COLLATZ CONJECTURE: AN ORDER MACHINE

MICHAEL WILLIAMS

Abstract. Collatz conjecture (3n+1 problem) is an application of Cantor's isomorphism theorem (Cantor-Bernstein) under recursion. The set of 3n+1 for all odd positive integers n, is an order isomorphism for (odd X, 3X+1). The other (odd X, 3X+1) linear order has been discovered as a bijective order-embedding, with values congruent to powers of four. This is demonstrated using a binomial series as a set rule, then showing the isomorphic structure, mapping, and cardinality of those sets. Collatz conjecture is representative of an order machine for congruence to powers of two. If an initial value is not congruent to a power of two, then the iterative program operates the (odd X, 3X+1) order isomorphism until an embedded value is attained. Since this value is a power of four, repeated division by two tends the sequence to one. Because this same process occurs, regardless of the initial choice for a positive integer, Collatz conjecture is true.

1. Introduction

Many problems in mathematics remain open and they provide the motivation for this work[1]. A well-reasoned argument using combinatorics and order theory has been found to reduce the mathematical abstraction known as Collatz conjecture ($3n + 1$ *problem*). The set of $3n + 1$ for all *odd* positive integers n, is an order isomorphism for ($odd\ X,\ 3X + 1$). By Cantor's isomorphism theorem (Cantor-Bernstein), any two unbounded countable dense linear orders are order-isomorphic. The other ($odd\ X,\ 3X+1$) linear order has been discovered as an embedding. Under recursion, it is the relationship between two structurally identical ($odd\ X,\ 3X + 1$) linear orders and the mappings between them that governs sequence behavior.

There exists a bijective order-embedding for ($odd\ X,\ 3X + 1$) that can be represented by the set of 4^q for all positive integers q. This is demonstrated using the binomial series expansion of $(1 + 3)^q$. Series rearrangement provides the necessary isomorphic structure and the cardinality to coalesce all sets under observation. Thus, the pair of order isomorphic linear orders share a countably infinite set of values. The *escape value* for any Collatz sequence comes from that denumerable set. These special values are recognized to be the *even-index terms of the Jacobsthal sequence*; which is the interpretation of *Rule 50* for the triangle read-by-row (successive states) generated by an elementary cellular automaton. Iteration continues until one of these values is attained by the sequence and then the sequence tends to

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[1]Michael Williams (0000-0002-2815-8163), Independent researcher, micewill@gmail.com

one. That is, the *up to isomorphism* loop is a part of, but not the entire machine under investigation.

Known to be Turing-complete[2], the statement of Collatz conjecture is equivalent to an order machine. An iterative program (i.e. while loop on n) operates under recursion until n is equivalent to an escape value. Since the escape value is *strictly odd* [*Lemma 2*], it guarantees on the next iteration that $n \mapsto 3n + 1$, which is equivalent to a value of 4^q for some positive integer q. And although the *up to isomorphism* loop has 'stopped', the recursive machine continues to operate until the 'off value' is reached.[3] That is, after 4^q is repeatedly divided by two, which *tends the sequence to one*. Since this same process occurs regardless of the initial choice for a positive integer n, and because cardinality guarantees an escape value will always be attained under the recursion, it will be shown that `Collatz conjecture is true`.

2. Background

2.1. **History of Collatz conjecture.** By all accounts, Jeff Lagarias is the ultimate authority on this problem. He has composed multiple papers on the subject [13, 14, 17, 18] and has compiled and maintained two annotated bibliographies spanning nearly 50 years of research, 1963-1999 [15] and 2000-2009 [16]. For background, history, and research related to this problem, these are excellent references.

Collatz conjecture is notorious for being simple to state, but impossible to prove. Though not a Millennium Prize Problem, there is a substantially similar reward being offered by Bakuage Co., Ltd. (Japan) for a solution [23]. The conjecture "concerns the behavior of the iterates of the function which takes odd integers n to $3n+1$ and even integers n to $n/2$" [13]. While largely attributed to Lothar Collatz, who is believed to have introduced the idea in 1937, the origin of this problem is obscure. Shrouded in mystery, it was largely circulated by word-of-mouth from known associates of Collatz and likely disseminated by others even prior to that time [18]. It is known as: the $3x + 1$ problem, Kakutani's problem, Syracuse problem, Thwaites conjecture, Ulam conjecture, and Hasse's algorithm. The sequences generated from the conjecture may be referred to as hailstone sequences or wondrous numbers, which is partly due to the wide range of values seen in many of the sequences. Notwithstanding name, the conjecture "asserts that starting from any positive integer n, repeated iteration of this function eventually produces the value 1." [13]

Proof of the conjecture has remained intractable. Paul Erdős[4] is often quoted as saying: "Mathematics is not yet ripe enough for such questions." And according to Lagarias (2010) [17]: "The track record on the $3x + 1$ problem so far suggests that this is an extraordinarily difficult problem, completely out of reach of present day mathematics." Although difficulty is a function of perspective, these sentiments mirror the observation that any Collatz sequence is randomly generated. While an onerous task, mathematicians have persevered. Research will be discussed after presenting the problem in detail.

[2] Can be used to simulate a Turing machine; most programming languages are Turing-complete.

[3] The function $n \mapsto 3n + 1$ will not be called again. The function $n \mapsto n/2$ will iterate until the sequence reaches one.

[4] Second-hand statement of Paul Erdős [11].

2.2. **Define the 3n+1 problem and Collatz conjecture.** A sequence is generated from the selection of a positive integer n. Once $c_0 = n$ is selected, two rules are applied to generate the sequence using function f and recursion c_i. If c_i is *even*, then that number is *divided by two*. If c_i is *odd*, then that number is *multiplied by 3 and one is added*. These rules are applied under recursion until, as it is conjectured, *the value of the sequence tends to one.*

Define the function f,

$$f(c_i) := \begin{cases} c_i/2, & \text{if } c_i \equiv 0 \ (mod2) \ \ [\textit{even number}] \\ 3c_i + 1, & \text{if } c_i \equiv 1 \ (mod2) \ \ [\textit{odd number}] \end{cases} \tag{2.1}$$

and define the sequence (c_i) recursively.

$$c_i := \begin{cases} c_0 \ = \ n, & \text{for } i = 0, \text{ where } n \geq 1, \ n \in \mathbb{N} \\ f(c_{i-1}), & \text{for } i > 0 \end{cases} \tag{2.2}$$

Example calculations and sequences follow before formally stating the conjecture.

Example 2.1. *Example calculation for a Collatz sequence.*
Let $n = 3$*, then* $c_0 = n = 3$.
Three is odd since $3 \equiv 1(mod2)$*, so* $c_1 = f(c_0 = 3) = 3(3) + 1 = 10$
Ten is even since $10 \equiv 0(mod2)$*, so* $c_2 = f(c_1 = 10) = 10 \div 2 = 5$
Five is odd since $5 \equiv 1(mod2)$*, so* $c_3 = f(c_2 = 5) = 3(5) + 1 = 16$
Sixteen is even since $16 = 4^2 \equiv 0(mod2)$
This process continues until the following Collatz sequence is obtained:
$(c_0 = 3, \ c_1 = 10, \ c_2 = 5, \ c_3 = 16, \ c_4 = 8, \ c_5 = 4, \ c_6 = 2, \ c_7 = 1)$

Example 2.2. *Example Collatz sequences.*
Let $n = 3 : \ (3, \ 10, \ 5, \ 16, \ 8, \ 4, \ 2, \ 1)$.
Let $n = 5 : \ (5, \ 16, \ 8, \ 4, \ 2, \ 1)$.
Let $n = 7 : \ (7, \ 22, \ 11, \ 34, \ 17, \ 52, \ 26, \ 13, \ 40, \ 20, \ 10, \ 5, \ 16, \ 8, \ 4, \ 2, \ 1)$.
Let $n = 21 : \ (21, \ 64, \ 32, \ 16, \ 8, \ 4, \ 2, \ 1)$.
Let $n = 85 : \ (85, \ 256, \ 128, \ 64, \ 32, \ 16, \ 8, \ 4, \ 2, \ 1)$.

