Convertible Bond and Credit Risk

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ABSTRACT

Convertible bond is issued mainly by start-up or small companies. It is susceptible to credit risk. This paper presents a model for valuing convertible bonds by taking credit risk into account. Testing results show that model prices fluctuate randomly around market prices, indicating the model is quite accurate. Convertible bond arbitrage usually employs delta-neutral hedging where an arbitrageur buys a convertible bond and sells the underlying equity at the current delta. Delta neutral hedging removes small directional risks and makes a profit on an explosive upside or downside breakout if the position's gamma is kept positive. Thus, delta neutral hedging is great for uncertain stocks that are expected to make large breakouts in either direction.

Key Words: convertible bond, convertible underpricing, convertible arbitrage, derivative valuation.

1. Introduction

A company can raise capital in financial markets either by issuing equities, bonds, or hybrids (such as convertible bonds). From an investor's perspective, convertible bonds with embedded optionality offer certain benefits of both equities and bonds.

There is a rich literature on the subject of convertible bonds. Arguably, the first widely adopted model among practitioners is the one presented by Goldman Sachs (1994) and then formalized by Tsiveriotis and Fernandes (1998). The Goldman Sachs' solution is a simple one factor model with an equity binomial tree to value convertible bonds.

Tsiveriotis and Fernandes (1998) argue that in practice one is usually uncertain as to whether the bond will be converted, and thus propose dividing convertible bonds into two components: a bond part that is subject to credit risk and an equity part that is free of credit risk.

Grimwood and Hodges (2002) indicate that the Goldman Sachs model is incoherent because it assumes that bonds are susceptible to credit risk but equities are not. Ayache *et al* (2003) conclude that the Tsiveriotis-Fernandes model is inherently unsatisfactory due to its unrealistic assumption of stock prices being unaffected by bankruptcy. To correct this weakness, Davis and Lischka (1999), Andersen and Buffum (2004), Bloomberg (2009), and Carr and Linetsky (2006) etc., propose a jump-diffusion model to explore defaultable stock price dynamics.

Although we agree that under a risk-neutral measure the market price of risk and risk preferences are irrelevant to asset pricing (Hull, 2003) and thereby the expectation of a risk-free[1] asset grows at the risk-free interest rate, we are not convinced that the expected rate of return on a defaultable asset must be also equal to the risk-free rate. We argue that unlike market risk, credit risk actually has a significant impact on asset prices.

Because of their hybrid nature, convertible bonds attract different type of investors. Especially, convertible arbitrage hedge funds play a dominant role in primary issues of convertible debt. In fact, it is

[1] Here, *risk-free* means free of credit risk, but not necessarily of market risk

believed that hedge funds purchase 70% to 80% of the convertible debt offered in primary markets. A prevailing belief in the market is that convertible arbitrage is mainly due to convertible underpricing (i.e., the model prices are on average higher than the observed trading prices) (see Ammann *et al* (2003), Calamos (2011), Choi *et al* (2009), Loncarski *et al* (2009), etc.).

We model both equities and bonds as defaultable in a consistent way. When a firm goes bankrupt, the investors who take the least risk are paid first. Secured creditors have the best chances of seeing the value of their initial investments come back to them. Bondholders have a greater potential for recovering some their losses than stockholders who are last in line to be repaid and usually receive little, if anything.

Valuation under our risky model can be solved by common numerical methods, such as, Monte Carlo simulation, tree/lattice approaches, or partial differential equation (PDE) solutions. The PDE algorithm is elaborated in this paper, but of course the methodology can be easily extended to tree/lattice or Monte Carlo.

The most important parameter to be determined is the volatility input for valuation. A common approach in the market is to use the at-the-money (ATM) implied Black-Scholes volatility to price convertible bonds. However, most liquid stock options have relatively short maturates (rarely more than 8 years). As a result, some authors, such as Ammann *et al* (2003), Loncarski *et al* (2009), Zabolotnyuk *et al* (2010), have to make do with historical volatilities.

The empirical results show that the model prices fluctuate randomly around the market prices, indicating the model is quite accurate. Our empirical evidence does not support a systematic underpricing hypothesis.

It is useful to examine the basics of the convertible arbitrage strategy. A typical convertible bond arbitrage employs delta-neutral hedging, in which an arbitrageur buys a convertible bond and sells the underlying equity at the current delta (see Choi *et al* (2009), Loncarski *et al* (2009), etc.).

We study the sensitivities of convertible bonds and find that convertible bonds have relatively large positive gammas, implying that convertible arbitrage can make a profit on a large upside or downside movement in the underlying stock price.

The rest of this paper is organized as follows: The model is presented in Section 2. Section 3 elaborates the PDE approach; Section 4 discusses the empirical results. The conclusions are provided in Section 5. Some numerical implementation details are contained in the appendices.

2 Model

Convertible bonds can be thought of as normal corporate bonds with embedded options, which enable the holder to exchange the bond asset for the issuer's stock. Despite their popularity and ubiquity, convertible bonds still pose difficult modeling challenges, given their hybrid nature of containing both debt and equity features.

Three sources of randomness exist in a convertible bond: the stock price, the interest rate, and the credit spread. As practitioners tend to eschew models with more than two factors, it is a legitimate question: How can we reduce the number of factors or which factors are most important? Grimwood and Hodges (2002) conduct a sensitivity study and find that accurately modeling the equity process appears crucial. This is why all convertible bond models in the market capture, at a minimum, the dynamics of the underlying equity price.

