Lecture Notes on Symmetry Optics

Lecture 5:
Critical Planes of Many-Slit Interference

To accompany https://youtu.be/1LsZw-pugP8
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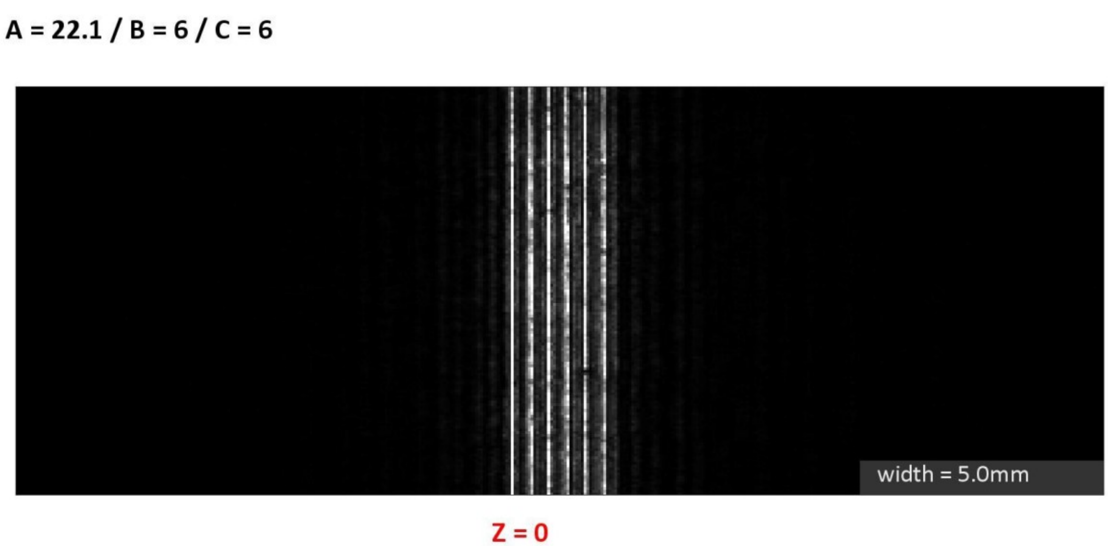
1 Introduction

1.1 Overview



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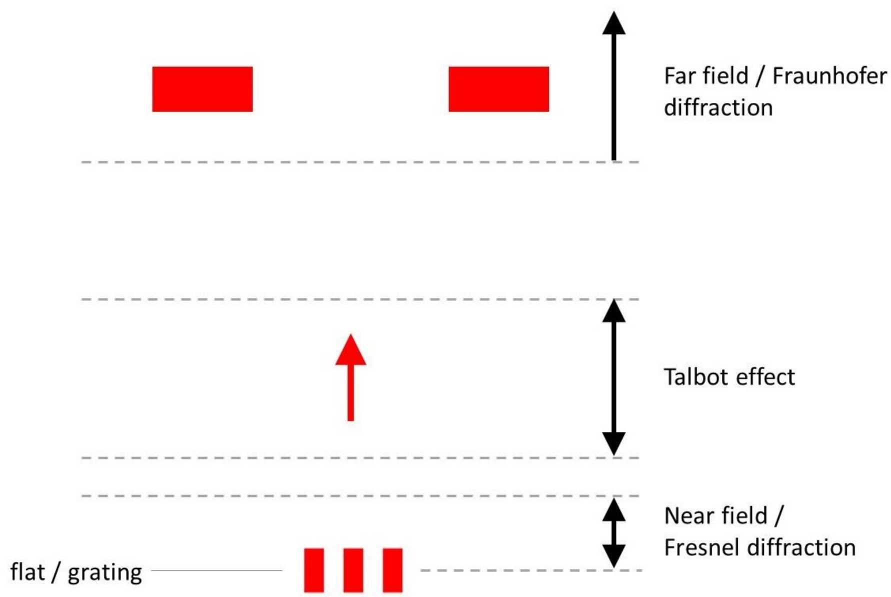
Welcome to Lectures on Symmetry Optics. I'm Paul Mirsky. This is Lecture 5 of the series, and the topic is: *The Critical Planes of Many-Slit Interference*.

The phenomenon of many-slit interference starts with a flat wavefront and a grating pattern of many identical stripes. As the light propagates, we see an intricate sequence of patterns. It defies easy description.

In this lecture, we'll see how symmetry optics reveals a remarkable order behind this jumble of lines.

1.2 Conventional view of many-slit interference



In conventional wave optics, the space is roughly divided into a few different regions:

The region very close to the grating is called the *near field*. The theoretical model that works best for analyzing this case is called *Fresnel diffraction*.

The region very far from the grating is called the *far field*. The model that works best for this case is called *Fraunhofer diffraction*.

In between, there's also a phenomenon called the *Talbot effect*, where images of the grating form in certain planes, even though there is no lens to create the image.

1.3 The critical planes

Plane	Z
far limit	$A^2 \cdot B^2 \cdot C^2 \cdot E$ *or* $A \cdot B \cdot C \cdot D$
late elbow	$A^2 \cdot B^2 \cdot C^2$
core end	$A^2 \cdot B^2 \cdot C$
breakout	$A^2 \cdot B^2 \cdot (C/B)$
interval elbow	$A^2 \cdot B^2$
core start	$A^2 \cdot B$
early elbow	A^2
flat	0

In symmetry optics, the structure of many-slit interference is described instead by the critical planes, which are listed in this table. These planes occur when live features grow to reach the same size as static features. Starting from the bottom of the table, the critical planes are: the flat, the early elbow, the core start, the interval elbow, breakout, the core end, the late elbow, and the far limit.

The right column shows the Z-distance where each plane occurs, and these are all expressed in terms of the factor sizes A, B, C, and D. Apart from the first two planes, every distance is some multiple of the previous distance. So, for example, the early elbow is at A^2, while the core start is at A^2B.

Note that the critical planes are not at all equally spaced. Two earlier planes might be only a fraction of a millimeter apart, but two later planes could be hundreds of meters apart.

The planes in the middle of the table form a region called the *core*. In the core, there are special planes called *coincidences*, which are closely related to the Talbot effect. The coincidences are special planes, but they don't quite rise to the level of critical planes.

We're going to study many-slit interference according to *stages*. Every stage starts at one critical plane, and ends at the next critical plane. We'll start with the first stage, which goes from the flat to the early elbow, and then we'll step through all the other stages, in turn. For each one, we'll go step-by-step and compute the patterns.

1.4 Note to the reader

This lecture will contain large amounts of detail, and it's going to deliver information much more quickly than any listener can possibly absorb it. So just focus on the parts that you can understand, and try to get the gist of it.

To actually learn the material, download and read the lecture notes. These contain the whole lecture in written form. Go over each step at your own pace, so that you can follow it in detail.

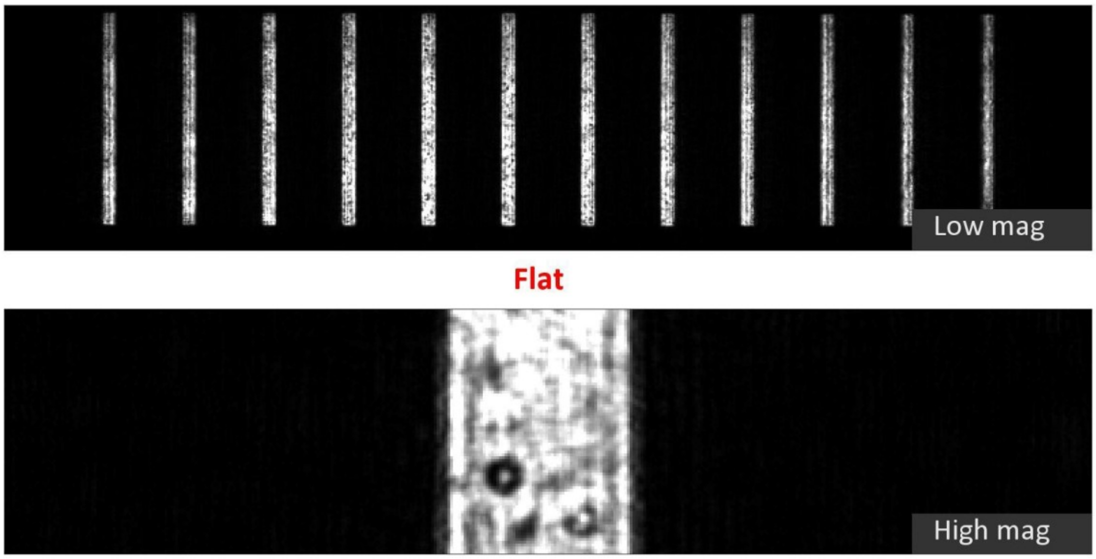
A middle path is to watch the video, but to make liberal use of the pause button.

Let's get started.

2 Flat to Early Elbow

2.1 Stage trend

A = 198.7 / B = 6 / C = 12

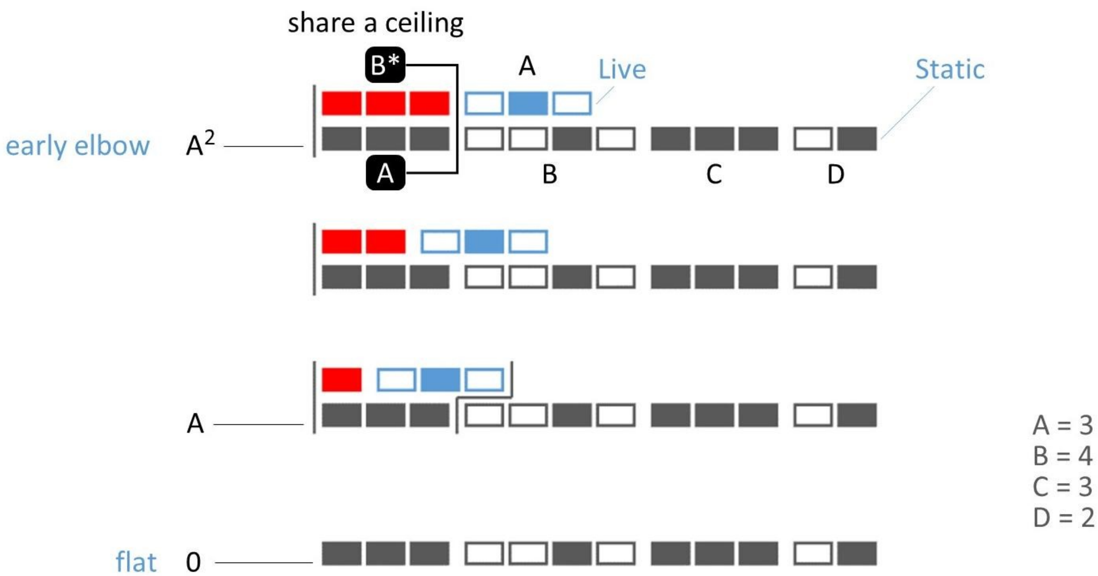


Here are some laboratory images of a grating at the flat. These are taken with a camera, so they show the screen pattern, but not the roundness. The top and bottom images show the same grating, but at two different magnifications.

Now look at the pattern at the early elbow. It's essentially unchanged from the flat. Of course, you can see some change, but we're concerned with strong, clear trends, and so we allow for small discrepancies.

Let's look at a couple more examples, with different factor sizes. They all behave the same basic way.

2.2 Factor chains

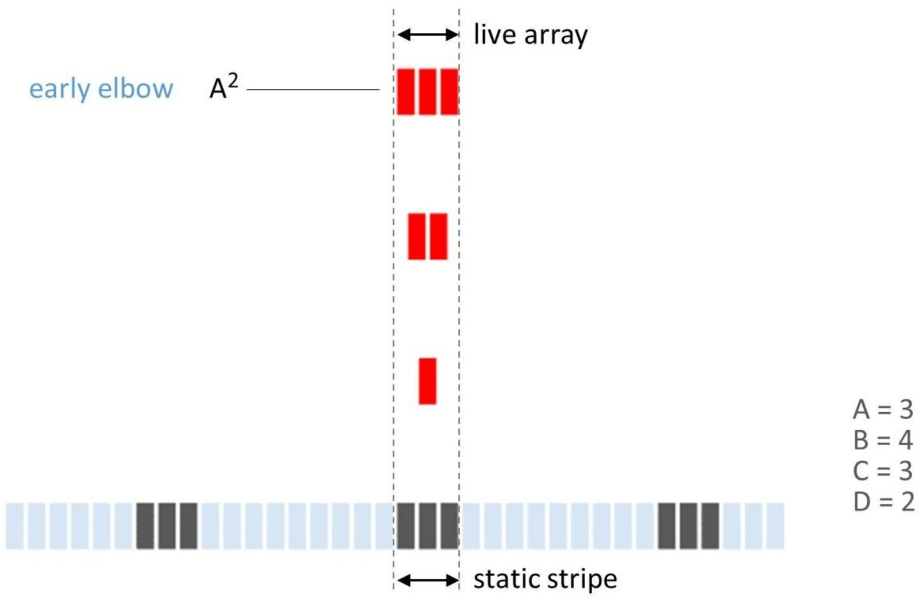


Let's see how the first stage looks in terms of factors. This diagram shows a sequence of four planes, from the flat to the early elbow. Recall that the dark grey factor chain represents the static, while the colored factor chain represents the live. The live grows larger, the further we get from the flat.