Example 2.3. *The (1, 4, 2, ...) loop.*

There is a closed loop sequence that occurs when c_i *is equivalent to 1. Let* $n = 1$*, then the generated sequence is:* $(1, 4, 2, 1, ...)$*. This sequence is an infinite loop under the conjecture recursion criteria. Therefore, it is agreeable that attaining* $c_i \equiv 1$ *is sufficient to stop the recursive process.*

Conjecture statement. The famous Collatz conjecture states, regardless of the initial choice of a positive integer, $c_0 = n$, iteration of f will always generate a sequence that will tend to one, that is $(c_i) \to 1$.

Remark 2.4. *A sufficient proof will show that any Collatz sequence goes to one for any positive integer starting value. This will not be as difficult as was once thought. A new perspective will reduce the situational complexity of the problem allowing for proof to proceed as an intelligible delineation.*

2.3. **Research and results.** The amount of computations performed for this problem is astonishing. As explained by Muller (2021) [19]: over 2^{68} sequences have been calculated for Collatz conjecture—and all of them eventually come back down to one. While that is a substantial amount of evidence, it doesn't prevent the existence of a counter-example. According to Kontorovich [19], counter-examples prove themselves auspicious and informative, so sometimes research needs to be so directed. In fact, Conway (1972) [6] proved that while Collatz conjecture is Turing-complete, it is still subject to the halting problem. That is, there could exist a sequence that enters an infinite loop and never stops running, providing yet another possibility for a counter-example.

Beyond data and algorithms, several types of analysis have been performed for this problem. Those studies are wide and varied, and include: sequence stopping time, coefficients for least iterate, linear transformations, periodicity, residue classes, density bounds, conjugacy maps, graphs and hypergraphs. This author concedes that most of these research methods are beyond his present technical understanding. Therefore, depending on the study of interest, readers should defer to the author or mathematics professional in that particular field or sub-field. Understanding of the aforementioned methods are not required to realize proof. Boundedness, heuristic probability, and the most recent research related to orbits will now be discussed.

In 1976, Terras [27] showed that almost all Collatz sequences reach a point below their initial value. The same result was discovered independently by Everett (1977) [10]. From those founding papers, that bound was significantly improved upon over the next several decades. Further validity was found by Barone (1999) [2] using a heuristic probabilistic argument, where the major finding was "that iterates of the $3x + 1$-function should decrease on average by a multiplicative factor $(3/4)^{1/2}$ at each step" [15]. These studies indicated that most Collatz sequences were bounded and decreasing.

More recently, Tao (2019) [26] showed the $3x+1$ problem follows a stricter set of criteria, proving that almost all Collatz orbits will attain a bounded value. During a presentation by Tao in 2020 he stated: "This is about as close as one can get to the Collatz conjecture without actually solving it" [19]. Although a wonderful result, and a testament to the power of mathematics, it was still not a complete proof. But, the research now indicated it was highly improbable that a counter-example could be found.

3. Preliminary

Standard set and order theory language and notation will be used. The most relevant definitions will be given in text, while others will be provided in appendix, or omitted. The sets, later to be defined and examined, are strictly monotone enumerations for all $n \in \mathbb{N}$. The mappings between them are central to proving the conjecture. This section will bring continuity to the manuscript.

3.1. **Notation.** Set $\mathbb{N}$ will represent the natural numbers with zero; defined as the non-negative integers starting with zero.

$$\mathbb{N} := \{0,\ 1,\ 2,\ 3,\ ...\} \tag{3.1}$$

Or equivalently,

$$\mathbb{N} = \{0\} \ \cup \ \mathbb{Z}^{+} \tag{3.2}$$

$$\mathbb{Z}^{+} = \{1,\ 2,\ 3,\ 4,\ ...\} \tag{3.3}$$

Defining $\mathbb{N}$ in this way will provide cohesiveness to all sets and mappings under investigation. This notation will also prevent superfluous set superscript and union notation for better readability.

3.2. **Cardinality.** Enumeration of $\mathbb{N}$ proceeds in the usual fashion. The function $f\colon \mathbb{N} \to \mathbb{Z}^+$ defined by $f(n) = n+1$ for $n \in \mathbb{N}$ is in one-to-one correspondence with the positive integers. To this end, the cardinality of $\mathbb{N}$, is recognized to be the first infinite cardinal number represented as aleph-naught, $\aleph_0$.

$$|\mathbb{N}| \;=\; \aleph_0 \tag{3.4}$$

Definition 3.1 (Cardinal numbers)**.** *The c.n. of the empty set is zero,* $|\varnothing| = 0$. *The c.n. of* $I_n = \{1, 2, 3, ..., n\}$ *is* n, $|I_n| = n$. *The c.n. of* $\mathbb{N}$ *is* $\aleph_0$, $|\mathbb{N}| = \aleph_0$.

3.3. **Order.** The term *order* [*Appendix 5.2*] is used throughout this manuscript and is meant to convey an ordering relation $\leq$ on a set. For example, all sets to be examined are strict (linear) orders that are countably infinite. The primary focus will be the two (*odd* X, $3X+1$) linear orders that are order isomorphic by Cantor's isomorphism theorem. This will be the general usage.

3.4. **Free and bounded variables.** The binomial theorem [*Appendix 5.1*] and binomial series expansions are fundamental to this presentation. Most equations are defined for a free variable, usually designated as n, and a variable bounded by n usually designated as k. These equations are shown to be valid for all $n \in \mathbb{N}$; therefore, a value may always be found for any chosen n.

This is both a blessing and a curse. A countably infinite set may be constructed from an equation, or its many equivalences, in a natural way. However, when attempting to show an equality or a comparison, in certain cases, the meaning of n can become obscured. For example, it is not desirable to have $n = E(n)$, when it is meant that n (LHS) is distinct from the n (RHS) of $E(n)$. Therefore, equations will be derived using n and k per the usual presentation given in most texts, but when set definitions are given, other free variable names (i.e. m, q) will be substituted. That way, the above example changes to $n = E(q)$ for distinct n and q where $n, q \in \mathbb{N}$, thus conveying the proper meaning.

3.5. **Zero point.** For further coherence, the Collatz conjecture function definition can be extended to include a value for $n = 0$. Since the conjecture statement *requires a choice*, selecting a positive integer value; this will give the system a state when *no choice has been made*. This adds completeness to the theory and further utilizes the definition for $\mathbb{N}$. Notice, $f(0) = 3(0) + 1$ is a suitable definition for this purpose.

$$f(0) := 1 \tag{3.5}$$

Thus, reaching the value of one is equivalent to 'turning off' the order machine and allowing for a new choice to be made.

3.6. **Methodology.** The purpose of this presentation is to provide a new perspective of the $3n + 1$ *problem*. A seemingly trivial observation leads to conjectural proof. First, equations are derived using the binomial theorem. Then, denumerable equinumerous sets are defined, using each equivalent equation as a set rule. Next, mappings and Cantor theorems are introduced to show the (*odd* X, $3X+1$)

order isomorphism. Finally, everything is put together under recursion to create the order machine.

The major insight is the coordination between two (*odd* $X,\ 3X+1$) linear orders, one of which is a bijectional order-embedding. Since any *even* X is *divided by two* until it becomes an *odd* X [*Lemma 4, Corollary 3*], this continues the recursion on the (*odd* $X,\ 3X+1$) order isomorphism. Once an *odd* X embedded value is attained, then the sequence tends to one, which leads to the observation that is Collatz conjecture. To best understand the end goal from the beginning, please see the figures for the order machine (OM) [*Figure 1*] and the pseudo-code for the OM algorithm [*Figure 2*].

