Full Free Motion of Celestial Bodies Around a Central Mass

Why do They Mostly Orbit in the Equatorial Plane?

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Abstract

After some research on the web, I was not able to find any study that answer some questions like:

- Why do secondary masses orbit in the equatorial plane of the central mass?
- Why do many galaxies have spiral shapes?
- What is the real orbit of a secondary mass around the central mass? *Even when Keppler laws are right, they only give geometrical relations. It can't be serious to continue with the use of these laws in the present days.*
- How to better predict motion's trajectories when some velocity component and/or mass change?
- Why study the Coriolis force or the Euler force separately, as isolated motions? They appear naturally in the general equations of motion and should not be considered as isolated forces. They are just components of the main force.

When we look at planets with their moons, our solar system, or distant galaxies, we see in general, that the secondary masses orbit around the central mass mostly at its equatorial plane. If the present analysis is correct, then it gives the answer by demonstrating why this happens. It also demonstrates the spiral nature of the motion, as well as the real orbit trajectories under diverse conditions. Moreover, it shows the motion modulation given by precession and/or nutation.

Introduction

The aim of this analysis is to provide a more efficient way to calculate and predict orbits and trajectories, not only for celestial bodies, but also for artificial satellites we send outside our atmosphere, and for ballistic. The equations presented here might also help improve weather forecasts and perhaps predict atmospheric behavior more precisely.

Nowadays, we know that gravity is an electromagnetic (EM) wave of atomic origin caused by a kind of dipole oscillation inside the atom [1]. This EM wave is always present in the universe as long as atoms exist and travels at the speed of light. When the originating mass of the EM gravity wave explodes or collapses, then a high amplitude EM gravity pulse is produced, which modulates the tail or final part of the regular wave. This pulse is the last message that we get from that mass.

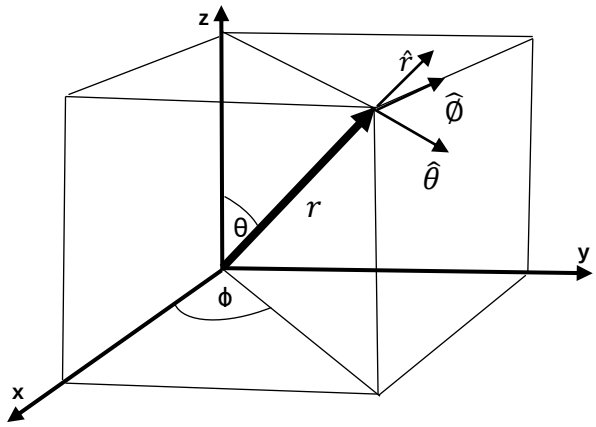
We also know that the gravity field produced by a spinning central mass induces a motion on a secondary mass [2]. This induced motion has similar characteristics to that of the central mass.

I don't have the knowledge to express gravity as a wave. Therefore, the present analysis is made based on the law of gravity according to Newton (instant action-at-a-distance). Just a pure mechanical study of the *related free motion of two centers of masses* in space, by considering all nine components of the gravity acceleration. Even though gravity in this study is not considered as a wave (where time delay appears), the results are astonishing. However, the low velocities

(compared with the speed of light) involved in the interaction of the free motion of two masses, make this study very suitable, *except* when the masses suddenly change, explode or collapse.

When I started the study, I did only analyze the motion in a single polar plane, without considering 3D spatial coordinates. I soon realized that this was a poor approach, not matching the real world. Then I switched to a spherical coordinate system to analyze the free motion between two centers of masses under the effects of gravity.

