational interactions is not only successfully quantized, but it's also *gauged* - whereas our best theory for gravitational interactions is neither; could this be a clue to explain the continued clash between GR and QFT - and possibly guide us to a better approach?

It was thinking on those lines that lead to the proposal here that this geometrodynamical (i.e. 'gravity-as-metric') view is fundamentally misguided, in that the metric need not be taken as a dynamical degree of freedom, and may be satisfactorily separated from the main machinery of the linear connection; such an arrangement not only brings about mathematical simplifications, but is also rich with physical implications, such as the restoration of the old 'gravity-as-force' outlook - a viewpoint we will refer to as *gravidynamics*, to distinguish it from the previous one. To the best of my knowledge, however, no attempt has ever been definitely forwarded in the literature to pursue such a theory (no doubt due to the said prestige accumulated by GR over the years, which made investigators wary of tinkering too much with it); the purpose of the present work, thus, is towards filling that gap.

In section II, we review some basic concepts of tensor calculus and develop an argument leading to the introduction and interpretation of the covariant derivative; in section III, we augment this covariant machinery by the introduction of some tensorial objects and discuss several different aspects of their structure. Section IV introduces physics by means of a transport formalism as well as a definite Lagrangean density; section V further builds on the resulting physical theory by building and investigating several models with real-world implications. Section VI, then, mentions several loose ends preventing the present treatment from being a complete theory of gravity, highlighting some challenges - particularly related to (minimal) coupling and quantization.

2 Basic Tensor Calculus

Since the reader is assumed to already have some familiarity with tensors and the relevant multilinear algebra, as well as exterior calculus, we will for the most part skip several technical definitions[1]; for our purposes, it will suffice to recall just a few. Given is a n-dimensional manifold M, with corresponding tangent space $T(M)$ and cotangent space $T^*(M)$; the meaning of these objects is readily intuited from observing that, for any point $p \in M$, $T_p(M)$ forms a n-dimensional vector space, and $T_p^*(M)$ is its dual. Vectors belonging to $T_p(M)$ can be expanded in terms of a basis $\{\mathbf{e}_i\}$ as $\mathbf{v} := v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + ... + v^n\mathbf{e}_n \equiv v^i\mathbf{e}_i$, with the v^i being the components of the vector, and the familiar Einstein summation convention is used in the second equality; likewise for covectors, which are expanded as $\mathbf{c} := c_i\mathbf{e}^i$, in terms of a (co)basis $\{\mathbf{e}^i\}$; furthermore, the outer product $\otimes$ allows us to write down the general expression of a tensor (that itself may be defined over any $p \in M$ or the whole M) as

$$\begin{aligned} \mathbf{T} &= T^{i_1...i_p}{}_{j_1...j_q}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q} \\ &=: T^{i_1...i_p}{}_{j_1...j_q} \bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m}, p, q \in \mathbb{N} \end{aligned}$$

We say that $\mathbf{T}$ is the tensor itself, the $T^{i_1...i_p}{}_{j_1...j_q}$ are its components, and the $\{\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q}\}$ are the generators of its basis[2]. It displays what is sometimes coloquially referred to as the 'tensorial property' of transforming under chartwise well-defined coordinate transformations $(x')^i := x^{i'} = x^{i'}(x^i)$ over M as follows:

$$T^{i'_1...i'_p}{}_{j'_1...j'_q} = J^{i'_1}{}_{i_1}...J^{i'_p}{}_{i_p}J^{j_1}{}_{j'_1}...J^{j_q}{}_{j'_q}T^{i_1...i_p}{}_{j_1...j_q} \tag{1}$$

with a similar expression holding for its generators; here we denote the components of the Jacobian matrix of the transformation $x^{i'} = x^{i'}(x^i)$ as $J^{i'_n}{}_{i_n} := \frac{\partial x^{i'_n}}{\partial x^{i_n}}$, whereas $J^{j_m}{}_{j'_m} := \frac{\partial x^{j_m}}{\partial x^{j'_m}}$ are the components of the inverse matrix, as easily checked using the chain rule of ordinary calculus. Strictly speaking, this is valid only for coordinate bases $\{\mathbf{e}_i = \frac{\partial}{\partial \mathbf{x}^i}, \mathbf{e}^i = d\mathbf{x}^i\}$, but the above is readily extended to noncoordinate bases as well, which we denote as $\{\mathbf{e}_{\tilde{i}} := e^i_{\tilde{i}}\frac{\partial}{\partial \mathbf{x}^i}, \mathbf{e}^{\tilde{i}} := e^{\tilde{i}}_i d\mathbf{x}^i\}$, with the $\{e^i_{\tilde{i}}, e^{\tilde{i}}_i\}$ assumed invertible ($e^k_{\tilde{i}}e^{\tilde{j}}_k = \delta^{\tilde{j}}_{\tilde{i}}, e^{\tilde{k}}_i e^j_{\tilde{k}} = \delta^j_i$). In this paper, unless explicitly stated, coordinate bases are always assumed when performing explicit computations - otherwise, we shall use the tilde notation, to emphasize that the bases in question are specifically noncoordinate (i.e., $e^i_{\tilde{i}} \neq \delta^i_{\tilde{i}}$).

This tensorial property, which may more properly be called coordinate-invariance, or *covariance*, makes tensors natural objects for the mathematical

[1] For a brush-up on these mathematical preliminaries, see, e.g., [13, 14, 15].

[2] In physics, we write differential equations involving tensors simply because of their nice, convenient, space-saving properties; as such, the ordering of each *individual* outer product $\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q}$ may, *a priori*, always be taken to be 'normal-ordered' in the manner shown here.

description of physical quantities; however, as easily checked from (1), partial derivatives $\partial_{k'}T^{i_1\ldots i_p}{}_{j_1\ldots j_q} = \frac{\partial}{\partial x^{k'}}T^{i_1\ldots i_p}{}_{j_1\ldots j_q}$ are, in general, nontensorial - which poses a problem for the use of tensors in differential equations. The problem is simply disposed of in the case of a Riemann space $(M, \mathbf{g})$; this new tensor $\mathbf{g} = g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$ we call the *metric*, and it has basically three uses: first, is it can be used to define a notion of distance in the manifold; this is easily illustrated with the special case of semi-Euclidian metrics (i.e., metrics that can be put as $g_{ij} = \eta_{ij} = diag\,[\pm 1, \pm 1, ..., \pm 1]$ in some global chart): given two points $\mathbf{x}, \mathbf{y}$, the distance between them may be written as $d(\mathbf{x}, \mathbf{y}) = \sqrt{\eta_{ij}\left(x^i - y^i\right)\left(x^j - y^j\right)}$ - which indeed corresponds to our ordinary notion of length for strictly Euclidian metrics (i.e., equal to $diag\,[+1, +1, ..., +1]$ in some global chart). A second one is that, along with its inverse, $\mathbf{g}^{-1} = g^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, it allows for raising and lowering indexes; for example:

$$T_i{}^{kl\ldots}{}_{j\ldots} = g_{in}g^{mk}T^n{}_{mj\ldots}^{\;\;l\ldots}$$

A third use will be the construction of the correcting factor we need, by the introduction of the operation defined by

$$\begin{aligned}\left(T^{i_1\ldots i_p}{}_{j\ldots j_q}\right)_{,k} \quad : \quad &= \nabla_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q} := \partial_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q} + \begin{Bmatrix} i_1 \\ ik \end{Bmatrix} T^{i\ldots i_p}{}_{j_1\ldots j_q} + \ldots \\ &\ldots + \begin{Bmatrix} i_p \\ ik \end{Bmatrix} T^{i_1\ldots i}{}_{j_1\ldots j_q} - \left(\begin{Bmatrix} j \\ j_1 k \end{Bmatrix} T^{i_1\ldots i_p}{}_{j\ldots j_q} + \ldots + \begin{Bmatrix} j \\ j_q k \end{Bmatrix} T^{i_1\ldots i_p}{}_{j_1\ldots j}\right)\end{aligned} \tag{2}$$

where the Christoffel symbol of the second kind $\left\{ {l \atop ij} \right\}$ is related to the symbol of the first kind $\{k|ij\}$ by

$$\begin{Bmatrix} l \\ ij \end{Bmatrix} := g^{lk}\{k|ij\} := \frac{1}{2}g^{lk}\left(\frac{\partial}{\partial x^j}g_{ik} + \frac{\partial}{\partial x^i}g_{kj} - \frac{\partial}{\partial x^k}g_{ij}\right) \tag{3}$$

and, contrary to common use, we employ a comma rather than a semicolon to denote the ∇-operation, for reasons that will be clear later on. After effecting a change of coordinates $g_{i'j'} = g_{ml}J^m{}_{i'}J^l{}_{j'}$ from some generic curvilinear system to another one, we can show the following by straightforward manipulation (mod standard analytical conditions):

$$\begin{aligned}\begin{Bmatrix} l' \\ i'j' \end{Bmatrix} &= \frac{1}{2}g^{l'k'}\left(\frac{\partial}{\partial x^{j'}}g_{i'k'} + \frac{\partial}{\partial x^{i'}}g_{k'j'} - \frac{\partial}{\partial x^{k'}}g_{i'j'}\right) \\ &= J^{l'}{}_l J^i{}_{i'} J^j{}_{j'}\begin{Bmatrix} l \\ ij \end{Bmatrix} + \frac{1}{2}J^{l'}{}_l\left(J^j{}_{j'}\frac{\partial}{\partial x^j}J^l{}_{i'} + J^i{}_{i'}\frac{\partial}{\partial x^i}J^l{}_{j'}\right) \\ &\quad + \frac{1}{2}g^{l'k'}g_{ij}\left[J^i{}_{i'}\left(J^k{}_{j'}\frac{\partial}{\partial x^k}J^j{}_{k'} - J^k{}_{k'}\frac{\partial}{\partial x^k}J^j{}_{j'}\right)\right. \\ &\quad \left. + J^j{}_{j'}\left(J^k{}_{i'}\frac{\partial}{\partial x^k}J^i{}_{k'} - J^k{}_{k'}\frac{\partial}{\partial x^k}J^i{}_{i'}\right)\right] \\ &= J^{l'}{}_l J^i{}_{i'} J^j{}_{j'}\begin{Bmatrix} l \\ ij \end{Bmatrix} + J^{l'}{}_l J^j{}_{j'}\frac{\partial}{\partial x^j}J^l{}_{i'}\end{aligned}$$

where in the last equality use was made of the analytical property $\frac{\partial}{\partial x^{i'}} J^l{}_{j'} = \frac{\partial}{\partial x^{j'}} J^l{}_{i'}$ to effect the simplification. This well-known transformation rule not only shows (explicitly) that the Christoffel symbols do not form a tensor, but also allow us to see (and prove by induction) why the ∇-operation works: it's because the extra piece $J^{l'}{}_l J^j{}_{j'} \frac{\partial}{\partial x^j} J^l{}_{i'}$ in the last equality exactly cancels out the one that appears due to the derivation of the tensor components.

At this juncture, one must have a very clear picture of what has been established, and why: that is, in order to maintain the covariance of our physical theories, we introduced a 'generalized partial derivative', or *covariant derivative*, for the *exclusive* purpose of bookkeeping coordinate changes in tensor components; this requirement, per se, has nothing to do with physics of any kind - it's just part of our *a priori* mathematical framework - in the same vein of number sets, algebraic structures, and so on.

As for the properties of the Christoffels, mere inspection of (3) shows that $\left\{ {l \atop ij} \right\} = \left\{ {l \atop ji} \right\}$; however, arguably more important is its metric compatibility:

$$\begin{aligned} g_{ab,c} &= \frac{\partial}{\partial x^c} g_{ab} - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{ak} + \frac{\partial}{\partial x^a} g_{kc} - \frac{\partial}{\partial x^k} g_{ac} \right) g_{lb} - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{bk} + \frac{\partial}{\partial x^b} g_{kc} - \frac{\partial}{\partial x^k} g_{bc} \right) g_{al} = 0, \\ \delta^b_{a,c} &= \frac{\partial}{\partial x^c} \delta^b_a + \frac{1}{2} g^{bk} \left(\frac{\partial}{\partial x^c} g_{lk} + \frac{\partial}{\partial x^l} g_{kc} - \frac{\partial}{\partial x^k} g_{lc} \right) \delta^l_a - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{ak} + \frac{\partial}{\partial x^a} g_{kc} - \frac{\partial}{\partial x^k} g_{ac} \right) \delta^b_l = 0 \end{aligned}$$

These relations suffice to show that $g^{ab}{}_{,c} = 0$ as well.

After the introduction of covariant differentiation, it is customary in GR and/or tensor calculus textbooks to define the Riemann–Christoffel (RC) tensor - typically in terms of transport of a vector along a closed circuit. Such an undertaking, however, is made considerably more expedient (not to mention geometrically clear) in the formalism of tensor-valued (multi)forms pioneered by Cartan [16]. With it, computation of the curvature, as well as other objects of geometrical interest, becomes quite efficient; indeed, ordinarily, this method "surpasses in efficiency every other known method for calculating the curvature 2-forms" [9].

A general tensor-valued r-form is defined as

$$\mathbf{T} := \mathbf{T}^{i_1 \dots i_p}{}_{j_1 \dots j_q} \bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} = T^{i_1 \dots i_p}{}_{j_1 \dots j_q k_1 \dots k_r} \left(\bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \otimes \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right), p, q, r \in \mathbb{N}$$

Since this formula reduces to our previous definition of tensor for $r = 0$, we see it is a straightforward generalization of the concept. Also, it is important to note that, albeit the newly introduced $\boldsymbol{\theta}^k$ are 'soldered' to the $\mathbf{e}^k$ in the sense that they transform identically under coordinate transformations (e.g., $\boldsymbol{\theta}^{\bar{k}} = \theta^{\bar{k}}_k \boldsymbol{\theta}^k \equiv e^{\bar{k}}_k \boldsymbol{\theta}^k$), nonetheless the spaces spanned by the $\boldsymbol{\theta}'s$ and $\mathbf{e}'s$ are to be treated differently - as we shall see below.

With these tensor-valued forms we define the covariant exterior derivative $\mathbf{d}$ by the operation

$$\begin{aligned}
\mathbf{dT} &= \left(\nabla_l T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r}\right)\left(\boldsymbol{\theta}^l \wedge \bigwedge_{s=1}^{r}\boldsymbol{\theta}^{k_s}\right)\otimes\left(\bigotimes_{n=1}^{p}\mathbf{e}_{i_n}\bigotimes_{m=1}^{q}\mathbf{e}^{j_m}\right) \quad (4)\\
&\equiv \left(\mathbf{d}T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r}\right)\wedge\left(\bigwedge_{s=1}^{r}\boldsymbol{\theta}^{k_s}\right)\otimes\left(\bigotimes_{n=1}^{p}\mathbf{e}_{i_n}\bigotimes_{m=1}^{q}\mathbf{e}^{j_m}\right)\\
&\quad+\left(T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r}\right)\left[\mathbf{d}\left(\bigotimes_{n=1}^{p}\mathbf{e}_{i_n}\bigotimes_{m=1}^{q}\mathbf{e}^{j_m}\right)\right]\wedge\left(\bigwedge_{s=1}^{r}\boldsymbol{\theta}^{k_s}\right)\\
&\quad+\left(T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r}\right)\left(\bigotimes_{n=1}^{p}\mathbf{e}_{i_n}\bigotimes_{m=1}^{q}\mathbf{e}^{j_m}\right)\otimes\left[\mathbf{d}\left(\bigwedge_{s=1}^{r}\boldsymbol{\theta}^{k_s}\right)\right]
\end{aligned}$$

In order to satisfy the second equality, one defines a 1-form

$$\boldsymbol{\gamma}_a^b := \gamma_{ak}^b\boldsymbol{\theta}^k \quad (5)$$

that in a coordinate basis is given by $\gamma_{ak}^b = \left\{ {b \atop ak} \right\}$, so that

$$\begin{aligned}
\mathbf{de}_a &: = \boldsymbol{\gamma}_a^b \otimes \mathbf{e}_b, && (6a)\\
\mathbf{de}^b &: = -\boldsymbol{\gamma}_a^b \otimes \mathbf{e}^a, && (6b)\\
\mathbf{d}\boldsymbol{\theta}^c &\equiv -\boldsymbol{\gamma}_i^c \wedge \boldsymbol{\theta}^i && (6c)
\end{aligned}$$

The last relation deserves some comment; in a coordinate basis, it is easily seen to be true: from Clairaut's theorem, we have $\mathbf{d}\boldsymbol{\theta}^c = d^2\mathbf{x}^c \equiv \mathbf{0}$ - but it is also true that $\boldsymbol{\gamma}_i^c \wedge \boldsymbol{\theta}^i = \left\{ {c \atop ik} \right\}\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^i \equiv \mathbf{0}$, because of $\left\{ {c \atop ik} \right\} = \left\{ {c \atop ki} \right\}$. To see that it holds even in noncoordinate bases, we will introduce components $\gamma_{\tilde{i}\tilde{j}}^{\tilde{l}}$ so that

$$\mathbf{dv} = \left[\mathbf{e}_j\left(v^l\right) + \gamma_{ij}^l v^i\right]\boldsymbol{\theta}^j \otimes \mathbf{e}_l \equiv \left[\mathbf{e}_{\tilde{j}}\left(v^{\tilde{l}}\right) + \gamma_{\tilde{i}\tilde{j}}^{\tilde{l}} v^{\tilde{i}}\right]\boldsymbol{\theta}^{\tilde{j}} \otimes \mathbf{e}_{\tilde{l}}$$

is covariant (and where the notation $\mathbf{e}_{\tilde{i}}(f) := e_{\tilde{i}}^i \frac{\partial}{\partial x^i} f$ was introduced). This simplifies to

$$\gamma_{\tilde{i}\tilde{j}}^{\tilde{l}} = e_l^{\tilde{l}} e_{\tilde{i}}^i e_{\tilde{j}}^j \gamma_{ij}^l + e_l^{\tilde{l}} e_{\tilde{j}}^j \left(\partial_j e_{\tilde{i}}^l\right)$$

from which we get the commutator

$$\gamma_{\tilde{i}\tilde{j}}^{\tilde{l}} - \gamma_{\tilde{j}\tilde{i}}^{\tilde{l}} = e_l^{\tilde{l}}\left[e_{\tilde{j}}^j\left(\partial_j e_{\tilde{i}}^l\right) - e_{\tilde{i}}^i\left(\partial_i e_{\tilde{j}}^l\right)\right] \equiv -c_{\tilde{i}\tilde{j}}^{\tilde{l}} \quad (7)$$

where the $c_{\tilde{i}\tilde{j}}^{\tilde{l}}$ are identified with the structure coefficients associated with the Lie bracket

$$\left[\mathbf{e}_{\tilde{i}}, \mathbf{e}_{\tilde{j}}\right] = e_{\tilde{i}}^i \frac{\partial}{\partial \mathbf{x}^i}\left(e_{\tilde{j}}^j \frac{\partial}{\partial \mathbf{x}^j}\right) - e_{\tilde{j}}^j \frac{\partial}{\partial \mathbf{x}^j}\left(e_{\tilde{i}}^i \frac{\partial}{\partial \mathbf{x}^i}\right) = \left[e_{\tilde{i}}^i\left(\partial_i e_{\tilde{j}}^l\right) - e_{\tilde{j}}^j\left(\partial_j e_{\tilde{i}}^l\right)\right] e_l^{\tilde{l}} \mathbf{e}_{\tilde{l}} =: c_{\tilde{i}\tilde{j}}^{\tilde{l}} \mathbf{e}_{\tilde{l}}$$

But this just happens to match the derivative of $\boldsymbol{\theta}^{\tilde{l}}$, as well:

$$\mathbf{d}\boldsymbol{\theta}^{\tilde{l}} = \left(\partial_i e_l^{\tilde{l}}\right)\boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^l = \left(\partial_i e_l^{\tilde{l}}\right)\left(e_{\tilde{i}}^i \boldsymbol{\theta}^{\tilde{i}} \wedge e_{\tilde{j}}^l \boldsymbol{\theta}^{\tilde{j}}\right) \equiv -e_{\tilde{i}}^i e_l^{\tilde{l}}\left(\partial_i e_{\tilde{j}}^l\right)\boldsymbol{\theta}^{\tilde{i}} \wedge \boldsymbol{\theta}^{\tilde{j}} \quad (8)$$

where in the third equality we used integration by parts and the fact that $\partial_i \left(e^{\tilde{l}}_l e^l_{\tilde{j}} \right) = \partial_i \delta^{\tilde{l}}_{\tilde{j}} = 0$. So, comparing eqs. (7) and (8), the result (6c) follows.