4. Proof of Collatz conjecture

4.1. **Combinatorics methods and results.** Perform a series expansion using the binomial theorem,

$$4^n = (1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k \qquad n \in \mathbb{N} \tag{4.1}$$

continue the equality, and notice the structure of the RHS.

$$\sum_{k=0}^{n} \binom{n}{k} 3^k = 3\left[\sum_{k=1}^{n} \binom{n}{k} 3^{k-1}\right] + 1 = 3[E(n)] + 1 \tag{4.2}$$

This is the major observation that leads to the proof of Collatz conjecture. From a value perspective, the above equality seems uninteresting. However, from a 'structural perspective' the RHS 'looks a lot like' $3n+1$. It can be seen that $E(n)$ is well-defined and ordered by $\mathbb{N}$. Later, the value of $E(n)$ is noted to be *strictly odd* [*Lemma 2*]. This is an important point. The observation of Collatz conjecture is the bijective coordination between a pair of (*odd* $X,\ 3X+1$) linear orders. Definitions, calculations, and properties of $E(n)$ follow. The relationship of $E(n)$ to the Jacobsthal sequence is also discussed.

$$E(n) := \sum_{k=1}^{n} \binom{n}{k} 3^{k-1} \qquad n \geq 1,\ n \in \mathbb{N} \tag{4.3}$$

Or equivalently,

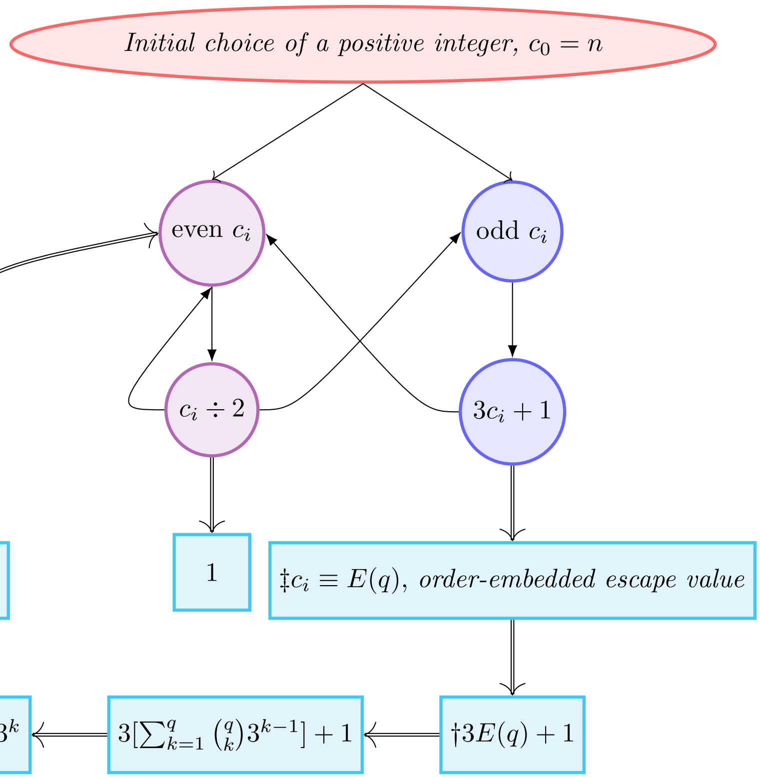
$$E(n) = \frac{1}{3}(4^n - 1) \tag{4.4}$$

Any free variable name (e.g. m, q, r, x) may be substituted for 'n' in this definition without loss of generality.

Example 4.1. *$E(n)$ example calculations (by definition).*

$$E(n) = \sum_{k=1}^{n} \binom{n}{k} 3^{k-1}$$

FIGURE 1. `Order machine (OM)`



Under recursion, *even integers are divided by two* (i.e. $c_i \div 2$), which generates *odd integers* for the $(odd\ X,\ 3X+1)$ *order isomorphism* [*Corollary 3*]. This continues until an *escape value*, $E(q)$, is attained‡. Since it is always true that $3E(q) + 1 = 4^q$, repeated division by two *tends the sequence to one.*

For example. Let, $c_i \equiv E(3) = 21$. Then, $3(21) + 1 = 64 = 4^3$. Since $4^3 \equiv 0(mod2)$, *the sequence tends to one.*

‡*Escape* is guaranteed by cardinality (Cantor-Bernstein theorem).

†The set of all $3[E(q)] + 1$ is a bijectional, order-embedding isomorphism for $(odd\ X,\ 3X+1)$.

(1) $E(q) := \sum_{k=1}^{q} \binom{q}{k} 3^{k-1} = \frac{1}{3}(4^q - 1) \quad$ for $q \geq 1,\ q \in \mathbb{N}$

(2) $E(q)$ is *strictly odd* [*Lemma 2*].

(3) $(E(q)) = (1,\ 5,\ 21,\ 85,\ 341,\ 1365,\ ...)$
**This sequence is recognized to be the even-index Jacobsthal numbers* `(OEIS A002450)`, *which is the interpretation of Rule 50* `(OEIS A071028)` *for an elementary cellular automaton.*

FIGURE 2. Pseudo code for OM algorithm

```
variable int n;

function DivTwo(n)
    return n/2;

function Primitive(n)
    return 3n + 1;

function PrintVal(n)    // outputs sequence
    print n;

Please choose a positive integer.    // input
    scan n;

if ( n > 1 ) {

    /* Iterate until an escape value E(q) is attained, then 3E(q) + 1 = 4^q */

    while ( n ≇ 2^r ) {              // 'up to isomorphism' loop

        if ( n ≡ 0(mod2) ) {
            n = DivTwo(n);       // even n, divide by two
            PrintVal(n);
        }
        else ( n ≡ 1(mod2) ) {
            n = Primitive(n);       // odd n, multiply by 3 and add one
            PrintVal(n);
        }
    }


    /* Sends any value of n ≅ 2^r to one */

    while( n > 1 ) {

        n = DivTwo(n);       // 2^r ≡ 0(mod2), n → 1
        PrintVal(n);
    }

    return 0;    // successful program execution
}
else(exit);    // initial choice of n is 1 or is invalid
```

‡The two *while loops* may be interpreted mathematically as: *any positive integer, greater than one, is either congruent to* $2^r \equiv 0(mod 2)$ *or it isn't* [*Lemma 4, Corollary 2*]. If it isn't, it remains in the '*up to isomorphism' loop* until it is, which has $3E(q) + 1 = 4^q$.

Let $n = 1$,

$$E(1) = \binom{1}{1}3^{1-1}$$
$$= 1$$

Let $n = 2$,

$$E(2) = \binom{2}{1}3^{1-1} + \binom{2}{2}3^{2-1}$$
$$= 2(1) + 1(3^1)$$
$$= 5$$

Let $n = 3$,

$$E(3) = \binom{3}{1}3^{1-1} + \binom{3}{2}3^{2-1} + \binom{3}{3}3^{3-1}$$
$$= 3(1) + 3(3^1) + 1(3^2)$$
$$= 21$$

Example 4.2. $E(n)$ *example calculations (by equivalence).*

$$E(n) = \frac{1}{3}(4^n - 1)$$

Let $n = 4$,

$$E(4) = \frac{1}{3}(4^4 - 1)$$
$$= \frac{1}{3}(256 - 1)$$
$$= 85$$

Let $n = 5$,

$$E(5) = \frac{1}{3}(4^5 - 1)$$
$$= \frac{1}{3}(1024 - 1)$$
$$= 341$$

Let $n = 6$,

$$E(6) = \frac{1}{3}(4^6 - 1)$$
$$= \frac{1}{3}(4096 - 1)$$
$$= 1365$$

The denumerable sequence of $E(n)$ [*See Figure 5*] may be defined for all $n \geq 1$ where $n \in \mathbb{N}$.

(4.5) $$(E(n)) = (1,\ 5,\ 21,\ 85,\ 341,\ 1365,\ 5461,\ ...)$$

This sequence is recognized to be (`OEIS A002450`) [21] the *even-index terms of the Jacobsthal sequence*. It is a special list of numbers, each having distinct properties attributable to unique factorization and various combinatoric compositions (e.g. congruent, cyclic, 3-Lehmer, etc.).

Jacobsthal sequence.