Motion in spherical coordinates



+θ when measured clockwise from +z
+ϕ when measured counterclockwise from +x
+r when measured outwards from the origin

$$x = r\sin\theta\cos\emptyset$$

$$y = r\sin\theta\sin\emptyset$$

$$z = r\cos\theta$$

The unit vectors are:

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x}\sin\theta\cos\emptyset + \hat{y}\sin\theta\sin\emptyset + \hat{z}\cos\theta$$

$$\hat{\theta} = \hat{\emptyset}\,\mathrm{x}\,\hat{r} = \hat{x}\cos\theta\cos\emptyset + \hat{y}\cos\theta\sin\emptyset - \hat{z}\sin\theta$$

$$\hat{\emptyset} = \frac{\hat{z}\,\mathrm{x}\,\hat{r}}{\sin\theta} = -\hat{x}\sin\emptyset + \hat{y}\cos\emptyset$$

Since the unit vectors are not attached to the coordinate system axes, they are functions of time and can point in any direction. Therefore, they are not constant, and their time derivatives are not zero.

The time derivative of the unit vectors

$$\dot{\hat{r}} = \frac{\partial\hat{r}}{\partial r}\dot{r} + \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\emptyset}\dot{\emptyset} = \dot{\theta}\hat{\theta} + \dot{\emptyset}\sin\theta\,\hat{\emptyset}$$

$$\dot{\hat{\theta}} = \frac{\partial\hat{\theta}}{\partial r}\dot{r} + \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\emptyset}\dot{\emptyset} = -\dot{\theta}\hat{r} + \dot{\emptyset}\cos\theta\,\hat{\emptyset}$$

$$\dot{\hat{\emptyset}} = \frac{\partial\hat{\emptyset}}{\partial r}\dot{r} + \frac{\partial\hat{\emptyset}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\emptyset}}{\partial\emptyset}\dot{\emptyset} = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\emptyset}$$

Velocity and Acceleration of a Point in Space (or a particle)

The position vector is given by: $\vec{r} = r\hat{r}$ (1)

Assuming that r is not constant, the expression for the velocity of a point located at the tip of the position vector is:

$$\vec{v} = \dot{\vec{r}} = \dot{r}r + \hat{r}\dot{r}$$

By replacing $\dot{\hat{r}}$ we obtain the final expression for the velocity with the three components:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\emptyset}\sin\theta\,\hat{\emptyset} \qquad (2)$$

Next, to obtain the expression for the acceleration, we differentiate the velocity with respect to time:

$$\vec{a} = \dot{r}\dot{\hat{r}} + \ddot{r}\hat{r} + r\dot{\theta}\dot{\hat{\theta}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\emptyset}\sin\theta\,\dot{\hat{\emptyset}} + \dot{r}\dot{\emptyset}\sin\theta\,\hat{\emptyset} + r\ddot{\emptyset}\sin\theta\,\hat{\emptyset} + r\dot{\emptyset}\dot{\theta}\cos\theta\,\hat{\emptyset}$$

By simplifying and ordering terms, we obtain the final expression for the acceleration:

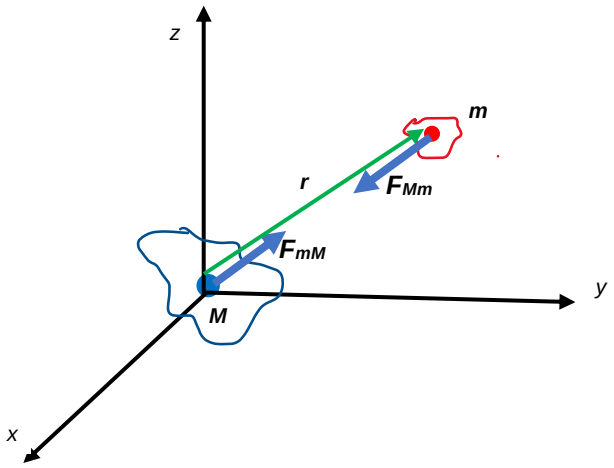
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\emptyset}^2 sin^2\theta\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\emptyset}^2 \sin\theta\cos\theta\right)\hat{\theta} + \left(r\ddot{\emptyset}\sin\theta + 2r\dot{\theta}\dot{\emptyset}\cos\theta + 2\dot{r}\dot{\emptyset}\sin\theta\right)\hat{\emptyset} \quad (3)$$

Equation (3) shows the nine components of the acceleration, three for each direction. In general terms, we have:

$$\vec{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_\emptyset\hat{\emptyset} \qquad (4)$$

Gravity as the Cause of the Acceleration

To analyze the free motion due to gravity of a secondary mass with respect to a central mass, we must find the components of the gravitational acceleration in all three directions: $\hat{r}$, $\hat{\theta}$, and $\hat{\emptyset}$.