As a sanity check, let us compute the metric compatibility in the new formalism:

$$\begin{aligned} \mathbf{dg} &= (\mathbf{d}g_{ab}) \otimes \mathbf{e}^a \otimes \mathbf{e}^b + g_{cb} (\mathbf{de}^c) \otimes \mathbf{e}^b + g_{ac} \mathbf{e}^a \otimes (\mathbf{de}^c) \\ &\equiv (\partial_c g_{ab}) \boldsymbol{\theta}^c \otimes \mathbf{e}^a \otimes \mathbf{e}^b - (g_{cb} \boldsymbol{\gamma}^c_a + g_{ac} \boldsymbol{\gamma}^c_b) \otimes \mathbf{e}^a \otimes \mathbf{e}^b \\ &\equiv (\nabla_c g_{ab}) \boldsymbol{\theta}^c \otimes \mathbf{e}^a \otimes \mathbf{e}^b \end{aligned} \tag{9}$$

We thus see the consistency with our previous computation; however, since this is a tensorial operation, we're able to rewrite the exact same thing in a noncoordinate basis

$$\mathbf{0} \equiv \mathbf{dg} = \mathbf{e}_{\tilde{c}} \left(g_{\tilde{a}\tilde{b}} \right) \boldsymbol{\theta}^{\tilde{c}} \otimes \mathbf{e}^{\tilde{a}} \otimes \mathbf{e}^{\tilde{b}} - \left(g_{\tilde{c}\tilde{b}} \boldsymbol{\gamma}^{\tilde{c}}_{\tilde{a}} + g_{\tilde{a}\tilde{c}} \boldsymbol{\gamma}^{\tilde{c}}_{\tilde{b}} \right) \otimes \mathbf{e}^{\tilde{a}} \otimes \mathbf{e}^{\tilde{b}}$$

from which it follows we can perform the linear combination

$$\begin{aligned} \left\{ \tilde{k} | \tilde{\imath}\tilde{\jmath} \right\} \quad : &= \frac{1}{2} \left[\mathbf{e}_{\tilde{\jmath}} \left(g_{\tilde{\imath}\tilde{k}} \right) + \mathbf{e}_{\tilde{\imath}} \left(g_{\tilde{k}\tilde{\jmath}} \right) - \mathbf{e}_{\tilde{k}} \left(g_{\tilde{\imath}\tilde{\jmath}} \right) \right] \\ &= \frac{1}{2} \left[\left(\gamma_{\tilde{k}\tilde{\imath}\tilde{\jmath}} + \gamma_{\tilde{\imath}\tilde{k}\tilde{\jmath}} \right) + \left(\gamma_{\tilde{\jmath}\tilde{k}\tilde{\imath}} + \gamma_{\tilde{k}\tilde{\jmath}\tilde{\imath}} \right) - \left(\gamma_{\tilde{\jmath}\tilde{\imath}\tilde{k}} + \gamma_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \right) \right] \end{aligned} \tag{10}$$

from which, after some algebra, we recuperate the expression of the Levi-Civita (LC) connection $\gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}}$ in *any* basis:

$$g_{\tilde{k}\tilde{l}} \gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}} = \gamma_{\tilde{k}\tilde{\imath}\tilde{\jmath}} = \left\{ \tilde{k} | \tilde{\imath}\tilde{\jmath} \right\} + \frac{1}{2} \left(c_{\tilde{\jmath}\tilde{k}\tilde{\imath}} + c_{\tilde{\imath}\tilde{k}\tilde{\jmath}} - c_{\tilde{k}\tilde{\imath}\tilde{\jmath}} \right) \tag{11}$$

The combinations of structure coefficients in parenthesis are often called the Ricci (rotation) coefficients; by inspection, they are seen to be antisymmetric in $\tilde{k}, \tilde{\imath}$.

As shown by these examples, the properties of $\mathbf{d}$ allow us to breeze through otherwise laborious calculations - for instance:

$$\begin{aligned} \mathbf{d}^2 (v^a \mathbf{e}_a) &= \mathbf{d} \left[(\mathbf{d}v^a) \otimes \mathbf{e}_a + v^a \mathbf{de}_a \right] = \left[\left(\mathbf{d}^2 v^a \right) \otimes \mathbf{e}_a - (\mathbf{d}v^a) \wedge \mathbf{de}_a \right] + \left[(\mathbf{d}v^a) \wedge \mathbf{de}_a + v^a \mathbf{d}^2 \mathbf{e}_a \right] \\ &\equiv \left[\mathbf{d} \left(\partial_j v^a \boldsymbol{\theta}^j \right) \right] \otimes \mathbf{e}_a + v^b \mathbf{d} \left(\boldsymbol{\gamma}^a_b \otimes \mathbf{e}_a \right) = \left[(\partial_i \partial_j v^a) \left(\boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \right) + v^b \left(\mathbf{d}\boldsymbol{\gamma}^a_b - \boldsymbol{\gamma}^c_b \wedge \boldsymbol{\gamma}^a_c \right) \right] \otimes \mathbf{e}_a \\ &= : v^b \mathbf{R}^a_b \otimes \mathbf{e}_a \end{aligned} \tag{12}$$

where in the last line the RC tensor is defined - from a tensor-valued 2-form. We can check that this is indeed the same quantity from the textbooks by simply writing it explicitly:

$$\begin{aligned} \mathbf{R}^a_b &= \mathbf{d} \left(\begin{Bmatrix} a \\ bj \end{Bmatrix} \boldsymbol{\theta}^j \right) - \begin{Bmatrix} c \\ bi \end{Bmatrix} \begin{Bmatrix} a \\ cj \end{Bmatrix} \boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \\ &= \left[\partial_i \begin{Bmatrix} a \\ bj \end{Bmatrix} - \begin{Bmatrix} c \\ bi \end{Bmatrix} \begin{Bmatrix} a \\ cj \end{Bmatrix} \right] \boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \end{aligned}$$

From the last line, we see the RC tensor is antisymmetric in i, j, meaning that, in n dimensions, it has $\frac{n^3(n-1)}{2}$ independent components; to diminish this, we can derive (9) again, and, using $\mathbf{d}^2\mathbf{e}^c = -\mathbf{d}\left(\boldsymbol{\gamma}^c_b \otimes \mathbf{e}^b\right) = -\mathbf{R}^c_b \otimes \mathbf{e}^b$, obtain the 'Ricci identity'

$$\mathbf{0} \equiv \mathbf{d}^2\mathbf{g} = g_{cb}\left(\mathbf{d}^2\mathbf{e}^c\right)\otimes\mathbf{e}^b + g_{ac}\mathbf{e}^a\otimes\left(\mathbf{d}^2\mathbf{e}^c\right) = -\left(g_{cb}\mathbf{R}^c_a + g_{ac}\mathbf{R}^c_b\right)\otimes\mathbf{e}^a\otimes\mathbf{e}^b \quad (13)$$

which lowers the components down to $\left[\frac{n(n-1)}{2}\right]^2$; we may, however, further down their number with the help of the algebraic identity, as computed from

$$\mathbf{d}^2\boldsymbol{\theta}^a = \mathbf{d}\left(\boldsymbol{\gamma}^a_b \wedge \boldsymbol{\theta}^b\right) \equiv \left(\mathbf{R}^a_b + \boldsymbol{\gamma}^c_b \wedge \boldsymbol{\gamma}^a_c\right)\wedge\boldsymbol{\theta}^b = \mathbf{R}^a_b \wedge \boldsymbol{\theta}^b - \boldsymbol{\gamma}^a_c \wedge \left(\boldsymbol{\gamma}^c_b \wedge \boldsymbol{\theta}^b\right) \quad (14)$$

otherwise known as $R^a{}_{[bij]} = 0$. This property further reduces the remaining independent components of $R^a{}_{bij}$ down to $\left[\frac{n(n-1)}{2}\right]^2 - n\left[\frac{n(n-1)(n-2)}{3!}\right] = \frac{n^2\left(n^2-1\right)}{12}$; so, for $n = 4$, this means we've made quite the economy, going from 256 components to just 20 - not too shabby. Finally, we get the so-called Bianchi identity by a similar procedure:

$$\mathbf{d}\mathbf{R}^a_b = \mathbf{d}^2\boldsymbol{\gamma}^a_b - (\mathbf{d}\boldsymbol{\gamma}^c_b)\wedge\boldsymbol{\gamma}^a_c + \boldsymbol{\gamma}^c_b\wedge(\mathbf{d}\boldsymbol{\gamma}^a_c) = -\left(\mathbf{R}^c_b + \boldsymbol{\gamma}^d_b \wedge \boldsymbol{\gamma}^c_d\right)\wedge\boldsymbol{\gamma}^a_c + \boldsymbol{\gamma}^c_b\wedge\left(\mathbf{R}^a_c + \boldsymbol{\gamma}^d_c \wedge \boldsymbol{\gamma}^a_d\right) \quad (15)$$

otherwise known (in this paper's notation) as $R^a{}_{b[ij,k]} = 0$; on its turn, this relation is famous as the starting point in the derivation of the Einstein tensor.

Before closing this section, a final word on notation: we can use the metric to freely lower and raise indexes and rewrite tensor-valued forms however we prefer, but we have to be careful when nontensorial objects are involved; for instance, in the case of the RC tensor, its defining expression is given by the structural eq. (12) - but, *a posteriori*, we may introduce $\mathbf{R}_{ab} = g_{ac}\mathbf{R}^c_b$, etc. As another example, consider the covariant derivative $\mathbf{U} = \mathbf{d}\mathbf{u}$ of a vector-valued 1-form $\mathbf{u}$; we can read off its components $\mathbf{U}^a$ from

$$\mathbf{U}^a \otimes \mathbf{e}_a = \mathbf{d}\left(\mathbf{u}^a \otimes \mathbf{e}_a\right) = \left(\mathbf{d}\mathbf{u}^a - \mathbf{u}^c \wedge \boldsymbol{\gamma}^a_c\right) \otimes \mathbf{e}_a$$

However, if we wish to treat $\mathbf{U}$ as a *co*vector-valued 2-form instead, its components will change to

$$\mathbf{U}_b \otimes \mathbf{e}^b = \mathbf{d}\left(\mathbf{u}_b \otimes \mathbf{e}^b\right) = \left(\mathbf{d}\mathbf{u}_b + \mathbf{u}_c \wedge \boldsymbol{\gamma}^c_b\right) \otimes \mathbf{e}^b$$

So, if we keep these distinctions in mind, there'll be no problem with the (admittedly language-abusing) notation $\mathbf{U} = \mathbf{U}^a \otimes \mathbf{e}_a = \mathbf{U}_b \otimes \mathbf{e}^b$ that'll be employed later on, because the ambiguity can be eliminated based on the context.

3 Differential Affine Geometry

The concepts thus introduced suffice to formulate a pragmatic, multipurpose tensor calculus framework fully integrated with exterior algebra, which is particularly important for problems involving integration and provides a modern, more elegant reformulation of the old vector calculus that can be readily generalized to any dimensionality. Nonetheless, up to now, no explicit mention has been made of any gravitational phenomena; in particular, the metric was introduced as an *ad hoc*, nondynamical mathematical device for the purposes of providing 1) a formalization of our intuition of 'length', 2) a means to raise and lower indices, and 3) an explicit expression of the covariant derivative, via its introduction in the Christoffels. None of these, we see, has any obvious gravitational connotation; in fact, since any physical system (whether under gravitational influences or not) may be described in terms of this formalism, we can appreciate their significance as being purely operational - as part of the general toolbox of mathematical concepts that we introduce in order to frame and quantify generic physical phenomena. How, then, can we characterize gravitational phenomena as separate from such a toolbox? To this problem we turn next.

Fortunately, a simple fix is available to us, thanks to the tensor-valued formalism: we propose introducing a new operator $\mathbf{D}$ that represents a slight generalization of our previous $\mathbf{d}$ by its effect on tensor-valued forms: for $\mathbf{d} \to \mathbf{D}$, substitute in eq. (4)

$$\begin{aligned} \nabla_l T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} &\to \overset{\omega}{\nabla}_l T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} \\ T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r, l} &\to T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r ; l} \end{aligned}$$

with $\mathbf{D} T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} = \mathbf{d} T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r}$, and where now

$$\overset{\omega}{\boldsymbol{\nabla}} \mathbf{e}_a \equiv \mathbf{D}\mathbf{e}_a := \mathbf{d}\mathbf{e}_a + \boldsymbol{\omega}_a^b \otimes \mathbf{e}_b + \boldsymbol{\omega}_a, \tag{16a}$$

$$\overset{\omega}{\boldsymbol{\nabla}} \mathbf{e}^b \equiv \mathbf{D}\mathbf{e}^b := \mathbf{d}\mathbf{e}^b - \boldsymbol{\omega}_a^b \otimes \mathbf{e}^a - \boldsymbol{\omega}^b, \tag{16b}$$

$$\mathbf{D}\boldsymbol{\theta}^c \; : \; = \mathbf{d}\boldsymbol{\theta}^c \tag{16c}$$

with the alternative notation for the components $T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r ; l} := \overset{\omega}{\nabla}_l T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r}$ (thus justifying our previous choice of notation). As seen from these definitions, the tensor-valued 1-form[3] $\boldsymbol{\omega}_a^b$ and the covector-valued 1-form $\boldsymbol{\omega}_a$ (dual to $\boldsymbol{\omega}^b$) account for all the deviation between $\mathbf{D}$ and $\mathbf{d}$ in a manifestly covariant way; furthermore, it will also prove useful to define from the general expression of $\mathbf{D}$ another operator $\mathbf{D_0}$, equivalent to the former but with $\boldsymbol{\omega}_a = \mathbf{0}$.

Now that $\mathbf{D}$ (respec. $\mathbf{D_0}$) has been defined, we proceed to once again calcu-

[3] Incidentally, this object has already appeared under the name "distortion 1-form" in the MAG literature at least as early as 1997 [18] - as well as the decomposition of the "total curvature" into the RC tensor and the "post-Riemannian pieces".

late the second derivative of the vector $\mathbf{v}$:

$$\begin{aligned}
\mathbf{D}^2\left(v^a\mathbf{e}_a\right) &= \mathbf{D}\left[(\mathbf{d}v^a)\otimes\mathbf{e}_a + v^a\mathbf{D}\mathbf{e}_a\right] \equiv (\mathbf{D}\mathbf{d}v^a)\otimes\mathbf{e}_a + v^b\mathbf{D}\left(\mathbf{d}\mathbf{e}_b + \boldsymbol{\omega}^a_b\otimes\mathbf{e}_a + \boldsymbol{\omega}_b\right)\\
&\equiv \left(\mathbf{d}^2 v^a\right)\otimes\mathbf{e}_a + v^b\left\{\left[\mathbf{d}\left(\boldsymbol{\gamma}^a_b + \boldsymbol{\omega}^a_b\right) - \left(\boldsymbol{\gamma}^c_b + \boldsymbol{\omega}^c_b\right)\wedge\left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right)\right]\otimes\mathbf{e}_a - \left(\boldsymbol{\gamma}^c_b + \boldsymbol{\omega}^c_b\right)\wedge\boldsymbol{\omega}_c + \mathbf{D}\boldsymbol{\omega}_b\right\}\\
&\equiv v^b\left\{\left[\mathbf{R}^a_b + \left(\mathbf{d}\boldsymbol{\omega}^a_b - \boldsymbol{\gamma}^c_b\wedge\boldsymbol{\omega}^a_c - \boldsymbol{\omega}^c_b\wedge\boldsymbol{\gamma}^a_c\right) - \boldsymbol{\omega}^c_b\wedge\boldsymbol{\omega}^a_c\right]\otimes\mathbf{e}_a + \left[\left(\mathbf{d}\boldsymbol{\omega}_b - \boldsymbol{\gamma}^c_b\wedge\boldsymbol{\omega}_c\right) - \boldsymbol{\omega}^c_b\wedge\boldsymbol{\omega}_c\right]\right\}\\
&=: v^b\left[\left(\mathbf{R}^a_b + \boldsymbol{\Omega}^a_b\right)\otimes\mathbf{e}_a + \boldsymbol{\Omega}_b\right] \qquad (17)
\end{aligned}$$

where in the last step we defined the tensor-valued 2-form, $\boldsymbol{\Omega}^a_b$, and the covector-valued 2-form, $\boldsymbol{\Omega}_b$. In keeping with our previous steps, it is straightforward to obtain new Bianchi-like identities associated with them: for instance,

$$\begin{aligned}
\mathbf{d}\boldsymbol{\Omega}^a_b &= \left(\mathbf{d}^2\boldsymbol{\omega}^a_b + \mathbf{d}\boldsymbol{\gamma}^a_c\wedge\boldsymbol{\omega}^c_b - \boldsymbol{\gamma}^a_c\wedge\mathbf{d}\boldsymbol{\omega}^c_b + \mathbf{d}\boldsymbol{\omega}^a_c\wedge\boldsymbol{\gamma}^c_b - \boldsymbol{\omega}^a_c\wedge\mathbf{d}\boldsymbol{\gamma}^c_b\right) + \mathbf{d}\boldsymbol{\omega}^a_c\wedge\boldsymbol{\omega}^c_b - \boldsymbol{\omega}^a_c\wedge\mathbf{d}\boldsymbol{\omega}^c_b\\
&\equiv \boldsymbol{\Omega}^a_c\wedge\left(\boldsymbol{\gamma}^c_b + \boldsymbol{\omega}^c_b\right) - \left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right)\wedge\boldsymbol{\Omega}^c_b + \left(\mathbf{R}^a_c\wedge\boldsymbol{\omega}^c_b - \boldsymbol{\omega}^a_c\wedge\mathbf{R}^c_b\right)
\end{aligned}$$

which, upon rearranging, can be written

$$\mathbf{D_0}\boldsymbol{\Omega} \equiv \left(\boldsymbol{\omega}^c_b\wedge\mathbf{R}^a_c - \boldsymbol{\omega}^a_c\wedge\mathbf{R}^c_b\right)\otimes\mathbf{e}_a\otimes\mathbf{e}^b \qquad (18)$$

where $\boldsymbol{\Omega} := \boldsymbol{\Omega}^a_b\otimes\mathbf{e}_a\otimes\mathbf{e}^b$. Likewise, noticing that with $\boldsymbol{\theta} := \boldsymbol{\omega}^a\otimes\mathbf{e}_a$

$$\mathbf{D_0}\boldsymbol{\theta} = \left[\left(\mathbf{d}\boldsymbol{\omega}^a + \boldsymbol{\gamma}^a_c\wedge\boldsymbol{\omega}^c\right) + \boldsymbol{\omega}^a_c\wedge\boldsymbol{\omega}^c\right]\otimes\mathbf{e}_a =: \boldsymbol{\Omega}^a\otimes\mathbf{e}_a \qquad (19)$$

for the dual of $\boldsymbol{\Omega}_b$, we have

$$\begin{aligned}
\mathbf{d}\boldsymbol{\Omega}^a &= \mathbf{d}^2\boldsymbol{\omega}^a + \left(\mathbf{d}\boldsymbol{\gamma}^a_c + \mathbf{d}\boldsymbol{\omega}^a_c\right)\wedge\boldsymbol{\omega}^c - \left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right)\wedge\mathbf{d}\boldsymbol{\omega}^c\\
&\equiv \left(\mathbf{R}^a_c + \boldsymbol{\Omega}^a_c\right)\wedge\boldsymbol{\omega}^c - \left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right)\wedge\boldsymbol{\Omega}^c
\end{aligned}$$

which, defining $\boldsymbol{\Theta} := \mathbf{d}\boldsymbol{\theta}$, rearranges to

$$\mathbf{d}\boldsymbol{\Theta} \equiv \left(\mathbf{R}^a_c\wedge\boldsymbol{\omega}^c\right)\otimes\mathbf{e}_a \qquad (20)$$