Definition 4.3 (Jacobsthal numbers)**.** *Like the Fibonacci numbers, the Jacobsthal numbers* `(OEIS A001045)` *[20] are a constant-recursive integer sequence where the recurrence relation is similarly defined,*

$$J_m = \begin{cases} 0, & \text{if } m = 0 \\ 1, & \text{if } m = 1 \\ J_{m-1} + 2J_{m-2}, & \text{if } m > 1 \end{cases} \tag{4.6}$$

and there exists a closed-form expression [24]*,*

$$J_m = \frac{2^m - (-1)^m}{3} \qquad m \in \mathbb{N} \tag{4.7}$$

which may be used to find the terms of the sequence.

$$J_m = (0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ ...) \tag{4.8}$$

From the closed-form expression, take only the *even-index terms of* J_m, by letting $m = 2q$ for all $q \geq 1$ where $q \in \mathbb{N}$.

$$J_{2q} = \frac{2^{2q} - (-1)^{2q}}{3} \tag{4.9}$$

Since the powers of negative one will always be even,

$$(-1)^{2q} = [(-1)^2]^q = [(-1) \cdot (-1)]^q = 1^q = 1 \tag{4.10}$$

the expression is able to be simplified, in this case.

$$J_{2q} = \frac{2^{2q} - 1}{3} \tag{4.11}$$

Rearrangement gives the desired result,

$$E(q) = J_{2q} = \frac{1}{3}(4^q - 1) \tag{4.12}$$

which provides for another statement of equality,

$$4^q = 3[E(q)] + 1 = 3[J_{2q}] + 1 \tag{4.13}$$

and demonstrates a little-known, closed-form expression for the even-index Jacobsthal numbers.

$$E(q) = J_{2q} = \sum_{k=1}^{q} \binom{q}{k} 3^{k-1} \qquad q \geq 1,\ q \in \mathbb{N} \tag{4.14}$$

These numbers are the interpretation of *Rule 50* [36] for the triangle read-by-row (successive states) generated by an elementary cellular automaton, which has the following binary sequence (`OEIS A071028`) [22].

$$(J_{2q})_2 = (1_2,\ 101_2,\ 10101_2,\ 1010101_2,\ 101010101_2,\ ...) \tag{4.15}$$

The binary representation of these numbers are simplistically elegant,

$$(J_{2q})_2 = (10)_{(q-1)}1_2 \qquad q \geq 1,\ q \in \mathbb{N} \tag{4.16}$$

and demonstrative of Hilbert's Hotel for a newly arriving guest '(10)'.

Example 4.4. *Example calculations of J_{2q} in binary.*

$$(J_2)_2 = (10)_0 1_2 = 1_2 = 1_d$$
$$(J_4)_2 = (10)_1 1_2 = (10)1_2 = 5_d$$
$$(J_6)_2 = (10)_2 1_2 = 10(10)1_2 = 21_d$$
$$(J_8)_2 = (10)_3 1_2 = 1010(10)1_2 = 85_d$$
$$(J_{10})_2 = (10)_4 1_2 = 101010(10)1_2 = 341_d$$

Remark 4.5. *The set of all even-index Jacobsthal numbers is an order-embedding of the odd positive integers. This will be demonstrated under the guise of $3X + 1$ for odd X, given the problem under investigation.*

Returning to the original series and expanding out a few terms, notice the structure of *the partial sum of the first two terms* (RHS).

$$\sum_{k=0}^{n} \binom{n}{k} 3^k = 1 + 3n + \sum_{k=2}^{n} \binom{n}{k} 3^k \qquad \{n \mid n \in \mathbb{N}\} \tag{4.17}$$

For every non-negative integer n, the above expression has a value. And by extension, the partial sum of the first two terms, which is equivalent to $3n + 1$, is well-defined. So too are the even and odd subsets. This demonstrates how the binomial series expansion of 4^n represented as $(1 + 3)^n$ is useful for obtaining both orders of ($odd\ X,\ 3X+1$) in a natural way. To be understood, the set of $3E(n)+1$ is the denumerable isomorphic order-embedding that governs sequence behavior. The required lemmas will now be given before proceeding to the combinatorics section summary.

Lemma 1. $4^n \equiv 0(mod2)$ *for all positive integers $n \in \mathbb{N}$.*

Proof. This will be demonstrated using mathematical induction.
Let $n = 1$, then $4^1 = 2^2$ and 2^2 is twice divisible by two. Thus, $2^2 \equiv 0(mod2)$ which implies $4 \equiv 0(mod2)$.
Let $n = k$, then $4^k = 2^{2k} = (2^2)^k$ and $(2^2)^k$ is twice divisible by two, k-times. Thus, $2^{2k} \equiv 0(mod2)$ which implies $4^k \equiv 0(mod2)$.
Let $n = k + 1$, then $4^{k+1} = 2^{2(k+1)} = 2^{2k+2} = 2^{2k}2^2$. From above, $2^{2k} \equiv 0(mod2)$ and $2^2 \equiv 0(mod2)$. Thus, the product implies $4^{k+1} \equiv 0(mod2)$. □

Lemma 2. *$E(n)$ is strictly odd for all positive integers $n \in \mathbb{N}$.*

Proof. Let $4^n = 3[E(n)] + 1$ as detailed through the binomial theorem. It is always true that 4^n is evenly divisible by two for all positive integers n [*Lemma 1*]. Thus, $3[E(n)] + 1$ is always even for $n \geq 1$. But, that implies $4^n - 1$ is an odd number.

So it must be true that $3[E(n)]$ is also odd. Since 3 is an odd number, the product $3[E(n)]$ cannot be odd unless $E(n)$ is also odd. □

Proof by contradiction. Assume $E(n)$ is even, then the product $3[E(n)]$ is also even. But, that implies $3[E(n)] + 1$ is odd, which is impossible by equivalence to 4^n [*Lemma 1*]. □

Lemma 3. $3X + 1$ *is always even for an odd positive integer* X*.*

Proof. Let X be an integer that is odd and positive. Then, $X + 1$ must be even. Since $2X$ is divisible by two, it is even. Thus, the sum of these two even numbers is also even, and observe $(X + 1) + (2X) = 3X + 1$. □

Lemma 4. *Any positive even integer is either congruent to* 2^r *for* $r \geq 1$ *where* $r \in \mathbb{N}$*, or it is not (dichotomy).*

Proof. Since 2^r is even *by definition* for all $r \geq 1$, then those values may be listed. Any even number not on the list may be readily found (i.e. 6), which must have a *unique factorization* (i.e. $2 \cdot 3$) that is different from 2^r by the *fundamental theorem of arithmetic.* □

Corollary 1. *No odd positive integer greater than one is congruent to* 2^r*.*

Corollary 2. *Any positive integer is congruent to* 2^r *or it is not.*

Corollary 3. *The unique factorization of any even integer, with the powers of two removed, is necessarily odd.*

Combinatorics results summary. The following list illustrates the interconnected relationships detailed above.

(1) For $n \geq 1$, it is always true that $4^n \equiv 0(mod 2)$ [*Lemma 1*].
(2) For $n \geq 1$, it is always true that $E(n)$ is *strictly odd* [*Lemma 2*].
(3) For *odd* positive X, it is always true that $3X + 1$ is *even* [*Lemma 3*].
(4) A proposed set of $3n + 1$ for all $n \geq 1$ where $n \in \mathbb{N}$ could be divided into even and odd subsets. [The definition of which may be taken directly from the binomial series expansion of $(1 + 3)^n$ as a partial sum, if so desired.]
(5) Collatz conjecture requires that an *even integer is divided by two.*
(6) The net effect of (5) under recursion is *to convert any even integer into an odd integer* [*Lemma 4, Corollary 3*]. Thus, any even integer remains in its own loop until it becomes odd and then $c_i \mapsto 3c_i + 1$ for that iterate value of c_i.
(7) When the value of an odd integer is equivalent to a value of $E(n)$, then *equivalence to* 4^n *is obtained on the next iteration.* That is, $3c_i + 1 = 3[E(n)] + 1 = 4^n$.
(8) The net effect of (7) under recursion is that $4^n \equiv 0(mod 2)$ sends $c_i \to 1$.
(9) From here on, n will be reserved for the initial choice of a positive integer, where $c_0 = n$ is used to start a sequence.