At the early elbow, live factor B shares a ceiling with static factor A. The early elbow is very similar to the elbow of the beam, which we saw in the previous lecture. By the way, the asterisk mark on the B indicates that the factor is only partially grown. Here, the full size of factor B is 4, but only 3 points have grown yet.

The next step will be to convert these factor chains into spatial diagrams. Whenever the factor diagram shows a shared ceiling between the static and the live, it means that in spatial terms, one of the static features has the same width as one of the live features.

2.3 Static and live features

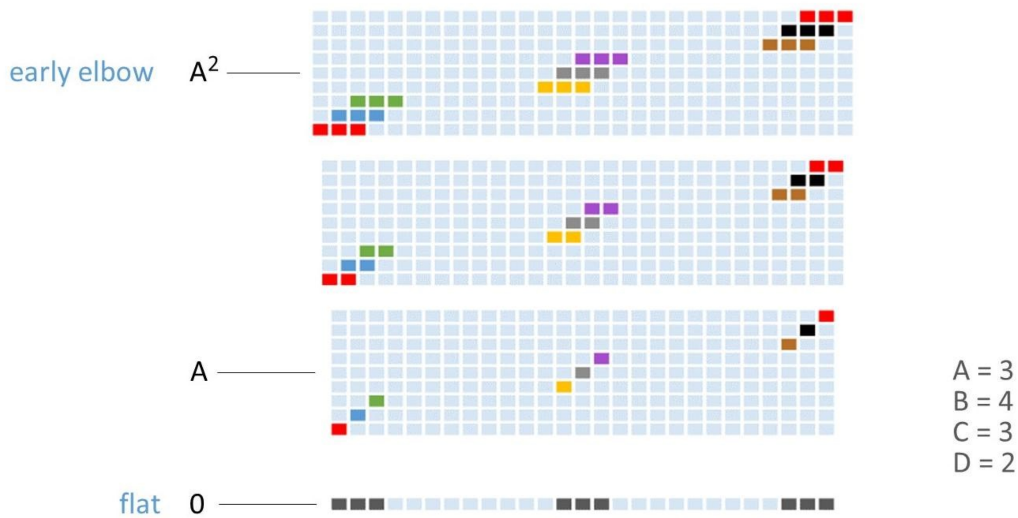


Here are the static and live, drawn spatially. By the way, just like last lecture, we're trimming off some excess dark space on either side.

Notice that at the early elbow, the live array is as wide as the static stripe. This is the essence of a critical plane – it's the point where a live feature grows to be as wide as a static feature.

By the way, notice that this dimension is the live *array*, not the live *stripe*. If you think this is the live stripe, hold that thought and we'll address it in a minute.

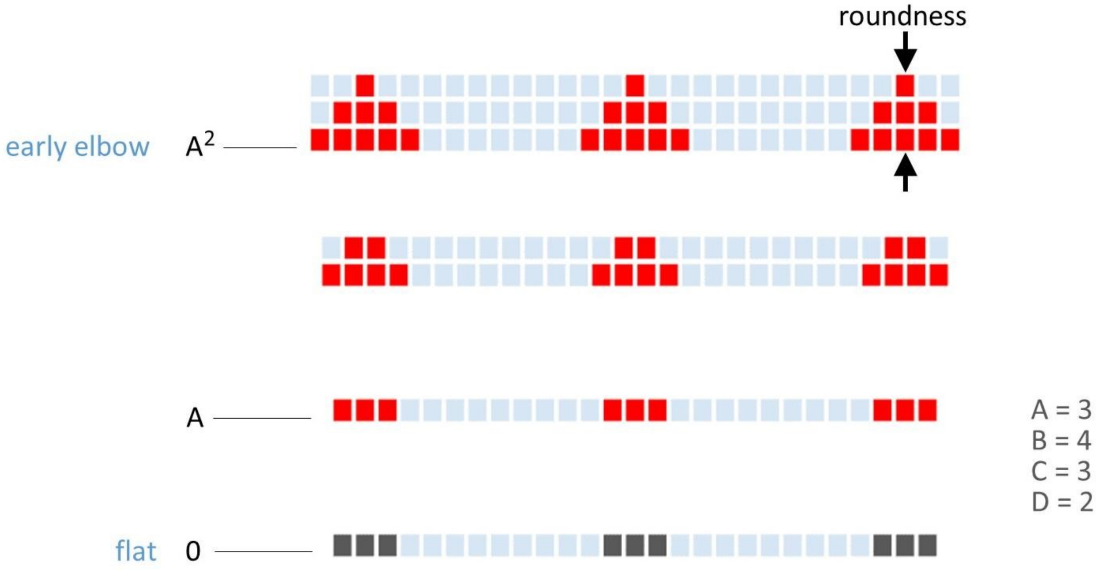
2.4 Source-target grids



Next, we tabulate the source-target grid, by aligning one instance of live with each individual source patch.

Also, remember – these are 3 separate source-target grids, for 3 different planes.

2.5 Distributions



And the final result comes when we collapse the grids to form the distributions.

This looks very much like a set of 3 identical beams. The light from each slit remains at a constant width, and we can neglect the small amount of spreading. This agrees with what we saw before in the laboratory images.

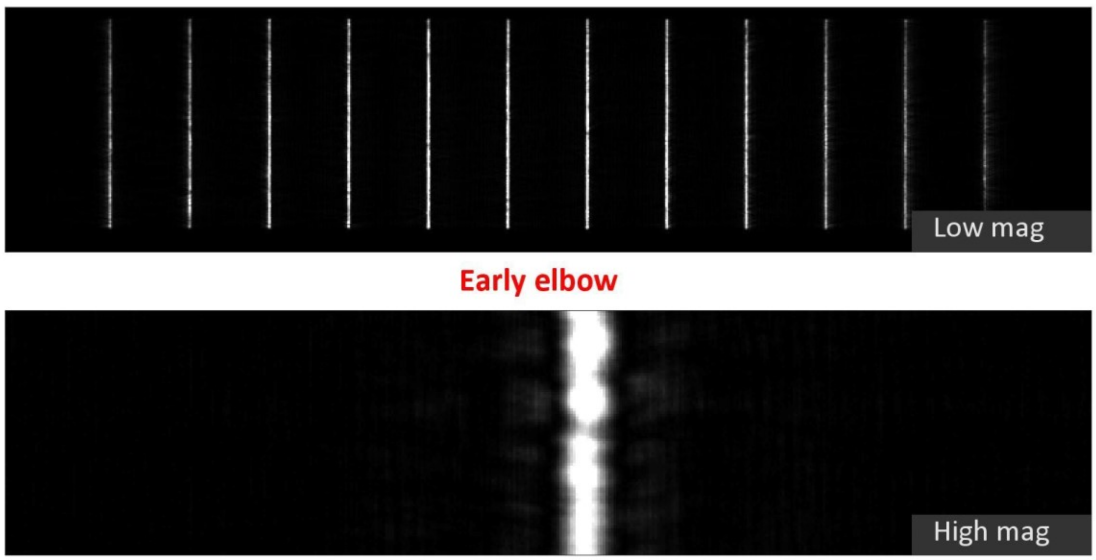
The trend in roundness is a slightly more complicated story. This one *particular* example behaves exactly like the beam does. In other words, the roundness increases linearly, and rises to A at the elbow. But this is not always the case for many-slit interference. Depending on the factor sizes, the roundness may plateau instead.

We will learn the general principles of roundness in the next lecture. But for this present lecture, we'll be concerned only with width, not roundness. All the width trends that we discuss here in this lecture will be valid in general, for any set of factor sizes.

3 Early Elbow to Core Start

3.1 Stage trend

A = 49.7 / B = 24 / C = 12

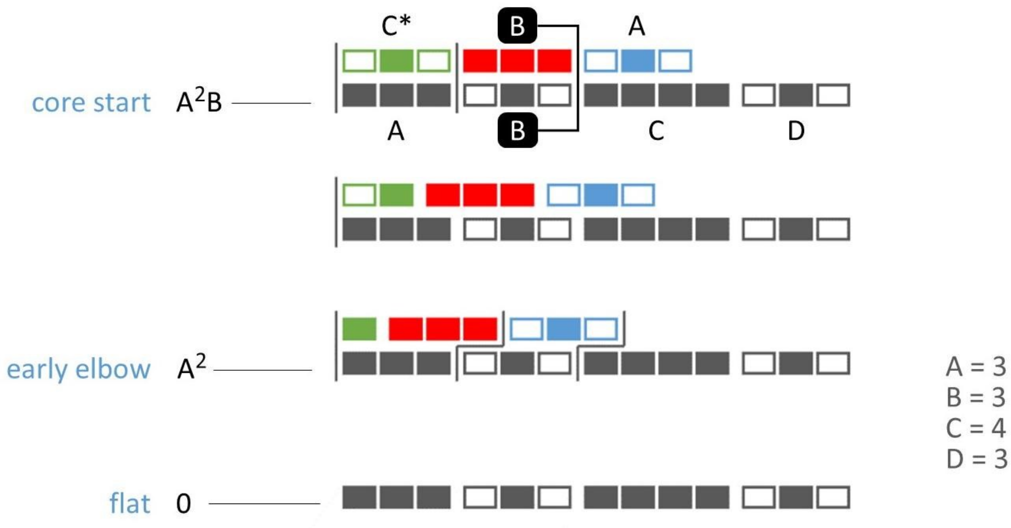


The next stage starts where the previous stage ended, which is at the early elbow. Then it goes up to the core start.

The general trend in this stage is that the individual beams grows wider, proportional to Z. They spread out into the dark space, until finally the beams touch one another at the core start, and form one large continuous region of bright space.

Here's another example, and another.

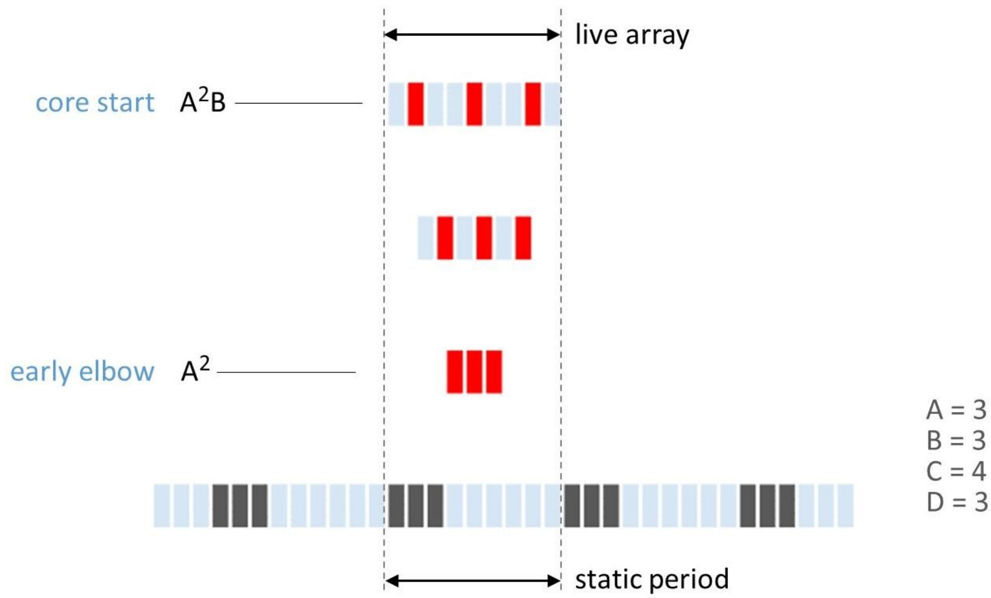
3.2 Factor chains, 1st shared ceiling



Again, we'll analyze an example. Note that this example uses different factor sizes from the previous one.

Again, we begin with the factor chains. The live grows, and all of its individual factors rise in rank, until the core start, where live B shares a ceiling with static B.

