# An intriguing integral

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Abstract

We study the integral: $I = \int_0^1 \frac{\tan^{-1} x}{(1+x)\sqrt{x(1-x)}}\, dx$

## Introduction

Recall that

$$I = \int_0^1 \frac{\tan^{-1} x}{(1+x)\sqrt{x(1-x)}}\, dx = \frac{\pi}{\sqrt{2}} \tan^{-1}\left( \frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}} \right) \tag{1}$$

$$I = \frac{\pi^2}{4\sqrt{2}} - \frac{\pi}{\sqrt{2}} \tan^{-1}\left( \sqrt{\frac{\sqrt{2}-1}{2}} \right) = \frac{\pi^2}{4\sqrt{2}} - \frac{\pi}{\sqrt{2}} \sin^{-1}\left(\sqrt{2}-1\right) \tag{2}$$

where

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) \tag{3}$$

The Gauss Hypergeometric function is defined by

$$F(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n \ , \ |x| < 1 \tag{4}$$

The Appell hypergeometric function $F_1$ is defined by

$$F_1(a, b, c, d, x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{n+m} (b)_n (c)_m}{(d)_{n+m}\, n!\, m!} x^n y^m \ , \ |x| < 1, |y| < 1 \tag{5}$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ ; $(a)_0 = 1$, See Olver et al., [3].

# Some formulas

Entry 1. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^n}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}2^{-k}\right)=$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+k}2^{-2m}a^{2n+k+m+1}}{(2n+1)(4n+2k+2m+3)}\binom{2m}{m}-$$
$$\frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\sum_{m=0}^{n-k}2^{-m}\binom{2m}{m}\binom{n-m+1}{n-2k-m} \tag{6}$$

Entry 2. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^n}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}2^{-k}\right)=$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+1}}{2n+3}F\left(\frac{1}{2},n+\frac{3}{2},n+\frac{5}{2},a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}-$$
$$\frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\sum_{m=0}^{n-k}2^{-m}\binom{2m}{m}\binom{n-m+1}{n-2k-m} \tag{7}$$

Entry 3. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^n}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}2^{-k}\right)=$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\frac{2^{-2n}a^{n+1}}{2n+3}F\left(1,n+\frac{3}{2},n+\frac{5}{2},-a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k 2^{4k}}{2k+1}\binom{2n-4k}{n-2k}-$$
$$\frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\sum_{m=0}^{n-k}2^{-m}\binom{2m}{m}\binom{n-m+1}{n-2k-m} \tag{8}$$

Entry 4. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^n}{2n+1}F\left(\frac{1}{2},n+\frac{1}{2},n+\frac{3}{2},1-a\right)\right)=$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+1}}{2n+3}F\left(\frac{1}{2},n+\frac{3}{2},n+\frac{5}{2},a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}- \tag{9}$$
$$\frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\sum_{m=0}^{n-k}2^{-m}\binom{2m}{m}\binom{n-m+1}{n-2k-m}$$

Entry 5. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\right.$$
$$\left.\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{2^{-2n}(1-a)^n}{2n+1}\binom{2n}{n}F\left(1,n+\frac{1}{2},n+\frac{3}{2},\frac{1-a}{2}\right)\right)=$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+1}}{2n+3}F\left(\frac{1}{2},n+\frac{3}{2},n+\frac{5}{2},a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}- \tag{10}$$
$$\frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\sum_{m=0}^{n-k}2^{-m}\binom{2m}{m}\binom{n-m+1}{n-2k-m}$$

Entry 6. If $0 < a < 1$, then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\right.$$
$$\left.\frac{\sqrt{1-a}}{4}\sum_{n=0}^{\infty}\frac{(2-2^{-n})(1-a)^n}{2n+1}F\left(-\frac{1}{2},n+\frac{1}{2},n+\frac{3}{2},1-a\right)\right)= \tag{11}$$
$$2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+1}}{2n+3}F\left(\frac{1}{2},n+\frac{3}{2},n+\frac{5}{2},a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}-$$

$$\frac{\sqrt{1-a}}{2} \sum_{n=0}^{\infty} \frac{2^{-n} (1-a)^{n+1}}{2n+3} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \sum_{m=0}^{n-k} 2^{-m} \binom{2m}{m} \binom{n-m+1}{n-2k-m}$$

Entry 7. If $0 < a < 1$, then

$$\pi \left( \frac{1}{\sqrt{2}} \tan^{-1}\left( \frac{\sqrt{2} - \sqrt{\sqrt{2} - 1}}{\sqrt{2} + \sqrt{\sqrt{2} - 1}} \right) - \frac{\sqrt{1-a}}{4} F_1\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1-a}{2}, 1-a\right) \right) =$$

$$2\sqrt{a} \sum_{n=0}^{\infty} \frac{(-1)^n a^{n+1}}{2n+3} F\left(\frac{1}{2}, n+\frac{3}{2}, n+\frac{5}{2}, a\right) \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} - \tag{12}$$

$$\frac{\sqrt{1-a}}{2} \sum_{n=0}^{\infty} \frac{2^{-n} (1-a)^{n+1}}{2n+3} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \sum_{m=0}^{n-k} 2^{-m} \binom{2m}{m} \binom{n-m+1}{n-2k-m}$$