4.2. **Set theory methods and results.** The sets to be examined are each defined explicitly by a set rule. They are enumerated by and cardinally equivalent to $\mathbb{N}$. As such, this presentation is an exposition delineating those sets. The main objective will be to show the two distinct (*odd* X, $3X + 1$) linear orders. Since these two denumerable sets are in one-to-one correspondence, they are able to be injectively mapped to one other. By the Cantor-Bernstein theorem, the injections between them guarantee a bijection exists, which embeds the ordered values of $3[E(q)] + 1$. These values are the mechanism controlling conjectural related sequence behavior. Some definitions will now be reviewed.

Definition 4.6 (Countable)**.** *Countable sets are those whose elements can be listed and indexed by the natural numbers. For example, let* $M = \{m_0,\ m_1,\ m_2,\ m_3\}$*, then set* M *is countable. If a set is not countable then it is uncountable.*

Definition 4.7 (Injection)**.** *The function* f *on set* M *is an injection (one-to-one), if for all* $m_1, m_2 \in M$*, if* $f(m_1) = f(m_2)$ *that implies* $m_1 = m_2$*. That is,* $f : M \hookrightarrow N$*.*

Definition 4.8 (Surjection)**.** *The function* f *on set* M *is a surjection (onto), if for all* $n \in N$*, there exists a unique* $m \in M$ *such that* $n = f(m)$*. That is,* $f : M \twoheadrightarrow N$*.*

Definition 4.9 (Bijection)**.** *The function* f *is a bijection, if each element of its codomain* N *is mapped to exactly one element of the domain* M*. A bijection is both injective and surjective. That is,* $f : M \to N$*.*

Definition 4.10 (Denumerable)**.** *A set* M *is termed denumerable (or countably infinite) if there exists a bijection* $f : \mathbb{N} \to M$*.*

Definition 4.11 (Equinumerous)**.** *Two sets* M *and* N *are said to be equinumerous (or cardinally equivalent) provided there is a one-to-one correspondence (a bijection) from* M *to* N*. That is,* $h : M \to N$*. This equivalence relation is expressed as* $M \approx N$ *or* $|M| = |N|$ *and is sometimes termed equipollent, equipotent, or simply equivalent* [8]*.*

From the binomial theorem, the following equivalence statement holds true for all non-negative integers q.

$$4^q = (1+3)^q = \sum_{k=0}^{q} \binom{q}{k} 3^k = 3\left[\sum_{k=1}^{q} \binom{q}{k} 3^{k-1}\right] + 1 = 3[E(q)] + 1 \tag{4.18}$$

Using each equivalent term, sets will be defined explicitly for all $q \in \mathbb{N}$. Since equinumerous sets are arguably more fundamental than the principle of counting[5], a theorem will be introduced.

Define A, the countably infinite set of values 4^q, where $4^q \equiv 0(mod2)$ for all $q \geq 1$ [*Lemma 1*].

$$A := 4^q \quad \{q \mid q \in \mathbb{N}\} \tag{4.19}$$

Or equivalently,

$$\begin{aligned} A &= \{4^0,\ 4^1,\ 4^2,\ 4^3,\ 4^4,\ 4^5,\ ...\} \\ &= \{1,\ 4,\ 16,\ 64,\ 256,\ 1024,\ ...\} \end{aligned}$$

$$B := (1+3)^q \quad \{q \mid q \in \mathbb{N}\} \tag{4.20}$$

[5]For instance, a small child is able to show one-to-one finger correspondence long before they are able to count to five [8]

Or equivalently,

$$B = \{(1+3)^0,\ (1+3)^1,\ (1+3)^2,\ (1+3)^3,\ (1+3)^4,\ (1+3)^5,\ ...\}$$
$$= \{1,\ 4,\ 16,\ 64,\ 256,\ 1024,\ ...\}$$

$$C := \sum_{k=0}^{q} \binom{q}{k} 3^k \qquad \{q \mid q \in \mathbb{N}\} \tag{4.21}$$

Or equivalently,

$$C = \left\{ \binom{0}{0} 3^0,\ \sum_{k=0}^{1} \binom{1}{k} 3^k,\ \sum_{k=0}^{2} \binom{2}{k} 3^k,\ \sum_{k=0}^{3} \binom{3}{k} 3^k,\ \sum_{k=0}^{4} \binom{4}{k} 3^k,\ \sum_{k=0}^{5} \binom{5}{k} 3^k,\ ... \right\}$$
$$= \{1,\ 4,\ 16,\ 64,\ 256,\ 1024,\ ...\}$$

Now define D, the set of all *escape values*, $E(q)$, recognized to be the *even-index Jacobsthal numbers*, J_{2q}. These are the *odd* X values used to make $3X+1$ congruent to 4^q for all $q \in \mathbb{N}$.

$$D := \begin{cases} 0, & q = 0 \\ E(q), & \{q \geq 1 \mid q \in \mathbb{N}\} \end{cases} \tag{4.22}$$

Or equivalently,

$$D := \frac{1}{3}(4^q - 1), \qquad \{q \mid q \in \mathbb{N}\} \tag{4.23}$$

And thus,

$$D = \{0\} \cup \{E(q)\}$$
$$= \{0,\ 1,\ 5,\ 21,\ 85,\ 341,\ ...\}$$

Now define G, the *embeddable* (*odd* $X,\ 3X+1$) linear order.

$$G := 3d + 1 \qquad \{d \mid d \in D\} \tag{4.24}$$

As a mapping,

$$\delta : D \to G \tag{4.25}$$

Or equivalently,

$$G := \begin{cases} 1, & q = 0 \\ 3[E(q)] + 1, & \{q \geq 1 \mid q \in \mathbb{N}\} \end{cases} \tag{4.26}$$

And thus,

$$G = \{1,\ 3(1)+1,\ 3(5)+1,\ 3(21)+1,\ 3(85)+1,\ 3(341)+1,\ ...\}$$
$$= \{1,\ 4,\ 16,\ 64,\ 256,\ 1024,\ ...\}$$

Remark 4.12. *Set G is a denumerable linear order for* (*odd* $X,\ 3X+1$) *since $E(q)$ is strictly odd [Lemma 2]. Set G, as it will be shown, is bijectional with and embeddable into the ordinal set of* (*odd* $X,\ 3X+1$)*, to be labelled set F.*

Theorem 4.13. ***Equinumerosity theorem*** *Equinumerosity is an equivalence relation $\approx$ on a family of sets* [3].

Proof. The equivalence relation $\approx$ is reflexive, symmetric, and transitive.

(1) ***Reflexive.*** For any set M, $M \approx M$ is a bijection $f : M \to M$.
(2) ***Symmetric.*** If $M \approx N$ is a bijection $f : M \to N$, then $f^{-1} : N \to M$ implies $N \approx M$.
(3) ***Transitive.*** If $M \approx N$ and $N \approx P$ are bijections $f : M \to N$ and $g : N \to P$, respectively, then $g \circ f : M \to P$ implies $M \approx P$.

□

Under set definition rules, A, B, C, and G are countably infinite sets. They each have a mapping, which is equivalent to the bijection ψ, where $X \equiv \mathbb{N}$ and $Y \equiv A$.

$$\psi :\ X \to Y \begin{cases} a :\ \mathbb{N} \to A \\ b :\ \mathbb{N} \to B \\ c :\ \mathbb{N} \to C \\ g :\ \mathbb{N} \to G \end{cases} \tag{4.27}$$

These mappings are congruent, $\psi(x) = y$, and invertible, $\psi^{-1}(y) = x$, for all $x \in X$ and $y \in Y$.

$$\psi^{-1} :\ Y \to X \begin{cases} a^{-1} :\ A \to \mathbb{N} \\ b^{-1} :\ B \to \mathbb{N} \\ c^{-1} :\ C \to \mathbb{N} \\ g^{-1} :\ G \to \mathbb{N} \end{cases} \tag{4.28}$$

These are equinumerous sets,

$$A \approx B \approx C \approx G \tag{4.29}$$

which are cardinally equivalent.

$$|A| = |B| = |C| = |G| = \aleph_0 \tag{4.30}$$

Each set is strictly ordered, monotonic, increasing, and denumerable. For any $x \in X$ it is always true that $x < \psi(x)$. For all distinct $x_1, x_2 \in X$, it is always true that $x_1 < x_2$ implies $\psi(x_1) < \psi(x_2)$ and $x_2 < x_1$ implies $\psi(x_2) < \psi(x_1)$. The set rule, which is enumerated by $\mathbb{N}$, is in one-to-one correspondence between the domain and codomain of ψ.