We consider a filtered probability space (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration.

The risk-free stock price process can be described as

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where $S(t)$ denotes the stock price, $r(t)$ denotes the risk-free interest rate, σ denotes the volatility, $W(t)$ denotes a Wiener process.

The expectation of equation (1) is

$$E\left(dS(t)\middle|\mathcal{F}_t\right) = r(t)S(t)dt \tag{2}$$

where $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$.

Next, we turn to a defaultable stock. The defaultable stock process proposed by Davis and Lischka (1999), Andersen and Buffum (2004), and Bloomberg (2009), etc., is given by

$$dS(t) = \left(r(t) + h(t)\right)S(t\_)dt + \hat{\sigma}S(t\_)dW(t) - S(t\_)dU(t) \tag{3}$$

where $U(t)$ is an independent Poisson process with $dU(t) = 1$ with probability $h(t)dt$ and 0 otherwise, $h(t)$ is the hazard rate or the default intensity,

The expectation of equation (3) is given by

$$E\left(dS(t)\middle|\mathcal{F}_t\right) = \left(r(t) + h(t)\right)S(t)dt - S(t)h(t)dt = r(t)S(t)dt \tag{4}$$

The jump-diffusion model was first proposed in the context of market risk, which naturally exhibits high skewness and leptokurtosis levels and captures the so-called implied volatility smile or skew effects. Ederington and Lee (1993) find that the markets tend to have overreaction and underreaction to the outside news.

However, we wonder whether it is appropriate to propagate the jump-diffusion model directly from the market risk domain to the credit risk domain, as credit risk actually impacts the valuation of assets. This is why financial institutions are required by regulators to report CVA.

The world of credit modeling is divided into two main approaches: structural models and reduced-form (or intensity) models. The structural models regard default as an endogenous event, focusing on the capital structure of a firm. The reduced-form models do not explain the event of default endogenously, but instead characterize it exogenously as a jump process.

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u)du\right) \tag{6}$$

The default probability for the period (*t, s*) in this framework is given by

$$q(t,s) := P(\tau \le s \mid \tau > t, Z) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u)du\right) \tag{7}$$

We consider a defaultable asset that pays nothing between dates t and T. Let $V(t)$ and $V(T)$ denote its values at t and T, respectively. Risky valuation can be generally classified into two categories: the *default time approach* (DTA) and the *default probability (intensity) approach* (DPA).

The DTA involves the default time explicitly. If there has been no default before time T (i.e., $\tau > T$), the value of the asset at T is $V(T)$. If a default happens before T (i.e., $t < \tau \leq T$), a recovery payoff is made at the default time τ as a fraction of the market value[2] given by $\varphi V(\tau)$ where φ is the default recovery rate and $V(\tau)$ is the market value at default. Under a risk-neutral measure, the value of this defaultable asset is the discounted expectation of all the payoffs and is given by

$$V(t) = E\left[\left(D(t,T)V(T)1_{\tau>T} + D(t,\tau)\varphi V(\tau)1_{\tau\leq T}\right) \mid \mathcal{F}_t\right] \tag{8}$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise, and $D(t,\tau)$ denotes the stochastic risk-free discount factor at t for the maturity τ given by

$$D(t,\tau) = \exp\left[-\int_t^\tau r(u)du\right] \tag{9}$$

The DPA relies on the probability distribution of the default time rather than the default time itself. We divide the time period (t, T) into n very small time intervals (Δt) and assume that a default may occur only at the end of each very small period. In our derivation, we use the approximation $\exp(y) \approx 1 + y$ for very small y. The survival and default probabilities for the period (t , $t + \Delta t$) are given by

$$\hat{p}(t) := p(t, t + \Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \tag{10}$$

$$\hat{q}(t) := q(t, t + \Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \tag{11}$$

The binomial default rule considers only two possible states: default or survival. For the one-period (t , $t + \Delta t$) economy, at time $t + \Delta t$ the asset either defaults with the default probability $q(t, t + \Delta t)$ or

[2] Here we use the recovery of market value (RMV) assumption.

survives with the survival probability $p(t, t+\Delta t)$. The survival payoff is equal to the market value $V(t+\Delta t)$ and the default payoff is a fraction of the market value: $\varphi(t+\Delta t)V(t+\Delta t)$. Under a risk-neutral measure, the value of the asset at *t* is the expectation of all the payoffs discounted at the risk-free rate and is given by

$$V(t) = E\left\{\exp\left(-r(t)\Delta t\right)\left[\hat{p}(t) + \varphi(t)\hat{q}(t)\right]V(t+\Delta t)\middle|\mathcal{F}_t\right\} \approx E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \tag{12}$$

where $y(t) = r(t) + h(t)\left(1 - \varphi(t)\right) = r(t) + c(t)$ denotes the risky rate and $c(t) = h(t)\left(1 - \varphi(t)\right)$ is called the (short) credit spread.