Let's suppose a mass ***M*** of any shape, whose center of mass is at the origin of the coordinate system, and a mass ***m*** of any shape, whose center of mass is at a distance ***r*** from ***M***. The attraction forces on them due to gravity are:

$$\boldsymbol{F_{mM} = -F_{Mm} = G\frac{m*M}{r^2}\hat{r}}$$

Where $\mathrm{G} = 6.67408 * 10^{-11}\ \frac{Nm^2}{Kg^2}$ is the universal gravitational constant.

Both centers of masses ***M*** and ***m*** may represent single bodies or distribution of several bodies.

The acceleration produced by ***M*** on ***m*** is given by: $a_{Mm} = -\mathrm{G}\frac{M}{r^2}\hat{r}$, and the acceleration produced by ***m*** on ***M*** is given by: $a_{mM} = \mathrm{G}\frac{m}{r^2}\hat{r}$.

We have two equal and opposite forces, but the accelerations are not equal. Working with two forces or two accelerations make things more complicated, and it is not helpful to analyze the interaction between the masses. How to find only one equation that describes the gravitational motion between the two bodies?

Gravity Interaction Between Two Masses Expressed With One Equation

We can make use of the second law of Newton, and express the force of gravity as:

$$\mathrm{F} = \mathrm{m}\vec{a} = \mathrm{m}\frac{d^2r}{dt^2}\hat{r} = -\mathrm{G}\frac{m*M}{r^2}\hat{r}$$

For mass ***M*** and ***m*** we can write:

$$\mathrm{M}\frac{d^2r}{dt^2}\hat{r} = -\mathrm{G}\frac{m*M}{r^2}\hat{r} \quad => \quad \frac{d^2r}{dt^2}\hat{r} = -\mathrm{G}\frac{m}{r^2}\hat{r} \qquad (5)$$

$$\mathrm{m}\frac{d^2r}{dt^2}(-\hat{r}) = -\mathrm{G}\frac{m*M}{r^2}(-\hat{r}) \quad => \quad \frac{d^2r}{dt^2}\hat{r} = -\mathrm{G}\frac{M}{r^2}\hat{r} \qquad (6)$$

Equation (5) is the acceleration produced by mass ***m***, and Eq. (6) is the acceleration produced by mass ***M***. By adding equations (5) and (6), we obtain:

$$2\frac{d^2r}{dt^2}\hat{r} = -\mathrm{G}\frac{(M+m)}{r^2}\hat{r} \quad => \quad \boldsymbol{\frac{d^2r}{dt^2} = -G\frac{(M+m)}{2}\frac{\hat{r}}{r^2} = a_{Gr}\hat{r}} \qquad (7)$$

Equation (7) gives the interaction or *relative motion* between the two masses. It's an average acceleration expressed by the average of the masses so that the relative acceleration produced by any one of the masses on the other has now the same value, which can be written as:

$$a_{avg} = -\mathrm{G}\frac{m_{avg}}{r^2}\hat{r} \qquad (8)$$

This way we can reduce our system to only one acceleration expression which accounts for the interaction of both masses.

For a big difference in masses (***M*>>*m*),** as the mass of the Sun compared with the mass of the Earth (the mass of the Sun is 333.000 times greater), then the acceleration is mainly given by the bigger mass (Sun):