Now define F, the *ordinal* (*odd* X, $3X + 1$) linear order. F is the primary set under observation from Collatz conjecture since *odd* $c_i \mapsto 3c_i + 1$.

$$F := \begin{cases} 1, & m = 0 \\ 3m + 1, & \{m \geq 1 \mid m = 2k + 1,\ k \in \mathbb{N}\} \quad (\textit{odd } m) \end{cases} \tag{4.31}$$

Or equivalently (by substitution of m),

$$F = \begin{cases} 1, & m = 0 \\ 6k + 4, & \{k \mid k \in \mathbb{N}\} \end{cases} \tag{4.32}$$

And thus,

$$\begin{aligned} F &= \{1,\ 3(1)+1,\ 3(3)+1,\ 3(5)+1,\ 3(7)+1,\ 3(9)+1,\ ...\} \\ &= \{1,\ 6(0)+4,\ 6(1)+4,\ 6(2)+4,\ 6(3)+4,\ 6(4)+4,\ ...\} \\ &= \{1,\ 4,\ 10,\ 16,\ 22,\ 28,\ ...\} \end{aligned}$$

The mapping $f:\ \mathbb{N} \to F$ is strictly ordered, monotonic, increasing, and denumerable. The set rule, which is enumerated by $\mathbb{N}$, is in one-to-one correspondence between the domain and codomain of f. This can be seen as a direct result after substituting the definition for m and simplifying. For any $k \in \mathbb{N}$ it is always true that $k < f(k)$. And for all distinct $k_1, k_2 \in \mathbb{N}$, it is always true that $k_1 < k_2$ implies $f(k_1) < f(k_2)$ and $k_2 < k_1$ implies $f(k_2) < f(k_1)$.

To obtain k for any *escape value*, notice that when $m = E(q)$, then $3m + 1 = 4^q$ and $6k + 4 = 4^q$. Thus, the value of k is given by solving the equality, $3E(q) + 1 = 6k + 4$.

$$k = \frac{1}{2}[E(q) - 1] \tag{4.33}$$

To obtain k directly from q, recall $E(q) = \frac{1}{3}(4^q - 1)$, and substitute $E(q)$ into the previous result for k and simplify.

$$k = \frac{2}{3}(4^{q-1} - 1) \tag{4.34}$$

Example 4.14. *Example calculation of k from $E(q)$.*
Let $q = 11$, then $E(11) = \frac{1}{3}(4^{11} - 1) = 1398101$.
Using the formula for k with $E(11)$ gives $k = \frac{1}{2}(1398101 - 1) = 699050$.
Since $m = 2k + 1$, $m = 2(699050) + 1 = 1398101$, which has $m \equiv E(11)$.

Example 4.15. *Example calculation for k from q.*
Let $q = 11$, as in the previous example.
Then, $k = \frac{2}{3}(4^{11-1} - 1) = \frac{2}{3}(1048575) = 699050$.
This result is in agreement with that of the previous example.

Remark 4.16. *The motivation now is to show that G is isomorphic to F by structure and order. By the Cantor theorems, the bijection between F and G is an order isomorphism that embeds G into F. The automorphism of F, which contains G, is the primary mapping under Collatz conjecture.*

4.3. **Cantor theorems.** The Cantor-Bernstein theorem and Cantor's isomorphism theorem (order theory) will be presented and applied to linear orders F and G. It will be shown that G is a bijectional order-embedding of F, which is a unique isomorphism [9] for (*odd* X, $3X + 1$). This is the mechanism controlling sequence behavior for Collatz conjecture. A sequence is generated by iteration on F. Any even number is made odd for continued iteration on F. Once an isomorphic embedded value of G contained in F is attained, then the sequence tends to one.

The Cantor-Bernstein theorem depends on the Zermelo-Fraenkel (ZF) axioms, but technically, not on the axiom of choice (C). However, it is from the well-ordering theorem, equivalent to the axiom of choice (*first-order logic*), that every set is susceptible to transfinite induction. According to Cantor(1883), this is a fundamental principle of thought used to guide intuition [4]. These theorems are useful for proving a bijection exists between two sets, which may otherwise be a difficult construction. They will simplify demonstrating the bijection between F and G, to only showing the two injections. Due to the (*odd* $X,\ 3X+1$) set structure of F and G, the one-to-one correspondence between these denumerable sets, and since $G \subset F$, this is the observation of Collatz conjecture.

Theorem 4.17. ***Cantor-Bernstein theorem*** *If each of two sets M and N can be mapped injectively into the other, $f : M \hookrightarrow N$ and $g : N \hookrightarrow M$, then there exists a bijection from M to N, $h : M \to N$, such that $|M| = |N|$ and $M \approx N$.*

Proof. Several proofs of this theorem exist and are attributable to Bernstein, Borel, Dedekind, Zermelo, König, and others. Some proofs rely on the axiom of choice, while others do not invoke it. The *back-and-forth method* (Silver, Huntington), the *going-forth method* (Cantor), and *chain theory* (i.e. chains of elements, König) can be used to show the bijection between M and N [1]. □

Recall set F, the `ordinal` (*odd* $X,\ 3X+1$) linear order,

$$F = \begin{cases} 1, & m = 0 \\ 3m+1, & \{m \geq 1 \mid m = 2k+1,\ k \in \mathbb{N}\} \end{cases}$$

and G, the `embeddable` (*odd* $X,\ 3X+1$) linear order.

$$G = \begin{cases} 1, & q = 0 \\ 3[E(q)]+1, & \{q \geq 1 \mid q \in \mathbb{N}\} \end{cases}$$

The denumerable sets of F and G are given by the following mappings,

$$f:\ \mathbb{N} \to F \tag{4.35}$$

$$g:\ \mathbb{N} \to G \tag{4.36}$$

which are one-to-one and invertible.

$$f^{-1}:\ F \to \mathbb{N} \tag{4.37}$$

$$g^{-1}:\ G \to \mathbb{N} \tag{4.38}$$

The injections as described by the Cantor-Bernstein theorem are given as,

$$\alpha:\ F \hookrightarrow G \tag{4.39}$$

$$\beta:\ G \hookrightarrow F \tag{4.40}$$

Thus, the bijection between F and G (*guaranteed by the theorem*) is given as,

$$\gamma:\ F \to G \tag{4.41}$$

Please see the figure enumerating the bijection between F and G [*Figure 3*].