Similarly, we have

$$V(t+\Delta t) = E\left\{\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right\} \tag{13}$$

Note that $\exp\left(-y(t)\Delta t\right)$ is $\mathcal{F}_{t+\Delta t}$-measurable. By definition, an $\mathcal{F}_{t+\Delta t}$-measurable random variable is a random variable whose value is known at time $t+\Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-y(t)\Delta t\right)E\left[\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right]\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\sum_{i=0}^{1} y(t+i\Delta t)\Delta t)\right)V(t+2\Delta t)\middle|\mathcal{F}_t\right\} \end{aligned} \tag{14}$$

By recursively deriving from *t* forward over *T* and taking the limit as Δt approaches zero, the risky value of the asset can be expressed as

$$V(t) = E\left\{\exp\left[-\int_t^T y(u)du\right]V(T)\middle|\mathcal{F}_t\right\} \tag{15}$$

Under a risk-neutral measure the market price of risk and risk preferences are irrelevant to asset pricing (Hull, 2003) and thereby the expectation of a risk-free asset grows at the risk-free interest rate. However, credit risk actually has a significant impact on asset prices.

In asset pricing theory, the fundamental no-arbitrage theorems do not require expected returns to be equal to the risk free rate, but only that prices are martingales after discounting under the numeraire.

If a company files bankruptcy, both bonds and stocks go into a default status. In other words, the default probabilities for both of them are the same (i.e., equal to the firm's probability of default). But the recovery rates are different because the stockholders are the lowest priority in the list of the stakeholders in the company, whereas the bondholders have a higher priority to receive a higher percentage of invested funds.

According to equation (15), we propose a risky model that embeds the probability of the default jump rather than the default jump itself into the price dynamics of an asset. The stochastic differential equation (SDE) of a defaultable stock is defined as

$$dS(t) = \left(r(t) + h(t)(1 - \varphi_s(t))\right)S(t)dt + \sigma S(t)dW(t) = y(t)S(t)dt + \sigma S(t)dW(t) \tag{16}$$

where φ_s is the recovery rate of the stock and $y(t) = r(t) + h(t)\left(1 - \varphi_s(t)\right)$ is the risky rate.

For most practical problems, zero recovery at default (or jump to zero) is unrealistic. For example, the stock of Lehman Brothers fell 94.3% on September 15, 2008 after the company filed for Chapter 11 bankruptcy.

Equation (16) is the direct derivation of equation (15). The formula allows different assumptions concerning recovery on default. In particular, $\varphi_s = 0$ represents the situation where the stock price jumps to 0, and $\varphi_s = 1$ corresponds to the risk-free case. The expectation of equations (16) is

$$E\left(dS(t)\middle|\mathcal{F}_t\right) = \left(r(t) + h(t)(1 - \varphi_s(t))\right)S(t)dt \tag{17}$$

3. PDE Algorithm

The defaultable stock price process is given by

$$dS(t) = \left(r(t) - q(t) + h(t)(1 - \varphi_s(t))\right)S(t)dt + \sigma S(t)dW(t) = \mu(t)S(t)dt + \sigma S(t)dW(t) \tag{18}$$

where $q(t)$ is the dividend and $\mu(t) = r(t) - q(t) + h(t)(1 - \varphi_s(t))$.

The valuation of a convertible bond normally has a backward nature since there is no way of knowing whether the convertible should be converted without knowledge of the future value. Only on the maturity date, the value of the convertible and the decision strategy are clear. If the convertible is certain to be converted, it behaves like a stock. If the convertible is not converted at an intermediate node, we are usually uncertain whether the continuation value should be treated as a bond or a stock, because in backward induction the current value takes into account the results of all future decisions and some future values may be dominated by the stock or by the bond or by both.

Suppose that $G(S,t)$ is some function of *S* and *t*. Applying Ito Lemma, we have

$$dG = \left(\mu S \frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \sigma S \frac{\partial G}{\partial S} dW \tag{19}$$

Since the Wiener process underlying *S* and *G* are the same, we can construct the following portfolio so that the Wiener process can be eliminated.

$$X = G - S \frac{\partial G}{\partial S} \tag{20}$$

Therefore, we have

$$dX = dG - \frac{\partial G}{\partial S} dS = \left(\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt \tag{21}$$

In contrast to all previous studies, we believe that the defaultable equity should grow at the risky rate of the equity including dividends, whereas the equity part of the convertible bond should earn the risky rate of the equity excluding dividends, i.e.,

$$\left(r + h(1-\varphi_s)\right) G dt - \left(r - q + h(1-\varphi_s)\right) \frac{\partial G}{\partial S} S dt = dX = \left(\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) dt \tag{22}$$

So that the PDE of the equity component is given by

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + \left(r - q + h(1-\varphi_s)\right) S \frac{\partial G}{\partial S} - \left(r + h(1-\varphi_s)\right) G = 0 \tag{23}$$

Similarly applying Ito Lemma to the bond part of the convertible $B(S,t)$, we obtain

$$dB = \left(\mu S \frac{\partial B}{\partial S} + \frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} \right) dt + \sigma S \frac{\partial B}{\partial S} dW \tag{24}$$

Let us construct a portfolio so that we can eliminate the Wiener process as follows

$$Y = B - S\frac{\partial B}{\partial S} \tag{25}$$

Thus, we have

$$dY = dB - \frac{\partial B}{\partial S}dS = \left(\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2}\right)dt \tag{26}$$

The defaultable equity should grow at the risky rate of the equity including dividends, while the bond part of the convertible bond grows at the risky rate of the bond. Consequently, we have

$$\left(r + h(1-\varphi_b)\right)Bdt - \left(r - q + h(1-\varphi_s)\right)\frac{\partial B}{\partial S}Sdt = dY = \left(\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2}\right)dt \tag{27}$$

where φ_b is the recovery rate of the bond.