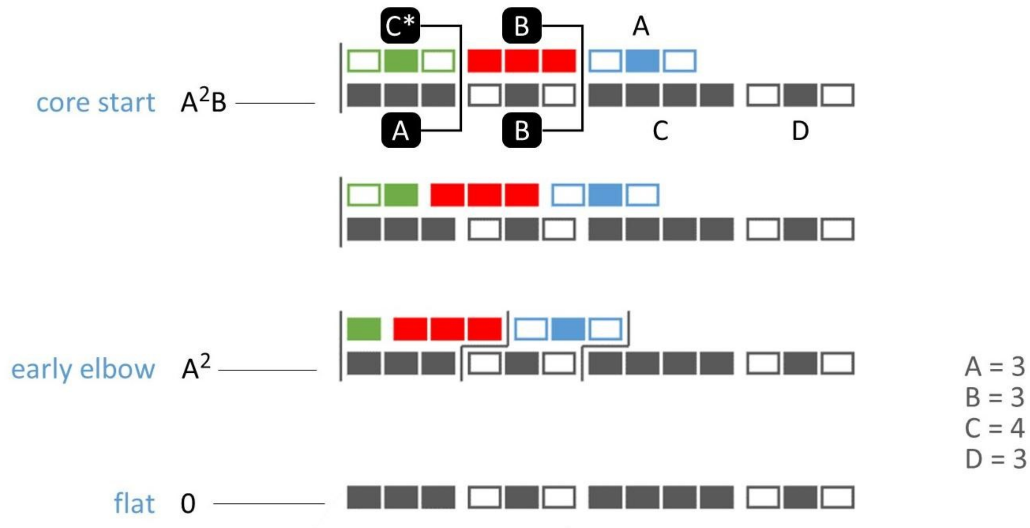
3.3 Static and live features, 1st equal width



Next, we convert the factor chains into spatial terms. And, we see that at the core start, the live array is the same width as the static period.

Also, now we're in a better position to answer the question from earlier. At the early elbow, we noted that this feature could easily be misinterpreted as the live stripe. It certainly looks a lot like the static stripe. But in the context of how it develops later, you can see that it is indeed the array – but its internal structure is not apparent, because the period and the stripe have not yet sprouted.

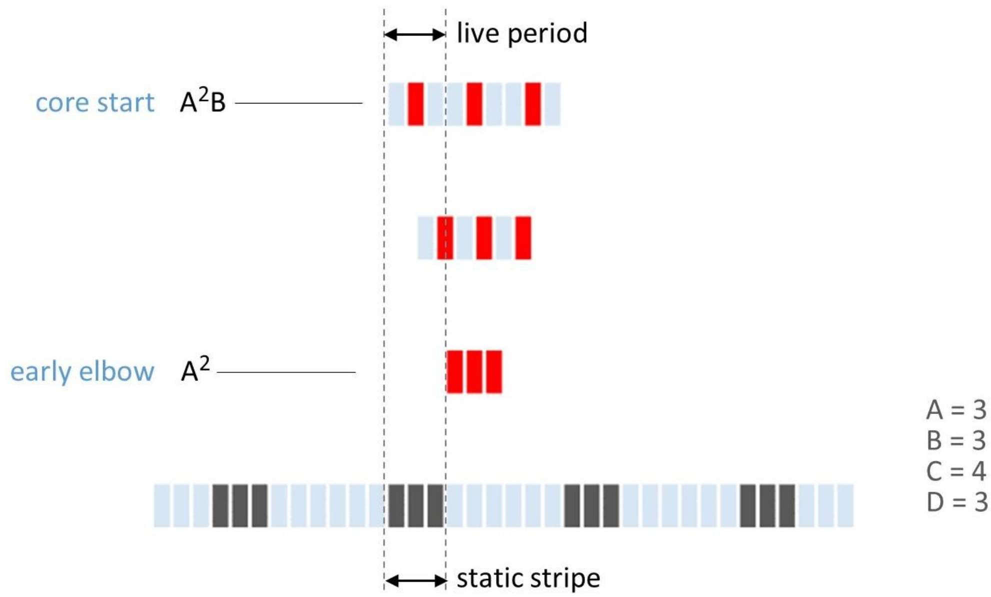
3.4 Factor chains, 2nd shared ceiling



Now let's return to the factor diagram.

Because live B and static B share a ceiling, and are also the same size, it means that they must also share a floor. And that floor is a ceiling for the next factors down the chain. Therefore, live C also shares a ceiling with static A.

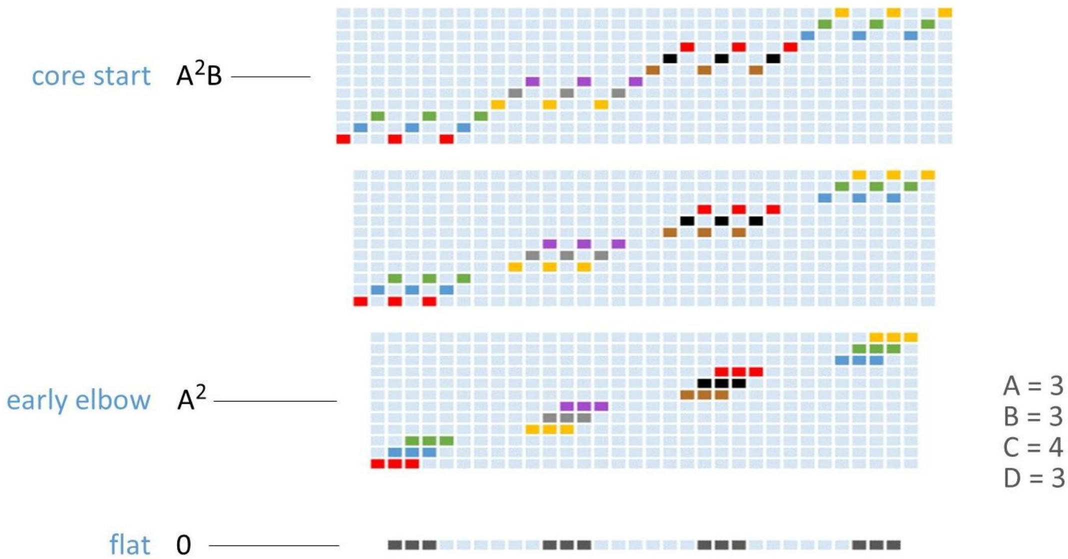
3.5 Static and live features, 2nd equal width



This also has a spatial interpretation, which is that the live period is the same width as the static stripe.

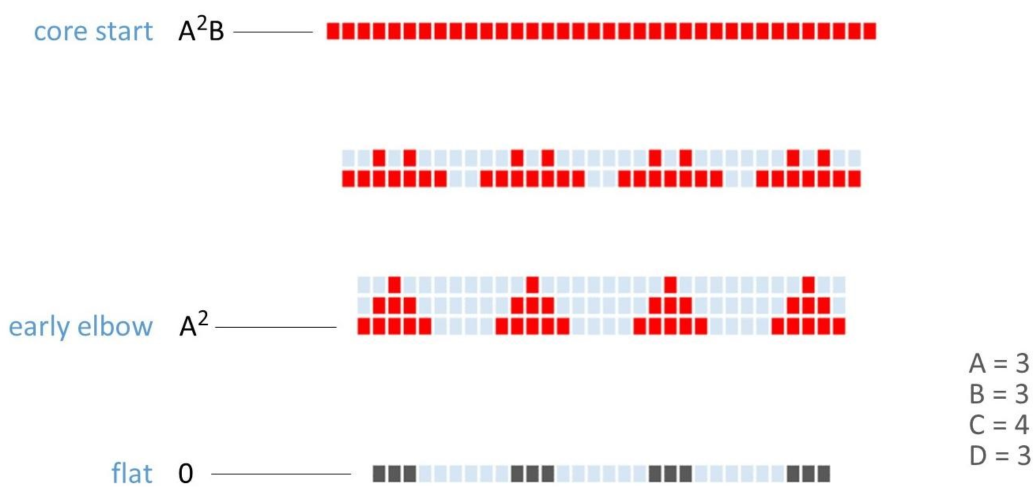
During this stage, more and more dark space appears in the live. Based on this, you might expect that the observable screen pattern would show more dark space, too.

3.6 Source-target grids



But actually, when the source-target grid is tabulated, we see that while each individual instance of live grows more and more dark space, the live from the neighboring source patches covers it up in just such a way that every target patch gets light from some source patch. So, the dark space is never actually observable in the screen pattern.

3.7 Distributions

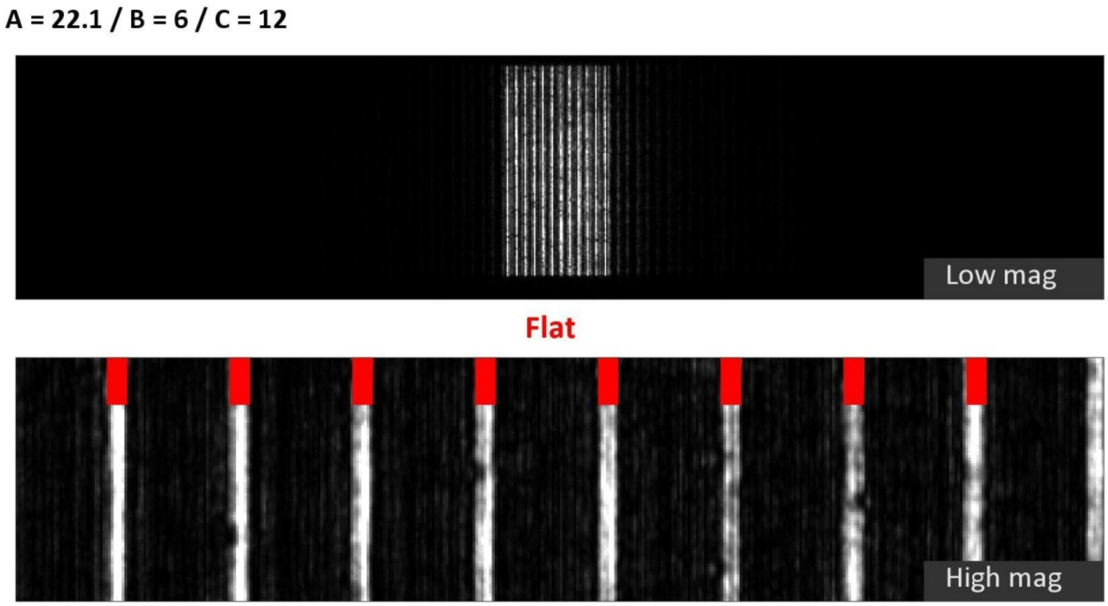


This is confirmed by looking at the distribution. The individual beams grow wider, and finally meet at the core start. Again, the example shows a generally good agreement with the laboratory data.

In this example, the roundness also decreases. But to emphasize again, this lecture is ignoring the trends in roundness, because they don't generalize well to all cases.

4 Core Start to Interval elbow

4.1 Stage trend



The next stage begins at the core start, and goes up to the interval elbow.

For reference, we'll begin by showing the pattern at the flat. The red tick marks indicate the slit locations and the slit widths.

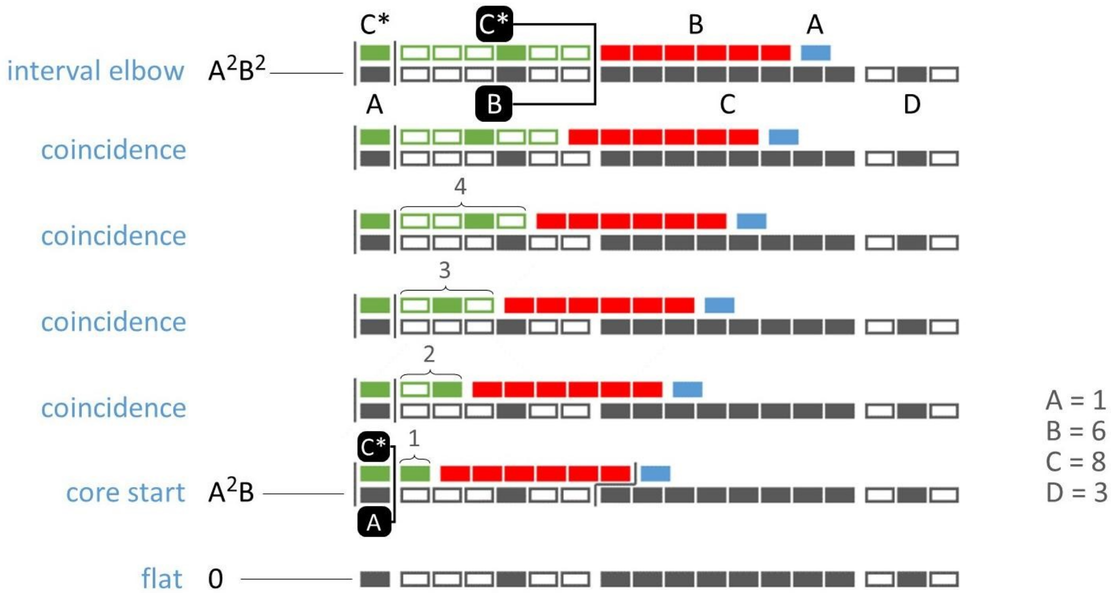
Now we'll jump to the core start, and then start moving towards the interval elbow. We see a rather complicated sequence of different patterns. The period seems to fluctuate, but the trend is not so obvious.

Eventually we reach the interval elbow, and we can see from the tick marks that the pattern is now back to the original pattern from the flat, except for an offset which we're not concerned with. The interval elbow is conventionally thought of as the first of the *Talbot images*, also called the *grating self-images*.

Let's see another example. Again, we jump from the flat to the core start, by moving one increment of A^2B. But this time, when we continue, we'll pause after every additional increment of A^2B. We'll find that we land on a special set of planes called the *coincidences*. They all have the original stripe width, but the periods fluctuate. The coincidences are a subset of what are conventionally called the *fractional Talbot images*.