Up to here, our analysis has led to three sets of objects: $\left\{\mathbf{e}_a, \mathbf{e}^b, \boldsymbol{\theta}^c\right\}$; $\left\{\mathbf{g}, \boldsymbol{\gamma}^a_b, \mathbf{R}^a_b\right\}$; and $\left\{\boldsymbol{\omega}^a_b, \boldsymbol{\omega}^a, \boldsymbol{\Omega}^a_b, \boldsymbol{\Omega}^a, \boldsymbol{\Theta}^a\right\}$ - all of which, with the exception of $\boldsymbol{\gamma}^a_b$, tensorial in nature. Consider the first of these: even in semi-Euclidian spaces, they can't be made to vanish nontrivially, because they're necessary to define the tensor-valued forms that'll be interpreted as physical objects (even in the absence of gravity); the situation is similar with the *Riemannian* objects $\left\{\mathbf{g}, \boldsymbol{\gamma}^a_b, \mathbf{R}^a_b\right\}$, because they're required to maintain the general covariance of all tensor(-valued) expressions. This suggests that these objects are not suitable candidates for being gravitational variables, which we'd like to be able to freely take to vanish in a manifestly covariant way; so, we're left with the *post-Riemannian* $\left\{\boldsymbol{\omega}^a_b, \boldsymbol{\omega}^a, \boldsymbol{\Omega}^a_b, \boldsymbol{\Omega}^a, \boldsymbol{\Theta}^a\right\}$ as prime candidates, and to their detailed geometrical properties we now turn, after introducing some nomenclature; without going to bundles [19], we shall find it convenient to simply refer to them as follows: a pair of given $\left(\boldsymbol{\omega}^a_b, \boldsymbol{\omega}^a\right)$ defines an *affine connection*, while a pair $\left(\boldsymbol{\Omega}^a_b, \boldsymbol{\Omega}^a\right)$ an *affine curvature*, whilst $\boldsymbol{\omega}^a_b$ and $\boldsymbol{\Omega}^a_b$ considered in isolation define a *linear connection* and *(linear) curvature*, respec.; finally, a given $\boldsymbol{\Theta}^a$ will be referred to as a *torsion*.

Let us first study the metric compatibility of $\mathbf{D}$ (which is identical to that of $\mathbf{D_0}$).

$$\begin{aligned}\mathbf{Dg} &= (\mathbf{d}g_{ab}) \otimes \mathbf{e}^a \otimes \mathbf{e}^b + g_{cb} (\mathbf{De}^c) \otimes \mathbf{e}^b + g_{ac}\mathbf{e}^a \otimes (\mathbf{De}^c) \\ &\equiv (\nabla_c g_{ab}) \boldsymbol{\theta}^c \otimes \mathbf{e}^a \otimes \mathbf{e}^b - (g_{cb}\boldsymbol{\omega}_a^c + g_{ac}\boldsymbol{\omega}_b^c) \otimes \mathbf{e}^a \otimes \mathbf{e}^b\end{aligned}$$

Since it was already established that $g_{ab,c} \equiv 0$, this shows that the *non-metricity* $\mathbf{n}$ has a simple dependence on $\boldsymbol{\omega}_{(ab)} = \frac{1}{2!} (\boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{ba})$:

$$\mathbf{n} := \boldsymbol{\omega}_{(ab)} \otimes \mathbf{e}^a \otimes \mathbf{e}^b \equiv -\frac{1}{2}\mathbf{Dg} \tag{21}$$

This observation induces the handy decomposition $\boldsymbol{\omega}_{ab} = \mathring{\boldsymbol{\omega}}_{ab} + \mathbf{n}_{ab}$ of the linear connection in terms of the antisymmetric $\mathring{\boldsymbol{\omega}}_{ab} := \boldsymbol{\omega}_{[ab]} = \frac{1}{2!} (\boldsymbol{\omega}_{ab} - \boldsymbol{\omega}_{ba})$ - which in turn leads to a similar decomposition of the curvature. After eq. (17), define

$$\mathring{\boldsymbol{\Omega}}_b^a := \mathbf{d}\mathring{\boldsymbol{\omega}}_b^a - \boldsymbol{\gamma}_b^c \wedge \mathring{\boldsymbol{\omega}}_c^a - \mathring{\boldsymbol{\omega}}_b^c \wedge \boldsymbol{\gamma}_c^a - \mathring{\boldsymbol{\omega}}_b^c \wedge \mathring{\boldsymbol{\omega}}_c^a \tag{22}$$

with $\mathring{\boldsymbol{\Omega}} := \mathring{\boldsymbol{\Omega}}_b^a \otimes \mathbf{e}_a \otimes \mathbf{e}^b$ - and substitute it back in that equation:

$$\begin{aligned}\boldsymbol{\Omega}_b^a &\equiv \mathring{\boldsymbol{\Omega}}_b^a + [\mathbf{dn}_b^a - (\boldsymbol{\gamma}_b^c + \mathring{\boldsymbol{\omega}}_b^c) \wedge \mathbf{n}_c^a - \mathbf{n}_b^c \wedge (\boldsymbol{\gamma}_c^a + \mathring{\boldsymbol{\omega}}_c^a) - \mathbf{n}_b^c \wedge \mathbf{n}_c^a] \\ &=: \mathring{\boldsymbol{\Omega}}_b^a + \mathbf{N}_b^a\end{aligned} \tag{23}$$

where in the last equality $\mathbf{N}_b^a$ was defined. The above development makes it natural to define a new operator $\mathring{\mathbf{D}}$ (respec. $\mathring{\mathbf{D}}_\mathbf{0}$) as being the same as $\mathbf{D}$ (respec. $\mathbf{D_0}$) but with the $\boldsymbol{\omega}_b^a$ restricted to $\mathring{\boldsymbol{\omega}}_b^a$ only; this will prove useful later on - though we can already adapt the Ricci identity argument of the previous section to $\mathring{\mathbf{D}}_\mathbf{0}$ and show that $\mathring{\boldsymbol{\Omega}}_{ab} = -\mathring{\boldsymbol{\Omega}}_{ba}$ too - just like the RC tensor.

Continuing this process, the nonmetricity can be further decomposed as

$$\boldsymbol{\phi} := \boldsymbol{\omega}_c{}^c \equiv \mathbf{n}_c{}^c, \tag{24a}$$

$$\mathbf{n}_{ab} =: \frac{1}{n} g_{ab}\boldsymbol{\phi} + \boldsymbol{\sigma}_{ab} \tag{24b}$$

from which the curvature can be further decomposed, as well:

$$\begin{aligned}\mathbf{N}_b^a &\equiv \frac{1}{n}\delta_b^a\mathbf{d}\boldsymbol{\phi} + [\mathbf{d}\boldsymbol{\sigma}_b^a - (\boldsymbol{\gamma}_b^c + \mathring{\boldsymbol{\omega}}_b^c) \wedge \boldsymbol{\sigma}_c^a - \boldsymbol{\sigma}_b^c \wedge (\boldsymbol{\gamma}_c^a + \mathring{\boldsymbol{\omega}}_c^a) - \boldsymbol{\sigma}_b^c \wedge \boldsymbol{\sigma}_c^a] \\ &=: \frac{1}{n}\delta_b^a\mathbf{d}\boldsymbol{\phi} + \boldsymbol{\Sigma}_b^a\end{aligned} \tag{25}$$

on account of the appearance of terms such as $\boldsymbol{\phi} \wedge \boldsymbol{\phi}$, $(\boldsymbol{\phi} \wedge \boldsymbol{\sigma}_b^a + \boldsymbol{\sigma}_b^a \wedge \boldsymbol{\phi})$, etc., which vanish identically.

This completes the description of our differential affine formalism. So far in this and the previous section, we've exclusively talked about pure mathematics; now is the time to transfer these theoretical results into the arena of physics - but before we do, a last digression will prove useful for bookkeeping purposes: if we reflect back on the literature [10, 11, 12], we see references to different 'geometries' or 'spaces' based on criteria such as the (non)vanishing of curvature, torsion, nonmetricity, etc. - i.e., starting from the most generic (metric-)affine description, one then picks some of the objects describing the geometry to be dynamical, and such a choice yields a physical theory of gravitation. In our present formalism, a similar approach can be pursued in terms of the four objects $\boldsymbol{\omega}_a, \mathring{\boldsymbol{\omega}}_{ab}, \boldsymbol{\phi}, \boldsymbol{\sigma}_{ab}$ comprising the affine connection; chosing whether or not any of these vanish yields a total of 16 geometries - some of which I've named in Table I.

Table I. Selected geometries for theories of gravitation.

geometry	non-dynamical object
Weyl–Cartan	$\boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weyl–Weitzenböck	$\mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weyl	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
pre-Weyl	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Cartan	$\boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weitzenböck	$\mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Ricci	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Riemann	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$

The attempt was made to introduce a nomenclature that mirrors the historical contributions of several eminent mathematicians; it is necessarily imperfect, due to the fact these authors did not employ the present schema - its value being mostly as a mnemonic device, that should be read with care (specially when comparing with the literature).

In the above list, we deliberately excluded, w.l.o.g., all eight geometries with $\boldsymbol{\sigma}_{ab} \neq \mathbf{0}$ guided by physical intuition; however, as the theoretical need for such geometries may rise, one can easily extend our naming conventions to include such fields - though in the present paper, we will not concern ourselves with them. (Also, the reader will notice the similarity of our taxonomy with that of the "MAGic cube" of [10]; in fact, by including the $\boldsymbol{\sigma}_{ab}$ field, we'd have a 'mAGic tesseract' - an amusing touch.)

At this point, finally, I will abandon mathematical generality and make physical commitments in order to obtain a theory of gravity in four-dimensional spacetime, which will be signaled by the switch to Greek indices. In order to obtain agreement with Special Relativity, the (nondynamical) metric is taken to be Minkowski's (with signature $\eta_{\mu\nu} = diag\,[-1, +1, +1, +1]$) - and as a consequence, we have $\mathbf{R}^{\alpha}_{\beta} = \mathbf{0}$, which notably simplifies the previous Bianchi-like identities obtained with a generic $\mathbf{g}$; as such, the role of *gravitational potential* is transferred wholly to the many pieces of the affine connection that have been introduced above. Before discussing their dynamics, it is of notice that, from a purely geometrical perspective, we can 'translate' the conventional GR picture into this framework - which may be surprising to some; but it can be simply achieved via the assignments

$$\boldsymbol{\omega}^{\beta} = \mathbf{0}, \boldsymbol{\omega}^{\beta}_{\alpha} = \boldsymbol{\Gamma}^{\beta}_{\alpha} - \boldsymbol{\gamma}^{\beta}_{\alpha} \tag{26}$$

where the $\boldsymbol{\Gamma}^{\beta}_{\alpha}$ refer to the LC connection computed from the Lorentzian metric $\mathbf{g}$ of Einstein's theory; it follows from these that the RC tensor in GR is equivalent to the curvature in this geometry. This alone might prove the usefulness of this formalism, for example, in the semiclassical regime - albeit it doesn't shed much light on the dynamics, as they're constrained by the Einstein field equations; henceforth we shall drop this equivalence formalism and instead deliberately experiment with an approach different from classic geometrodynamics, yet closer in philosophy to the gauge field theories used in modern physics.

4 Gravidynamics

The basic requirements for the statement of a gravitational theory are vividly expressed by the Wheelerian maxim [9]: "Space tells matter how to move; matter tells space how to curve". In GR, the first part essentially alludes to the geodesic equation, while the second is a shorthand for the EFE. The general conceptual foundation here, however, is no more sophisticated or mysterious than that of the Lorentz force equation and the Maxwell field equations in classical electrodynamics; thus, what is needed is a framework for the *transport* of matter and radiation, as well as a set of *field equations* that govern the strength of gravitational influences affecting said transport. To the question of formulating such a framework based on the geometries discussed thus far we now turn.

Space tells matter how to move. Our starting point is the recognition that $\mathbf{dv} = \mathbf{0}$ is essentially the geodesic equation expressed in covariant language; to see this, write its components w.r.t. a 'mixed' basis:

$$\mathbf{0} = \mathbf{dv} = \left(\frac{\partial}{\partial x^\nu} v^\alpha + \gamma^\alpha_{\beta\nu} v^\beta\right) \boldsymbol{\theta}^\nu \otimes \mathbf{e}_\alpha \equiv \left(e^\nu_{\tilde{\nu}} \frac{\partial}{\partial x^\nu} v^\alpha + \gamma^\alpha_{\beta\nu} v^\beta e^\nu_{\tilde{\nu}}\right) \boldsymbol{\theta}^{\tilde{\nu}} \otimes \mathbf{e}_\alpha$$

Then, choosing this basis so that $e^\nu_{\tilde{0}} = v^\nu =: \frac{dx^\nu}{d\tau}$, and using the chain rule to write $\frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}\frac{dx^\lambda}{d\tau} \equiv \frac{d^2x^\lambda}{d\tau^2}$, the $\left(\boldsymbol{\theta}^{\tilde{0}} \otimes \mathbf{e}_\alpha\right)$-components of the above may be expressed as

$$\frac{d^2x^\alpha}{d\tau^2} + \gamma^\alpha_{\beta\nu}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{27}$$

This is the familiar equation of a geodesic in Minkowski spacetime; naturally, one obtains the GR version by the substitution $\gamma^\alpha_{\beta\nu} \to \Gamma^\alpha_{\beta\nu}$ - and by the same token, it is not difficult to conceive a generalization $\gamma^\alpha_{\beta\nu} \to \gamma^\alpha_{\beta\nu} + \omega^\alpha_{\beta\nu}$ within the present scheme; however, let us pause here for a moment, and think about this physically: since the above expression can be interpreted simply as that of the acceleration of a particle in the absence of external forces, we see the $\gamma^\alpha_{\beta\nu}$ term must correspond to pseudoforces[4], which appear due to the choice of coordinate system, and which, by the equivalence principle, must be (locally, at least) indistinguishable from *actual* gravitational forces for a free fall. This would seem to peg the $\omega^\alpha_{\beta\nu}$ down as the components of the gravitational force;

[4] We're all familiar with the concept that gravity can be artificially simulated with a centrifuge.

however, from Newton's theory, we know those are given by the *gradient* of a potential, and this presents a conundrum. It is thus suggestive to reflect on the ambiguity we encountered when defining the affine derivative: we found an expression for the translation potentials, but no natural way to have them influence a vector like the one above; what, then, if we make the heuristic substitution $\omega^{\alpha}_{\beta\nu} \to \omega^{\alpha}_{\beta\nu} - f^{\alpha}_{\beta\nu}$? In that case, we'd be led to a geodesic-like *transport equation (TE)*

$$\frac{d^2x^\alpha}{d\tau^2} + \left(\gamma^{\alpha}_{\beta\nu} + \omega^{\alpha}_{\beta\nu}\right)\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = f^{\alpha}_{\beta\nu}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} \tag{28}$$

If we then surmise the $f^{\alpha}_{\beta\nu}$ terms to be related to those of the *torsion* (which happens to have the same number of indices as the linear connection), we'd recover the Newtonian notion of the gravitational force as gradient of a potential, and we'd make good use of the affine machinery developed earlier - two birds with one stone. Thus, let us write for the equation of motion

$$\mathbf{D_0 v} = \mathbf{a} := v^\beta f^{\alpha}_{\beta\nu} e^{\nu}_{\tilde{\nu}} \boldsymbol{\theta}^{\tilde{\nu}} \otimes \mathbf{e}_\alpha \tag{29}$$

which, we'll note, is strongly reminiscent of the Euler equation of inviscid fluid dynamics, provided we interpret $\mathbf{D_0}$ as the material derivative - whereas the 'specific force' reminds one of the Lorentz force of electrodynamics. Conversely, one may write this exact same expression in terms of a *Newton(–Cartan) operator* $\mathbf{D_N}$, which action on a vector $\mathbf{v}$ is given by

$$\mathbf{D_N v} := \mathbf{D_0 v} - \mathbf{a} = \mathbf{0} \tag{30}$$

and resembles Fermi–Walker differentiation; thus, one may in analogy say a free-falling particle is being 'Newton-transported'.