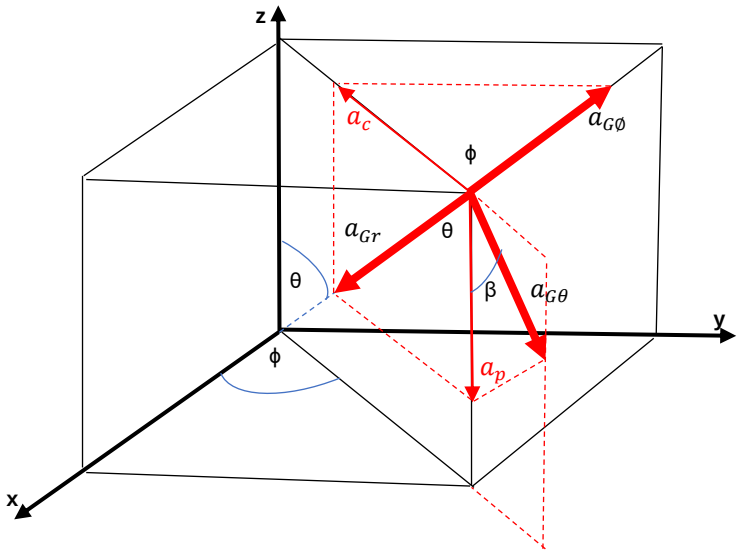
$$\frac{d^2r}{dt^2} = -\mathrm{G}\frac{M}{2}\frac{\hat{r}}{r^2}$$

Components of the Gravity Acceleration in Spherical Coordinates

Equation (7) gives the acceleration of gravity in just one direction ($\hat{r}$). The general expression of the acceleration of gravity in spherical coordinates is:

$$\vec{a}_G = a_{Gr}\hat{r} + a_{G\theta}\hat{\theta} + a_{G\emptyset}\hat{\emptyset} \qquad (9)$$

Let's find the remaining components of $\vec{a}_G$.



The centripetal acceleration a_c is the horizontal component of a_{Gr} parallel to the x-y plane and points to the z-axis, which will be rotating in counterclockwise direction.

$$a_c = a_{Gr}\sin\theta$$

The projection of a_c in the direction of $\hat{\emptyset}$ is the component $a_{G\emptyset}$:

$$a_{G\emptyset} = a_c\cos\emptyset$$

By replacing a_c we obtain the final expression for $a_{G\emptyset}$:

$$\boldsymbol{a_{G\emptyset} = a_{Gr}\sin\theta\cos\emptyset} \qquad (10)$$

By replacing a_{Gr} from Eq. (7) in (10), we obtain the gravitational acceleration in the $\hat{\emptyset}$ direction:

$$\boldsymbol{a_{G\emptyset} = -G\frac{(M+m)}{2\,r^2}\sin\theta\cos\emptyset} \qquad (11)$$

The polar acceleration a_p is the vertical component of a_{Gr} parallel to the z-axis and ***points downwards*** when the vector $\vec{r}$ is above the x-y plane or ***points upwards*** when the vector $\vec{r}$ is below the x-y plane. This component is responsible for the pendulum-like motion of the center of mass ***m*** around the equatorial plane of the central mass ***M***.

$$a_p = a_{Gr}\cos\theta$$

The projection of a_p in the direction of $\hat{\theta}$ is the component $a_{G\theta}$:

$$a_{G\theta} = a_p\cos\beta \qquad \text{with } \beta = \frac{\pi}{2} - \theta \qquad \Rightarrow \qquad a_{G\theta} = a_p\cos\left(\frac{\pi}{2} - \theta\right) = a_p\sin\theta$$

By replacing a_p we obtain the final expression for $a_{G\theta}$:

$a_{G\theta} = a_{Gr}\cos\theta\sin\theta$ => $\boldsymbol{a_{G\theta} = a_{Gr}\frac{\sin(2\theta)}{2}}$ (12)

By replacing a_{Gr} from Eq. (7) in (12), we obtain the gravitational acceleration in the $\hat{\theta}$ direction:

$$\boldsymbol{a_{G\theta} = -G\frac{(M+m)}{4\,r^2}\sin(2\theta)} \quad (13)$$

Inserting the components of the gravitational acceleration in Eq. (9), we get:

$$\vec{a}_G = -G\frac{(M+m)}{2\,r^2}\hat{r} - G\frac{(M+m)}{4\,r^2}\sin(2\theta)\,\hat{\theta} - G\frac{(M+m)}{2\,r^2}\sin\theta\cos\emptyset\,\hat{\emptyset} \quad (14)$$