The PDE of the bond component is

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + \left(r - q + h(1-\varphi_s)\right)S\frac{\partial B}{\partial S} - \left(r + h(1-\varphi_b)\right)B = 0 \tag{28}$$

The final conditions at maturity *T* can be generalized as

$$G_T = \begin{cases} \eta S_T, & if\ \eta S_T > \min\left[P_c, \max\left(P_p, N+C\right)\right] \\ 0, & otherwise \end{cases} \tag{29}$$

$$B_T = \begin{cases} \min\left[P_c, \max\left(P_p, N+C\right)\right], & if\ \eta S_T \le \min\left[P_c, \max\left(P_p, N+C\right)\right] \\ 0, & otherwise \end{cases} \tag{30}$$

where *N* denotes the bond principal, *C* denotes the coupon, P_c denotes the call price, P_p denotes the put price and η denotes the conversion ratio. The final conditions tell us that the convertible bond at the maturity is either a debt or an equity.

The upside constraints at time $t \in [0,T]$ are

$$\begin{cases} G_t = \eta S_t,\ B_t = 0 & if\ \eta S_t > \min\left[P_c, \max\left(P_p, \tilde{L}_t\right)\right] \\ G_t = 0,\ B_t = P_p & else\ \ if\ \ \tilde{L}_t \le P_p \\ G_t = 0,\ B_t = P_c & else\ \ if\ \ \tilde{L}_t \ge P_c \\ G_t = \tilde{G}_t,\ B_t = \tilde{B}_t & else \end{cases} \tag{31}$$

where $\tilde{L}_t = \tilde{B}_t + \tilde{G}_t$ is the continuation value of the convertible bond, $\tilde{B}_t$ is the continuation value of the bond component and $\tilde{G}_t$ is the continuation value of the equity component.

4. Empirical results

This section presents the empirical results. We use two years of daily data from September 10, 2010 to September 10, 2012, i.e., a total of 522 observation days. This proprietary data are obtained from an investment bank.

We only consider the convertibles outstanding during the period and with sufficient pricing information. As a result, we obtain a final sample of 164 convertible bonds and a total of 164 × 522 = 85,608 observations.

As of September 10, 2012, the sample represents a family of convertible bonds with maturities ranging from 2 months to 36.6 years, and has an average remaining maturity of 4.35 years. The histogram of contracts on September 10, 2012 for various maturity classes is given in Figure 1.

Convertible bond prices observed in the market will be compared with theoretical prices under different volatility assumptions. The sample is segmented into two sets according to maturities: a short-maturity class (0 ~ 8 years) and a long-maturity class (> 8 years).

Figure 1. Maturity distribution of convertible bonds

This histogram splits the total number of convertible bonds of the sample into different classes according to the maturity of each convertible bond. The x-axis represents maturities in years and the y-axis represents the number of convertibles in each class. The *n* maturity class covers contracts with maturities ranging from *n*-1 years to *n* years.

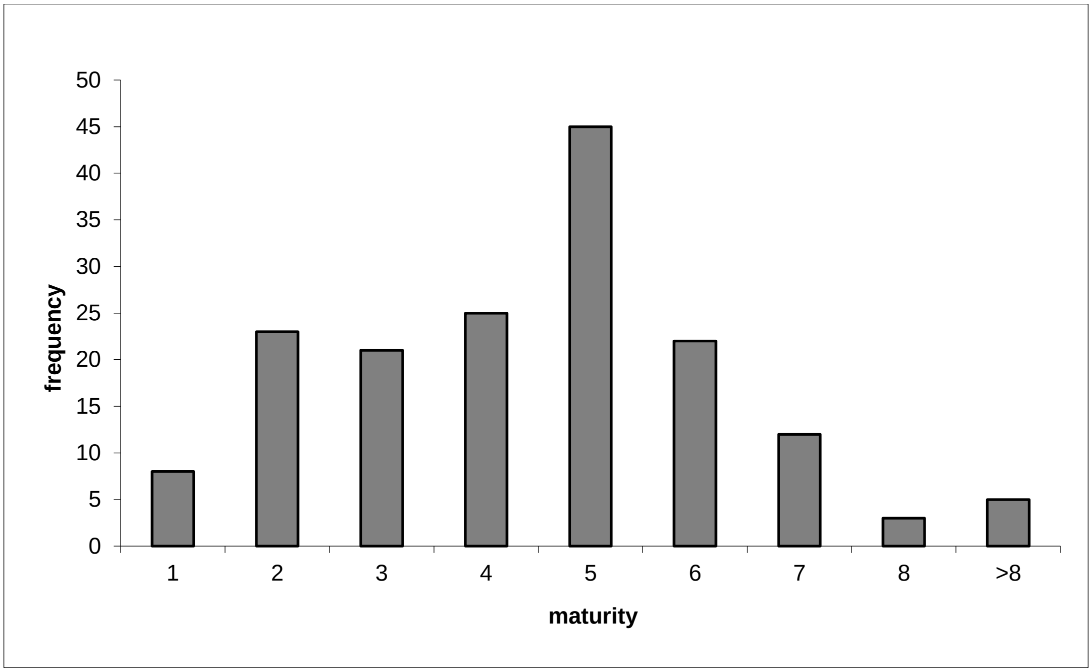


Table 1. Convertible bond examples

We hide the issuer names according to the security policy of the investment bank, but everything else is authentic. In the market, either a conversion price or a conversion ratio is given for a convertible bond, where conversion ratio = (face value of the convertible bond) / (conversion price).