But what is the explicit form of $f^{\alpha}_{\beta\nu}$? Specializing the transport eq. (28) to $\gamma^{\alpha}_{\beta\nu} = \omega^{\alpha}_{\beta\nu} = 0$, and making use of $x^0 := ct$ and $\frac{d\tau}{dt} = \left(\frac{dt}{d\tau}\right)^{-1}$, we can express the 3-acceleration of a particle as

$$\begin{aligned} \frac{d^2x^k}{dt^2} &= \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{d\tau}{dt}\frac{dx^k}{d\tau}\right) = \left(\frac{d\tau}{dt}\right)^2\left(\frac{d}{d\tau}\frac{dx^k}{d\tau}\right) + \frac{dx^k}{dt}\left(\frac{d}{d\tau}\frac{d\tau}{dt}\right) \\ &\doteq f^{k}_{\beta\nu}\frac{dx^\beta}{dt}\frac{dx^\nu}{dt} - \frac{dx^k}{dt}\frac{1}{c}\left(f^{0}_{\beta\nu}\frac{dx^\beta}{dt}\frac{dx^\nu}{dt}\right) \end{aligned} \tag{31}$$

If we want the Newtonian limit, we must ignore all contributions in powers of the 3-velocity components $\frac{dx^k}{dt}$ in the r.h.s. of the last line as well as time derivatives, leaving us with $\frac{d^2x^k}{dt^2} \approx f^{k}_{00}c^2$ - which tells us that the f^{k}_{00} are essentially the three components of the Newtonian force - in turn a gradient of the one potential in the theory. Collecting all this information, let us finally introduce[5] the *Ansatz*

$$f_{\alpha\beta\nu} = \left(p^{\mu}_{\alpha} p^{\rho}_{\beta} \Theta^{\gamma}_{\mu\rho} p^{\lambda}_{\gamma}\right)\eta_{\lambda\nu} \tag{32}$$

[5] Although here the force term was essentially guessed on physical grounds, I've later found there are remarkable similarities with the teleparallel approach, as seen, e.g., in [20].

where the p's are (unity) projection operators whose sole job is to match indexes with the appropriate vector spaces; henceforth their presence shall be implicitly assumed, for brevity. As we see, the only dynamical object in this formula is the torsion, which in turn depends on the first-derivatives of the connection components ω^{α}_{μ}. Our justification for this is by substitution in the Newtonian formula together with the stationary condition $\frac{\partial}{\partial x^0}\omega^{\alpha}_{\mu} = 0$, which then yields

$$\frac{d^2 x^k}{dt^2} \approx -\frac{\partial}{\partial x^k}\left(\frac{\omega^0_0 c^2}{2}\right) \tag{33}$$

Similarly to GR then, where the Newtonian potential is associated with $\frac{(g_{00}+1)c^2}{2}$, here we see this role is filled by $\frac{\omega^0_0 c^2}{2}$ more naturally; confirmation of this is found in the FE, which as we'll see, for only ω^0_0 nonvanishing, reduce to Poisson's law. Furthermore, as we can see, one may also develop a post-Newtonian treatment following the lines of, e.g., the bookkeeping formalism of Weinberg [21]. Before proceeding, we pause to more closely inspect the relation between $\omega^{\alpha}_{\beta\nu}$ and $f^{\alpha}_{\beta\nu}$: for in particular, from (32), we see that $f_{\beta\alpha\nu} = -f_{\alpha\beta\nu}$ - just as with $\mathring{\omega}_{\alpha\beta\nu}$. Bearing this in mind, it is suggestive to introduce the notation $F^{\alpha}_{\beta\nu} := f^{\alpha}_{\beta\nu} - \mathring{\omega}^{\alpha}_{\beta\nu}$, which encapsulates the ambiguity inherent in telling the two contributions apart: one may choose the former as to simulate the effects of the latter, and *vice versa*, depending on one's convenience. This remarkable feature is itself another manifestation of the equivalence principle - which at this juncture already enables one to incorporate a notion of 'gravitoelectromagnetism' into the theory: from the discussion just above, we recognize the $f_{0k\nu} = -f_{k0\nu}$ terms as basically gravitoelectric contributions.

A further test of the physical content of these ideas is in the anholonomy of a vector $\mathbf{X} = X^{\alpha}\mathbf{e}_{\alpha}$ as measured by the quantity $\mathbf{A} := \boldsymbol{\nabla}_{\mathbf{0}}\boldsymbol{\nabla}_{\mathbf{0}}\mathbf{X}$, where $\boldsymbol{\nabla}_{\mathbf{0}}$ is just the nonsymmetrized version of $\mathbf{D_0}$. From our work above, we have

$$\boldsymbol{\nabla}^2_{\mathbf{0}}\mathbf{X} = \boldsymbol{\nabla}_{\mathbf{0}}\left(e^{\nu}_{\tilde{\nu}} X^{\alpha}_{;\nu}\boldsymbol{\theta}^{\tilde{\nu}} \otimes \mathbf{e}_{\alpha}\right) \equiv e^{\mu}_{\tilde{\mu}}\left(e^{\nu}_{\tilde{\nu}} X^{\alpha}_{;\nu}\right)_{;\mu}\boldsymbol{\theta}^{\tilde{\mu}} \otimes \boldsymbol{\theta}^{\tilde{\nu}} \otimes \mathbf{e}_{\alpha} \tag{34}$$

In order to calculate those components, let us first define the quantity

$$B^{\alpha}_{\tilde{\nu}} := e^{\nu}_{\tilde{\nu}} X^{\alpha}_{;\nu} - e^{\alpha}_{\tilde{\nu};\nu} X^{\nu} \tag{35}$$

so that

$$\begin{aligned}
A^{\alpha}_{\tilde{\mu}\tilde{\nu}} &= e^{\mu}_{\tilde{\mu}}\left(B^{\alpha}_{\tilde{\nu}} + e^{\alpha}_{\tilde{\nu};\nu} X^{\nu}\right)_{;\mu} \\
&\equiv e^{\mu}_{\tilde{\mu}} B^{\alpha}_{\tilde{\nu};\mu} + e^{\mu}_{\tilde{\mu}}\left(e^{\alpha}_{\tilde{\nu};\nu\mu} - e^{\alpha}_{\tilde{\nu};\mu\nu}\right) X^{\nu} + e^{\mu}_{\tilde{\mu}} e^{\alpha}_{\tilde{\nu};\mu\nu} X^{\nu} + \left(B^{\nu}_{\tilde{\mu}} + e^{\nu}_{\tilde{\mu};\mu} X^{\mu}\right) e^{\alpha}_{\tilde{\nu};\nu} \\
&\equiv \left(e^{\mu}_{\tilde{\mu}} B^{\alpha}_{\tilde{\nu};\mu} + B^{\nu}_{\tilde{\mu}} e^{\alpha}_{\tilde{\nu};\nu}\right) + e^{\mu}_{\tilde{\mu}}\left(e^{\alpha}_{\tilde{\nu};\nu\mu} - e^{\alpha}_{\tilde{\nu};\mu\nu}\right) X^{\nu} + \left(e^{\mu}_{\tilde{\mu}} e^{\alpha}_{\tilde{\nu};\mu}\right)_{;\nu} X^{\nu}
\end{aligned} \tag{36}$$

By the reasoning developed previously, we're specifically interested only in the $A^{\alpha}_{\tilde{0}\tilde{0}}$, so that we can put

$$A^{\alpha}_{\tilde{0}\tilde{0}} = v^{\mu}\left(v^{\alpha}_{;\nu\mu} - v^{\alpha}_{;\mu\nu}\right) X^{\nu} + \left(v^{\mu} v^{\alpha}_{;\mu}\right)_{;\nu} X^{\nu} + \left(v^{\mu} B^{\alpha}_{\tilde{0};\mu} + B^{\nu}_{\tilde{0}} v^{\alpha}_{;\nu}\right) \tag{37}$$

This expression is valid for *any* $\mathbf{X}$; from this point onwards, we restrict attention to the particular vectors for which $B^{\alpha}_{\tilde{0}} = 0$ - a condition that, for given v^{ν}, may be seen as a system of first-order PDEs on the X^{ν}'s. Physically speaking, we can take such vectors to form a basis for a vector that measures the separation between two neighbouring free-fall curves, so that the quantity $\mathbf{A}$ can be interpreted as the 'relative acceleration' between them. With this restriction, one recognizes this immediately as the Jacobi formula

$$A^{\alpha}_{\tilde{0}\tilde{0}} \doteq -R^{\alpha}_{\beta\mu\nu} v^{\beta} v^{\mu} X^{\nu} \tag{38}$$

from GR (a.k.a. geodesic deviation), since in that theory the equation of motion is given by $v^{\mu} v^{\alpha}_{;\mu} = v^{\mu} v^{\alpha}_{,\mu} \doteq 0$ and the only anholonomical contribution is from the RC tensor. Taking this to be a satisfactory sanity check of the formalism, let us extend this formula together with its interpretation to the present scheme, so that the modified formula

$$A^{\alpha}_{\tilde{0}\tilde{0}} \doteq -\left[\Omega^{\alpha}_{\beta\mu\nu} v^{\beta} v^{\mu} - \left(f^{\alpha}_{\beta\mu} v^{\beta} v^{\mu}\right)_{;\nu}\right] X^{\nu} \tag{39}$$

is now seen to govern tidal displacements instead, and rather naturally: in Cartesian coordinates and with vanishing linear connection, the r.h.s. reduces to $X^{\nu} \frac{\partial}{\partial x^{\nu}} \left(f^{\alpha}_{\beta\mu} v^{\beta} v^{\mu}\right)$; for a distribution of matter at rest but stressed, we have $\frac{dx^0}{d\tau} = c$, $\frac{dx^k}{d\tau} = 0$, and for stationary potentials, $\frac{\partial}{\partial x^0} \omega^{\alpha}_{\mu} = 0$, so that

$$A^{0}_{\tilde{0}\tilde{0}} \doteq 0, A^{k}_{\tilde{0}\tilde{0}} \doteq -\eta^{kj} X^{i} \partial_i \partial_j \left(\frac{\omega^0_0 c^2}{2}\right) \tag{40}$$

Now, from Newtonian theory, we know that the tidal forces depend on the Hessian of the potential - so the preservation of that behavior is seen to be yet another consistency check of our previous identification of $\frac{\omega^0_0 c^2}{2}$ with the classic Newtonian potential.

As we move to the field-theoretical discussion, an interlude on an important domain of interest is in order. So far, we've focused on the transport of matter (i.e., of time-like vectors); presumably, one may extend the treatment to null-like vectors in order to obtain the transport of radiation as well, by taking advantage of the geometric optics approximation [9]: given that the wave vectors for light rays propagating in the absence of background gravitation do trace out null-like geodesics, one takes that the affine equivalent will be traced in the presence of *weak* gravitational fields. While this may be good enough for, say, Solar System-bound applications, in general we must hold that radiation is in the final analysis the province of the theory of electrodynamics. That theory was developed by Maxwell and others for the special case of a Minkowski, gravity-less spacetime; it can be extended as to be compatible with GR - the extension being thus broadly referred to as Maxwell–Einstein theory. Based on the analogies we've drawn with our affine geometry, it isn't difficult to see that there must be an analogous formalism which, for the sake of clarity, one may refer to as the *Maxwell–Newton(–Cartan) theory*; such a theory, however, lies beyond the scope of the present article. We'll hence treat it as an open-ended question, and briefly revisit it later on.

Matter tells space how to curve. Let's now shift attention to the Lagrangean density aspect of gravitational dynamics. After all the trouble separating ourselves from the Riemannian formalism, it is now quite natural to turn to a Lagrangean quadratic in the field strengths, rather than just linear, after all the desirable features that made them a mainstay in the Standard Model of Particle Physics - not the least of which being agreeable to quantization[6]. With that in mind, we postulate the (free) gravitational Lagrangean scalar-valued 4-form

$$\mathbf{Lag}_{GD} = \frac{1}{2\kappa_0}\left(\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \boldsymbol{\Omega}^\alpha_\beta \wedge \star\boldsymbol{\Omega}^\beta_\alpha\right) \tag{41}$$

where κ_0 is the gravitational constant, and $\star$ is the Hodge star[7]. Researchers interested in the rigorous variational treatment are referred to Bleecker [22]. Here, a quick heuristic will suffice: with $\tilde{\boldsymbol{\Omega}}$ as shorthand for the affine curvature, Taylor-expanding the Lagrangean up to first order in ϵ around a perturbation $\delta\tilde{\boldsymbol{\Omega}}$ yields

$$\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}} + \epsilon\delta\tilde{\boldsymbol{\Omega}}\right) =: \mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}\right) + \epsilon\delta\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}, \delta\tilde{\boldsymbol{\Omega}}\right) + \mathbf{O}\left(\epsilon^2\right) \tag{42}$$

from which we can immediately read off the first variation $\delta\mathbf{Lag}_{GD}$ and simplify it further:

$$\begin{aligned}
\delta\mathbf{Lag}_{GD} &= \frac{1}{\kappa_0}\left[\frac{1}{2}\left(\boldsymbol{\Omega}_\beta \wedge \star\delta\boldsymbol{\Omega}^\beta + \delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta\right) + \frac{1}{2}\left(\boldsymbol{\Omega}^\alpha_\beta \wedge \star\delta\boldsymbol{\Omega}^\beta_\alpha + \delta\boldsymbol{\Omega}^\alpha_\beta \wedge \star\boldsymbol{\Omega}^\beta_\alpha\right)\right] \\
&\equiv \frac{1}{\kappa_0}\left[\delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \delta\left(\mathring{\boldsymbol{\Omega}}^\alpha_\beta + \frac{1}{4}\delta^\alpha_\beta \mathbf{d}\boldsymbol{\phi}\right) \wedge \star\left(\mathring{\boldsymbol{\Omega}}^\beta_\alpha + \frac{1}{4}\delta^\beta_\alpha \mathbf{d}\boldsymbol{\phi}\right)\right] \\
&\equiv \frac{1}{\kappa_0}\left(\delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \delta\mathring{\boldsymbol{\Omega}}^\alpha_\beta \wedge \star\mathring{\boldsymbol{\Omega}}^\beta_\alpha + \frac{1}{4}\delta\mathbf{d}\boldsymbol{\phi} \wedge \star\mathbf{d}\boldsymbol{\phi}\right)
\end{aligned}$$

where in the second equality we explicitly constrained ourselves to Weyl–Cartan geometry, for reasons that'll be considered later. Now, in order to proceed, we have to rearrange this quantity to

$$\delta\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}, \delta\tilde{\boldsymbol{\Omega}}\right) =: \delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\omega}}, \mathbf{d}\tilde{\boldsymbol{\omega}}\right) \wedge \delta\tilde{\boldsymbol{\omega}} \tag{43}$$

where the newly-introduced notation is another self-evident shorthand. Thus, to obtain $\delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD}$, there is a need for the explicit expression for the variations of the field strengths

$$\begin{aligned}
\delta\boldsymbol{\Omega}_\beta &= \left(\delta\mathbf{d}\boldsymbol{\omega}_\beta - \boldsymbol{\gamma}^\sigma_\beta \wedge \delta\boldsymbol{\omega}_\sigma\right) - \left(\mathring{\boldsymbol{\omega}}^\sigma_\beta + \frac{1}{4}\delta^\sigma_\beta\boldsymbol{\phi}\right) \wedge \delta\boldsymbol{\omega}_\sigma - \left(\delta\mathring{\boldsymbol{\omega}}^\sigma_\beta + \frac{1}{4}\delta^\sigma_\beta\delta\boldsymbol{\phi}\right) \wedge \boldsymbol{\omega}_\sigma \\
\delta\mathring{\boldsymbol{\Omega}}^\alpha_\beta &= \delta\mathbf{d}\mathring{\boldsymbol{\omega}}^\alpha_\beta - \boldsymbol{\gamma}^\sigma_\beta \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\sigma - \delta\mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \boldsymbol{\gamma}^\alpha_\sigma - \delta\mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \mathring{\boldsymbol{\omega}}^\alpha_\sigma - \mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\sigma
\end{aligned}$$

[6] Another important aspect of such Lagrangeans pertains to *gauge-invariance*; it is pretty clear that the general affine group $GA(4,\mathbb{R}) = GL(4,\mathbb{R}) \ltimes T(4)$ and its subgroups are intimately related to the symmetries here - although the exact details remain to be ellucidated.

[7] The Hodge operator requires a metric which is not to be confused with $\mathbf{g} = g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b$ - but thanks to the soldering, we may map this $\mathbf{g}$ into $\mathbf{g}_{\boldsymbol{\theta}} := g_{ab}\boldsymbol{\theta}^a \otimes \boldsymbol{\theta}^b$.

