Explicit Approximate Formula for the Critical Exponent in Orthogonal Class using the Multi-points Summation Method

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Abstract

I suggest a new explicit formula for dimensional dependence of the critical exponent of the Anderson transition considering high dimensional asymptotic behavior and using the multi-points summation method. Asymptotic expansion at infinite dimension is estimated from numerical data. Combining known asymptotic series at two dimension and infinite dimension using the multi-points summation method, I obtained useful approximation formula for the critical exponent in the Orthogonal class.

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The Anderson transition[1, 2] is the quantum phase transition between localized state and extended state. At and near the critical point, universal phenomena is observed such as the value of the critical exponent ν of the localization length. In previous studies, the critical exponents were estimated both numerically and analytically. In this paper, I give new approximate formula for the critical exponent for arbitrary dimensionality.

Asymptotic series at the lower critical dimension $d = 2$ is calculated by S. Hikami[3].

First, I perform high dimensional asymptotic expansion of the critical exponent. From previous numerical studies, I fit those data with following asymptotic series,

$$\nu \sim \nu_\infty \equiv \frac{1}{2} + \frac{a_1}{\epsilon} + \frac{a_2}{\epsilon^2} + \frac{a_3}{\epsilon^3} \tag{1}$$

Here,$\epsilon = d - 2$. In our previous study, only first term is considered[4].

By fitting numerical data of $d = 2.22, 3, 4, 5, 6$[5, 4, 6], I obtained

$$\begin{aligned} a_1 &= 1.482 \ \pm 0.151 \\ a_2 &= -0.480 \pm 0.211 \\ a_3 &= 0.075 \pm 0.040 \end{aligned} \tag{2}$$

Second, I perform the multi-points summation method for the critical exponent. A known asymptotic series at two dimension is

$$\nu \sim \nu_\infty \equiv \frac{1}{\epsilon} - \frac{9}{4}\zeta(3)\epsilon^2 + \frac{27}{16}\zeta(4)\epsilon^3 + O(\epsilon^4) \tag{3}$$

The multi-points summation method uses both asymptotic series in Eqns.(3) and (1) By subracting asymptotic terms at infinite dimension from the one at two dimension, I obtain

$$f(\epsilon) \equiv \nu_0 - \nu_\infty \tag{4}$$

Multiplying ϵ^3, I obtain power series,

$$\epsilon^3 f(\epsilon) = 1.826\epsilon^6 - 2.705\epsilon^5 - 0.5\epsilon^3 - 0.482\epsilon^2 + 0.480\epsilon - 0.075 \tag{5}$$

Then perform Padé approximant of order $[1/5](t)$(This order of Padé approximant gives best estimate. I also tried Borel Padé analysis, but just using Padé gives better estimate of the critical exponents.),

$$\epsilon^3 f(\epsilon) \simeq \frac{-0.075 + 0.354\epsilon}{1 + 1.681\epsilon + 4.347\epsilon^2 + 10.39\epsilon^3 + 27.47\epsilon^4 + 44.11\epsilon^5} \tag{6}$$

By multiplying $1/\epsilon^3$ and summing ν_∞, I obtain final approximate expression,

$$\nu \simeq \nu_\infty + \frac{1}{\epsilon^3} \cdot \frac{-0.075 + 0.354\epsilon}{1 + 1.681\epsilon + 4.347\epsilon^2 + 10.39\epsilon^3 + 27.47\epsilon^4 + 44.11\epsilon^5} \tag{7}$$

This formula is consistent with asymptotic series at two limit of $\epsilon \to 0$ and $\epsilon \to \infty$. In this respect, MPS is different from just fitting. For the sake of readability, it should be mentioned that the numbers are rounded to 4 significant digits.

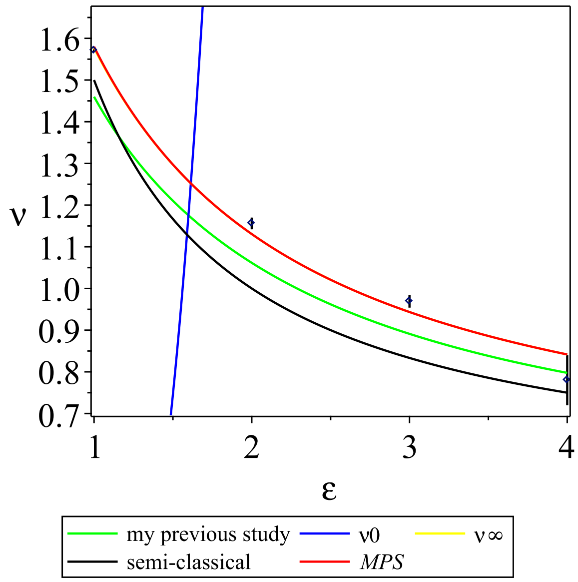

Figure 1: dimensional dependence of the critical exponent ν vs $\epsilon = d - 2$.