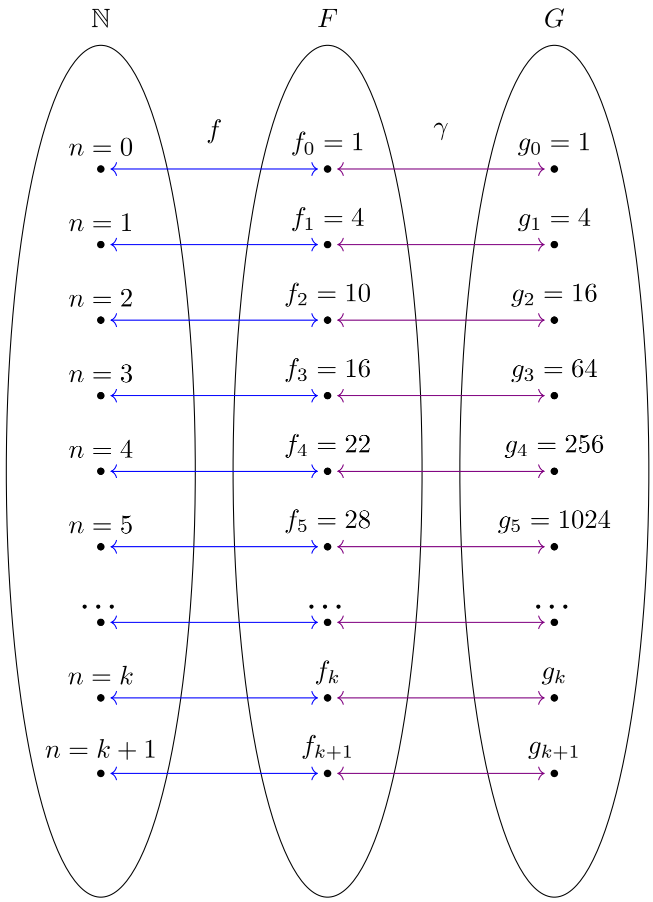
Since F and G are bijectional with each other, then they must also each have a bijectional mapping to themselves.

$$\sigma:\ F \to F \tag{4.42}$$

$$\tau:\ G \to G \tag{4.43}$$

Thus, σ and τ are the automorphisms of F and G, respectively; that is, $\sigma = Aut(F)$ and $\tau = Aut(G)$. Since $G \subset F$, $Aut(G)$ is contained in $Aut(F)$. The denumerable set of *escape values* of $E(q)$ contained in set D has the mapping $\delta : D \to G$, from the set definition. Please see the figure demonstrating the embedding of G, containing D, in F [*Figure 4*].

FIGURE 3. Bijection between F and G.



Cantor's isomorphism theorem is the Cantor-Bernstein theorem from the viewpoint of order theory [*Appendix 5.2*]. From the Cantor-Bernstein theorem it was shown that $\gamma : F \to G$ is a denumerable bijection. However, it is the ordering relation that lends perspective to the embedding of G into F. As such, it is presented here in the general spirit and verbiage of this manuscript.

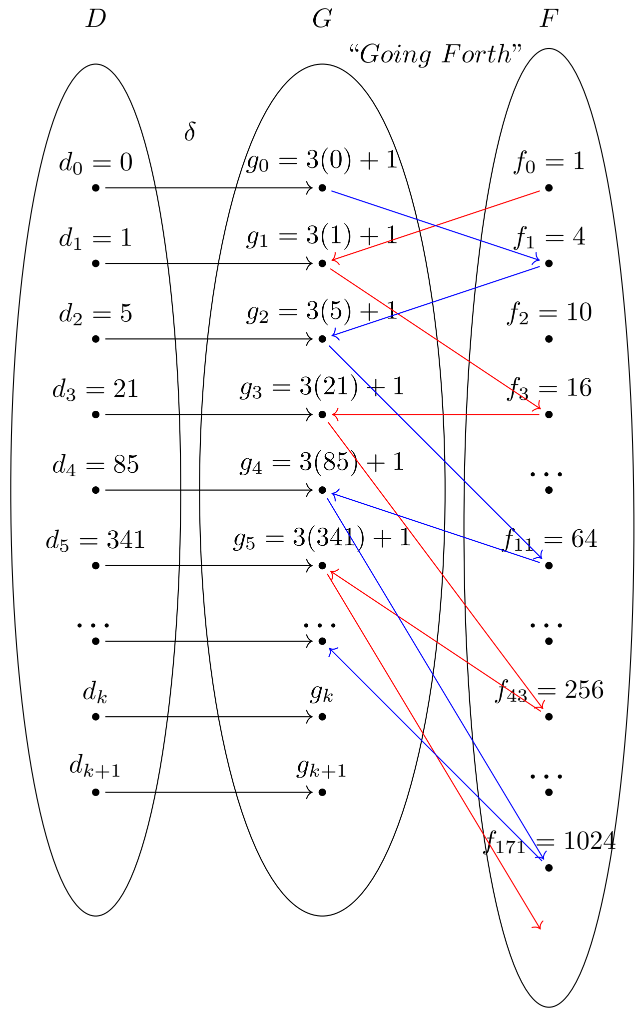
Definition 4.18 (Order isomorphism)**.** *Given two partially ordered sets (posets)* $(M, \leq_M)$ *and* $(N, \leq_N)$*, an order isomorphism is a bijection* $f : M \to N$ *having the property that for every* $x, y \in M$*,* $x \leq_M y$ *if and only if* $f(x) \leq_N f(y)$ [33]. That is, it is a bijective order-embedding [5].

Theorem 4.19. ***Cantor's isomorphism theorem*** *Every two countable dense unbounded linear orders are order-isomorphic [30].*

Theorem 4.20. ***Cantor's isomorphism theorem (alternate version)*** *Any two countably dense linear orders with no endpoints are isomorphic.*

Proof. The standard proof uses the *back-and-forth method* to build up an isomorphism between any two given orders using a *greedy algorithm* [30]. In this context,

Figure 4. Embedding of G in F.



F and G have one fixed endpoint. G may be obtained as a proper subset of elements of F and G [Figure 4] by *going forth* from that fixed endpoint. Since G is bijectional with F [Figure 3], any element of G may be renamed to an element of F, and vice versa. □

4.4. **Recursion and the order machine.** Up until now, even integer *division by two* has largely been ignored. However, the purpose of the mapping $c_i \mapsto c_i/2$ is important and twofold under the conjecture. Since any positive even integer is either congruent to $2^r \equiv 0(mod2)$ or it isn't [*Lemma 4*]; under recursion, this congruence determines if the sequence tends to one from that value or it doesn't. If it doesn't, then the even integer is *divided by two* until it becomes an odd integer [*Corollary 3*]. As previously discussed, odd integers map to the (*odd* X, $3X + 1$) order isomorphism until an escape value is reached, then the sequence tends to one.

The mechanism described above will be referred to as an *order machine* (OM): it is the algorithmic interpretation of Collatz conjecture using an (*odd* X, $3X+1$) order isomorphism. Please review the OM [*Figure 1*] and OM algorithm [*Figure 2*] figures. The initial choice of a positive integer, $c_0 = n$, starts the machine. That integer, if not congruent to 2^r, enters the first *up to isomorphism* while loop. Depending if the integer is even or odd, the corresponding function DivTwo() or Primitive() is called. This process continues until a value of $3E(q)+1$ is attained, which has the necessary congruence. Then, the second while loop sends the sequence to one.

The sets on which Collatz conjecture operates can now be understood in a tangible way. If an initial choice for an integer is not congruent to 2^r, the recursive machine operates the (*odd* X, $3X+1$) order isomorphism until it attains a value of 4^q for some $q \in \mathbb{N}$. The order isomorphism is composed of two (*odd* X, $3X+1$) linear orders. These denumerable sets are obtained through the binomial series expansion of 4^q as $(1+3)^q$ in two different ways [*Combinatorics*]. Both losets contain the subset of values where $3E(q)+1 = 4^q$. Since all odd integers are represented by the order isomorphism, and since any even integer not congruent to 2^r can be made odd, the OM is able to accept and process any positive integer.

4.5. **Discussion.** A series of observations has unravelled the mystery surrounding Collatz conjecture. There is an (*odd* X, $3X+1$) order isomorphism between a pair of (*odd* X, $3X+1$) linear orders that share a denumerable set of embedded values. If an initial choice for a positive integer is not congruent to 2^r, then values for *odd* X are created under recursion to operate the machine. When one of these *odd* X values is equivalent to an embedded value, then the sequence is able to escape by tending to one. Since cardinality guarantees that an escape value will always be found for any positive integer starting value, the proclamation of `Collatz conjecture is true`.