Convertible bond	Case 1 (a 7-year convertible)	Case 2 (a 20-year convertible)
Issuer	*X* company	*Y* company
Principal of bond	100	100
Annual coupon rate	2.625	5.5
Payment frequency	Semiannual	Semiannual
Issuing date	June 9, 2010	June 15, 2009
Maturity date	June 15, 2017	June 15, 2029
Conversion price	30.288	13.9387
Currency	USD	USD
Day count	30/360	30/360
Business day convention	Following	Following
Put price	-	100 at June 20, 2014

Let valuation date be September 10, 2012. An interest rate curve is the term structure of interest rates, derived from observed market instruments that represent the most liquid and dominant interest rate products for certain time horizons.

Table 2: A list of instruments used to build a USD interest rate curve

This table displays the closing prices as of September 10, 2012. These instruments are used to construct an interest rate curve.

Instrument Name	Price
September 19, 2012 LIBOR	0.6049%
September 2012 Eurodollar 3 month	99.6125
December 2012 Eurodollar 3 month	99.6500
March 2013 Eurodollar 3 month	99.6500
June 2013 Eurodollar 3 month	99.6350
September 2013 Eurodollar 3 month	99.6200
December 2013 Eurodollar 3 month	99.5900
March 2014 Eurodollar 3 month	99.5650
2 year swap rate	0.3968%
3 year swap rate	0.4734%
4 year swap rate	0.6201%
5 year swap rate	0.8194%
6 year swap rate	1.0537%
7 year swap rate	1.2738%
8 year swap rate	1.4678%
9 year swap rate	1.6360%
10 year swap rate	1.7825%
12 year swap rate	2.0334%
15 year swap rate	2.2783%
20 year swap rate	2.4782%
25 year swap rate	2.5790%
30 year swap rate	2.6422%

The equity information and recovery rates are provided in Table 3. To determine hazard rates, we need to know the observed market prices of corporate bonds or CDS premia, as the market standard practice is to fit the implied risk-neutral default intensities to these credit sensitive instruments. The corporate bond prices are unfortunately not available for companies *X* and *Y*, but their CDS premia are observable as shown in Table 4.

Unlike other studies that use bond spreads for pricing (see Tsiveriotis and Fernandes (1998), Ammann *et al* (2003), Zabolotnyuk *et al* (2010), etc.), we perform risky valuation based on credit information extracted from CDS spreads.

Table 3. Equity and recovery information

This table displays the closing stock prices and dividend yields on September 10, 2012, as well as the recovery rates

	Company X	**Company Y**
Stock price	34.63	23.38
Dividend yield	2.552%	3.95%
Bond recovery rate	40%	36.14%
Equity recovery rate	2%	1%

Table 4. CDS premia

This table displays the closing CDS premia as of September 10, 2012.

Name	**Company X**	**Company Y**
6 month CDS spread	0.00324	0.01036
1 year CDS spread	0.00404	0.01168
2 year CDS spread	0.00612	0.01554
3 year CDS spread	0.00825	0.01924
4 year CDS spread	0.01027	0.02272
5 year CDS spread	0.01216	0.02586
7 year CDS spread	0.01388	0.02851
10 year CDS spread	0.01514	0.03003

15 year CDS spread	0.01544	0.03064
20 year CDS spread	0.01559	0.03101

The most important input parameter to be determined is the volatility for valuation. A common approach in the market is to use ATM implied Black-Scholes volatilities to price convertible bonds. For the 5-year outstanding convertible bond (case 1 in Table 1), we find the ATM implied Black-Scholes volatility is 31.87%, and then price the convertible bond accordingly.

For the 17-year outstanding convertible bond (case 2 in Table 1), however, most liquid stock options have relatively short maturates (rarely more than 8 years). Therefore, some authors, such as Ammann *et al* (2003), Loncarski *et al* (2009), Zabolotnyuk *et al* (2010), have to make do with historical volatilities.

Table 5. Model prices vs. market prices

This table shows the differences between the model prices and the market prices of the convertible bonds under different volatility assumptions, where Difference = (Model price) / (Market observed price) – 1. The convertible bonds are defined in Table 1.

	Case 1 (a 7-year convertible)	**Case 2 (a 20-year convertible)**
Type of volatility	ATM implied Black-Scholes volatility	Annualized historical volatility
Value of volatility	31.87%	18.07%
Model price	134.32	171.58
Market observed price	134.88	169.77
Difference	-0.42%	1.07%

We repeat this exercise for all contracts on all observation dates. For any short-maturity convertible bond, we use the ATM implied Black-Scholes volatility for pricing, whereas for any long-maturity convertible bond, we perform valuation via the historical volatility. The results are presented in Tables 6.

Table 6. Underpricing statistics for different maturity classes

An observation corresponds to a price snapshot of a convertible bond at a certain valuation date. Underpricing is referred to as the model price minus the market price.

Maturity	**Observations**	**Underpricing**			
		Mean (%)	**Std (%)**	**Max (%)**	**Min (%)**
≤ 8 years	82998	-0.13	1.37	0.79	-1.08
> 8 years	2610	1.67	2.03	2.24	0.58

Next, our sample is partitioned into subsamples according to the moneyness of convertibles. The moneyness is measured by the ratio of the conversion value to the equivalent straight bond value or the investment value.