I plotted obtained expression in Fig.1 with numerical data[5, 4, 7]. Red curve is MPS. Green curve is our previous formula. Black curve is semi-classical theory[8]. Blue and Yellow curves are asymptotic series at two dimension and infinite dimension. The red curve for MPS and the yellow for asymptotic series at $d \to \infty$ curve almost overlap. Self-consistent theory of the Anderson transition[9] gives following formula,

$$\nu = \left\{ \tfrac{1}{\epsilon}(\epsilon < 2) \; \frac{1}{2}(\epsilon \geq 2) \right. \tag{8}$$

This formula gives far estimate from numerical data.

The values of the critical exponent at integer dimensions and spectral dimensions of real number are listed in Table.1 and 2.

d	$Eq.$(7)	Previous formula[4]	numerical estimate
3	1.580	1.460	1.571 ± .004[5]
4	1.130	1.061	1.156 ± .014[4]
5	0.944	0.891	0.969 ± .015[4]
6	0.842	0.798	0.78 ± .06[7]

Table 1: Estimated critical exponents for the orthogonal symmetry class for $d = 3, 4, 5$ and 6 obtained from Eq.(7).

These estimated values are almost consistent with numerical estimates.

d	$Eq.$(7)	Previous formula[4]	Refs.[6, 10, 11]
2.22	4.508	4.41	4.402 ± .18
2.226	4.389	4.28	2.82 ± .05
2.32	3.152	3.02	2.59 ± .19
2.33	3.068	2.94	2.92 ± .14
2.365	2.818	2.68	2.27 ± .06
2.41	2.575	2.44	2.50 ± .21
2.54	2.154	2.01	2.24 ± .31

Table 2: Same as for Table 1 but for fractals with spectral dimension $2 < d < 3$. Values of critical exponents in Ref. [6] were provided by M. Schreiber.

These estimated values are consistent with numerical estimates by 2σ except for some outliers.

I plotted obtained expression in Fig.2 for integer dimensions.

I plotted obtained expression in Fig.2 around $d = 2.1$.

From Fig.3,it is obvious that fitting extrapolation failed to describe asymptotic series near $d = 2$. On the other hand MPS succeeds to describe asymptotic series near $d = 2$.

I perform the MPS with asymptotic series obtained from high dimensional numerical results. Obtained formula almost agree with all numerical estimates except for outliers and consistent with asymptotic series at both $d = 2$ and $d = \infty$. This formula for dimensional dependence of the critical exponent in the Orthogonal class is the best formula obtained in the past.

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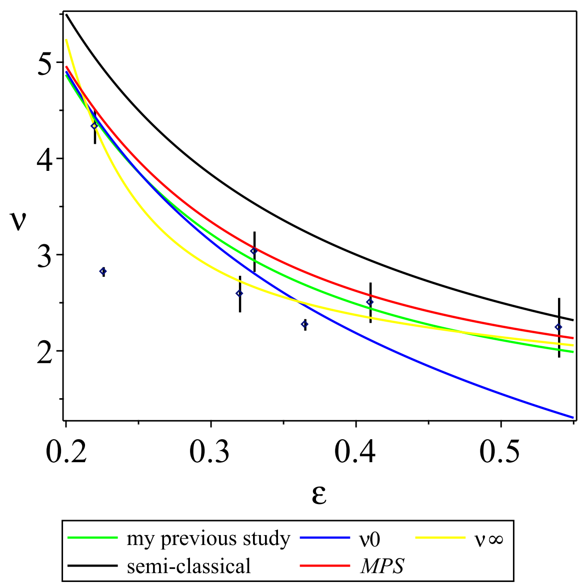

Figure 2: dimensional dependence of the critical exponent ν vs $\epsilon = d - 2$.

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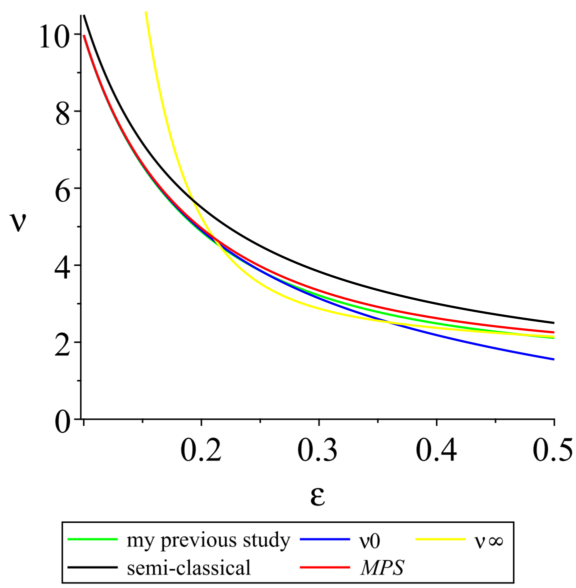

Figure 3: dimensional dependence of the critical exponent ν vs $\epsilon = d - 2$.