Equinumerosity and the Cantor theorems are powerful devices. A once thought intractable problem has been proven, but intriguingly, any sequence generated by the conjecture remains random. This methodology may serve as a guide for other analogously similar problems, where situational awareness can be used to provide proof, but random variation is still present. While the Cantor theorems guarantee an escape value for every sequence starting value, it is not known in advance which value will be used. However, these escape values are the even-index Jacobsthal numbers, each of which have special numerical and combinatoric properties, so there may be an argument that links a sequence starting value to its corresponding escape value. But that question, and many others, provide the motivation for more work to be done!

5. Appendix

5.1. **The Binomial Theorem.** The binomial theorem [29] describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand any non-negative integer power of $x+y$ into a sum,

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \\ &\cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n\end{aligned} \tag{5.1}$$

where $\binom{n}{k}$ is the familiar binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{5.2}$$

for n choose k.

This gives the general result,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{5.3}$$

in compact summation notation.

Proof A1. Proof of the Binomial Theorem.

The proof of this theorem will be demonstrated using mathematical induction. Assume the general result is true.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{5.4}$$

Let $n = 0$ and $x + y = z$. Evaluating the LHS gives $(x + y)^0 = z^0 = 1$. And evaluating the RHS gives $\binom{0}{0}x^0y^0 = 1$. Thus, it is true for $n = 0$.

Let $n = 1$. Evaluating the LHS gives $(x + y)^1 = z^1 = z$. And evaluating the RHS gives $\binom{1}{0}x^1 + \binom{1}{1}y^1 = x + y = z$. Thus, it is true for $n = 1$.

Now perform a series expansion of $(x + y)^n$,

$$\begin{aligned}(x+y)^n \;=\; & x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \\ & \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n-1}xy^{n-1} + y^n\end{aligned} \tag{5.5}$$

and let $n = m$, where m is a positive integer for which the statement is true.

$$\begin{aligned}(x+y)^m \;=\; & x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \\ & \cdots + \binom{m}{k}x^{m-k}y^k + \cdots + \binom{m}{m-1}xy^{m-1} + y^m\end{aligned} \tag{5.6}$$

Induction step. Multiply $(x + y)^m$ by $(x + y)$ to show $(x + y)^{m+1}$.

(5.7)

$$\begin{aligned}(x+y)(x+y)^m = \;& (x+y)\Bigg(x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \\ & \cdots + \binom{m}{k}x^{m-k}y^k + \\ & \cdots + \binom{m}{m-1}xy^{m-1} + y^m\Bigg) \\ = \;& x^{m+1} + \left[1 + \binom{m}{1}\right]x^m y + \left[\binom{m}{1} + \binom{m}{2}\right]x^{m-1}y^2 + \\ & \cdots + \left[\binom{m}{k-1} + \binom{m}{k}\right]x^{m-k+1}y^k + \\ & \cdots + \left[\binom{m}{m-1} + 1\right]xy^m + y^{m+1}\end{aligned}$$

Simplify the above expression using ***Pascal's identity***.

$$\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}, \quad \textit{where} \quad 0 < k \le m \tag{5.8}$$

Applying the identity

$$\begin{aligned} & x^{m+1} \;+\; \binom{m+1}{1} x^m y \;+\; \binom{m+1}{2} x^{m-1} y^2 \;+ \\ & \cdots \;+\; \binom{m+1}{k} x^{(m+1)-k} y^k \;+ \\ & \cdots \;+\; \binom{m+1}{m} x y^m \;+\; y^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{(m+1)-k} y^k \\ &= (x+y)^{m+1} \end{aligned} \tag{5.9}$$

shows the result is true for $m + 1$. And by induction, the result is true for all positive integers m and n. □

Example 5.1. *Application of the binomial theorem.*

Using the generalized binomial theorem, let $x = 1$ and $y = 3$. Then, perform the necessary algebraic rearrangement.

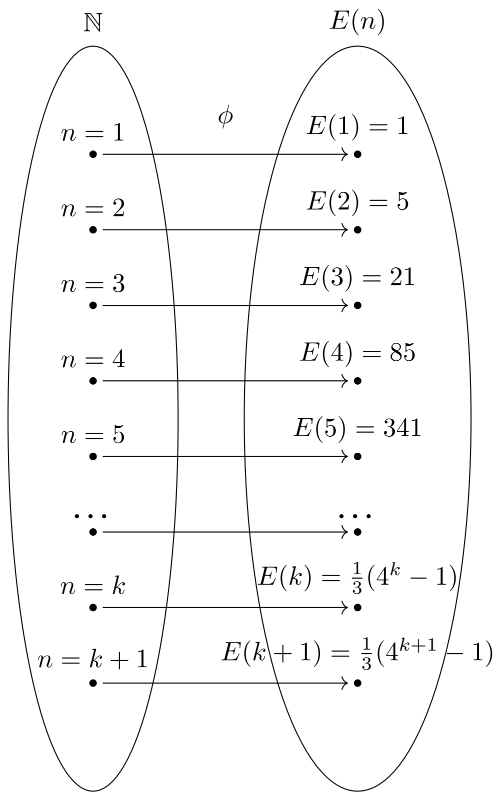
$$\begin{aligned} 4^n &= (1+3)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 3^k \\ &= \sum_{k=0}^{n} \binom{n}{k} 3^k \\ &= 1 \;+\; 3n \;+\; \sum_{k=2}^{n} \binom{n}{2} 3^k \\ &= 1 \;+\; 3 \left[n \;+\; \sum_{k=2}^{n} \binom{n}{2} 3^{k-1} \right] \\ &= 1 + 3 \left[\sum_{k=1}^{n} \binom{n}{k} 3^{k-1} \right] \\ &= 1 + 3[E(n)] \end{aligned} \tag{5.10}$$

This is an identical structure to $3X + 1$, but for *odd* X since $E(n)$ is *strictly odd* [*Lemma 2*]. This was the major observation used to recognize the (*odd* X, $3X + 1$) order isomorphism. To be understood, the values of $E(n)$ for any positive integer n are the order-embedded *escape values* controlling the sequence behavior of Collatz conjecture. Please see the figure demonstrating the enumeration of $E(n)$ [*Figure 5*].

5.2. **Set and Order Theory.**

Definition 5.2 (Trichotomy law)**.** *For any arbitrary real numbers $x, y \in \mathbb{R}$,* ***exactly one*** *of the the following relations is true.*

FIGURE 5. Enumeration of E(n)



(1) $x < y$
(2) $x > y$
(3) $x = y$

Definition 5.3 (Comparability)**.** *Two elements $x, y \in M$ are said to be comparable with respect to a binary relation $\leq$ if at least one of $x \leq y$ or $y \leq x$ is true. The elements x, y are incomparable if they are not comparable.*

Definition 5.4 (Partial order)**.** *A relation $\leq$ is a partial order on a nonempty set M if it satisfies these three properties for all $x, y, z \in M$:*

(1) *Reflexivity: $x \leq x$ for all $x \in M$.*
(2) *Antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$.*
(3) *Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$.*

A partially ordered set is also called a poset [35].

Definition 5.5 (Bounded)**.** *A subset M of a partially ordered set N is called bounded below if there is an element b in N such that $b \leq m$ for all m in M. The subset M is bounded above if there is an element c in N such that $c \geq m$ for all m in M.*

Definition 5.6 (Unbounded)**.** *Let $(N, \leq_N)$ be an ordered set. A subset $M \subseteq N$ is unbounded in N if and only if it is not bounded.*

Definition 5.7 (Linear order)**.** *A linear order is a partial order that is strongly connected in which any two elements are comparable.*

(1) *Reflexive, antisymmetric, and transitive for all $x, y, z \in M$.*
(2) *Comparability: either $x \leq y$ or $y \leq x$ is true for all $x, y \in M$.*

A linearly ordered set is also called a loset [34].

Definition 5.8 (Dense order)**.** *A linear order $\leq$ on a set M is said to be dense, if for all $m_1, m_2 \in M$ for which $m_1 < m_2$, there exists an $m_3 \in M$ such that $m_1 < m_2 < m_3$. That is, for any two distinct elements, one less than the other, there is another element between them.*

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Independent researcher, Private address
Email address: `micewill@gmail.com`