Here's yet another example. This time we'll pause on just a subset of coincidences. We'll see that despite all the fluctuations, there actually is a cumulative underlying trend, in which the period is getting wider and wider.

4.2 Factor chains

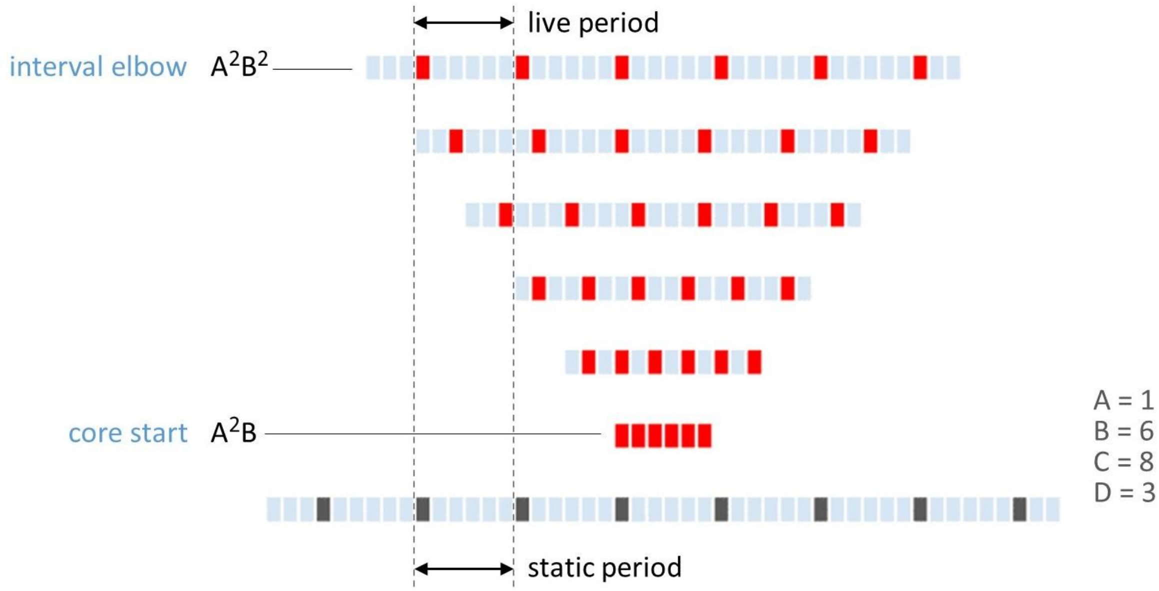


Let's see the factor chains.

At first, live C is sharing a ceiling with static A, but during this stage, it rises in rank. Every time one new point of live C rises above factor A, there is a coincidence. This occurs with every new distance increment of A^2B.

Finally, at the interval elbow, live C shares a ceiling with static B.

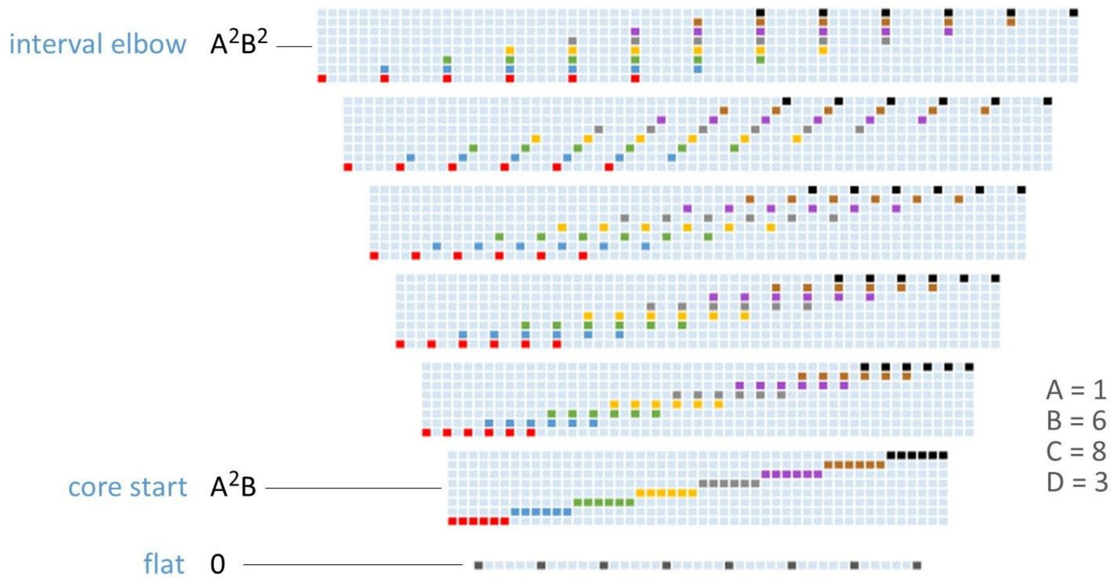
4.3 Static and live features



This is more meaningful when we convert to spatial features, and see the interpretation that the live period is the same width as the static period at the interval elbow.

Let's examine the trend in the live. The period always increases monotonically -- in this example, from 1, to 2, to 3, etc. up to B, which in this case is 6. This corresponds to the cumulative underlying trend that we saw before in the laboratory images.

4.4 Source-target grids

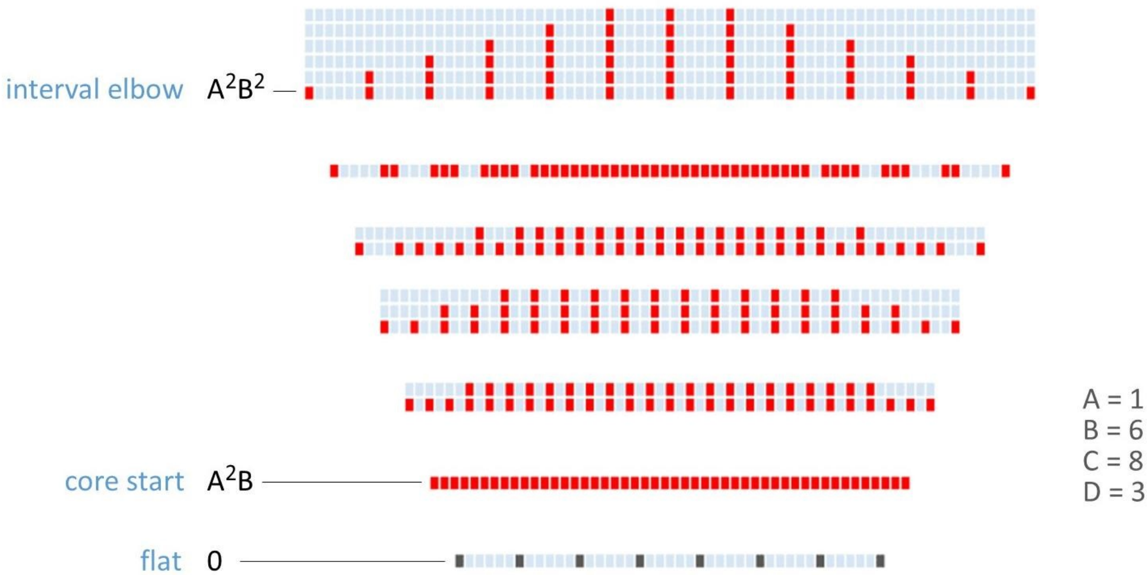


But when the various instances of live are tabulated into the Source-Target Grid, we see why the live period is not always observable in the screen pattern.

In some planes, like this one (4th from the top), the bright space from neighboring source patches tends to line up. In this case, the screen pattern will have the same period as the live. It's planes like this one that show the cumulative underlying trend.

But in other planes, like this one (2nd from the top), one source's bright space lines up with another source's dark space, and in yet other planes (3rd from the top) it's a combination of bright and dark. In these cases, the observable period does not match the underlying trend.

4.5 Distributions



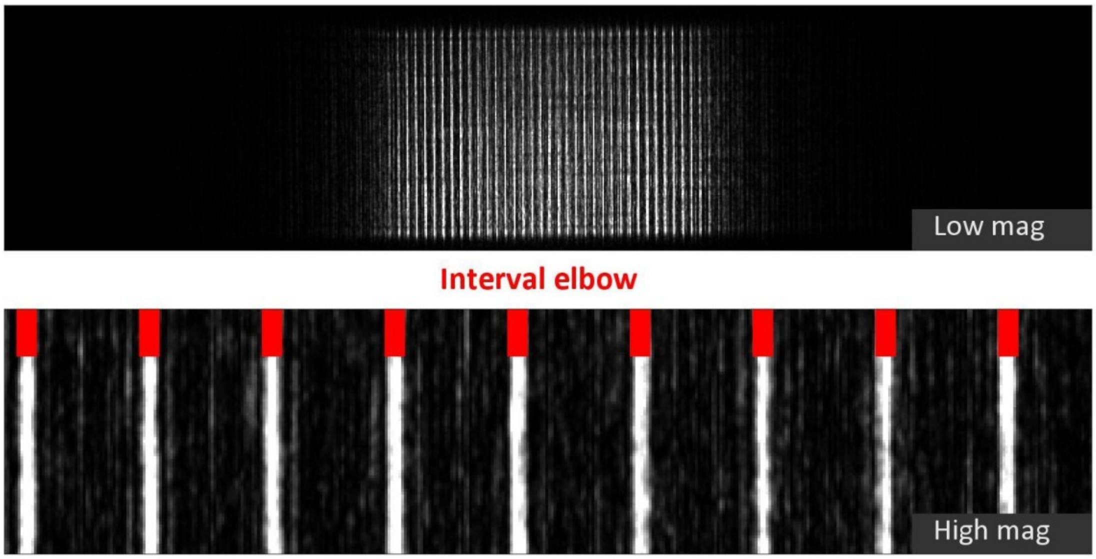
The distribution shows the pattern of coincidences. The period oscillates – 1, 2, 3, 2, 1, 6. Four of those – 1, 2, 3, and 6 – embody the underlying trend. The others do not.

Regardless of the period, the stripe width remains constant.

5 Interval Elbow to Breakout

5.1 Stage trend

A = 22.1 / B = 6 / C = 36



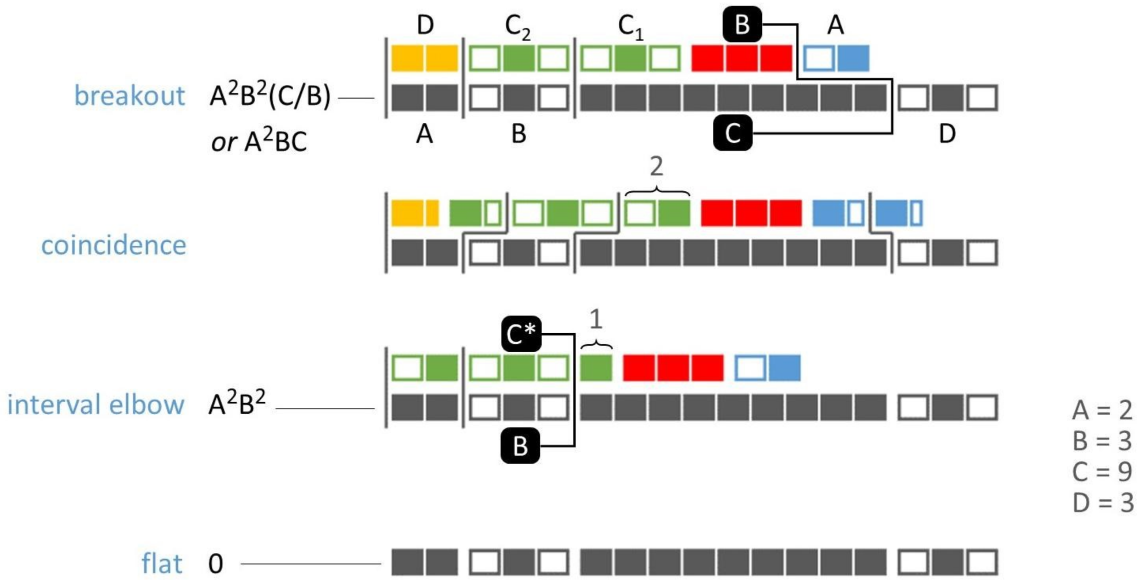
The next stage begins at the interval elbow, and goes up to breakout.

This stage is a series of cycles. We begin at the distance A^2B^2, and each cycle takes another incremental step of A^2B^2. Each individual cycle is like a repetition of the entire progression so far, from the flat, through the early elbow and the core start, to the interval elbow. Each cycle ends at a coincidence that looks just like the original pattern at the flat. The last of these is breakout. Conventionally, these are considered the Talbot images.

It may appear that the light doesn't actually make any progress in this stage, because it always returns to its starting point. In fact, there is a cumulative trend, but it is a change in roundness, and so we can't observe it in the screen pattern.

Here's another example.