Notice how, consistent with our philosophy, no term like $\delta\boldsymbol{\gamma}^{\alpha}_{\beta}$ appears above, as such variations are physically meaningless. The road is now clear: putting as usual $\delta\mathbf{d}\mathring{\boldsymbol{\omega}}^{\alpha}_{\beta} = \mathbf{d}\delta\mathring{\boldsymbol{\omega}}^{\alpha}_{\beta}$, etc., then integrating by parts, putting the field variations in evidence and dropping the boundary terms, we're left, after algebraic manipulation, with

$$\begin{aligned}
\delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD} \wedge \delta\tilde{\boldsymbol{\omega}} \quad \equiv \quad & -\frac{1}{\kappa_0}\frac{1}{4}\left(\mathbf{d}\star\mathbf{d}\boldsymbol{\phi} - \boldsymbol{\omega}_{\beta}\wedge\star\boldsymbol{\Omega}^{\beta}\right)\wedge\delta\boldsymbol{\phi} - \frac{1}{\kappa_0}\left[\mathbf{d}\star\boldsymbol{\Omega}^{\sigma} + \left(\boldsymbol{\gamma}^{\sigma}_{\beta} + \mathring{\boldsymbol{\omega}}^{\sigma}_{\beta} + \frac{1}{4}\delta^{\sigma}_{\beta}\boldsymbol{\phi}\right)\wedge\star\boldsymbol{\Omega}^{\beta}\right]\wedge\delta\boldsymbol{\omega}_{\sigma} \\
& -\frac{1}{\kappa_0}\left[\mathbf{d}\star\mathring{\boldsymbol{\Omega}}^{\beta}_{\alpha} - \left(\boldsymbol{\gamma}^{\sigma}_{\alpha} + \mathring{\boldsymbol{\omega}}^{\sigma}_{\alpha}\right)\wedge\star\mathring{\boldsymbol{\Omega}}^{\beta}_{\sigma} + \left(\boldsymbol{\gamma}^{\beta}_{\sigma} + \mathring{\boldsymbol{\omega}}^{\beta}_{\sigma}\right)\wedge\star\mathring{\boldsymbol{\Omega}}^{\sigma}_{\alpha} - \boldsymbol{\omega}_{\alpha}\wedge\star\boldsymbol{\Omega}^{\beta}\right]\wedge\delta\mathring{\boldsymbol{\omega}}^{\alpha}_{\beta}
\end{aligned}$$

The Euler–Lagrange (EL) equations are finally obtained by equating the above term with

$$\mathbf{J}_{\tilde{\boldsymbol{\omega}}}\wedge\delta\tilde{\boldsymbol{\omega}} := -\frac{1}{4}\mathbf{C}\wedge\delta\boldsymbol{\phi} - \mathbf{T}^{\sigma}\wedge\delta\boldsymbol{\omega}_{\sigma} - \mathbf{L}^{\beta}_{\alpha}\wedge\delta\mathring{\boldsymbol{\omega}}^{\alpha}_{\beta} \tag{44}$$

where $\mathbf{C}, \mathbf{T} := \mathbf{T}^{\sigma}\otimes\mathbf{e}_{\sigma}, \mathbf{L} := \mathbf{L}^{\beta}_{\alpha}\otimes\mathbf{e}_{\beta}\otimes\mathbf{e}^{\alpha}$ are obviously the current 3-forms associated with the potentials. All of this can be put in the compact (and elegant) form:

$$\mathbf{d}\star\mathbf{d}\boldsymbol{\phi} - \boldsymbol{\omega}_{\beta}\wedge\star\boldsymbol{\Omega}^{\beta} \quad = \quad \kappa_0\mathbf{C} \tag{45a}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}}\star\boldsymbol{\Theta} + \frac{1}{4}\boldsymbol{\phi}\wedge\star\boldsymbol{\Theta} \quad = \quad \kappa_0\mathbf{T} \tag{45b}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}}\star\mathring{\boldsymbol{\Omega}} - \boldsymbol{\theta}\wedge\star\boldsymbol{\Theta} \quad = \quad \kappa_0\mathbf{L} \tag{45c}$$

The equations above showcase the degree of coupling between the different pieces of the connection - therefore presenting an opportunity to examine their asymptotic flatness (i.e., the regimes under which one or more potentials are taken to zero), which in turn effectively ('weakly') change the underlying geometry from Weyl–Cartan to one of the other geometries listed in Table 1. It is apparent from our dynamical laws that this procedure will consistently leave the currents associated with the vanishing fields to vanish also - unless $\boldsymbol{\omega}_a \neq \mathbf{0}$ (thus, for the Weyl–Weitzenböck, Cartan and Weitzenböck cases). Because of this difficulty, we suggest the following (provisional) workaround: referring back to the above calculation, if we define

$$\delta\boldsymbol{\Omega}_{\beta}\wedge\star\boldsymbol{\Omega}^{\beta} =: \delta\boldsymbol{\Theta}_{\beta}\wedge\star\boldsymbol{\Theta}^{\beta} - \kappa_0\left(\mathbf{J}_{grav}\right)_{\tilde{\boldsymbol{\omega}}}\wedge\delta\tilde{\boldsymbol{\omega}} \tag{46}$$

where $\mathbf{J}_{grav}$ is interpreted as the current piece that arises due to gravity itself, the previous $\mathbf{J}$ being that from matter (i.e., non-gravitational sources), $\mathbf{J}_{mat}$, the FE may be put in the form

$$\mathbf{d}\star\mathbf{d}\boldsymbol{\phi} \quad = \quad \kappa_0\left(\mathbf{C}_{mat} + \mathbf{C}_{grav}\right) \tag{47a}$$

$$\mathbf{d}\star\boldsymbol{\Theta} \quad = \quad \kappa_0\left(\mathbf{T}_{mat} + \mathbf{T}_{grav}\right) \tag{47b}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}}\star\mathring{\boldsymbol{\Omega}} \quad = \quad \kappa_0\left(\mathbf{L}_{mat} + \mathbf{L}_{grav}\right) \tag{47c}$$

for which, provided the *net* currents in the r.h.s. vanish, we do get appropriate vacuum behavior; furthermore, if the theoretical need arises, one may introduce Lagrange multipliers so as to properly take care of the $\mathbf{J}_{grav}$ at the Lagrangean level ('strongly'). This is ultimately the reason for our introduction of $\boldsymbol{\Theta}^{\beta}$ as field strength, instead of $\boldsymbol{\Omega}^{\beta}$.

Having said this, if we keep to a Weitzenböck geometry (for which $\boldsymbol{\Omega}^{\beta} \equiv \boldsymbol{\Theta}^{\beta}$), we get simply

$$\mathbf{d} \star \boldsymbol{\Theta} = \kappa_0 \mathbf{T} \tag{48a}$$

$$\mathbf{d}\boldsymbol{\Theta} = \mathbf{0} \tag{48b}$$

Hopefully, the formal similarity with the Maxwell equations of electrodynamics will not be lost on the reader - specially as we show the Bianchi identity alongside the EL law. It is an unsurprising result, actually, due to the well-known fact that both Coulomb's and Newton's gravitational laws are derived from the same differential equation (namely, Poisson's), on the one hand, and the fact that both the translation subgroup $T(4)$ and the unitary $U(1)$ of quantum electrodynamics are Abelian, on the other. This showcases the prospects of the framework we're advocating, as it allows the immediate application of familiar electrodynamical techniques [23], which are relevant to establishing the Newtonian limit, the existence of gravitational waves, and even hint at hitherto unexplored possibilities, such as analogues of a 'macroscopic' formulation similar to that used for dielectrics, or of magnetic monopoles (which in this case would signal the breakdown of the Bianchi-like identity). From these arguments, this (sub)theory is seem to offer a rich phenomenological testbed that can be explored with known theoretical tools, as well as a rather convenient starting point for a quantum theory of gravity; for these reasons, it'll be convenient to give this special case its own name: we'll call it *teledynamics*, to honor also the old teleparallelism theory.

Another important consequence of the laws thus formulated pertains to the phenomenology of the (relativistic) stress-momentum $\mathbf{T}$: just as we did to compute the anholonomy associated with a (co)vector in the first section, we easily verify that

$$\mathbf{d}^2 \star \boldsymbol{\Theta} \equiv (\star\boldsymbol{\Theta})^{\alpha} \wedge \mathbf{d}^2 \mathbf{e}_{\alpha} \equiv (\star\boldsymbol{\Theta})^{\alpha} \wedge \mathbf{R}_{\alpha}^{\beta} \otimes \mathbf{e}_{\beta}$$

Since we restrict attention to $\mathbf{R}_{\alpha}^{\beta} \equiv \mathbf{0}$, it follows from the FE that $\mathbf{dT} \doteq \mathbf{0}$ - or to put in words: the statement of the conservation of the stress-momentum follows as a consequence of the field equations - just like in GR! Indeed, we can confirm that this expression gives a set of (four) conservation laws by writing the components explicitly:

$$\begin{aligned} \mathbf{dT} &= \frac{1}{6} \left(\nabla_{\mu} T^{\alpha}_{\nu\rho\sigma} \right) \boldsymbol{\theta}^{\mu} \wedge \boldsymbol{\theta}^{\nu} \wedge \boldsymbol{\theta}^{\rho} \wedge \boldsymbol{\theta}^{\sigma} \otimes \mathbf{e}_{\alpha} \\ &= \left(\nabla_0 T^{\alpha}_{[123]} - \nabla_1 T^{\alpha}_{[023]} - \nabla_2 T^{\alpha}_{[031]} - \nabla_3 T^{\alpha}_{[012]} \right) \boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}^3 \otimes \mathbf{e}_{\alpha} \end{aligned} \tag{49}$$

Analogous considerations for the case of only $\mathring{\boldsymbol{\omega}}_{\beta}^{\alpha}$ nonvanishing lead us to the formulae

$$\mathring{\mathbf{D}}_{\mathbf{0}} \star \mathring{\mathbf{\Omega}} = \kappa_0 \mathbf{L} \tag{50a}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}} \mathring{\mathbf{\Omega}} = \mathbf{0} \tag{50b}$$

which are simply the Yang–Mills (YM) equations associated with the (non-Abelian) Lorentz group; for this reason, one may call this subtheory *orthodynamics*. Contrasted with the Maxwellian nature of teledynamics, this orthodynamics doesn't have an immediate interpretation in Newtonian terms, but at the same time may be tied to the presence of a conserved current associated with, e.g., rotations. Unsurprising, again, if we interpret this in light (*mutatis mutandis*) of the pioneering works of Uchiyama [24] and Kibble [25]; doubly so, if we also heed the no less prophetic words of Cartan [17]: "La translation révèle la *torsion*, la rotation révèle la *courbure* de la variété donnée". This is then the physical motivation, or justification, for keeping to Weyl–Cartan geometry: it is large enough to allow us to formulate a gauge theory for the Poincaré group (prominent in SR), from which the gravitational force is seen to emerge as a natural consequence thereof. All that is left to interpretation is the single extra field $\boldsymbol{\phi}$, which may always be taken to vanish identically if no theoretical necessity of it is apparent; more interestingly, however, it might be associated with the subgroup of scalings of the conformal group, which is a symmetry group for the source-free Maxwell equations [26], and/or be made to perform a function similar to that of the Higgs field in the Standard Model, which at present seems to be the only elementary boson that is not the carrier of any force, being introduced in *ad hoc* fashion in order to spontaneously break the symmetry of the electroweak theory. In a similar vein, another possible speculative use of this field might be in the context of inflationary cosmology.

5 Gravitational mechanics

We shall now study several different teledynamical models in order to explore the consequences of the theory. For what follows, it will prove convenient to introduce the following: for a given solution of the TE with velocity components $\frac{dx^\mu}{d\tau}$, we define the (dimensionless) *time dilation factor* Φ

$$-\Phi c^2 := \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{51}$$

to best keep track of its effects on transport. From the definition, it follows that Φ must be always locally nonnegative (depending on whether the curve is time-like or null-like); furthermore, one would like to check that it can be made constant everywhere the curve is evaluated - so that the curve may be regarded as physically realistic.

As our first (and simplest) example of a solution of the FE (48a), we consider plane-waves; taking a cue from Maxwell's theory, we also notice there's a gauge freedom associated with the torsion, in that we're free to add the derivative of a vector $\boldsymbol{\chi}$ to the potential:

$$\begin{aligned} \boldsymbol{\Theta} &= \mathbf{d}\left(\boldsymbol{\omega}^a \otimes \mathbf{e}_a\right) \to \mathbf{d}\left[\boldsymbol{\omega}^a \otimes \mathbf{e}_a + \mathbf{d}\boldsymbol{\chi}\right] \\ &= \boldsymbol{\Theta} + \chi^b \mathbf{R}^a_b \otimes \mathbf{e}_a \end{aligned}$$

This leaves $\boldsymbol{\Theta}$ invariant due to our choice of Minkowski geometry; thus, we may impose a further condition in $\boldsymbol{\omega}^a$ without altering the field strength, which governs observable outcomes. A particularly attractive condition would be the PDEs compactly written as $\mathbf{d} \star \boldsymbol{\theta} = \mathbf{0}$; this reads, in components,

$$|-\eta|^{-\frac{1}{2}} \partial_\lambda \left(|-\eta|^{\frac{1}{2}} \eta^{\lambda\sigma} \omega^\alpha_\sigma\right) + \eta^{\lambda\sigma} \omega^\beta_\sigma \gamma^\alpha_{\beta\lambda} = 0 \tag{52}$$

which are now recognizable: in Cartesians, they reduce simply to the 'fourfold Lorenz condition' $\eta^{\lambda\sigma}\partial_\lambda \omega^\alpha_\sigma = 0$ - as expected from the electrodynamical analogy. The nice thing about this 'gauge' is that it condenses the teledynamical FE down to the single formula

$$\Box^2 \omega^\alpha_\sigma \equiv \kappa_0 \, |-\eta|^{-\frac{1}{2}} \left(\frac{1}{6} \, |-\eta|^{\frac{1}{2}} \eta^{\mu\zeta} \eta^{\nu\xi} \eta^{\lambda\rho} \varepsilon_{\sigma\zeta\xi\rho} T^\alpha_{\mu\nu\lambda}\right) =: \kappa_0 \, |-\eta|^{-\frac{1}{2}} \, T^\alpha_\sigma \tag{53}$$

where $\Box^2 := \nabla^2 - \partial_0^2 := \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_0^2$ is the d'Alembertian operator and T^α_σ the Hodge dual[8] of the stress-momentum. From such manipulations, it is evident that in a vacuum plane-waves derived from linear combinations of $e^\alpha_\sigma \exp\left(\pm i k_\lambda x^\lambda\right)$ with $e^\alpha_\lambda k^\lambda = 0$ satisfy the field equations, provided $k_\lambda k^\lambda = 0$ (i.e., they travel at the speed of light, as in GR).

Next, let us consider a perfect (isotropic) fluid in Cartesians, for which the stress-momentum reads in full

$$\begin{aligned} \mathbf{T}_{pf} \quad : \quad &= \left(\rho_m c^2\right) \mathbf{e}_0 \otimes d\mathbf{x}^x \wedge d\mathbf{x}^y \wedge d\mathbf{x}^z + (P)\, \mathbf{e}_x \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^y \wedge d\mathbf{x}^z + \\ &\quad (P)\, \mathbf{e}_y \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^z \wedge d\mathbf{x}^x + (P)\, \mathbf{e}_z \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^x \wedge d\mathbf{x}^y \end{aligned} \tag{54}$$

As an immediate consequence of this definition, it follows from the conservation laws (49) that such a system has of necessity stationary specific density (i.e., $\partial_0 \rho_m = 0$) and uniform hydrostatic pressure (i.e., $\partial_k P = 0$). In keeping to these two restrictions, and putting $\omega^0_0 = \omega^0_0(x, y, z)$ and $\omega^x_x = \omega^y_y = \omega^z_z = a\left(x^0\right)$ as the only nonvanishing components, the field equations reduce to the pair[9]

$$\nabla^2 \omega^0_0 \doteq \kappa_0 \rho_m c^2, \tag{55a}$$

$$-\partial_0^2 a \doteq \kappa_0 P \tag{55b}$$

which offers no mathematical difficulty; indeed, as mentioned previously, it even contains Poisson's law as a special case. If, however, we're interested in

[8] Cf. [27].

[9] We're not working in the Lorenz gauge - not here, nor in any further example.

more realistic models for systems of astrophysical interest like collapsing stars and polytropes, or even cosmologies, it becomes clear that $\mathbf{T}_{pf}$ must be suitably modified. As an illustration of the general method, consider adding a $\tilde{\mathbf{T}}$ to $\mathbf{T}_{pf}$, and extending the dependency of ω_0^0 to $\omega_0^0 = \omega_0^0\left(x^0, x, y, z\right)$; provided $\tilde{\mathbf{T}}$ does not modify the density nor the pressure, the only other conditions it needs to meet in order for the given Newtonian potential to satisfy the FE are

$$\partial_0\left(\partial_k \omega_0^0\right) \doteq \kappa_0 \tilde{T}_k^0 \tag{56}$$

So, we choose the $\tilde{T}_k^0$ as to satisfy this relation 'weakly', and put the remaining ones to zero. The effect of this can be seen in the conservation laws directly, for $\alpha = 0$:

$$\begin{aligned} 0 &\doteq \partial_0\left(\rho_m c^2\right) - \partial_x\left(\tilde{T}^0_{[0yz]}\right) - \partial_y\left(\tilde{T}^0_{[0zx]}\right) - \partial_z\left(\tilde{T}^0_{[0xy]}\right) \\ &\doteq \partial_0\left(\rho_m c^2\right) - \frac{1}{\kappa_0}\partial_0\left(\nabla^2 \omega_0^0\right) \end{aligned}$$

But the last line vanishes identically, due to Poisson's law, and no new relations are introduced. We can actually do better, and interpret this little trick physically: if the above represents a statement of the conservation of *mass*, then the $\tilde{T}_k^0$ just introduced correspond naturally to a current responsible for dynamically displacing such a mass, in much the same way as electric current displaces charge; if there were *no* such currents, it'd be impossible to transport it continuously across any given spatial boundary. Similar justifications may equally be invoked in extending the dependency of a to spatial coordinates, as well; in that case, they have to do with the transport of momentum.

Be as it may, we can now consider cosmological models for which $\rho_m = \rho_m\left(x^0\right)$ and $P = P\left(x^0\right)$: separating $\omega_0^0 = \kappa_0 \rho_m c^2 f\left(x, y, z\right)$, the sole nonvanishing torsion components are seen to be $\Theta^0_{k0} = \frac{\kappa_0 \rho_m c^2}{2}\partial_k f$ and $\Theta^k_{0k} = \frac{1}{2}\partial_0 a$. We're not interested in the detailed evolution of the Cosmos, though; instead, let us assume that, at some epoch in the past, we have $\rho_m, P \to 0$. The reasoning here is physically self-explanatory: low-density fluids tend to have low internal pressure as well, and both conditions seem adequate for treating the highly diluted intergalactic medium. Substituting these quasivacuum conditions into the pressure equation, we have that $a = \frac{2H_0}{c}x^0$ is a solution, where H_0 is a constant with dimensions of time^{-1}. Under these general conditions, we also have $\Theta^0_{k0} \approx 0$, leading us to the transport equations

$$\frac{d^2x^0}{d\tau^2} - F^0_{xx}\frac{dx^x}{d\tau}\frac{dx^x}{d\tau} - F^0_{yy}\frac{dx^y}{d\tau}\frac{dx^y}{d\tau} - F^0_{zz}\frac{dx^z}{d\tau}\frac{dx^z}{d\tau} = 0 \tag{57a}$$

$$\frac{d^2x^x}{d\tau^2} - F^x_{0x}\frac{dx^0}{d\tau}\frac{dx^x}{d\tau} = 0 \tag{57b}$$

$$\frac{d^2x^y}{d\tau^2} - F^y_{0y}\frac{dx^0}{d\tau}\frac{dx^y}{d\tau} = 0 \tag{57c}$$

$$\frac{d^2x^z}{d\tau^2} - F^z_{0z}\frac{dx^0}{d\tau}\frac{dx^z}{d\tau} = 0 \tag{57d}$$

The three spatial components are numerically identical, and readily integrated after multiplication by $\exp\left(-\int^{x^0} F^i_{0i} dx^{0'}\right)$ to

$$K_i := \exp\left(-\int^{x^0} F^i_{0i} dx^{0'}\right) \frac{dx^i}{d\tau} \tag{58}$$

which may in turn be substituted into the temporal equation after multiplication by $2\frac{dx^0}{d\tau}$ to yield

$$\frac{d}{d\tau}\left[\left(\frac{dx^0}{d\tau}\right)^2 - \left(K_x^2 + K_y^2 + K_z^2\right)\exp\left(2\int^{x^0} F^i_{0i} dx^{0'}\right)\right] = 0 \tag{59}$$

Thus, defining the term in brackets as K_0, we finally obtain $-\Phi c^2 = -K_0$; since $\Phi = 0$ for a light-beam, we can solve for the frequency shift in terms of $\left|\frac{dx^0}{d\tau}\right|$:

$$\frac{\Delta\nu}{\nu} := \frac{\left|\frac{dx^0}{d\tau}\right|_{then} - \left|\frac{dx^0}{d\tau}\right|_{now}}{\left|\frac{dx^0}{d\tau}\right|_{then}} = 1 - \exp\left(\int_{x^0_{then}}^{x^0_{now}} F^i_{0i} dx^{0'}\right) \tag{60}$$

We need now solve for the F^i_{0i}, in order to obtain an empirically observable result. If we take advantage of the Minkowski geometry to put $-\left(x^0_{now} - x^0_{then}\right)^2 + D^2 = 0$, where D is the spatial distance between emitter and receiver as determined from observed luminosities, and Taylor-expand the exponential in the above formula, we get

$$\frac{\Delta\nu}{\nu} = 1 - \exp\left(-\frac{H_0}{c}D\right) \approx \frac{H_0}{c}D \tag{61}$$

which closely mirrors Hubble's law for distances small compared to a 'Hubble radius' $\frac{c}{H_0}$. This is rather convenient, because it gives free range to cosmogonical models such as a Big Bang-like scenario to fit parameters related to the relative elemental abundances of H and He, the temperature of the cosmic blackbody radiation, and the distribution of galaxies and other astronomical structures, independently of cosmographical considerations such as the redshift-luminosity dependence; one need only be careful when matching the current dusty epoch with the earlier, denser one (as it is currently thought to have been).