Relating The Motion With Its Cause

Now we can equate the equation of motion (4) with its cause (9):

$\vec{a} = \vec{a}_G$ => $a_r\hat{r} + a_\theta\hat{\theta} + a_\emptyset\hat{\emptyset} = a_{Gr}\hat{r} + a_{G\theta}\hat{\theta} + a_{G\emptyset}\hat{\emptyset}$

We obtain a system of three differential equations that describe the *full* motion of the center of mass ***m*** relative to ***M***:

$a_r = a_{Gr}$; $a_\theta = a_{G\theta}$; $a_\emptyset = a_{G\emptyset}$

Replacing all components:

$$\ddot{r} - r\dot{\theta}^2 - r\dot{\emptyset}^2 sin^2\theta = -G\frac{(M+m)}{2\,r^2} \quad (15)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\emptyset}^2\sin\theta\cos\theta = -G\frac{(M+m)}{4\,r^2}\sin(2\theta) \quad (16)$$

$$\mathrm{r}\ddot{\emptyset}\sin\theta + 2\mathrm{r}\dot{\theta}\dot{\emptyset}\cos\theta + 2\dot{r}\dot{\emptyset}\sin\theta = -\mathrm{G}\frac{(M+m)}{2\quad^2}\sin\theta\cos\emptyset \quad (17)$$

These equations describe the radial motion (15), the polar motion (16), and the azimuthal motion (17).

Radial Motion Differential Equation

We can write Eq. (15) as:

$$\ddot{\mathrm{r}}_{(\mathrm{t})} - \mathrm{r}_{(\mathrm{t})}\dot{\theta}^2_{(\mathrm{t})} - \mathrm{r}_{(\mathrm{t})}\dot{\emptyset}^2_{(\mathrm{t})}\sin^2\left(\theta_{(\mathrm{t})}\right) + \mathrm{G}\frac{(\mathrm{M}+\mathrm{m})}{2\,\mathrm{r}^2} = 0$$

$$\ddot{r}_{(t)} - \left[\dot{\emptyset}^2_{(t)}sin^2\left(\theta_{(t)}\right) + \dot{\theta}^2_{(t)}\right]r_{(t)} + G\frac{(M+m)}{2r^2_{(t)}} = 0 \quad (18)$$

Polar Motion Differential Equation

We can write Eq. (16) as:

$$r_{(t)}\ddot{\theta}_{(t)} + 2\dot{r}_{(t)}\dot{\theta}_{(t)} - r_{(t)}\dot{\emptyset}^2_{(t)}\sin\theta_{(t)}\cos\theta_{(t)} + G\frac{(M+m)}{4\,r^2_{(t)}}\sin\left(2\theta_{(t)}\right) = 0$$

$$\ddot{\theta}_{(t)} + \frac{2\dot{r}_{(t)}\dot{\theta}_{(t)}}{r_{(t)}} - \dot{\emptyset}^2_{(t)}\frac{\sin\left(2\theta_{(t)}\right)}{2} + G\frac{(M+m)}{4\,r^3_{(t)}}\sin\left(2\theta_{(t)}\right) = 0$$

$$\ddot{\theta}_{(t)} + \frac{2\dot{r}_{(t)}\dot{\theta}_{(t)}}{r_{(t)}} + \left[G\frac{(M+m)}{4\,r^3_{(t)}} - \frac{\dot{\emptyset}^2_{(t)}}{2}\right]\sin\left(2\theta_{(t)}\right) = 0 \quad (19)$$