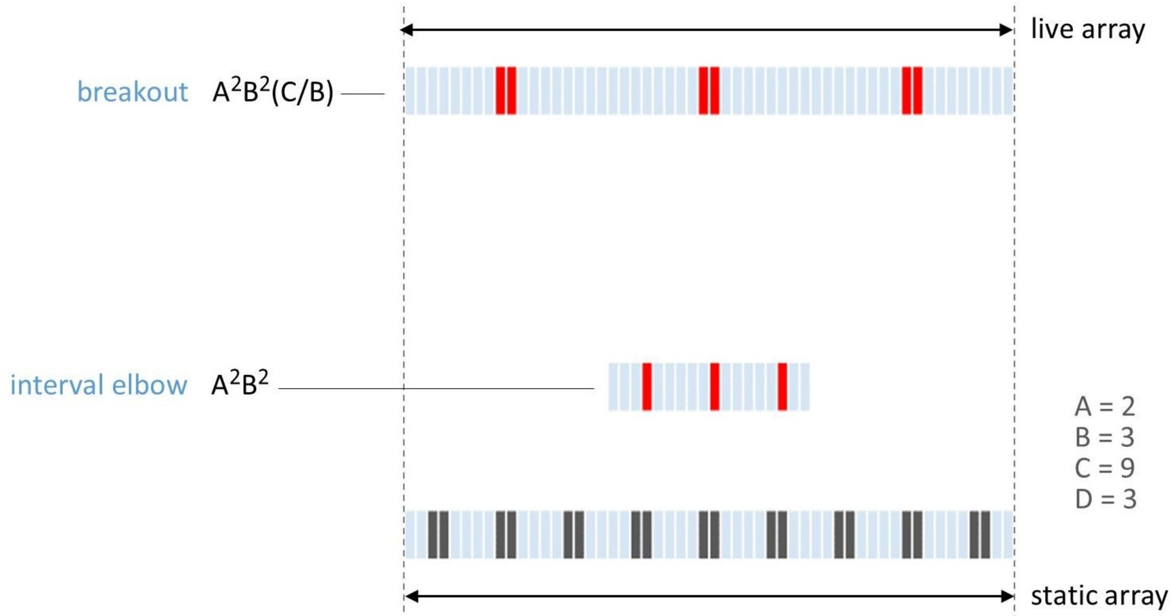
5.2 Factor chains, 1st shared ceiling



First, we'll look at the trend in terms of factor chains.

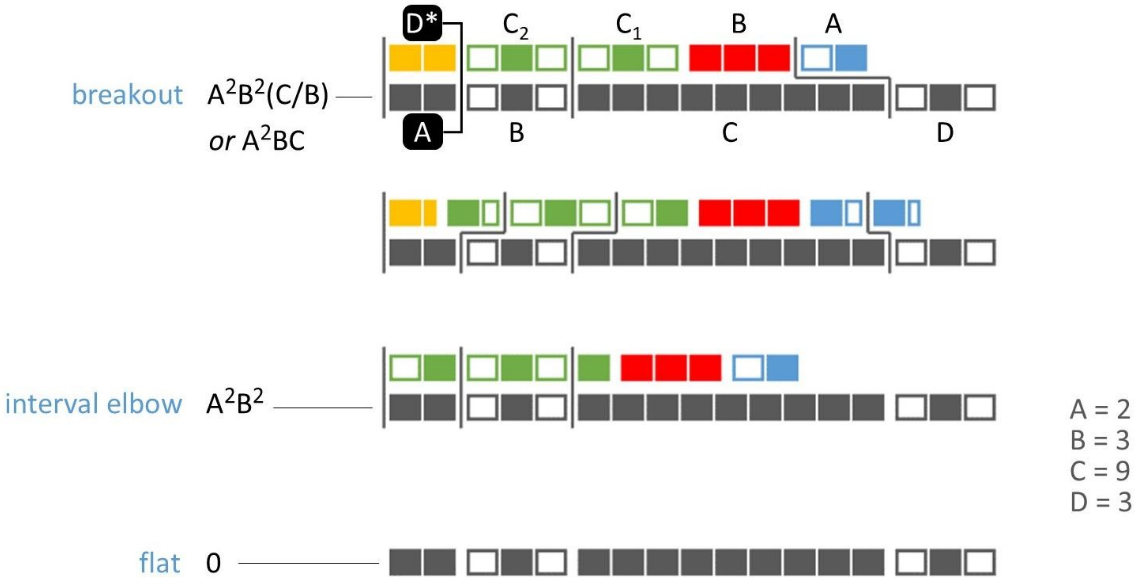
At first, live C shares a ceiling with static B. But, with every new Z increment of A^2B^2, one new point of live C rises above static B. Each time this occurs, there is a coincidence. Eventually, at breakout, live B shares a ceiling with static C.

5.3 Static and live features, 1st equal width



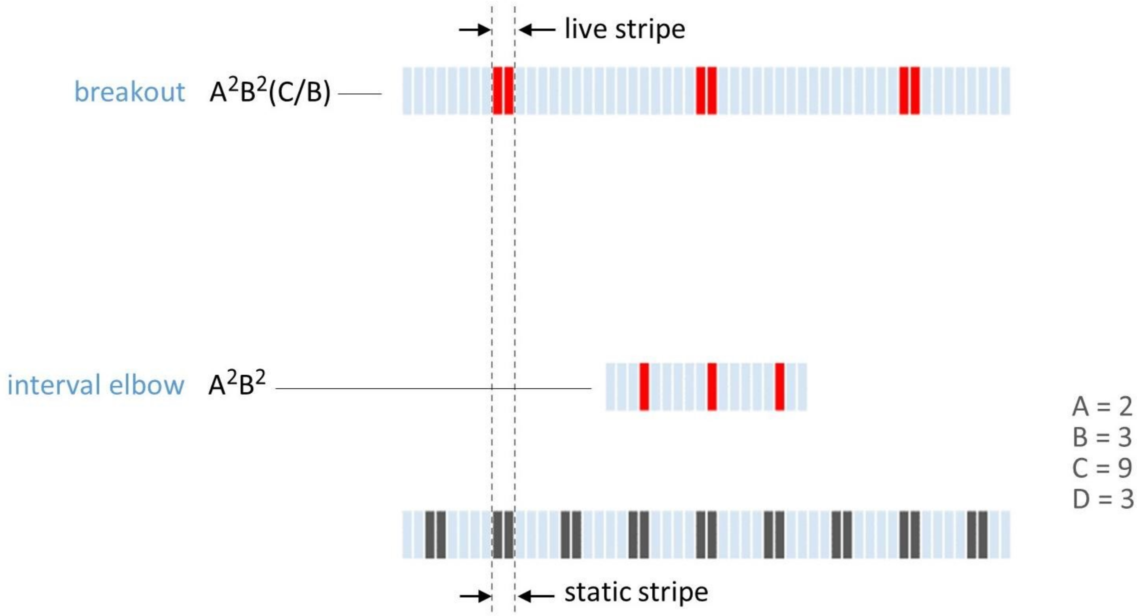
This has an equivalent in terms of feature sizes; namely, the live array is the same width as the static array.

5.4 Factor chains, 2nd shared ceiling



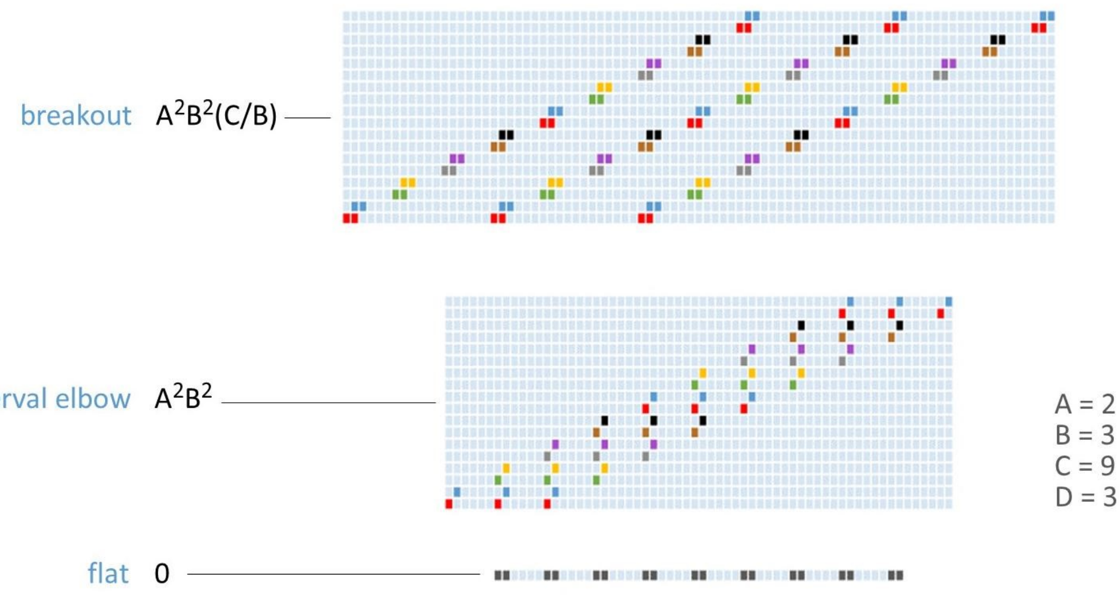
But also, if we return to the factor diagram, there is a second shared ceiling at breakout, where live D shares a ceiling with static A.

5.5 Static and live features, 2^{nd} equal width



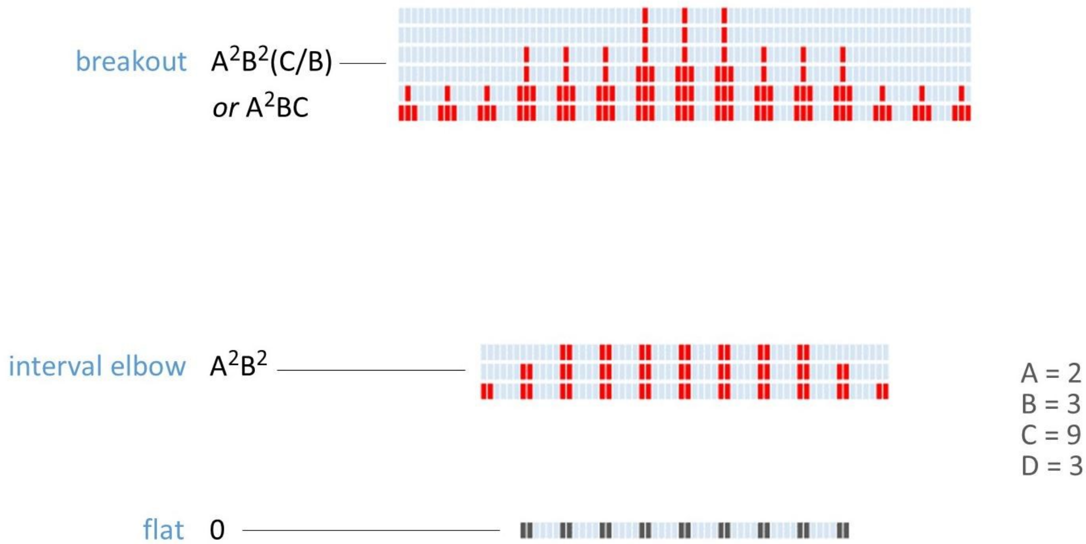
When we convert this to spatial terms, we find that the live stripe is the same width as the static stripe.

5.6 Source-target grids



We tabulate the source-target grid, and finally…

5.7 Distributions

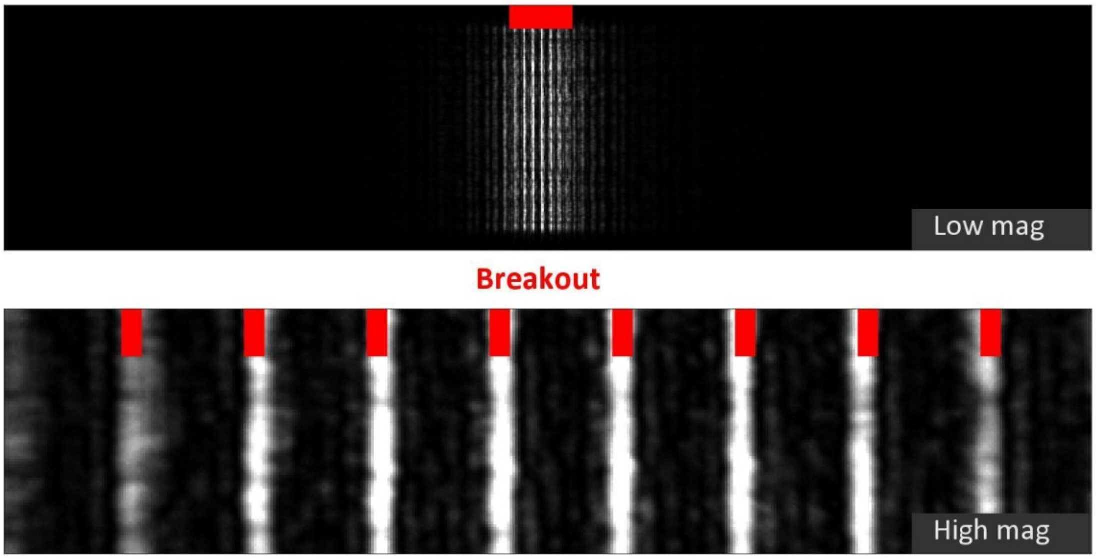


… collapse it to get the distribution, which again, matches the laboratory images to a decent approximation.