Table 7. Underpricing statistics for different moneyness classes

The moneyness is measured by dividing the conversion value through the associated straight bond value. An observation corresponds to a snapshot of the market used to price a convertible bond at a certain valuation date.

Moneyness	**Observations**	**Underpricing**	
		Mean (%)	**Std (%)**
< 0.5	5794	0.72	2.23
0.5 – 0.7	10595	-0.87	2.37
0.7 – 0.9	19850	0.51	1.64
0.9 – 1.1	14737	0.45	1.12
1.1 – 1.3	14379	-0.55	1.89
1.3 – 1.5	11631	-0.42	2.04
> 1.5	8622	-0.62	1.72

From Tables 7, it can be seen that the model prices fluctuate randomly around the market prices (sometimes overpriced and sometimes underpriced), indicating the model is quite accurate. Empirically, we do not find support for presence of a systematic underpricing as indicated in previous.

In a typical convertible bond arbitrage strategy, the arbitrageur entails purchasing a convertible bond and selling the underlying stock to create a delta neutral position. The number of shares sold short usually reflects a delta-neutral or market neutral ratio. It is well known that delta neutral hedging not only removes small directional risks but also is capable of making a profit on an explosive upside or downside breakout if the position's gamma is kept positive.

We calculate the delta and gamma values for the two deals described in table 1. The Greek vs. spot equity price graphs are plotted in Figures 2~ 5. It can be seen that the deltas increase with the underlying stock prices in Figures 2 and 4. At low market levels, the convertibles behave like their straight bonds with very small deltas.

Figure 2. Delta vs spot price graph for a 7-year convertible bond

This graph shows how the delta of the 7-year convertible bond (described in Table 1) changes as the underlying stock price changes.

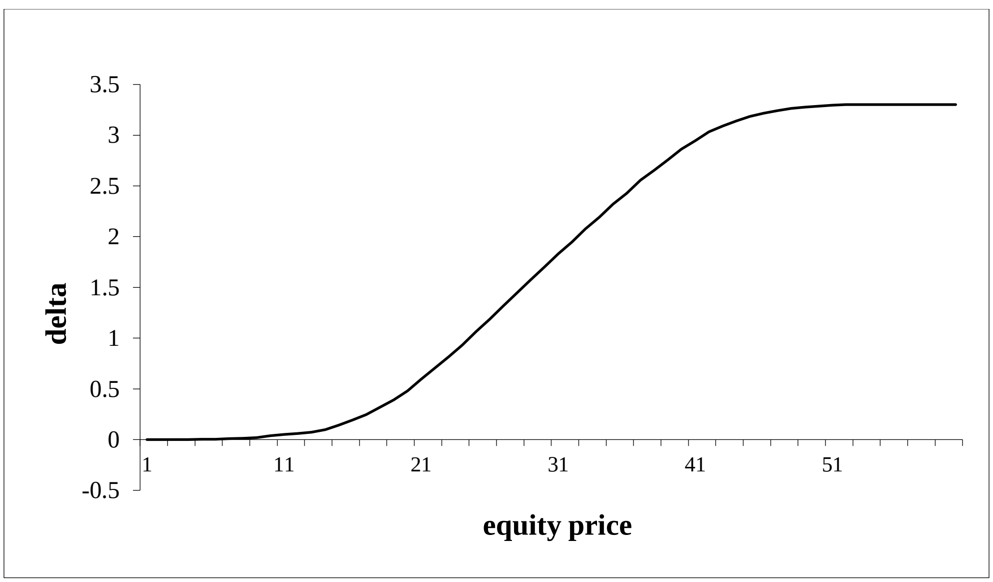


Figure 3. Variation in gamma vs. spot price for a 7-year convertible bond

This graph shows how the gamma of the 7-year convertible bond (described in Table 1) changes as the underlying stock price changes.

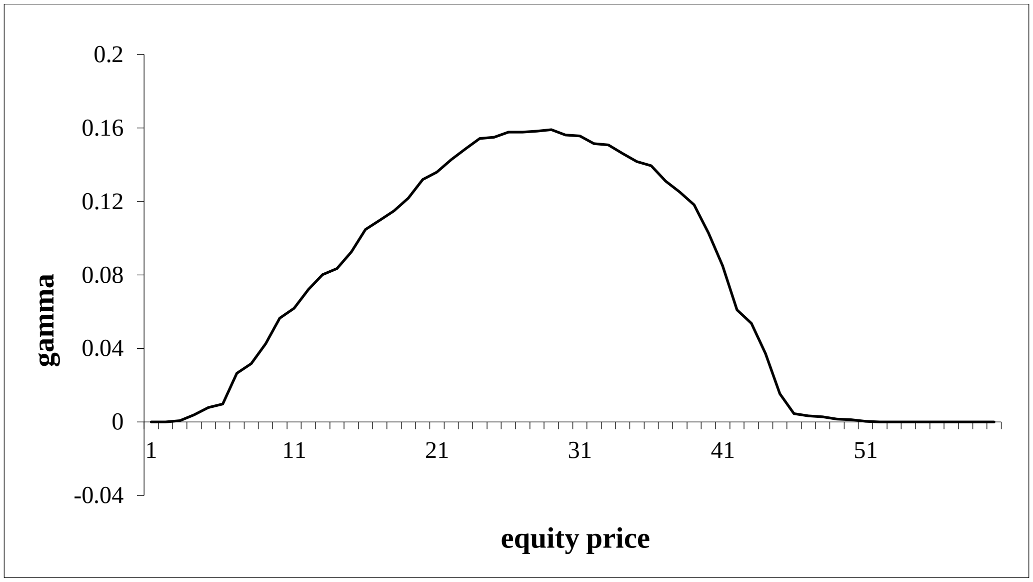


Figure 4. Delta plotted against changing spot price for a 20-year convertible bond

This graph shows how the delta of the 20-year convertible bond (described in Table 1) changes as the underlying stock price changes.