Azimuthal Motion Differential Equation

We can write Eq. (17) as:

$$r_{(t)}\ddot{\emptyset}_{(t)}\sin\theta_{(t)} + 2r_{(t)}\dot{\theta}_{(t)}\dot{\emptyset}_{(t)}\cos\theta_{(t)} + 2\dot{r}_{(t)}\dot{\emptyset}_{(t)}\sin\theta_{(t)} + G\frac{(M+m)}{2\,r^2}\sin\theta_{(t)}\cos\emptyset_{(t)} = 0$$

$$\ddot{\emptyset}_{(t)} + \frac{2r_{(t)}\dot{\theta}_{(t)}\dot{\emptyset}_{(t)}\cos\theta_{(t)}}{r_{(t)}\sin\theta_{(t)}} + \frac{2\dot{r}_{(t)}\dot{\emptyset}_{(t)}\sin\theta_{(t)}}{r_{(t)}\sin\theta_{(t)}} + G\frac{(M+m)}{2r_{(t)}^2}\frac{\sin\theta_{(t)}\cos\emptyset_{(t)}}{r_{(t)}\sin\theta_{(t)}} = 0$$

$$\ddot{\emptyset}_{(t)} + \frac{2\dot{\theta}_{(t)}\dot{\emptyset}_{(t)}\cos\theta_{(t)}}{\sin\theta_{(t)}} + \frac{2\dot{r}_{(t)}\dot{\emptyset}_{(t)}}{r_{(t)}} + G\frac{(M+m)}{2r_{(t)}^3}\cos\emptyset_{(t)} = 0$$

$$\ddot{\emptyset}_{(t)} + 2\left[\frac{\dot{\theta}_{(t)}}{\tan\theta_{(t)}} + \frac{\dot{r}_{(t)}}{r_{(t)}}\right]\dot{\emptyset}_{(t)} + G\frac{(M+m)}{2r_{(t)}^3}\cos\emptyset_{(t)} = 0 \qquad (20)$$

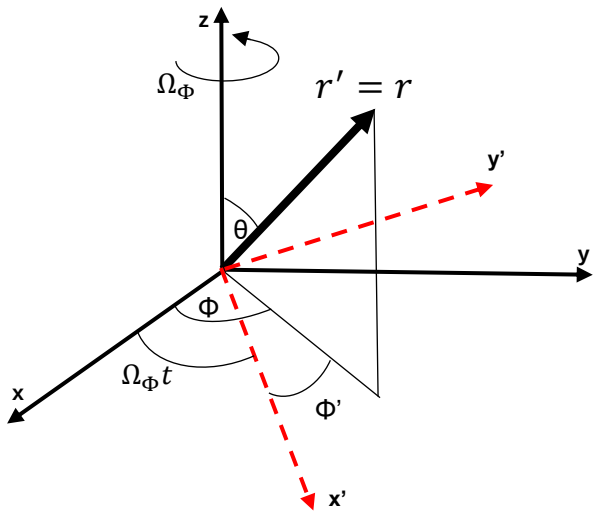
The System Of Differential Equations Of The Free Motion

Now we have a system of three differential equations that describe the free motion in space resulting from the interaction of two masses.

$\ddot{\boldsymbol{r}}_{(t)} - \left[\dot{\emptyset}_{(t)}^2 \boldsymbol{sin}^2\left(\boldsymbol{\theta}_{(t)}\right) + \dot{\boldsymbol{\theta}}_{(t)}^2\right]\boldsymbol{r}_{(t)} + \boldsymbol{G}\frac{(\boldsymbol{M}+\boldsymbol{m})}{2\boldsymbol{r}_{(t)}^2} = \boldsymbol{0}$	(21)	=>	*Radial motion*
$\ddot{\boldsymbol{\theta}}_{(t)} + \frac{2\dot{\boldsymbol{r}}_{(t)}\dot{\boldsymbol{\theta}}_{(t)}}{\boldsymbol{r}_{(t)}} + \left[\boldsymbol{G}\frac{(\boldsymbol{M}+\boldsymbol{m})}{4\,\boldsymbol{r}_{(t)}^3} - \frac{\dot{\emptyset}_{(t)}^2}{2}\right]\boldsymbol{\sin}\left(\boldsymbol{2\theta}_{(t)}\right) = \boldsymbol{0}$	(22)	=>	*Polar motion*
$\ddot{\emptyset}_{(t)} + \boldsymbol{2}\left[\frac{\dot{\boldsymbol{\theta}}_{(t)}}{\tan\boldsymbol{\theta}_{(t)}} + \frac{\dot{\boldsymbol{r}}_{(t)}}{\boldsymbol{r}_{(t)}}\right]\dot{\emptyset}_{(t)} + \boldsymbol{G}\frac{(\boldsymbol{M}+\boldsymbol{m})}{2\boldsymbol{r}_{(t)}^3}\boldsymbol{\cos}\,\emptyset_{(t)} = \boldsymbol{0}$	(23)	=>	*Azimuthal motion*