6 Breakout to Core End

6.1 Stage trend

A = 22.1 / B = 6 / C = 6



The next stage begins at breakout, and goes up to the core end.

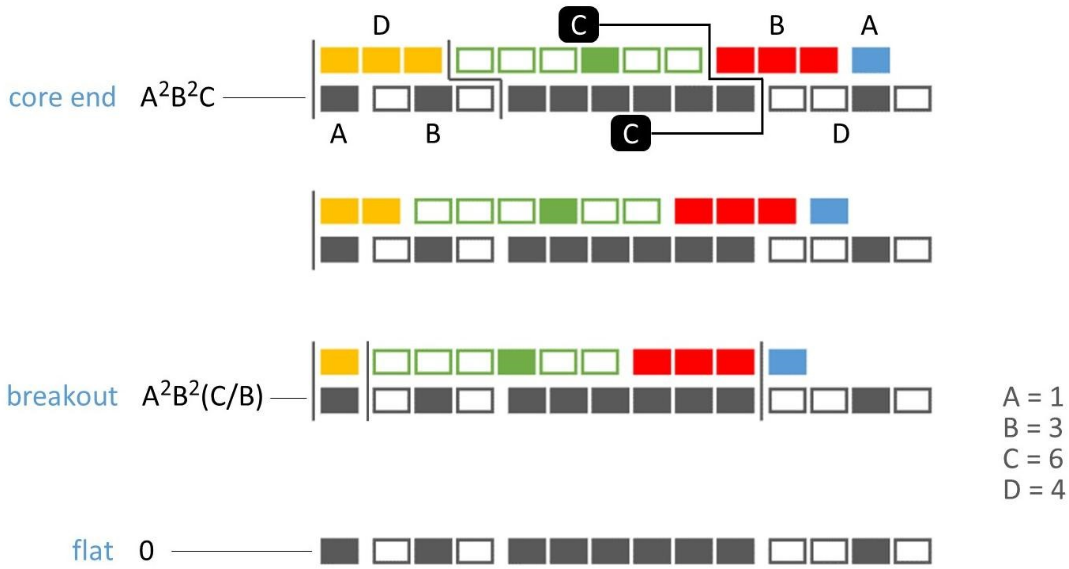
Now that we're getting to later and later stages, the distances involved are getting longer and longer. This camera has a very limited travel range, so it becomes more and more challenging to show good examples, but we'll do the best we can.

The red bar on top indicates the array size at breakout. Up to now, we haven't paid much attention to the array, because it's been staying roughly the same, but now it will start to change. We'll also see trends at both magnifications.

Initially, the stripe is much narrower than the period. But, while the period always cycles back to a constant, the stripe grows continually wider. Eventually, the stripe fills the entire period, and the pattern becomes a solid continuous bright region. Meanwhile, on the large scale, the array has grown much wider.

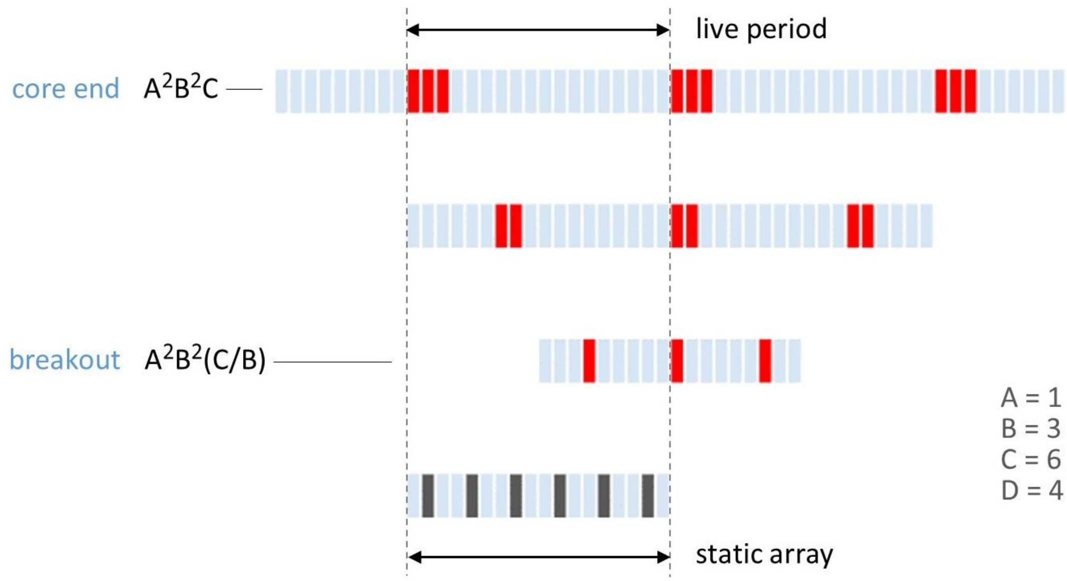
Let's see another example.

6.2 Factor chains, 1st shared ceiling



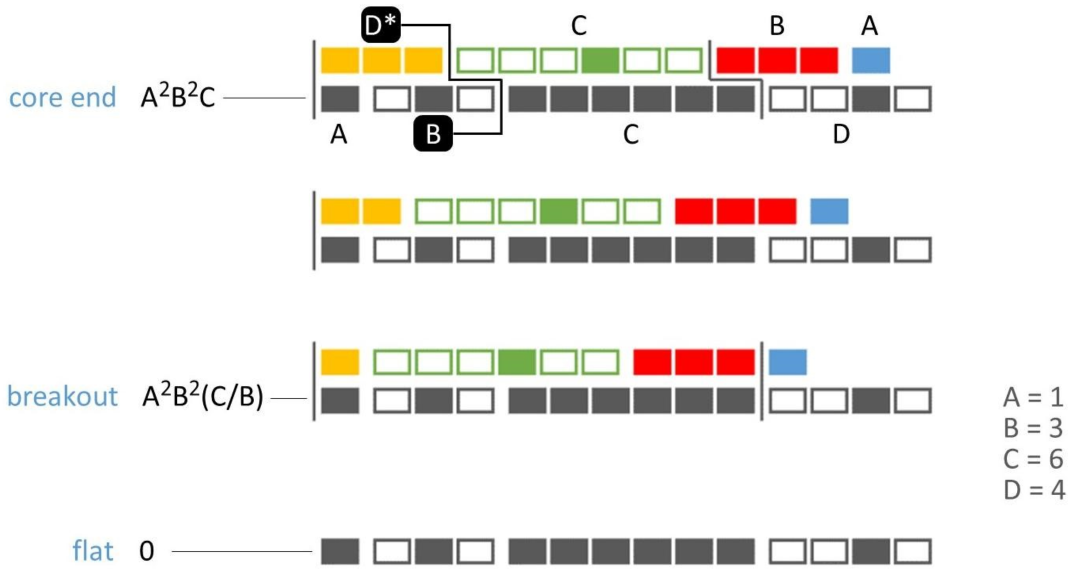
In terms of factors, the live chain continues to rise to the core end, where there are two shared ceilings. First, live C shares a ceiling with static C.

6.3 Static and live features, 1st equal width



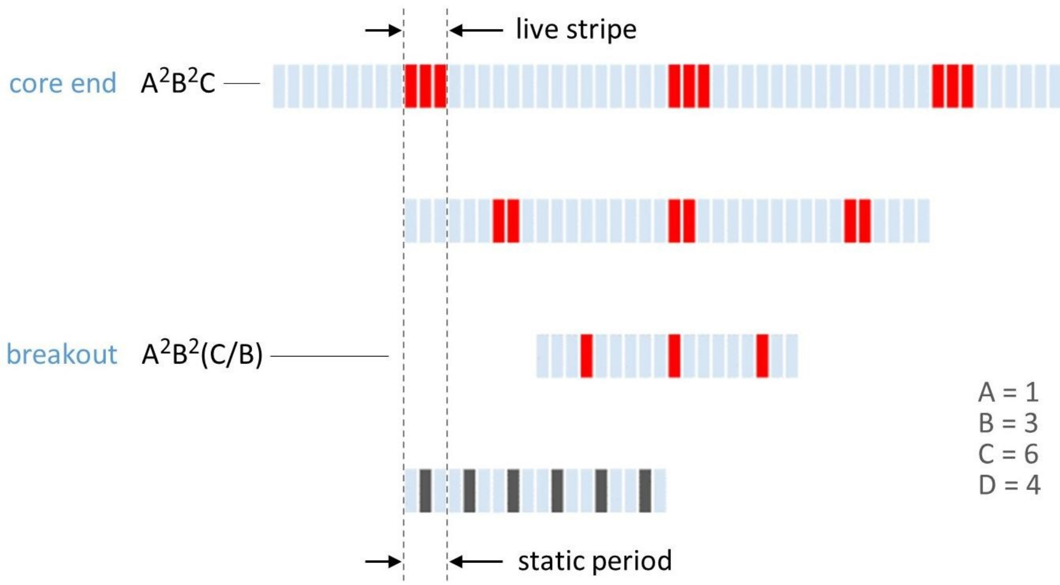
In spatial terms, this means the live period is as wide as the static array.

6.4 Factor chains, 2nd shared ceiling



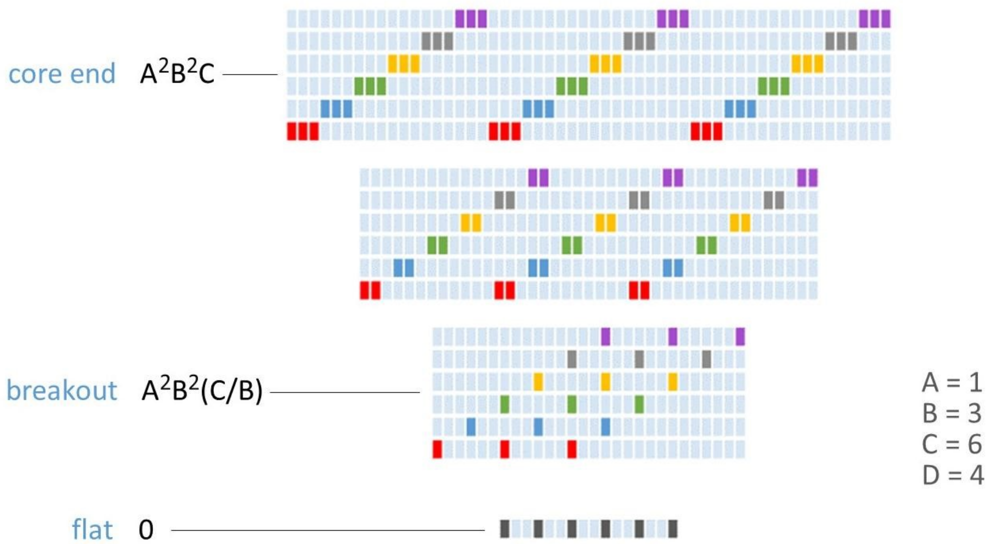
And second, live D shares a ceiling with static B.

6.5 Static and live features, 2nd equal width



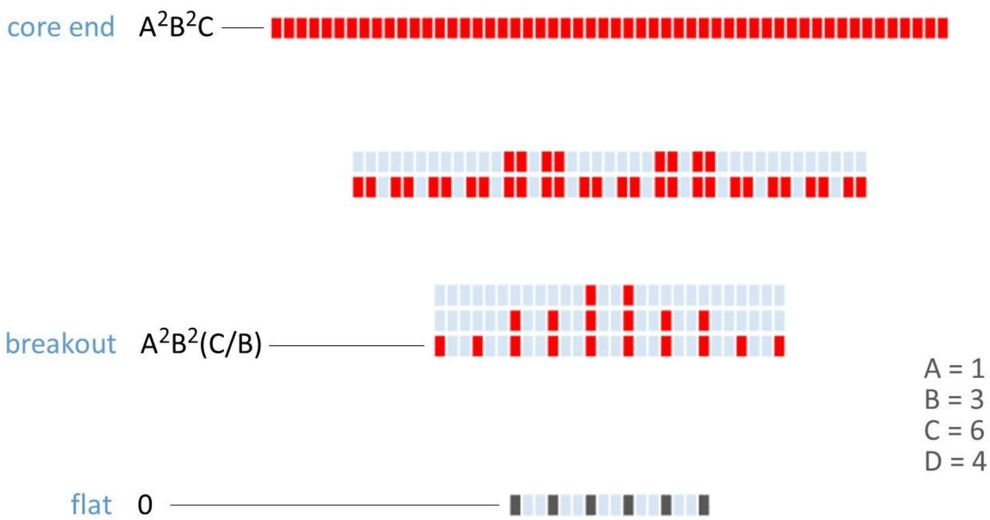
In spatial terms, this corresponds to the live stripe being as wide as the static period

6.6 Source-target grids



In the source-target grid, you can see the growing live stripe filling in all the dark space caused by the original space between slits.