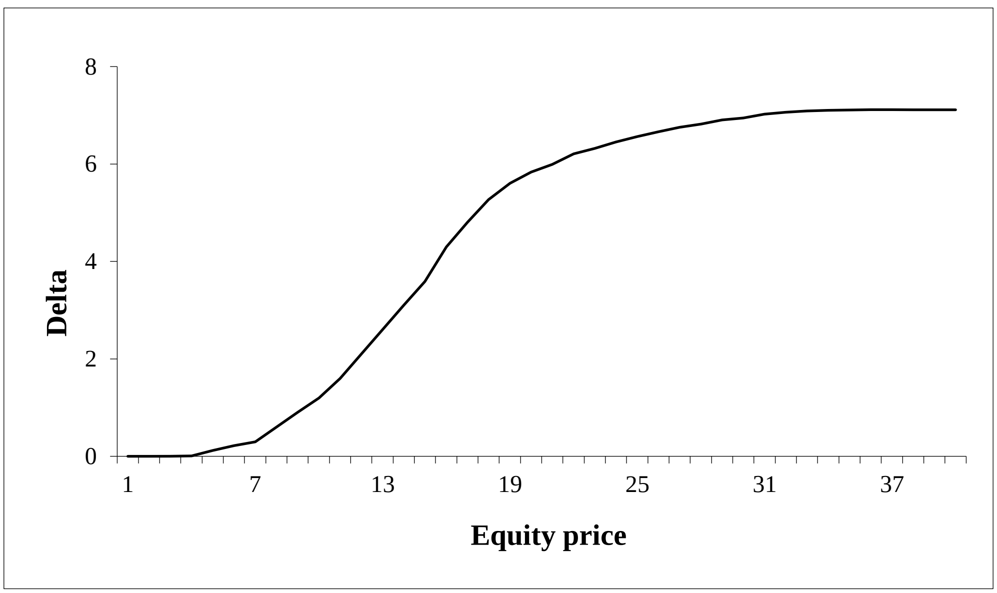
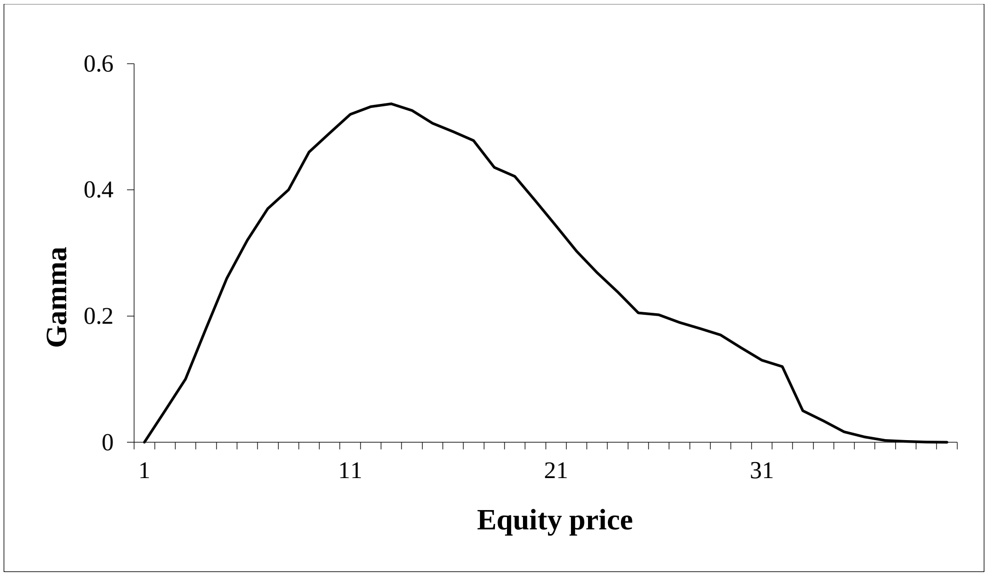


Figure 5. Gamma variation with spot price for a 20-year convertible bond

This graph shows how the gamma of the 20-year convertible bond (described in Table 1) changes as the underlying stock price changes.



The gamma diagrams in Figures 3 and 5 have a frown shape. The gammas are the highest when the convertibles are at-the-money. It is intuitive that when the stock prices rise or fall, profits increase because of favorably changing deltas

5. Conclusion

This paper aims to price hybrid financial instruments (e.g., convertible bonds) whose values may simultaneously depend on different assets subject to credit risk in a proper and consistent way. The motivation for our model is that if a company goes bankrupt, all the securities (including the equity) of the company default.

Our study shows that risky asset pricing is quite different from risk-free asset pricing. In fact, the expectation of a defaultable asset actually grows at a risky rate rather than the risk-free rate. We propose a hybrid framework to value risky equities and debts in a unified way. The model relies on the probability distribution of the default jump rather than the default jump itself. As such, the model can achieve a high order of accuracy with a relatively easy implementation.