Entry 8. If $0 < a < 1$, then

$$\pi \left( \frac{1}{\sqrt{2}} \tan^{-1}\left( \frac{\sqrt{2} - \sqrt{\sqrt{2} - 1}}{\sqrt{2} + \sqrt{\sqrt{2} - 1}} \right) - \frac{\sqrt{1-a}}{4} F_1\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1-a}{2}, 1-a\right) \right) =$$

$$2\sqrt{a} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)(4n+3)} F_1\left(2n+\frac{3}{2}, 1, \frac{1}{2}, 2n+\frac{5}{2}, -a, a\right) - \tag{13}$$

$$\frac{\sqrt{1-a}}{2} \sum_{n=0}^{\infty} \frac{2^{-n} (1-a)^{n+1}}{2n+3} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \sum_{m=0}^{n-k} 2^{-m} \binom{2m}{m} \binom{n-m+1}{n-2k-m}$$

Entry 9. If $0 < a < 1$, then

$$\pi \left( \frac{1}{\sqrt{2}} \tan^{-1}\left( \frac{\sqrt{2} - \sqrt{\sqrt{2} - 1}}{\sqrt{2} + \sqrt{\sqrt{2} - 1}} \right) - \frac{\sqrt{1-a}}{4} F_1\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1-a}{2}, 1-a\right) \right) =$$

$$2\sqrt{a} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)(4n+3)} F_1\left(2n+\frac{3}{2}, 1, \frac{1}{2}, 2n+\frac{5}{2}, -a, a\right) - \tag{14}$$

$$\sqrt{1-a} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n-1} (1-a)^{2n+1}}{(2n+1)(4n+3)} F_1\left(2n+\frac{3}{2}, 2n+2, \frac{1}{2}, 2n+\frac{5}{2}, \frac{1-a}{2}, 1-a\right)$$

Entry 10. If $0 < a < 1$ , then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}F_1\left(\frac{1}{2},1,\frac{1}{2},\frac{3}{2},\frac{1-a}{2},1-a\right)\right)= 2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)(4n+3)}F_1\left(2n+\frac{3}{2},1,\frac{1}{2},2n+\frac{5}{2},-a,a\right)-\frac{\sqrt{1-a}}{2} \sum_{n=0}^{\infty}\frac{2^{-2n}(1-a)^{n+1}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k 2^{2k}}{2k+1}\binom{2n-4k}{n-2k}F\left(2k+2,n+\frac{3}{2},n+\frac{5}{2},\frac{1-a}{2}\right) \tag{15}$$

Entry 11. If $0 < a < 1$ , then

$$\pi\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right)-\frac{\sqrt{1-a}}{4}F_1\left(\frac{1}{2},1,\frac{1}{2},\frac{3}{2},\frac{1-a}{2},1-a\right)\right)= 2\sqrt{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)(4n+3)}F_1\left(2n+\frac{3}{2},1,\frac{1}{2},2n+\frac{5}{2},-a,a\right)- \frac{\sqrt{1-a}}{2}\sum_{n=0}^{\infty}\frac{2^{-n}(1-a)^{n+1}}{2n+3}F\left(\frac{1}{2},n+\frac{3}{2},n+\frac{5}{2},1-a\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n+1}{n-2k} \tag{16}$$

# Endnote

$$\int_0^1\frac{1}{\sqrt{1-x^2}}\left(\frac{1}{3-x}\tan^{-1}\left(\frac{1-x}{2}\right)+\frac{1}{3+x}\tan^{-1}\left(\frac{1+x}{2}\right)\right)dx= \frac{\pi}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right) \tag{17}$$

$$\int_1^{\infty}\frac{\tan^{-1}x}{(x+1)\sqrt{x-1}}dx=\frac{\pi^2}{2\sqrt{2}}-\frac{\pi}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{2}-1}}{\sqrt{2}+\sqrt{\sqrt{2}-1}}\right) \tag{18}$$

$$\int_0^1 \frac{\tan^{-1} x}{\sqrt{x(1-x)}}\,dx = \frac{\pi^2}{4} - \pi \tan^{-1}\left(\frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}}\right) \tag{19}$$

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}} \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = \frac{\pi^2}{2} - 2\pi \tan^{-1}\left(\frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}}\right) \tag{20}$$

$$\int_0^{\pi/2} \tan^{-1}\left(\frac{1}{1-\sin^2 x + \sin^4 x}\right) dx = \frac{\pi^2}{4} - \pi \tan^{-1}\left(\frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}}\right) \tag{21}$$

$$\int_0^{\pi/2} \tan^{-1}\left(1-\sin^2 x + \sin^4 x\right) dx = \pi \tan^{-1}\left(\frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}}\right) \tag{22}$$

$$8\pi - 64\tan^{-1}\left(\frac{\sqrt{2} - \sqrt{\sqrt{2}-1}}{\sqrt{2} + \sqrt{\sqrt{2}-1}}\right) = \sum_{n=0}^{\infty} 2^{-5n} \binom{2n+2}{n+1} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{n-2k} \tag{23}$$

## References


[1] Boros, G., and Moll, V., Irresistible Integrals, Cambridge University Press, 2004.

[2] Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series and Products, 7th. ed., Academic Press, 2007.

[3] Olver, F.W.J., et al., NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.