6.7 Distributions



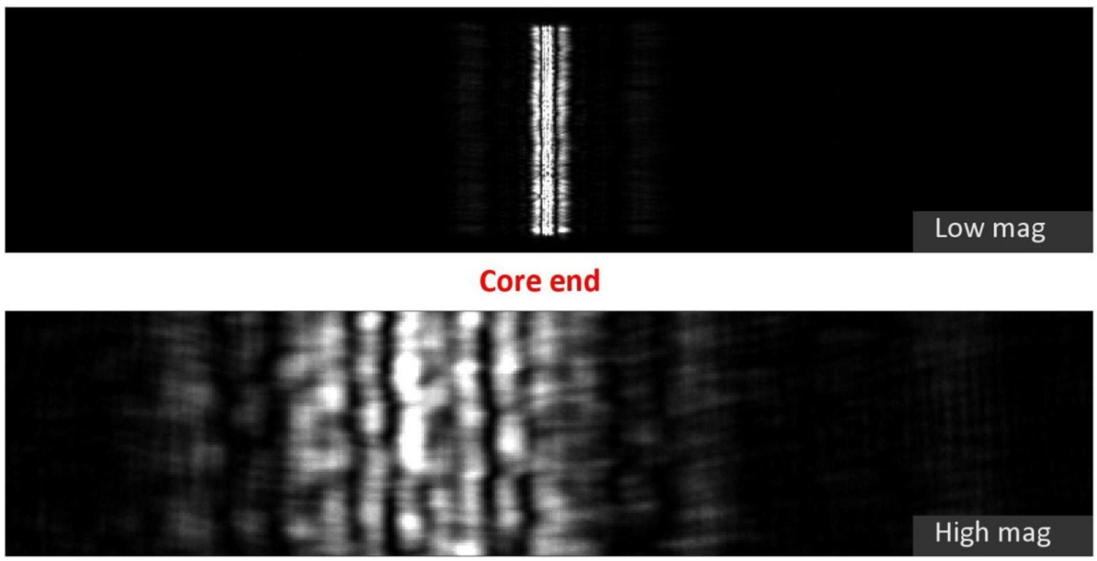
And the distribution confirms the trends from the laboratory images.

Breakout is the final plane that resembles the flat. Then, the stripe grows B times as wide to fill the entire width of the period, and the array also grows to be B times as wide as the array at the flat.

7 Core End to Late Elbow

7.1 Stage trend

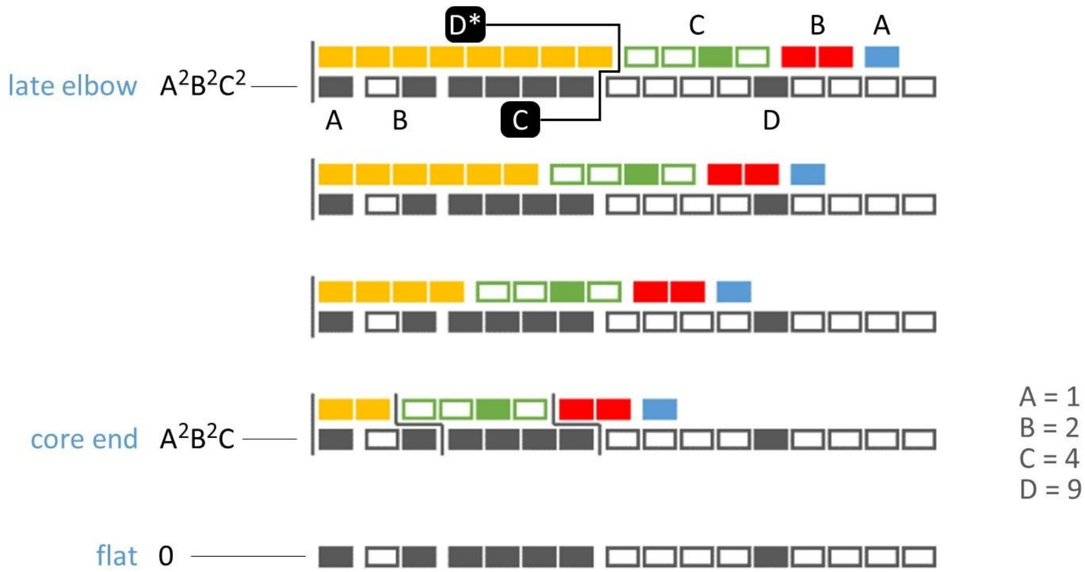
A = 22.1 / B = 2 / C = 4



The next stage begins at the core end, and goes up the late elbow

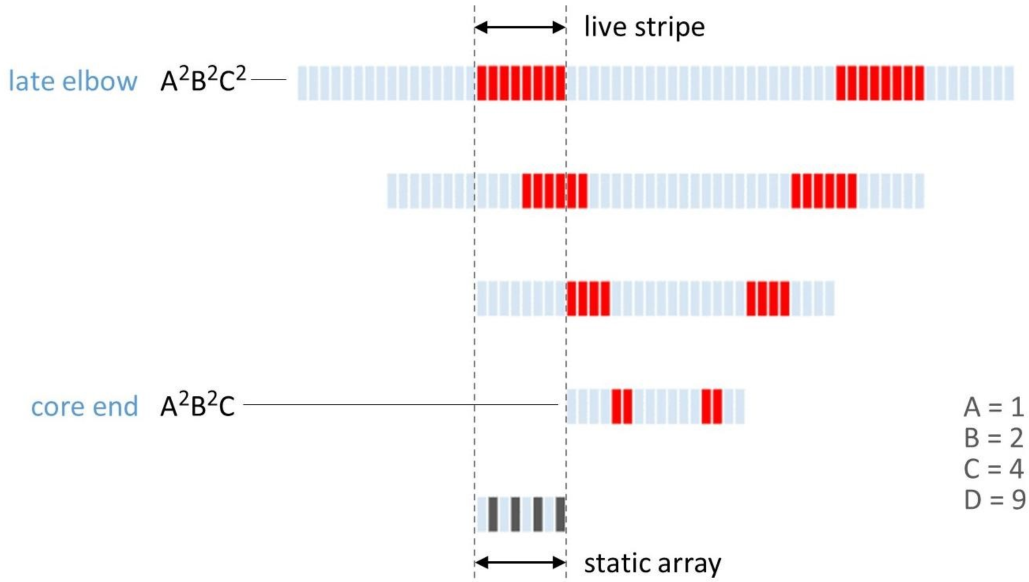
At the core end, the pattern is one continuous bright region. During this stage, that bright region separates into individual beams, which move apart from one another. But while dark space grows between beams, each individual beam stays at a constant width.

7.2 Factor chains



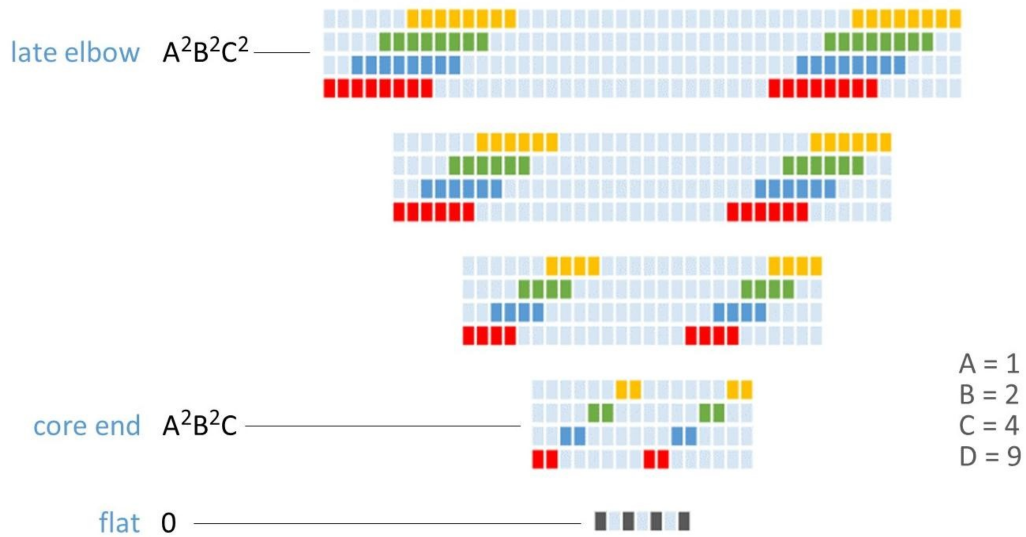
In terms of factor chains, the late elbow occurs when live D shares a ceiling with static C.

7.3 Static and live features



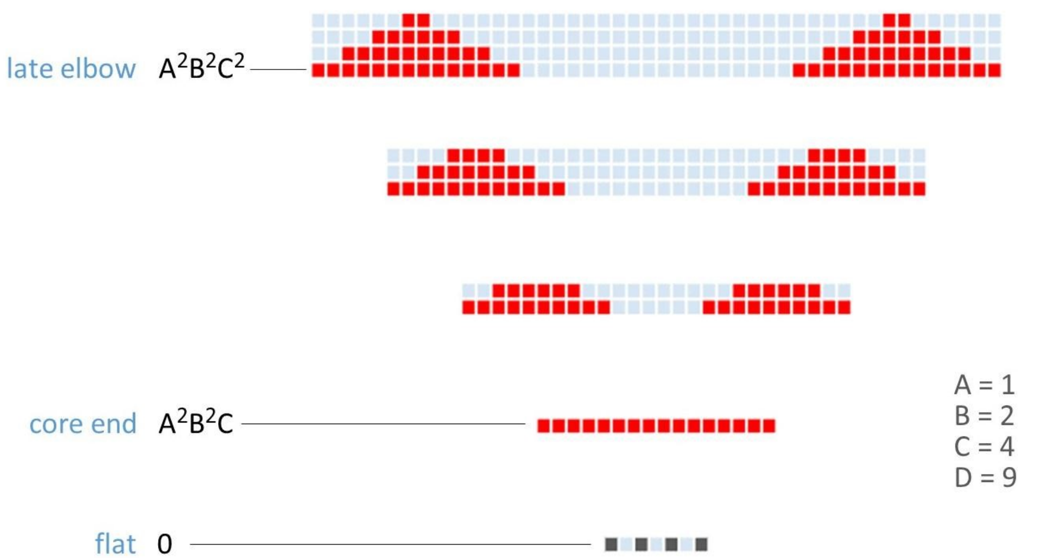
In terms of spatial features, this corresponds to the live stripe being as wide as the static array.

7.4 Source-target grids



In the Source-Target Grid, the bright space from one source patch overlaps with the bright space from other source patches.

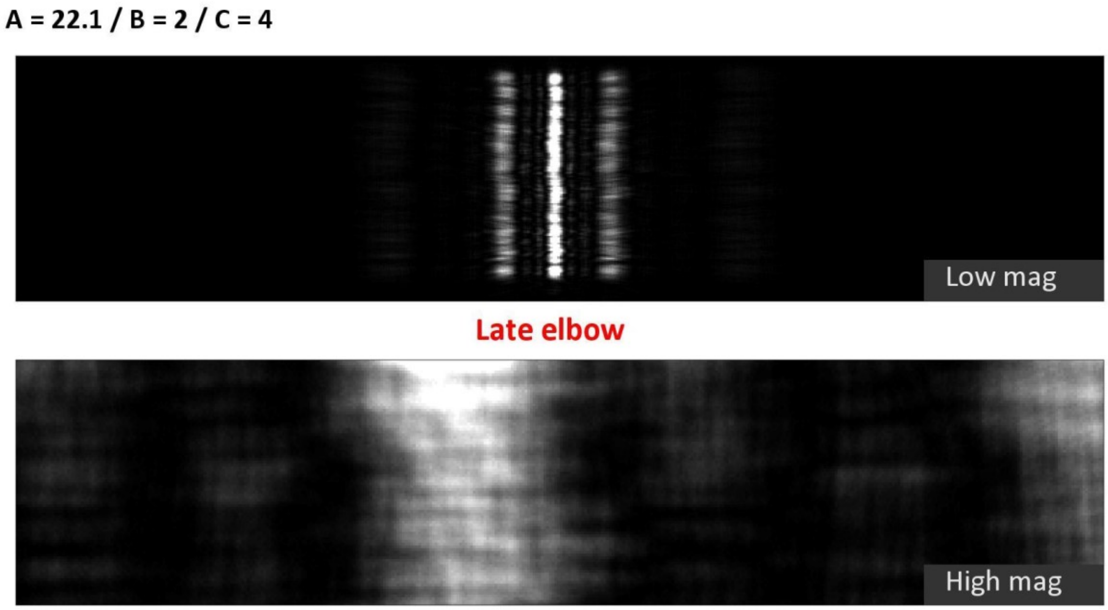
7.5 Distributions



And the distribution shows the result, that the individual beams separate from one another, but with minimal widening.