***Obs**.: this is a system with three unknowns and must obviously be solved as such. Attempts to solve any equation as an isolated one, by just giving values to two unknown variables, is non-sense.*

The angular velocity of translation of the center of mass ***m*** is $\dot{\emptyset}_{(t)} = \Omega$. We'll assume that ***m*** doesn't spin, but only revolves around ***M***. We'll also assume that the center of mass ***M*** at the origin of the coordinate system is spinning at an angular velocity Ω_ϕ, which means that our frame of reference is rotating at that speed as shown in the figure:



In the primed reference frame, we have:

$t' = t \quad r' = r \quad \theta' = \theta$ and $\phi' = \phi - \Omega_\phi t$

Since the only difference is the relative angle, by differentiating we get the new relative angular velocity:

$\dot{\phi}' = \dot{\phi} - \Omega_\phi = \Omega - \Omega_\phi$

The relative angular velocity could be zero, positive or negative.

$\dot{\boldsymbol{\phi}}' = \boldsymbol{0}$ *could mean that the spin of M = translation velocity of m, or both angular velocities = 0.*

$\dot{\boldsymbol{\phi}}' < \mathbf{0}$ *is the case Sun-Earth or Earth-Moon. The spin of the Sun is faster than the translation's angular speed of the Earth. The spin of the Earth is faster than the translation's angular speed of the Moon.*

$\dot{\boldsymbol{\phi}}' > \mathbf{0}$ *another case for which I don't have an example.*

In the system of differential equations (where only $\dot{\phi}$ appears), **we have just to change the initial conditions by replacing it with the proper relative angular velocity for each case**. The variable $\dot{\phi}$ represents the same as $\dot{\boldsymbol{\phi}}'$. The only difference is the value to be used in its initial condition.

If the mass ***M*** represents the mass of the Sun, we know that the period of one spin is ~ 25 days. Then its angular velocity is: $\Omega_{\phi} = 2.9\ 10^{-6}[\frac{1}{s}]$. The angular velocity of the Earth translation is:

$\Omega = 2\ 10^{-7}[\frac{1}{s}]$. Then the relative angular velocity between both masses is: $\Omega - \Omega_{\phi} = -2.7\ 10^{-6}[\frac{1}{s}]$

Remarks About The Polar Motion

The differential equation of the polar motion (Eq. 22), has a term in brackets that can have three different values:

$$\left[\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{4}\,\boldsymbol{r}_{(t)}^{3}} - \frac{\dot{\phi}_{(t)}^{2}}{\mathbf{2}}\right] = \mathbf{0} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{3}} = \dot{\phi}_{(t)}^{2} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{2}} = \boldsymbol{r}_{(t)}\dot{\phi}_{(t)}^{2}$$

$$\left[\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{4}\,\boldsymbol{r}_{(t)}^{3}} - \frac{\dot{\phi}_{(t)}^{2}}{\mathbf{2}}\right] > \mathbf{0} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{3}} > \dot{\phi}_{(t)}^{2} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{2}} > \boldsymbol{r}_{(t)}\dot{\phi}_{(t)}^{2}$$

$$\left[\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{4}\,\boldsymbol{r}_{(t)}^{3}} - \frac{\dot{\phi}_{(t)}^{2}}{\mathbf{2}}\right] < \mathbf{0} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{3}} < \dot{\phi}_{(t)}^{2} \quad => \quad \boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\boldsymbol{r}_{(t)}^{2}} < \boldsymbol{r}_{(t)}\dot{\phi}_{(t)}^{2}$$

We see that the cases above happen when the *relative gravitational acceleration* between the masses is equal to, greater than, or less than the *centripetal acceleration* of the center of mass ***m***.