For our third case study, we shift to spherical coordinates. A more extensive discussion of a generalized fluid model and the associated FE was relegated to an Appendix; here, we focus on some consequences for the special case of a stationary, spherically symmetrical vacuum: taking only ω^0_0 and ω^φ_0 to be nonvanishing, the FE reduce to the pair

$$\partial_r\left(r^2\partial_r\omega^0_0\right) = 0, \tag{62a}$$

$$\partial_r\left(r^2\partial_r\omega^\varphi_0\right) + 2r\partial_r\omega^\varphi_0 = 0 \tag{62b}$$

for the two unknowns, whose asymptotically admissible nontrivial solutions are given by $\omega^0_0 = -\frac{\alpha}{r}, \omega^\varphi_0 = \frac{\beta}{r^3}$, α and β being constants of integration; the first of these corresponds, naturally, to the Schwarzschild radius, as we'd expect from Newtonian theory - while the other is a new parameter, tentatively identified as a measure of the rotation of the point-like source generating the field. Now, since the only nonvanishing components of the field strength are given by

$$
\begin{aligned}
\Theta^0_{0r} &= -\frac{1}{2}\partial_r \omega^0_0, \\
\Theta^r_{0\varphi} &= -\frac{1}{2}\gamma^r_{\varphi\varphi}\omega^\varphi_0, \\
\Theta^\theta_{0\varphi} &= -\frac{1}{2}\gamma^\theta_{\varphi\varphi}\omega^\varphi_0, \\
\Theta^\varphi_{0r} &= -\frac{1}{2}\partial_r \omega^\varphi_0 - \frac{1}{2}\gamma^\varphi_{\varphi r}\omega^\varphi_0, \\
\Theta^\varphi_{0\theta} &= -\frac{1}{2}\gamma^\varphi_{\varphi\theta}\omega^\varphi_0,
\end{aligned}
$$

we can explicitly calculate the nonvanishing force components appearing in the TE (28), ending up with the system

$$
\begin{aligned}
\frac{d^2x^0}{d\tau^2} - \left(F^0_{r0}\frac{dx^r}{d\tau}\frac{dx^0}{d\tau} + F^0_{\varphi r}\frac{dx^\varphi}{d\tau}\frac{dx^r}{d\tau} + F^0_{\varphi\theta}\frac{dx^\varphi}{d\tau}\frac{dx^\theta}{d\tau} + F^0_{r\varphi}\frac{dx^r}{d\tau}\frac{dx^\varphi}{d\tau} + F^0_{\theta\varphi}\frac{dx^\theta}{d\tau}\frac{dx^\varphi}{d\tau}\right) &= 0 \\
\frac{d^2x^r}{d\tau^2} + \gamma^r_{\theta\theta}\frac{dx^\theta}{d\tau}\frac{dx^\theta}{d\tau} + \gamma^r_{\varphi\varphi}\frac{dx^\varphi}{d\tau}\frac{dx^\varphi}{d\tau} - \left(F^r_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + F^r_{0\varphi}\frac{dx^0}{d\tau}\frac{dx^\varphi}{d\tau}\right) &= 0 \\
\frac{d^2x^\theta}{d\tau^2} + \gamma^\theta_{\varphi\varphi}\frac{dx^\varphi}{d\tau}\frac{dx^\varphi}{d\tau} + 2\gamma^\theta_{r\theta}\frac{dx^r}{d\tau}\frac{dx^\theta}{d\tau} - \left(F^\theta_{0\varphi}\frac{dx^0}{d\tau}\frac{dx^\varphi}{d\tau}\right) &= 0 \\
\frac{d^2x^\varphi}{d\tau^2} + 2\gamma^\varphi_{r\varphi}\frac{dx^r}{d\tau}\frac{dx^\varphi}{d\tau} + 2\gamma^\varphi_{\varphi\theta}\frac{dx^\varphi}{d\tau}\frac{dx^\theta}{d\tau} - \left(F^\varphi_{0r}\frac{dx^0}{d\tau}\frac{dx^r}{d\tau} + F^\varphi_{0\theta}\frac{dx^0}{d\tau}\frac{dx^\theta}{d\tau}\right) &= 0
\end{aligned}
$$

Rather than considering this generally, we'll restrict ourselves as usual to $x^\theta \doteq \frac{\pi}{2}$, which we see is indeed a solution for the third equation, leaving then the remaining three simplified down to

$$
\frac{d^2x^0}{d\tau^2} - \left(F^0_{r0}\frac{dx^r}{d\tau}\frac{dx^0}{d\tau} + F^0_{\varphi r}\frac{dx^\varphi}{d\tau}\frac{dx^r}{d\tau} + F^0_{r\varphi}\frac{dx^r}{d\tau}\frac{dx^\varphi}{d\tau}\right) = 0 \tag{63a}
$$

$$
\frac{d^2x^r}{d\tau^2} + \gamma^r_{\varphi\varphi}\frac{dx^\varphi}{d\tau}\frac{dx^\varphi}{d\tau} - \left(F^r_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + F^r_{0\varphi}\frac{dx^0}{d\tau}\frac{dx^\varphi}{d\tau}\right) = 0 \tag{63b}
$$

$$
\frac{d^2x^\varphi}{d\tau^2} + 2\gamma^\varphi_{r\varphi}\frac{dx^r}{d\tau}\frac{dx^\varphi}{d\tau} - \left(F^\varphi_{0r}\frac{dx^0}{d\tau}\frac{dx^r}{d\tau}\right) = 0 \tag{63c}
$$

Before proceeding, we wish to check that $-\Phi c^2$ is indeed an integral of motion; to do so, first we solve the $\beta = 0$ problem to obtain the constants of integration

$$
\begin{aligned}
K_0 &: = \exp\left(-\int^{x^r} F^0_{r0} dx^{r'}\right)\frac{dx^0}{d\tau}, && \text{(64a)}\\
K_r &: = \left(\frac{dx^r}{d\tau}\right)^2 + \frac{K_\varphi^2}{(x^r)^2} - K_0^2 \exp\left(2\int^{x^r} F^0_{r0} dx^{r'}\right), && \text{(64b)}\\
K_\theta &: = \frac{dx^\theta}{d\tau} = 0, && \text{(64c)}\\
K_\varphi &: = (x^r)^2 \frac{dx^\varphi}{d\tau} && \text{(64d)}
\end{aligned}
$$

from which we see that $-\Phi c^2 \equiv K_r$; then, for $\beta \neq 0$, we employ the same notation but this time around without assuming K_0, K_r, K_φ to be constant. Rewriting the equations in terms of these K's, we end up with

$$
\begin{aligned}
\frac{dK_0}{d\tau} &= \exp\left(-\int^{x^r} F^0_{r0} dx^{r'}\right)\left(F^0_{\varphi r} + F^0_{r\varphi}\right)\frac{dx^\varphi}{d\tau}\frac{dx^r}{d\tau}, && \text{(65a)}\\
\frac{dK_r}{d\tau} &= 2\frac{dx^r}{d\tau}\left[\left(F^r_{00} - F^0_{r0}\right)\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \left(F^r_{0\varphi} - F^0_{r\varphi}\right)\frac{dx^0}{d\tau}\frac{dx^\varphi}{d\tau}\right.\\
&\quad \left. + \left(F^\varphi_{0r}(x^r)^2 - F^0_{\varphi r}\right)\frac{dx^0}{d\tau}\frac{dx^\varphi}{d\tau}\right], && \text{(65b)}\\
\frac{dK_\varphi}{d\tau} &= (x^r)^2 F^\varphi_{0r}\frac{dx^0}{d\tau}\frac{dx^r}{d\tau} && \text{(65c)}
\end{aligned}
$$

But notice that because of the symmetry $F_{r00} = -F_{0r0}$, etc. of the force terms, the curly brackets in the r.h.s. of (65b) vanishes identically, leaving us with $\frac{dK_r}{d\tau} \equiv -\frac{d}{d\tau}\Phi c^2 = 0$, as we wished to show. Additionally, the scheme presents an approach to treating the $\beta \neq 0$ problem perturbatively, by expanding

$$
\begin{aligned}
K_0 &= K_0^{(0)} + \epsilon K_0'(\tau), && \text{(66a)}\\
K_\varphi &= K_\varphi^{(0)} + \epsilon K_\varphi'(\tau) && \text{(66b)}
\end{aligned}
$$

and treating the r.h.s. of (65a) and (65c) as first-order in the perturbation parameter ϵ; then, the resulting equations for the perturbations of the K's can be immediately integrated to

$$
\begin{aligned}
K_0' &\approx \int \left[\left(F^0_{\varphi r} + F^0_{r\varphi}\right) \frac{K^{(0)}_\varphi}{\left(x^r_{(0)}\right)^2} \exp\left(- \int^{x^r_{(0)}} F^0_{r0} dx^{r'}_{(0)} \right) \frac{dx^r_{(0)}}{d\tau} \right] d\tau \\
&\approx \frac{\beta K^{(0)}_\varphi}{2} \frac{1}{\left(x^r_{(0)}\right)^3}, && (67a) \\
K_\varphi' &\approx \int \left[K^{(0)}_0 \left(x^r_{(0)}\right)^2 \exp\left(\int^{x^r_{(0)}} F^0_{r0} dx^{r'}_{(0)} \right) F^\varphi_{0r} \frac{dx^r_{(0)}}{d\tau} \right] d\tau \\
&\approx \frac{\beta K^{(0)}_0}{2} \frac{1}{x^r_{(0)}}, && (67b)
\end{aligned}
$$

where we take for brevity $x^r_{(0)}$ to refer to the unperturbed solution, and expanded the exponentials to zeroth order in the small (dimensionless) quantity $\frac{\alpha}{x^r_{(0)}}$, since this yields the leading contributions.

Alternatively, if one is interested in exact solutions for eqs. (63a-63c), it is easily checked that

$$
x^0 = c\tau, x^r = R, x^\varphi = \frac{2\pi}{T}\tau \tag{68}
$$

for R, T constant give one such solution, provided the constraint

$$
\left(\frac{2\pi}{T}\right)^2 = -\frac{1}{R}\left(F^r_{00}c^2 + F^r_{0\varphi}\frac{2\pi}{T}c\right) = \frac{1}{R^3}\left(\frac{\alpha c^2}{2} + \beta c \frac{2\pi}{T}\right) \tag{69}
$$

is met (which incidentally, for $\beta = 0$, we recognize as a particular instance of Kepler's third law); in this case, we compute

$$
-\Phi c^2 \doteq -c^2 + R^2 \left(\frac{2\pi}{T}\right)^2 = -\left[1 - \frac{1}{Rc^2}\left(\frac{\alpha c^2}{2} + \beta c \frac{2\pi}{T}\right)\right] c^2 \tag{70}
$$

so that the expression in the square brackets is seen to provide a constraint on the radius of a timelike solution, such as a planetary orbit.

Having laid all this groundwork for the transport across our simple vacuum two-parameter model, we can now produce estimates of effects relevant to observation and of interest to astronomy; for example, if we ignore the small quantity K_0', (64a) can be used to model the gravitational redshift observed between a location at radius R and one at the slightly larger radius $R + h$:

$$
\frac{\Delta\nu}{\nu} := \frac{\left.\left(\frac{dx^0}{d\tau}\right)\right|_R - \left.\left(\frac{dx^0}{d\tau}\right)\right|_{R+h}}{\left.\left(\frac{dx^0}{d\tau}\right)\right|_R} \approx 1 - \exp\left[\frac{\alpha}{2(R+h)} - \frac{\alpha}{2R}\right] \approx \frac{\alpha h}{2R^2} \tag{71}
$$

This result seems to match the GR prediction as famously tested by the Pound–Rebka experiment [28].

Next, we return to the exact solution (68). Our interest isn't in this trajectory *per se*, but rather in the transport of a spacelike vector $\mathbf{s}$ along it. Based on the discussion in the previous section, such a transport can be expressed by the formula

$$u^{\mu}\frac{\partial}{\partial x^{\mu}}s^{\lambda}+u^{\mu}s^{\nu}\gamma^{\lambda}_{\nu\mu}=u^{\mu}s^{\nu}F^{\lambda}_{\nu\mu} \tag{72}$$

where $\gamma^{\lambda}_{\nu\mu}$ and $F^{\lambda}_{\nu\mu}$ are both evaluated w.r.t. the components of the free-fall curve, with tangent $u^{\mu}=\frac{dx^{\mu}}{d\tau}$. Putting $\frac{ds^{\lambda}}{d\tau}=\frac{dx^{\mu}}{d\tau}\frac{\partial s^{\lambda}}{\partial x^{\mu}}$, this can then be written in components as follows:

$$\begin{aligned}
\frac{ds^0}{d\tau}-\left(F^0_{r0}s^r\frac{dx^0}{d\tau}+F^0_{r\varphi}s^r\frac{dx^{\varphi}}{d\tau}\right) &= 0 && \text{(73a)}\\
\frac{ds^r}{d\tau}+\gamma^r_{\varphi\varphi}s^{\varphi}\frac{dx^{\varphi}}{d\tau}-\left(F^r_{00}s^0\frac{dx^0}{d\tau}+F^r_{0\varphi}s^0\frac{dx^{\varphi}}{d\tau}\right) &= 0 && \text{(73b)}\\
\frac{ds^{\theta}}{d\tau} &= 0 && \text{(73c)}\\
\frac{ds^{\varphi}}{d\tau}+\gamma^{\varphi}_{r\varphi}s^r\frac{dx^{\varphi}}{d\tau} &= 0 && \text{(73d)}
\end{aligned}$$

By deriving the radial equation and substituting in the others, we get

$$\frac{d^2s^r}{d\tau^2}+\Phi\left(\frac{2\pi}{T}\right)^2s^r=0 \tag{74}$$

For $\Phi>0$, we see from this that v^r is sinusoidal; for definiteness, let us pick $s^r=S\cos\left(\Phi^{\frac{1}{2}}\frac{2\pi}{T}\tau\right)$. Substituting this into the remaining equations, we can integrate them to

$$s^0=-\Phi^{-\frac{1}{2}}\frac{2\pi R}{cT}S\sin\left(\Phi^{\frac{1}{2}}\frac{2\pi}{T}\tau\right),\, s^{\theta}=s^{\theta}_0,\, s^{\varphi}=-\Phi^{-\frac{1}{2}}\frac{S}{R}\sin\left(\Phi^{\frac{1}{2}}\frac{2\pi}{T}\tau\right) \tag{75}$$

and compute (for general τ)

$$\eta_{\mu\nu}s^{\mu}s^{\nu}=S^2+R^2\left(s^{\theta}_0\right)^2>0 \tag{76}$$

thus corroborating that $\mathbf{s}$ is indeed spacelike. The significance of this result lies in the fact that, for an observer falling freely alongside the curve with components x^{μ}, it takes a period T as measured in local time to complete a full revolution - but by the time that is accomplished, s^r and s^{φ} don't return to their respective initial values; this occurs only a little later, after

$$T'=\Phi^{-\frac{1}{2}}T\approx\left[1+\frac{1}{2Rc^2}\left(\frac{\alpha c^2}{2}+\beta c\frac{2\pi}{T}\right)\right]T=\left[1+\frac{(1-\Phi)}{2}\right]T, \tag{77}$$

which in turn translates to an excess angle of

$$\delta\approx 2\pi\left[1+\frac{(1-\Phi)}{2}\right]-2\pi=(1-\Phi)\,\pi \tag{78}$$

swept by the orbit. This can be interpreted as a form of frame-dragging, analogous to the Lense–Thirring effect in GR - although the presence of β already suggests the value to be at variance with that expected from Einstein's theory, or at least an experimental constraint on the parameters of this model.

Finally, as our third application of (63a-63c), we consider the anomalous perihelion precession. We assume for simplicity $\beta \approx 0$, and treat the orbit variationally, by assuming it can be suitably approximated by the Keplerian expression

$$\frac{1}{x^r} = \frac{\{1 + e\cos[(1-\Delta)\,x^\varphi - \omega]\}}{p} \tag{79}$$

where the eccentricity e, the semi-*latus rectum* p, the longitude of perihelion ω and the quantity Δ are all constants that parameterize it. Since it's possible to rewrite $\left(\frac{dx^r}{d\tau}\right)^2$ in terms of this $\frac{1}{x^r}$, substitution in (64b) yields

$$\begin{aligned}
K_r &\doteq \left[\left(e^2-1\right) - \left(\frac{p}{x^r}\right)^2 + 2\frac{p}{x^r}\right]\left(\frac{K_\varphi}{p}\right)^2(1-\Delta)^2 - K_0^2\exp\left(\frac{\alpha}{x^r}\right) + \frac{K_\varphi^2}{(x^r)^2} \\
&= -\left[1 + \left(1-e^2\right)\left(\frac{K_\varphi}{pK_0}\right)^2(1-\Delta)^2\right]K_0^2 + \left[\frac{2K_\varphi^2}{\alpha p}(1-\Delta)^2 - K_0^2\right]\frac{\alpha}{x^r} \\
&\quad + \left[\left(\frac{K_\varphi}{\alpha}\right)^2\left(2\Delta-\Delta^2\right) - \frac{K_0^2}{2}\right]\left(\frac{\alpha}{x^r}\right)^2 + O\left(\left(\frac{\alpha}{x^r}\right)^3\right)
\end{aligned} \tag{80}$$

after expanding the exponential. In this expansion lies the nature of the approximation being invoked: for suppose we require the r.h.s. to be a constant, only up to first order in $\frac{\alpha}{x^r}$; in that case we may put $\Delta = 0$ and solve for p by taking the second brackets to vanish, obtaining the quasi-Newtonian result

$$p = p_0 := \frac{2K_\varphi^2}{\alpha K_0^2} \tag{81}$$

for $K_0 = c$. Likewise, extension to second order in $\frac{\alpha}{x^r}$ would obtain a pair of algebraic equations on the two unknowns Δ and p via the vanishing of the second and third brackets; these in turn immediately reduce to the conditions $p = p_0 - \alpha$ and

$$\begin{aligned}
0 &= \Delta^2 - 2\Delta + \frac{\alpha}{p_0} \\
\therefore &\quad \Delta = 1 \pm \sqrt{1 - \frac{\alpha}{p_0}} \approx 1 \pm \left(1 - \frac{\alpha}{2p_0}\right)
\end{aligned} \tag{82}$$

Taking the negative-sign root, then, leaves us with $\Delta \approx \frac{\alpha}{2p}$ for $p \gg \alpha$. In the case of an elliptic orbit, $2\pi\Delta$ would seem to give us the anomalous amount of precession after one revolution; this result is roughly a third of what is predicted by GR [21]. Keep in mind, though, that the above calculations were all done in a purely relativistic context which does not immediately translate to what is observed astronomically; based on (64a), putting $x^0 = ct$ implies that $\frac{dt}{d\tau} = \exp\left(\int^{x^r} F^0_{r0} dx^{r'}\right)$ - so that there's a correction that needs to be taken into account when writing the components of the orbit as a function of t. Arguably[10], this correction might just be what is needed to obtain the full empirical effect (technical details nonwithstanding).