8 Late Elbow to Far Limit

8.1 Stage trend

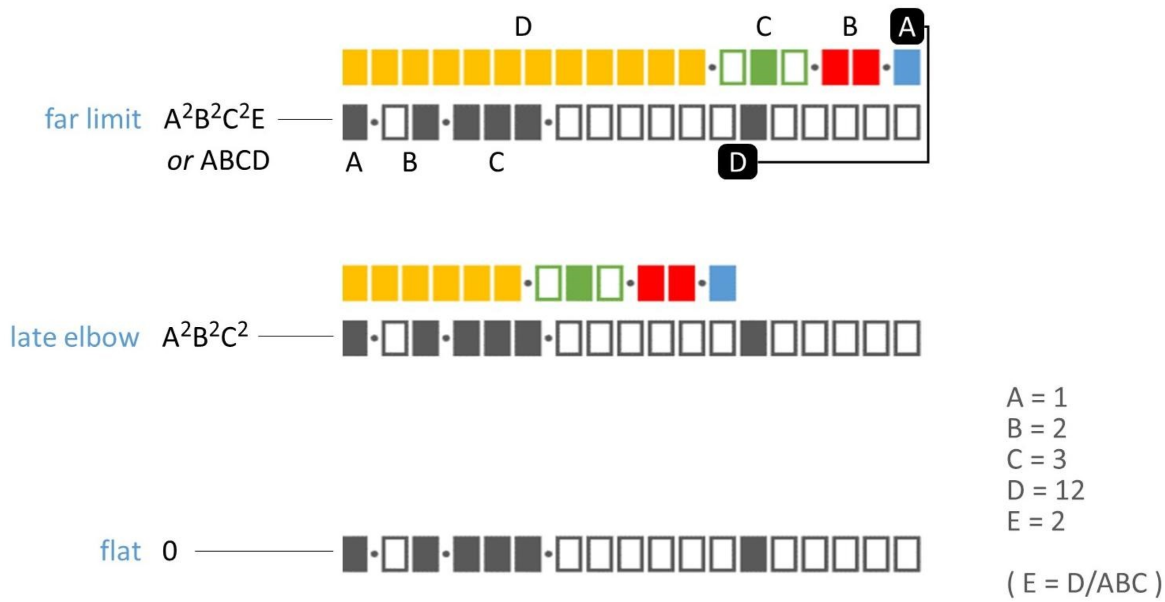


The very final stage of many-slit interference begins at the late elbow, and continues to the far limit. This final stage corresponds to the far field, or the Fraunhofer region, in conventional optics.

Here, the pattern has reached a steady state. It keeps the same shape, except for a simple uniform expansion. So for instance, in this case, at the late elbow, the stripe is roughly one-fourth of the period. Then, the entire pattern grows wider, but the ratio of stripe to period remains at about 1-to-4, and the number of stripes also stays constant.

This can continue indefinitely, as long as the light keeps propagating.

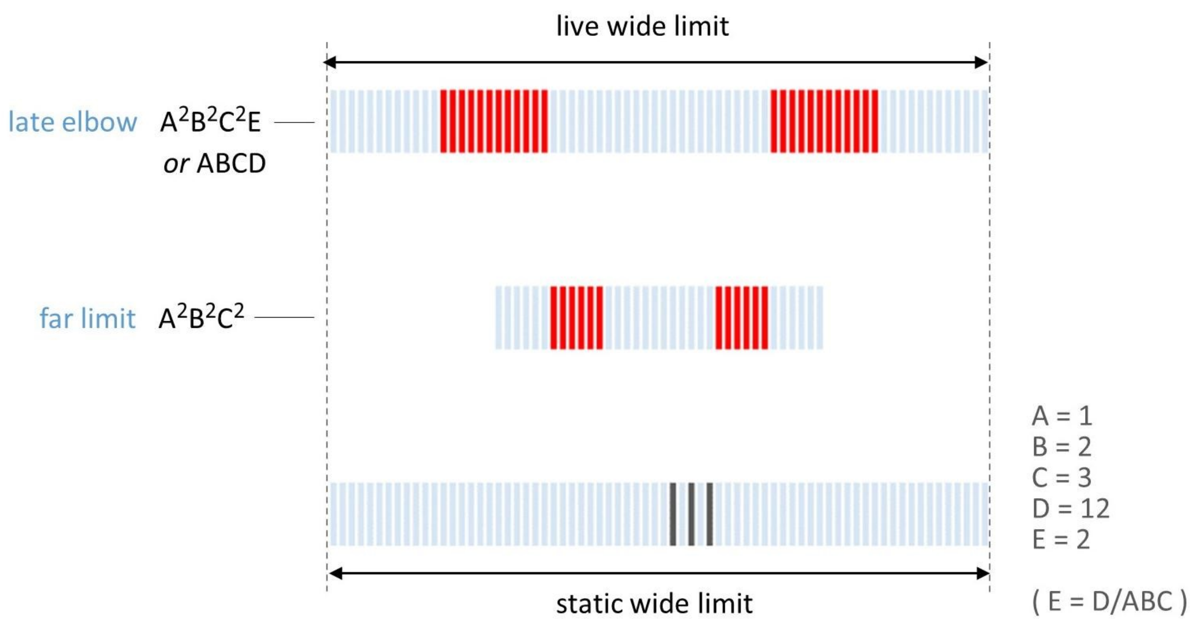
8.2 Factor chains



The additional distance past the late elbow can be referred to as factor E, for expansion. But it's not really a new parameter, because E is just equal to D / ABC.

The live chain reaches its full size at the far limit, where live A shares a ceiling with static D.

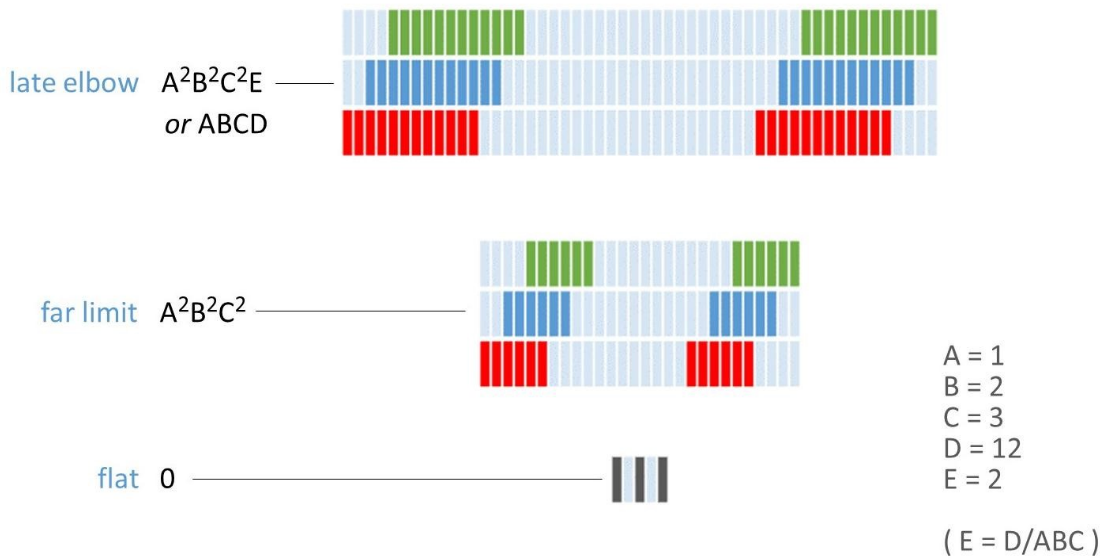
8.3 Static and live features



In spatial terms, the live wide limit is the same size as the static wide limit.

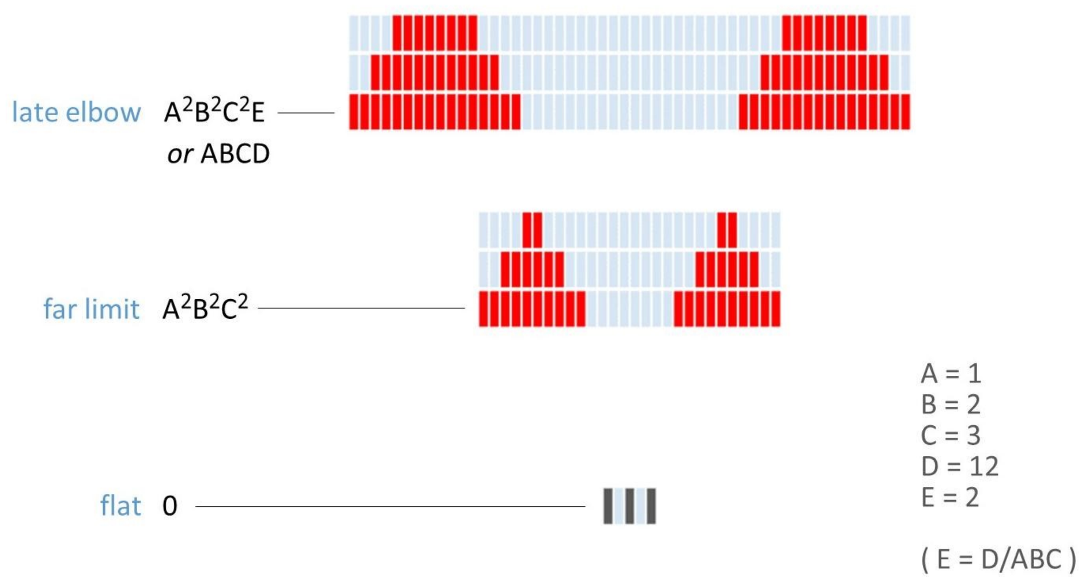
Note that in this diagram, we have *not* trimmed off the excess dark space, as we had been doing.

8.4 Source-target grids



The source-target grid shows that all the instances of live reinforce one another. The offset from one source patch to the next has relatively little impact, because it's smaller than the smallest feature of the live.

8.5 Distributions



And the distribution shows a uniform expansion of the pattern, without distortion of the shape.

We've now gone through all the stages, and analyzed all of the critical planes.

9 Conclusions

9.1 Almost-critical early planes

<table>
<tr><th>Plane</th><th>Z</th><th colspan="2">Increase</th></tr>
<tr><td>far limit</td><td>$A^2 \cdot B^2 \cdot C^2 \cdot E$ *or*
$A \cdot B \cdot C \cdot D$</td><td colspan="2">E</td></tr>
<tr><td>late elbow</td><td>$A^2 \cdot B^2 \cdot C^2$</td><td colspan="2">C</td></tr>
<tr><td>core end</td><td>$A^2 \cdot B^2 \cdot C$</td><td rowspan="2">C</td><td>B</td></tr>
<tr><td>breakout</td><td>$A^2 \cdot B^2 \cdot (C/B)$</td><td>C/B</td></tr>
<tr><td>interval elbow</td><td>$A^2 \cdot B^2$</td><td colspan="2">B</td></tr>
<tr><td>core start</td><td>$A^2 \cdot B$</td><td colspan="2">B</td></tr>
<tr><td>early elbow</td><td>A^2</td><td colspan="2">A</td></tr>
<tr><td>-</td><td>A</td><td colspan="2">A</td></tr>
<tr><td>-</td><td>1</td><td colspan="2">-</td></tr>
<tr><td>flat</td><td>0</td><td colspan="2">-</td></tr>
</table>

One final point: the distances of the critical planes follow a very clear trend, where each distance is some multiple of the previous distance. So, for example, to get from the core start at A^2B to the interval elbow at A^2B^2, there is an increase factor of B. But this trend breaks in the other direction, because the early elbow distance, at A^2 is not any multiple of 0.

So, in a sense, there are two other critical planes that we haven't discussed: the planes at Z = A, at Z = 1. These planes don't have much physical significance, so we haven't discussed them, and we haven't given them any names. But they help us to see two things:

First, if we ignore breakout, the increase factors go A-A-B-B-C-C-E. In the next lecture, we'll discuss why breakout is different from the other planes.

And secondly, while we can extrapolate the trend all the way back to one wavelength from the flat, the trend abruptly ends there. Surprisingly, it never actually gets all the way to the flat.

9.2 Reviewing key points

That's all for this lecture, so let's review the key points:

- Symmetry optics describes many-slit interference in terms of critical planes.
- At every critical plane, at least one live factor shares a ceiling with a static factor.
- When a shared ceiling is translated into spatial terms, it means that some live feature is as wide as some static feature.
- When the distributions are calculated, each stage has a specific, distinctive trend.

9.3 Outro

Wherever you found this video, you'll also find links to more resources for learning symmetry optics.

More than any other lecture in this series, this one really needs to be studied carefully in the form of the lecture notes. This will take some mental effort, but it will let you appreciate the power and the elegance of the symmetry-optical model.

Also, check out the companion code. Particularly if you find that you are at all skeptical about the trends we summarized here, you may need to try more examples, and to try larger examples, in order to be convinced. When you use larger factor sizes, the diagrams become a little bit unwieldy, but the trends are more obvious.

Finally, try the problem set to test your understanding.

In the next lecture, we'll learn about the continuous factor model, which is an alternative way of computing patterns in symmetry optics. So please join me.

I'm Paul Mirsky; thanks for listening.