Empirically, we do not find evidence supporting a systematic underpricing hypothesis. We also find that convertible bonds have relatively large positive gammas, implying that convertible arbitrage can make a significant profit on a large upside or downside movement in the underlying stock price.

Appendix

In this section, we describe the numerical method used to solve discrete forms of (23) and (28). Let $x = \ln\left(\frac{S_t}{S_0}\right)$ and define backward time as $\delta = T - t$. The equations (23) and (28) can be rewritten as

$$\frac{\partial B}{\partial \delta} - \frac{1}{2}\sigma^2 \frac{\partial^2 B}{\partial x^2} - \left(r - q + h(1-\varphi_s) - \frac{\sigma^2}{2}\right)\frac{\partial B}{\partial x} + \left(r + h(1-\varphi_b)\right)B = 0 \qquad \text{(A1)}$$

$$\frac{\partial G}{\partial \delta} - \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2} - \left(r - q + h(1-\varphi_s) - \frac{\sigma^2}{2}\right)\frac{\partial G}{\partial x} + \left(r + h(1-\varphi_s)\right)G = 0 \qquad \text{(A2)}$$

The equations (A1) and (A2) can be approximated using Crank-Nicolson rule. We discretize the *x* to be equally spaced as a grid of nodes 0 ~ *M*. At the maturity, G_T and B_T are determined according to (29) and (30). At any time i+1, the boundary conditions are

$$\begin{cases} B_0^{i+1}(1+0.5(r+h(1-\varphi_b)\Delta\tau) = B_0^i(1-0.5(r+h(1-\varphi_b))\Delta\tau) \\ G_0^{i+1}(1+0.5(r-q+h(1-\varphi_s)\Delta\tau) = G_0^i(1-0.5(r-q+h(1-\varphi_s))\Delta\tau) \end{cases} \quad when\ x \approx 0 \qquad \text{(A3)}$$

$$\begin{cases} B_M^{i+1} = 0 \\ G_M^{i+1} = \eta S_M \end{cases} \quad when\ x \approx \infty \qquad \text{(A4)}$$

References

Agarwal, V., Fung, W., Loon, Y. and Naik, N. (2007) Liquidity provision in the convertible bond market: analysis of convertible arbitrage hedge funds. CFR-working paper.

Ammann, M., Kind, A., and Wilde, C. (2003) Are convertible bonds underpriced? An analysis of the French market. *Journal of Banking & Finance* 27: 635-653.

Andersen, L. and Buffum, D. (2004) Calibration and implementation of convertible bond models. *Journal of Computational Finance* 7: 1-34.

Ayache, E., Forsyth, P. A., and Vetzal, K. R. (2003) The valuation of convertible bonds with credit risk. *Journal of Derivatives* 11: 9-30.

Batta, G., Chacko, G. and Dharan, B. (2007) Valuation consequences of convertible debt issuance. Working paper.

Brennan, M. and Schwartz, E. (1980) Analyzing convertible bonds. *Journal of Financial and Quantitative Analysis* 15: 907-929.

Calamos, N. P. (2011) *Convertible arbitrage: Insights and techniques for successful hedging*. John Wiley & Sons.

Carr, P. and Linetsky, V. (2006) A jump to default extended CEV model: an application of Bessel processes. *Finance and Stochastics* 10: 303-330.

Carayannopoulos, P. and Kalimipalli, M. (2003) Convertible bond prices and inherent biases. *Journal of Fixed Income* 13: 64-73.

Choi, D., Getmansky, M. and Tookes, H. (2009) Convertible bond arbitrage, liquidity externalities, and stock prices. *Journal of Financial Economics* 91: 227-251.

Davis, M. and Lischka, F. R. (1999) Convertible bonds with market risk and credit risk. Working paper, Tokyo-Mitsubishi International plc.

Duffie, D., and Huang, M. (1996) Swap rates and credit quality. *Journal of Finance* 51: 921-949.

Ederington, L. and Lee, H. (1993) How markets process information: News releases and volatility. *Journal of Finance* 48: 1161-1191.

FinPricing, 2013, Pricing tool, https://finpricing.com/lib/EqBarrier.html

Grimwood, R., and Hodges, S. (2002) The valuation of convertible bonds: a study of alternative pricing models. Working paper, Warwick University.

Goldman Sachs (1994) Valuing convertible bonds as derivatives. Quantitative Strategies Research Notes, Goldman Sachs.

Hull, J. (2003) *Options, Futures and Other Derivatives (5th ed.)*. Prentice Hall, Upper Saddle River, NJ.

Jarrow, R. A. and Protter, P. (2004) Structural versus reduced form models: a new information based perspective. *Journal of Investment Management* 2: 34-43.

Loncarski, I., Horst, J. and Veld C. (2009) The rise and demise of the convertible arbitrage strategy. *Financial Analysts Journal* 26: 35-50.

Merton, R. C. (1976) Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economy* 3: 125-144.

J.P. Morgan (2001) Par credit default swap spread approximation from default probabilities. Risk Publications.

Somanath, V.S. (2011) *International financial management*. I.K. International Publishing House Pvt. Ltd.

Tsiveriotis, K. and Fernandes, C. (1998) Valuing convertible bonds with credit risk. *Journal of Fixed Income* 8: 95–102.

Zabolotnyuk, Y., Jones, R., and Veld, C., (2010), “An empirical comparison of convertible bond valuation models,” Financial Management, 39, 2, 675-705.