At any rate, for the readers tempted to rush to the conclusion that this purported discrepancy falsifies the model from the outset (not to mention the *entire* theory so far discussed!), I would suggest the exercise of caution: for it may be noted that the result was obtained under extremely simple assumptions, as well as the use of a number of approximations; between inclusion of rotational (i.e., $\beta \neq 0$) effects and post-Newtonian corrections, as well as a careful comparison with observational data (all of which falling beyond the scope of this article), there is still a lot of room within the theory to accomodate the apparent disparity - being thus at this juncture unseemly to outright dismiss the effort, just yet. Similar considerations naturally extend to other testable effects of gravitational theories we haven't addressed here, like lensing and echo delay.

6 Miscellaneous questions

The paradigm here presented and explored is relatively new, whereas General Relativity has the benefit of a century of peer-reviewed publications, and of astronomical and Earthbound observations, which were (and are) interpreted by the lens of the theory. In absence of a working 'theory of everything', I'd like to address a few objections that the (rightfully so) skeptic may level against this framework, before it even gets off the ground. The first (possibly more stringent even than the 'classic tests' touched upon in the last section) is that, even at the classical level, the formalism we've developed is incomplete: one needs to show how all the varied subjects such as thermodynamics, hydrodynamics[11] and electrodynamics would adapt or otherwise be made compatible with the tensor-valued formalism, in such a way as to include a proper variational treatment, suitable interpretation of the gauge aspects, and miscellaneous items of interest formulated covariantly; to see how these interconnect, consider that in the case of the Maxwell–Newton theory one wants not only the form of the Maxwell FE and Lorentz force equation in the presence of gravity while guaranteeing the former will satisfy the geometric optics limit, but also the Maxwell–Poynting stress tensor written as a vector-valued 3-form that may be used as a source for the *gravitational* field. For the moment such details present a theoretical corner that remains to be explored; said exploration amounts to a non-negligible undertaking, true, and I do not purport to have completed such a large-scale project. Nonetheless, such difficulties should not distract us from the fact that the incompleteness in display is not of a fundamental nature, but rather an artifact of ignorance.

[10] Another thing to keep in mind is that in the end of the Appendix it is pointed out that a 'quadrupole' contribution to the Newtonian potential may also be a solution to this problem.

[11] For an approach to continuum mechanics similar in spirit to my take here, cf. [29].

Moving to specific issues in the previous section; first, gravitational waves. Although the solutions were obtained in elementary fashion, it is not entirely clear how cogent they are to the quantization program: for it can be objected that the 'polarizations' e^{α}_{λ} are not compatible with spin-2 waves, and this raises further questions. Gravitational-wave astronomy, however, does not seem to provide a severe challenge, because agreement with interferometric observation might be met just by constraining the stress-momentum of the matter that sources the waves, and applying appropriate boundary conditions; in principle, one simply refurbishes the GR calculations for, say, the luminosity due to the orbital decay of close binaries, and accounts for discrepancies by adjusting T^{α}_{σ} accordingly for one's model. This might even be useful as an analytical tool for probing astrophysical systems.

Speaking of quantizing: the current proposal is compatible with standard quantum field-theoretical techniques for dealing with non-Abelian groups; this can be seen with the help of a simple recipe: first, one writes down a gauge theory of the Lorentz group that keeps only the (antisymmetric) curvature-bearing piece of the free Lagrangean (41). Naturally, it is the associated potential that is quantized as well as treated as a gauge field; in order to understand what is being 'gauged' here, however, we remind ourselves that in particle physics one deals with 'internal' symmetries associated with abstract labels that we attach to quantum states [30]. What kind of label should we have here, given everything is already Lorentz-invariant by construction? Since it's also the case that each elementary particle already belongs to some spinorial representation [31], unitary operators for infinitesimal transformations in spin-space can be constructed with appropriate generators of the Lie algebra of the Lorentz group, so as to leave the state invariant; gauging this theory then means making such spin-transformations spacetime-dependent (i.e., local) via a minimal coupling prescription.

The general procedure delineated in the previous paragraph is familiar to particle physicists - meaning it probably won't cost much in terms of theoretical machinery to implement it. Once this Lorentz gauge theory is laid down, one then proceeds to the next step: invoking the equivalence principle, make the substitution $\mathring{\omega}^{\alpha}_{\beta\nu} \rightarrow -F^{\alpha}_{\beta\nu}$ for the field components of the gauge field - where the torsion is naturally also present in the now full-blown Lagrangean (41), whose associated potential is also quantized. This can be tentatively treated as a form of gauging the Poincaré group - and while the nonminimal coupling will likely introduce challenges in the description of particle interactions, it seems to be the most straightforward route towards a quantum theory of gravity based on the present framework and established particle physics. If, however, it proves to be nonrenormalizable, or otherwise intractable, one is still free to look for other quantization schemes; the situation would be no worse than the many current attempts at quantizing GR.

Let us now comment on the concept of black holes, since it can lend itself to some confusion and possible equivocation. At peril of sounding pedantic, I'd like to draw some logical distinctions for clarity: first, one may talk of a *black hole* as a particular kind of vacuum solution of the gravitational field equations;

secondly, one may talk of a *collapsar* as a particular kind of compact object akin to neutron stars; and thirdly, one may talk of certain dynamical phenomena observed by astronomers that hint at high concentrations of unobserved matter.

In regards to the first one, we'd also like to briefly comment on the curious happenstance that a black hole model also doubles as the theoretical underpinning for the classical tests of General Relativity, in the form of the Schwarzschild metric. It is a rather unfortunate circumstance that seemingly only a minority of authors are vocal (or even cognizant) about the fact that the vacuum solution published by Schwarzschild in 1915 is not actually the same black hole metric featured ubiquitously and prominently in textbooks under his name. Developing the claim[12] would only serve to stretch this article further and lose focus on the bigger picture, so we'll concentrate on just two points: first one is that, in modern textbook notation (and in my favored signature convention), Schwarzschild's original solution [34] can be written

$$\mathbf{g}_\rho = -\left[1-\alpha\left(r^3+\rho\right)^{-\frac{1}{3}}\right]c^2dt^2+\left[\frac{r^4\left(r^3+\rho\right)^{-\frac{4}{3}}}{1-\alpha\left(r^3+\rho\right)^{-\frac{1}{3}}}\right]dr^2+\left(r^3+\rho\right)^{\frac{2}{3}}\left(d\theta^2+\sin^2\theta d\varphi^2\right) \tag{83}$$

where α and ρ are constants of integration. By introducing the coordinate transformation $R_\rho := \left(r^3+\rho\right)^{\frac{1}{3}}$, it is readily shown that $dR_\rho = r^2\left(r^3+\rho\right)^{-\frac{2}{3}}dr$, so the above may be put as

$$\mathbf{g}_\rho = -\left(1-\frac{\alpha}{R_\rho}\right)c^2dt^2+\left(1-\frac{\alpha}{R_\rho}\right)^{-1}dR_\rho^2+R_\rho^2\left(d\theta^2+\sin^2\theta d\varphi^2\right) \tag{84}$$

which is quite familiar to physics students everywhere. So, is this a black hole? Not *necessarily* - but it is understandable why one would interpret it as such; part of the confusion seems to stem from a misappreciation of the concept of diffeomorphism invariance by more physics-minded researchers. One way to see this is the following: if the range of the coordinate r is $[0,+\infty)$, that makes the range of R_ρ to be $\left[\rho^{\frac{1}{3}},+\infty\right)$ - meaning that, for $\rho=\alpha^3$, the metric singularity $R_\rho=\alpha$ is identified with the origin $r=0$ of the coordinate system; this was noticed by Schwarzschild himself, and is the basis of his choice for this parameter, as well as of his claim that the resulting solution represents physically a *point-mass* (as explicited in the title of the paper). To insist that the range of R_ρ be extended to $[0,\alpha)$ as well in this case would be tantamount to glueing a 'pocket universe' at the origin - and needless to say, there is no physically compelling, observational reason to do so; nonetheless, it's important to realize that, incidentally, putting $\rho=0$ *also* obtains a (different) solution to the vacuum EFE, the Hilbert–Droste metric. That they can be both made to *look* the same via coordinate transformation is ultimately irrelevant, because form-invariance is not the same as general covariance, is the moral of this story.

[12] For a historical overview, see sec. 2 and refs therein of [32]; for an opposing viewpoint, see [33].

Another important point, which to my knowledge is not sufficiently emphasized (or even mentioned) anywhere in the literature, is that, from a post-Newtonian perspective, both solutions reduce to the same expression when expanded to low order[13] in $\frac{\alpha}{r}$:

$$
\begin{aligned}
& - \left[1 - 1\frac{\alpha}{r} + 0\frac{\alpha^2}{r^2} + O\left(\frac{\alpha^3}{r^3}\right)\right] c^2 dt^2 + \left[1 + 1\frac{\alpha}{r} + O\left(\frac{\alpha^2}{r^2}\right)\right] dr^2 \quad (85) \\
+ \quad & r^2 \left[1 + 0\frac{\alpha}{r} + O\left(\frac{\alpha^2}{r^2}\right)\right] \left(d\theta^2 + \sin^2\theta d\varphi^2\right)
\end{aligned}
$$

Thus, to the extent that Solar System experiments probe only gravitational influences up to this order in the metric components (and ignoring frame-dragging effects), we see the classical tests cannot distinguish between the two solutions - therefore providing a mathematical justification for the use of the Hilbert–Droste vacuum to model the solar gravitational field, in spite of the fact that the Sun is obviously not a black hole in any sensible use of that nomenclature. That this is not a mere philosophical digression is due to the fact that the distinction just pointed is more poignant in the present proposal: it is not clear at all what (if any) solution would correspond to a black hole in the Hilbert–Droste sense, whereas the vacuum exterior to a point-mass in the Schwarzschild sense does not present any difficulty of principle, and is the one more directly related to the classical solar tests, anyway. If we go beyond those tests, however, we can still entertain the astrophysical idea of the collapsar by a more careful study of the FE for the interior of a compact object - and indeed an idealized model of the equations that might govern such an object has been delineated in the Appendix. Finally, one can also ask whether Penrose's argument [35] may not be adapted to the present discussion, with appropriate modifications as suggested by the generalized form of the Jacobi equation (39); this complements the collapsar discussion not only insofar the simplification of spherical symmetry is concerned, but also in terms of observed phenomenology, given astronomers don't observe collapsars (or black holes) directly, but only infer their presence via dynamical effects that are interpreted in gravitational terms, such as stellar motion around a seemingly empty region that can be explained by packing a large mass into a small volume therein.

This line of reasoning segues naturally into another open puzzle of astronomical interest - namely, the collection of phenomena that fall under the dark matter umbrella. As an illustration of the possibilities afforded by the proposed framework, we notice one such phenomenon explained by the dark matter hypothesis, that of the anomalous galactic rotation curves, may be framed not in terms of 'missing' mass (presumably of matter not accounted for in the Standard Model of Particle Physics), but rather as missing *acceleration* [36]; based on the transport equation (28), this in turn suggests two simple explanatory routes: either by a gravito(electro)magnetic effect (which has indeed been entertained [37] in the GR context), or the presence of a nonvanishing linear connection that

[13] Cf. [21], p. 183.

adds to the Newtonian expectation, thanks to the equivalence principle - and there is always, of course, the combination of both. There may be even more possibilities to explore, before the necessity of postulating a hitherto undetected form of matter to explain observations imposes itself.

In view of this development, one may want to ask as to other modern mysteries such as the effect that came to be referred as 'dark energy'; at this point, however, we find it more fitting to conclude this section with a rather poignant quote by Sachs & Wu [38]: "Cosmology (like the rest of physics) is circular reasoning in the following sense: one cannot really discuss the empirical data coherentIy without using, explicitIy or implicitly, some tentative theoretical model; one cannot sensibly choose even a tentative theoretical model without some reference to the empirical data".

7 Concluding remarks

Since the introduction of GR in the beginning of the 20^{th} century, the Riemannian paradigm has dominated the theoretical framework of classical gravitation - partly because it was then the state-of-art of differential geometry, and partly because of the tremendous empirical successes it undoubtedly enjoyed since. The geometrical alternative defined and discussed in this paper stands on its own mathematical merits regardless of any further physical consideration, and indeed may be studied on its own right - however, as we've shown, it also displays a list of potentially desirable features for a gauge theory of gravity - perhaps even as a competitor to the Einstein theory. Nonetheless, a complete understanding of how it can be used to describe real-world gravitational phenomena is still wanting and riddled with open problems which, in particular, currently preclude us from contrasting it empirically with GR. While this might at first sight seem like a fatal weakness dooming the whole enterprise, one should remind oneself that the perspective being advocated here is rather new and as such should not be readily dismissed without due consideration of its contents, incomplete they may be; indeed, I wanted to leave the matter of the gauge content as open-ended as possible for now to showcase the phenomenological power of the formalism - questions of symmetry pose a rich scenario to explore from here (not least those touching subjects such as reframing the 'cosmological principle' in this language). Furthermore, the very nature of this incompleteness represents a definiteness in terms of a research programme exploring these questions systematically - not the least being that (apart from the caveats already addressed) these ideas do not seem to require a substantial revision of previous physics, as it seems to be the case with several modern attempts at quantizing gravity.

Given the current status of research, it is of interest to explore as many different avenues as possible. The revision of our geometric intuition being here proposed comes with pros and cons, but it is hoped that the impression will be that the pros outweigh the cons. It is in this sense that we posit the theory presents, at the current stage of inquiry, a contender for the Einsteinian paradigm; the simple fact we obtained results that agree with experiment qualitatively and/or within an order of magnitude, which is no mean feat, should at the very least raise some eyebrows - thus encouraging a more rigorous study of the problem; where there's smoke, often one finds fire.

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8 Appendix: Axisymmetrical, rigidly rotating, time-dependent fluid model

For spherical coordinates

$$
\begin{aligned}
& x^0, \\
x &= r\sin\theta\cos\varphi, \\
y &= r\sin\theta\sin\varphi, \\
z &= r\cos\theta
\end{aligned}
$$

the metric reads

$$\boldsymbol{\eta} = -d\mathbf{x}^0 \otimes d\mathbf{x}^0 + d\mathbf{x}^r \otimes d\mathbf{x}^r + r^2\left(d\mathbf{x}^\theta \otimes d\mathbf{x}^\theta + \sin^2\theta d\mathbf{x}^\varphi \otimes d\mathbf{x}^\varphi\right)$$

so that $|-\eta|^{\frac{1}{2}} = r^2\sin\theta$, the Laplacean operator ∇^2 is given by

$$\nabla^2 f = \frac{1}{r^2}\partial_r\left(r^2\partial_r f\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta f\right) + \frac{1}{r^2\sin^2\theta}\partial^2_\varphi f$$

and the nonvanishing Christoffel symbols are given by

$$
\begin{aligned}
\begin{Bmatrix} r \\ \theta\theta \end{Bmatrix} &= -r, \\
\begin{Bmatrix} r \\ \varphi\varphi \end{Bmatrix} &= -r\sin^2\theta, \\
\begin{Bmatrix} \theta \\ \varphi\varphi \end{Bmatrix} &= -\sin\theta\cos\theta, \\
\begin{Bmatrix} \theta \\ \theta r \end{Bmatrix} &= \frac{1}{r} = \begin{Bmatrix} \theta \\ r\theta \end{Bmatrix}, \\
\begin{Bmatrix} \varphi \\ \varphi r \end{Bmatrix} &= \frac{1}{r} = \begin{Bmatrix} \varphi \\ r\varphi \end{Bmatrix}, \\
\begin{Bmatrix} \varphi \\ \varphi\theta \end{Bmatrix} &= \frac{\cos\theta}{\sin\theta} = \begin{Bmatrix} \varphi \\ \theta\varphi \end{Bmatrix}
\end{aligned}
$$

Based on this, we propose the following *Ansatz* for the potentials ω^α_σ that vanish from the get-go (in addition to demanding that $\partial_\varphi \omega^\alpha_\sigma = 0$):

$$\begin{pmatrix} \omega^0_0 & \omega^0_r & \omega^0_\theta = 0 & \omega^0_\varphi \\ \omega^r_0 & \omega^r_r & \omega^r_\theta = 0 & \omega^r_\varphi = 0 \\ \omega^\theta_0 = 0 & \omega^\theta_r = 0 & \omega^\theta_\theta & \omega^\theta_\varphi = 0 \\ \omega^\varphi_0 & \omega^\varphi_r = 0 & \omega^\varphi_\theta = 0 & \omega^\varphi_\varphi \end{pmatrix}$$

