Summation of multiple times of a geometric series and its binomial series

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Abstract: This paper presents a binomial series of summation of multiple times of a geometric series. This will be useful for the researchers who are involving to solve the scientific problems.

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Summation of multiple times of a geometric series:

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \qquad (1)$$

The left side of Equ.(1) is the summation of multiple times of a geometric series[1-5] and the right side of Equ.(1) is a binomial series derived from the computation of multiple times of a geometric series. Here, the optimized combination [1-5] is shown below:

$$V_r^n = \frac{(r+1)(r+2)\cdots\cdots(r+n)}{n!} = \frac{(n+1)(n+2))\cdots\cdots(n+r)}{r!} = V_n^r,$$

$$i.e., \qquad V_r^n = \prod_{i=1}^{n}\frac{r+i}{n!} = \prod_{i=1}^{r}\frac{n+i}{r!} = V_n^r \quad (n,r \in N),$$

where $N = \{0, 1, 2,3, \cdots\cdots\}, V_r^n$ is a binomial coefficient, and $n!$ is the factorial of n.

Some results [1, 2] of the optimized combination are provided below:

i). $V_n^0 = V_0^n = 1\ (n \geq 1 \ \ \& \ \ n \in N),$

where V_n^0 alaway implies V_0^n , $i.e., V_n^0 \Longrightarrow V_0^n$.

Note that $V_r^n = V_n^r = (n+r)C_r = (n+r)C_n = \dfrac{(n+r)!}{n!\,r!}$ and $V_0^0 = 1$.

ii). $V_r^n = V_n^r \ (n, r \geq 1 \ \& \ n, r \in N) \ \& \ V_n^0 = V_0^n$.

When substituting r = 1, Equ. (1) becomes the summation of two times of a geometric series,

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n} x^{i_2} = \sum_{i_2=0}^{n} x^{i_2} + \sum_{i_2=1}^{n} x^{i_2} + \sum_{i_2=2}^{n} x^{i_2} + \cdots + \sum_{i_2=n}^{n} x^{i_2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n.$$

$$1 + 2x + 3x^2 + \cdots + (n+1)x^n = \sum_{i=0}^{n}(i+1)x^i = \sum_{i=0}^{n} V_i^1 x^i.$$

When substituting r = 2, Equ. (1) becomes the summation of three times of a geometric series,

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i_2=0}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=2}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \cdots + \sum_{i_2=n}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i=0}^{n} V_i^2 x^i.$$

Similarly, if the above process continues upto r times, the r^{th} equation becomes as follows:

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n} \cdots\cdots\cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r} \qquad (2)$$

If substituting r = 0, Equ. (2) becomes the actual geometric series,

$$\sum_{i=0}^{n} V_i^0 x^i = \sum_{i_1=0}^{n} x^{i_1} = 1 + x + x^2 + x^3 + \cdots + x^n \left(\because\ V_0^0 = V_i^0 = v_0^i = 1, i \in N\right).$$

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