Substitution into (48a) yields the final expression for the field equations

$$\begin{aligned} \kappa_0 \mathbf{T}^0 \quad \doteq \quad & r^2 \sin\theta \left\{ \frac{1}{r^2} \partial_r \left[r^2 \left(\partial_r \omega^0_0 - \partial_0 \omega^0_r \right) \right] + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \partial_\theta \omega^0_0 \right) \right\} d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\ & r^2 \sin\theta \left[\partial_0 \left(\partial_r \omega^0_0 - \partial_0 \omega^0_r \right) + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \partial_\theta \omega^0_r \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\ & \sin\theta \left[\partial_0 \left(\partial_\theta \omega^0_0 \right) - \partial_r \left(\partial_\theta \omega^0_r \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^\varphi \wedge d\mathbf{x}^r + \\ & \frac{1}{\sin\theta} \left[-\partial_0 \left(\partial_0 \omega^0_\varphi \right) + \partial_r \left(\partial_r \omega^0_\varphi \right) + \sin\theta \partial_\theta \left(\frac{1}{r^2 \sin\theta} \partial_\theta \omega^0_\varphi \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^r \wedge d\mathbf{x}^\theta \end{aligned}$$

and

$$\begin{aligned} \kappa_0 \mathbf{T}^r \quad \doteq \quad & r^2 \sin\theta \left\{ \frac{1}{r^2} \partial_r \left[r^2 \left(\partial_r \omega^r_0 - \partial_0 \omega^r_r \right) \right] - \frac{2}{r^2} \omega^r_0 \right. \\ & \left. + \frac{1}{r^2} \left[\partial_\theta \left(\partial_\theta \omega^r_0 \right) + \frac{\cos\theta}{\sin\theta} \partial_\theta \omega^r_0 \right] + \frac{1}{r} \partial_0 \left(\omega^\theta_\theta + \omega^\varphi_\varphi \right) \right\} d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\ & r^2 \sin\theta \left\{ \partial_0 \left(\partial_r \omega^r_0 - \partial_0 \omega^r_r \right) + \frac{1}{r^2} \partial_\theta \left(\partial_\theta \omega^r_r \right) + \frac{\cos\theta}{r^2 \sin\theta} \partial_\theta \omega^r_r \right. \\ & \left. + \frac{1}{r} \left[\partial_r \left(\omega^\theta_\theta + \omega^\varphi_\varphi \right) - \frac{1}{r} \left(2\omega^r_r - \omega^\theta_\theta - \omega^\varphi_\varphi \right) \right] \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\ & \sin\theta \left\{ \partial_0 \left(\partial_\theta \omega^r_0 \right) - \partial_r \left(\partial_\theta \omega^r_r \right) + \frac{1}{r} \left[\partial_\theta \omega^\varphi_\varphi + \frac{\cos\theta}{\sin\theta} \left(\omega^\varphi_\varphi - \omega^\theta_\theta \right) \right] \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^\varphi \wedge d\mathbf{x}^r + \\ & \frac{1}{\sin\theta} \left\{ \partial_0 \left(-r \sin^2\theta \omega^\varphi_0 \right) \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^r \wedge d\mathbf{x}^\theta \end{aligned}$$

and

$$
\begin{aligned}
\kappa_0 \mathbf{T}^\theta \doteq\ & r^2 \sin\theta \left\{ \frac{1}{r^2} \partial_\theta \left(\frac{1}{r} \omega_0^r - \partial_0 \omega_\theta^\theta \right) + \frac{\cos\theta}{r^2 \sin\theta} \partial_0 \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) \right\} d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\
& r^2 \sin\theta \left\{ \frac{1}{r^3} \left(\partial_\theta \omega_r^r \right) + \frac{\cos\theta}{r^2 \sin\theta} \left[\partial_r \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) + \frac{1}{r} \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) \right] \right. \\
& \left. - \frac{1}{r^2} \partial_\theta \left[\partial_r \omega_\theta^\theta + \frac{1}{r} \left(\omega_\theta^\theta - \omega_r^r \right) \right] \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\
& \sin\theta \left\{ \partial_0 \left(\frac{1}{r} \omega_0^r - \partial_0 \omega_\theta^\theta \right) + \partial_r \left(\partial_r \omega_\theta^\theta \right) + \frac{1}{r} \partial_r \omega_\theta^\theta + \frac{1}{r} \partial_r \left(\omega_\theta^\theta - \omega_r^r \right) \right. \\
& \left. + \frac{\cos\theta}{r^2 \sin\theta} \left[\partial_\theta \omega_\varphi^\varphi + \frac{\cos\theta}{\sin\theta} \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) \right] \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^\varphi \wedge d\mathbf{x}^r + \\
& \frac{1}{\sin\theta} \left[-\sin\theta \cos\theta \left(\partial_0 \omega_0^\varphi \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^r \wedge d\mathbf{x}^\theta
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa_0 \mathbf{T}^\varphi \doteq\ & r^2 \sin\theta \left\{ \frac{1}{r^2} \partial_r \left[r^2 \left(\partial_r \omega_0^\varphi \right) \right] + \frac{2}{r} \partial_r \omega_0^\varphi + \frac{1}{r^2 \sin\theta} \left[\partial_\theta \left(\sin\theta \partial_\theta \omega_0^\varphi \right) + 2 \cos\theta \partial_\theta \omega_0^\varphi \right] \right\} d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\
& r^2 \sin\theta \left[\partial_0 \left(\partial_r \omega_0^\varphi + \frac{1}{r} \omega_0^\varphi \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\
& \sin\theta \left[\partial_0 \left(\partial_\theta \omega_0^\varphi + \frac{\cos\theta}{\sin\theta} \omega_0^\varphi \right) \right] d\mathbf{x}^0 \wedge d\mathbf{x}^\varphi \wedge d\mathbf{x}^r + \\
& \frac{1}{\sin\theta} \left\{ \partial_0 \left(\frac{1}{r} \omega_0^r - \partial_0 \omega_\varphi^\varphi \right) + \frac{1}{r} \partial_r \left(r \partial_r \omega_\varphi^\varphi \right) + \frac{1}{r} \partial_r \left(\omega_\varphi^\varphi - \omega_r^r \right) \right. \\
& \left. + \frac{1}{r^2} \left[\partial_\theta \left(\partial_\theta \omega_\varphi^\varphi \right) + \frac{\cos\theta}{\sin\theta} \partial_\theta \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) - \frac{1}{\sin^2\theta} \left(\omega_\varphi^\varphi - \omega_\theta^\theta \right) \right] \right\} d\mathbf{x}^0 \wedge d\mathbf{x}^r \wedge d\mathbf{x}^\theta
\end{aligned}
$$

We now construct a model for the stress-momentum: starting from the perfect fluid with components

$$
\begin{aligned}
\mathbf{T}_{pf} =\ & \left(\rho_m c^2 r^2 \sin\theta \right) \mathbf{e}_0 \otimes d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \left(P r^2 \sin\theta \right) \mathbf{e}_r \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \\
& \left(P \sin\theta \right) \mathbf{e}_\theta \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^\varphi \wedge d\mathbf{x}^r + \left(\frac{1}{\sin\theta} P \right) \mathbf{e}_\varphi \otimes d\mathbf{x}^0 \wedge d\mathbf{x}^r \wedge d\mathbf{x}^\theta
\end{aligned}
$$

we effect a transformation $x^0 = x^{0'}, r = r', \theta = \theta', \varphi = \varphi' - \Omega t'$ into a primed coordinate system rotating at constant speed Ω; the perfect fluid stress-momentum changes to

$$
\begin{aligned}
\mathbf{T}_{pf} =\ & \left(\rho_m c^2 r^2 \sin\theta \right) \mathbf{e}_0 \otimes d\mathbf{x}^r \wedge d\mathbf{x}^\theta \wedge d\mathbf{x}^\varphi + \mathbf{T}_{pf}^k \otimes \mathbf{e}_k \\
\mapsto\ & \left(\rho_m c^2 r'^2 \sin\theta' \right) \left(\mathbf{e}_{0'} + \frac{\Omega}{c} \mathbf{e}_{\varphi'} \right) \otimes d\mathbf{x}^{r'} \wedge d\mathbf{x}^{\theta'} \wedge \left(d\mathbf{x}^{\varphi'} - \frac{\Omega}{c} d\mathbf{x}^{0'} \right) + \mathbf{T}_{pf}^{k'} \otimes \mathbf{e}_{k'}
\end{aligned}
$$

where the $\mathbf{T}_{pf}^{k'} \otimes \mathbf{e}_{k'}$ piece is form-invariant w.r.t. the unprimed system. Based on this, we *define* a new $\mathbf{T}_\Omega$ by the components of the above expression but with the primes removed (i.e., as in the original, untransformed system). The rationale for this procedure is that an *active* rigid rotation can be simulated by a *passive* one - i.e., we interpret $\mathbf{T}_\Omega$ to be the stress-momentum of an (idealized) rigidly rotating perfect fluid. Finally, in order to account for more complex fluid behavior, we introduce a 'correction' term $\tilde{\mathbf{T}}$, so that the stress-momentum to be considered in the FE can be put in the form $\mathbf{T} = \mathbf{T}_\Omega + \tilde{\mathbf{T}}$.

Since all the $\tilde{T}^\alpha_{\nu\rho\sigma}$ are in principle arbitrary (or rather, problem-dependent), we find it useful to specify the diagonal components $\tilde{T}^0_{[r\theta\varphi]}, \tilde{T}^r_{[0\theta\varphi]}, \tilde{T}^\theta_{[0\varphi r]}, \tilde{T}^\varphi_{[0r\theta]}$ in the following manner: specializing to $\omega^0_r = \omega^r_0 = 0$ and $\omega^\varphi_\varphi = \omega^\theta_\theta = \omega^r_r$, if we put

$$
\begin{aligned}
r^2 \sin\theta \left(\partial^2_0 \omega^0_0\right) &\doteq \kappa_0 \tilde{T}^0_{[r\theta\varphi]}, &\text{(86a)}\\
-r^2 \sin\theta \left(\partial^2_r \omega^r_r\right) &\doteq \kappa_0 \tilde{T}^r_{[0\theta\varphi]}, &\text{(86b)}\\
-\frac{\sin\theta}{r^2}\left(r\partial_r \omega^\theta_\theta + \partial^2_\theta \omega^\varphi_\varphi\right) &\doteq \kappa_0 \tilde{T}^\theta_{[0\varphi r]}, &\text{(86c)}\\
-\frac{1}{\sin\theta}\left(\frac{1}{r}\partial_r \omega^\varphi_\varphi + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\partial_\theta \omega^\varphi_\varphi\right) &\doteq \kappa_0 \left(\tilde{T}^\varphi_{[0r\theta]} - \frac{\Omega^2}{c^2}\rho_m c^2 r^2 \sin\theta\right) &\text{(86d)}
\end{aligned}
$$

and substitute these into the FE (while keeping in mind that $\partial_\varphi \omega^\alpha_\sigma \equiv 0$), the latter are seen to simplify to the two comprehensive formulae

$$
\begin{aligned}
\Box^2 \omega^0_0 &\doteq \kappa_0 \rho_m c^2, &\text{(87a)}\\
\Box^2 \omega^r_r &\doteq \kappa_0 P &\text{(87b)}
\end{aligned}
$$

To check whether this simple result is compatible with the conservation law $\mathbf{dT} \doteq \mathbf{0}$, we just substitute eqs. (86) alongside the FE in:

$$
\begin{aligned}
0 &\doteq \partial_0 \left(r^2 \sin\theta \rho_m c^2 + \tilde{T}^0_{[r\theta\varphi]}\right) - \partial_r \tilde{T}^0_{[0\theta\varphi]} - \partial_\theta \tilde{T}^0_{[0\varphi r]} - \partial_\varphi \left(-\frac{\Omega}{c}\rho_m c^2 r^2 \sin\theta + \tilde{T}^0_{[0r\theta]}\right)\\
&\doteq -r^2 \sin\theta \partial_0 \left(\frac{1}{\kappa_0}\Box^2 \omega^0_0 - \rho_m c^2\right),\\
0 &\doteq \partial_0 \tilde{T}^r_{[r\theta\varphi]} - \partial_r \left(r^2 \sin\theta P + \tilde{T}^r_{[0\theta\varphi]}\right) - \left[\partial_\theta \tilde{T}^r_{[0\varphi r]} + \gamma^r_{\theta\theta}\left(\sin\theta P + \tilde{T}^\theta_{[0\varphi r]}\right)\right]\\
&\quad - \left[\partial_\varphi \tilde{T}^r_{[0r\theta]} + \gamma^r_{\varphi\varphi}\left(\frac{1}{\sin\theta}P - \frac{\Omega^2}{c^2}\rho_m c^2 r^2 \sin\theta + \tilde{T}^\varphi_{[0r\theta]}\right)\right]\\
&\doteq r^2 \sin\theta \partial_r \left(\frac{1}{\kappa_0}\Box^2 \omega^r_r - P\right),
\end{aligned}
$$

$$
\begin{aligned}
0 &\doteq \partial_0 \tilde{T}^{\theta}_{[r\theta\varphi]} - \left(\partial_r \tilde{T}^{\theta}_{[0\theta\varphi]} + \gamma^{\theta}_{\theta r}\tilde{T}^{\theta}_{[0\theta\varphi]}\right) - \left[\partial_\theta\left(\sin\theta P + \tilde{T}^{\theta}_{[0\varphi r]}\right) + \gamma^{\theta}_{r\theta}\tilde{T}^{r}_{[0\varphi r]}\right] \\
&\quad - \left[\partial_\varphi \tilde{T}^{\theta}_{[0r\theta]} + \gamma^{\theta}_{\varphi\varphi}\left(\frac{1}{\sin\theta}P - \frac{\Omega^2}{c^2}\rho_m c^2 r^2 \sin\theta + \tilde{T}^{\varphi}_{[0r\theta]}\right)\right] \\
&\doteq \sin\theta\,\partial_\theta\left(\frac{1}{\kappa_0}\Box^2\omega^r_r - P\right), \\
0 &\doteq \partial_0\left(\frac{\Omega}{c}\rho_m c^2 r^2 \sin\theta + \tilde{T}^{\varphi}_{[r\theta\varphi]}\right) - \left(\partial_r \tilde{T}^{\varphi}_{[0\theta\varphi]} + \gamma^{\varphi}_{\varphi r}\tilde{T}^{\varphi}_{[0\theta\varphi]}\right) - \left(\partial_\theta \tilde{T}^{\varphi}_{[0\varphi r]} + \gamma^{\varphi}_{\varphi\theta}\tilde{T}^{\varphi}_{[0\varphi r]}\right) \\
&\quad - \left[\partial_\varphi\left(\frac{1}{\sin\theta}P - \frac{\Omega^2}{c^2}\rho_m c^2 r^2\sin\theta + \tilde{T}^{\varphi}_{[0r\theta]}\right) + \gamma^{\varphi}_{r\varphi}\tilde{T}^{r}_{[0r\theta]} + \gamma^{\varphi}_{\theta\varphi}\tilde{T}^{\theta}_{[0r\theta]}\right] \\
&\doteq \partial_0\left(\frac{\Omega}{c}\rho_m c^2 r^2\sin\theta + \tilde{T}^{\varphi}_{[r\theta\varphi]}\right) - \frac{1}{\kappa_0} r^2 \sin\theta\,\partial_0\left\{\frac{1}{r^2}\partial_r\left[r^2\left(\partial_r\omega^{\varphi}_0\right)\right] + 2\frac{1}{r}\partial_r\omega^{\varphi}_0\right. \\
&\quad \left. + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\omega^{\varphi}_0\right) + 2\frac{\cos\theta}{\sin\theta}\left(\partial_\theta\omega^{\varphi}_0\right)\right]\right\}
\end{aligned}
$$

Thus, our choice (86) is seen to be consistent with eqs. (87); that takes care of the specific density and pressure.

While the above treatment provides a way to study more complex situations such as stellar collapse, it is also useful to consider more idealized cases of interest - in particular, something that we might recognize as more or less equivalent to the Kerr solution in GR. Such a solution is, presumably, simple in nature, as well as stationary; this reduces the problem, then, to the system

$$
\begin{aligned}
\kappa_0\rho_m c^2 &\doteq \frac{1}{r^2}\partial_r\left(r^2\partial_r\omega^0_0\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\omega^0_0\right), && \text{(88a)} \\
\kappa_0 P &\doteq \frac{1}{r^2}\partial_r\left(r^2\partial_r\omega^r_r\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\omega^r_r\right), && \text{(88b)} \\
\kappa_0\left(-\frac{\Omega}{c}\rho_m c^2 r^2\sin\theta\right) &\doteq \frac{1}{\sin\theta}\left[\partial_r\left(\partial_r\omega^0_\varphi\right) + \sin\theta\,\partial_\theta\left(\frac{1}{r^2\sin\theta}\partial_\theta\omega^0_\varphi\right)\right] && \text{(88c)} \\
\kappa_0\left(\frac{\Omega}{c}\rho_m c^2 r^2\sin\theta\right) &\doteq r^2\sin\theta\left\{\frac{1}{r^2}\partial_r\left(r^2\partial_r\omega^{\varphi}_0\right) + \frac{2}{r}\partial_r\omega^{\varphi}_0\right. && \text{(88d)} \\
&\quad \left. + \frac{1}{r^2\sin\theta}\left[\partial_\theta\left(\sin\theta\,\partial_\theta\omega^{\varphi}_0\right) + 2\cos\theta\,\partial_\theta\omega^{\varphi}_0\right]\right\}
\end{aligned}
$$

where for simplicity we assumed that $\tilde{T}^{0}_{[0r\theta]} = \tilde{T}^{\varphi}_{[r\theta\varphi]} = 0$. The problem now bifurcates into an interior solution and an exterior vacuum ($\rho_m = P = \Omega = 0$) that may be matched at the boundary of the distribution of matter; one can solve the former by introducing a specific equation of state such as the one for a relativistic gas of free fermions [39], but we'll confine our interest to the latter case. This is a simple matter of using the method of separation of variables to write

$$\begin{aligned}
\omega_0^0 &= A_l\,(r)\,C_l^{\frac{1}{2}}\,(\cos\theta)\,, \\
\omega_\varphi^0 &= Q_l\,(r)\,C_l^{-\frac{1}{2}}\,(\cos\theta)\,, \\
\omega_0^\varphi &= R_l\,(r)\,C_l^{\frac{3}{2}}\,(\cos\theta)
\end{aligned}$$

with the help of the Gegenbauer polynomials [40] C_l^λ (and ignoring ω_r^r since it has the same equation as ω_0^0), so as to obtain from (88) the ODEs

$$\partial_r^2 A_l + \frac{2}{r}\partial_r A_l = \frac{l\,(l+1)}{r^2}A_l, \tag{89a}$$

$$\partial_r^2 Q_l = \frac{l\,(l-1)}{r^2}Q_l, \tag{89b}$$

$$\partial_r^2 R_l + \frac{4}{r}\partial_r R_l = \frac{l\,(l+3)}{r^2}R_l \tag{89c}$$

The above is then easily solved using Euler's method: putting $A_l \propto r^k, Q_l \propto r^m$ and $R_l \propto r^n$ we get

$$\begin{aligned}
k\,(k+1) &= l\,(l+1)\,, \\
m\,(m-1) &= l\,(l-1)\,, \\
n\,(n+3) &= l\,(l+3)
\end{aligned}$$

Consideration of the asymptotic behavior leads us to pick off $k = -\,(l+1)$ for $l \geq 0$, $m = -\,(l-1)$ for $l \geq 2$, and $n = -\,(l+3)$ for $l \geq 0$. Notice that in the special case of spherical symmetry (i.e., $l = 0$) the potential ω_φ^0 doesn't drop off with radial distance; this in turn justifies modeling the exterior vacuum of a spherically symmetrical (but rotating) point-like source by putting $\omega_\varphi^0 = 0$. If, furthermore, we also take $\omega_r^r = 0$, we end with a two-parameter model, comparable with Kerr's.

Incidentally, an interesting alternative to this α,β-model is presented by the above discussion: suppose we take only ω_0^0 nonzero, but this time make it a combination of $\frac{C_0^{\frac{1}{2}}(\cos\theta)}{r}$ and $\frac{C_2^{\frac{1}{2}}(\cos\theta)}{r^3}$ (for definiteness, call this the α,α'-model). In this case new terms $F_{\theta 0}^0, F_{00}^\theta$ dependent on the torsion component $\Theta_{\theta 0}^0$ will arise; however, since it can be checked that the latter vanishes for $\theta = \frac{\pi}{2}$, it follows that we can still use eqs. (64)! The only difference is that F_{r0}^0 will now include a term proportional to r^{-4}, and this seems to agree with the GR transport, for judicious choice of α'.