For a given relative angular velocity (translation of ***m***-rotation of ***M***), the equations above will give the distance:

$$\boldsymbol{r}_{(t)} = \sqrt[3]{\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\dot{\phi}_{(t)}^{2}}} \qquad \boldsymbol{r}_{(t)} < \sqrt[3]{\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\dot{\phi}_{(t)}^{2}}} \qquad \boldsymbol{r}_{(t)} > \sqrt[3]{\boldsymbol{G}\frac{(\boldsymbol{M+m})}{\mathbf{2}\,\dot{\phi}_{(t)}^{2}}} \qquad (24)$$

The first term of the acceleration in the brackets **gives a curious result** when we substitute the values of the mass of the Sun in ***M***, the mass of the Earth in ***m***, and the distance Sun-Earth in ***r***.

$$\mathbf{G}\frac{(\mathbf{M}+\mathbf{m})}{\mathbf{4}\,\mathbf{r}_{(t)}^{3}} = \frac{6.67408\ 10^{-11}(2\ 10^{30} + 5.9\ 10^{24})}{4\ (150\ 10^{9})^{3}} = \mathbf{9.887555\ 10^{-15}}\ \left[\frac{1}{s^{2}}\right]$$

This angular acceleration is a small fraction of the gravitational acceleration on the Earth's surface.

Solving The System of Three Differential Equations

Finding the solutions of the system of differential equations (21), (22), and (23) is not an easy task, and requires the use of some software for Mathematics and Physics applications. Numerical solutions were obtained by using a Runge-Kutta method, as well as several plots that describe the

behavior of the free motion in space due to the gravitational interaction between two centers of masses under diverse initial conditions. The results are astonishing.

However, as some calculations might take days in a regular PC, in order to confirm some conclusions, a much more powerful computer is needed to evaluate the motion under certain conditions, and for very long times.

Nevertheless, in most of the cases, the calculations and the resolution of the graphics were satisfactory for the present analysis.

The system of Differential equations (21), (22), and (23), describe the relative free motion of ANY two centers of masses ***M*** and ***m*** under the action of gravity acceleration. The centers of masses can represent single bodies or a group of bodies of any shape.

To make calculations with real values, the masses of the **Sun** and **Earth**, as well as distances like Sun-Earth were used in the system of differential equations. It could also have been the interaction between Andromeda and the Milky Way, or a human artificial satellite and Earth, or whatever two (centers of) masses one may think about.

Therefore, in the present study we have a central mass ***M*** which is much bigger than the secondary mass ***m,*** that is ***M*** >> ***m***.

Results show that in most of the cases the center of mass ***m*** *reaches equilibrium around the equatorial plane* of the central mass ***M***, i.e., $\boldsymbol{\theta \approx 90°}$.

There are only a few cases where ***m*** reaches equilibrium at $\theta \approx 0°$ *(+ z-axis),* $\theta \approx 180°$ *(- z-axis)*, and other polar angular positions.

Results

First, general results were obtained with initial conditions for three different relative azimuthal angular velocities ($\dot{\phi}$), which give three different initial positions of the vector $r_{(t)}$ as given by equations (24). Thus, we can check the behavior of the motion by giving the term in brackets of Eq. (22) three different values: zero, positive or negative. See Table 1.

Next, a similar analysis was performed, but exclusively for polar initial positions different of zero. See Table 2.

The final analysis shown in Table 3 was made to check the motion behavior at rather short and long distances between the two centers of masses.

Parameters and Common Initial Conditions

$G = 6.67408 * 10^{-11}\ \frac{Nm^2}{Kg^2}$ *(the universal gravitational constant)*

$M = 2\ 10^{30}\ Kg$ *(the mass of the Sun)*

$m = 5.9\ 10^{24}\ Kg$ *(the mass of the Earth)*

$\dot{r}_{(0)} = 0\ [m/s]$ *(linear velocity of the center of mass **m** at t=0)*

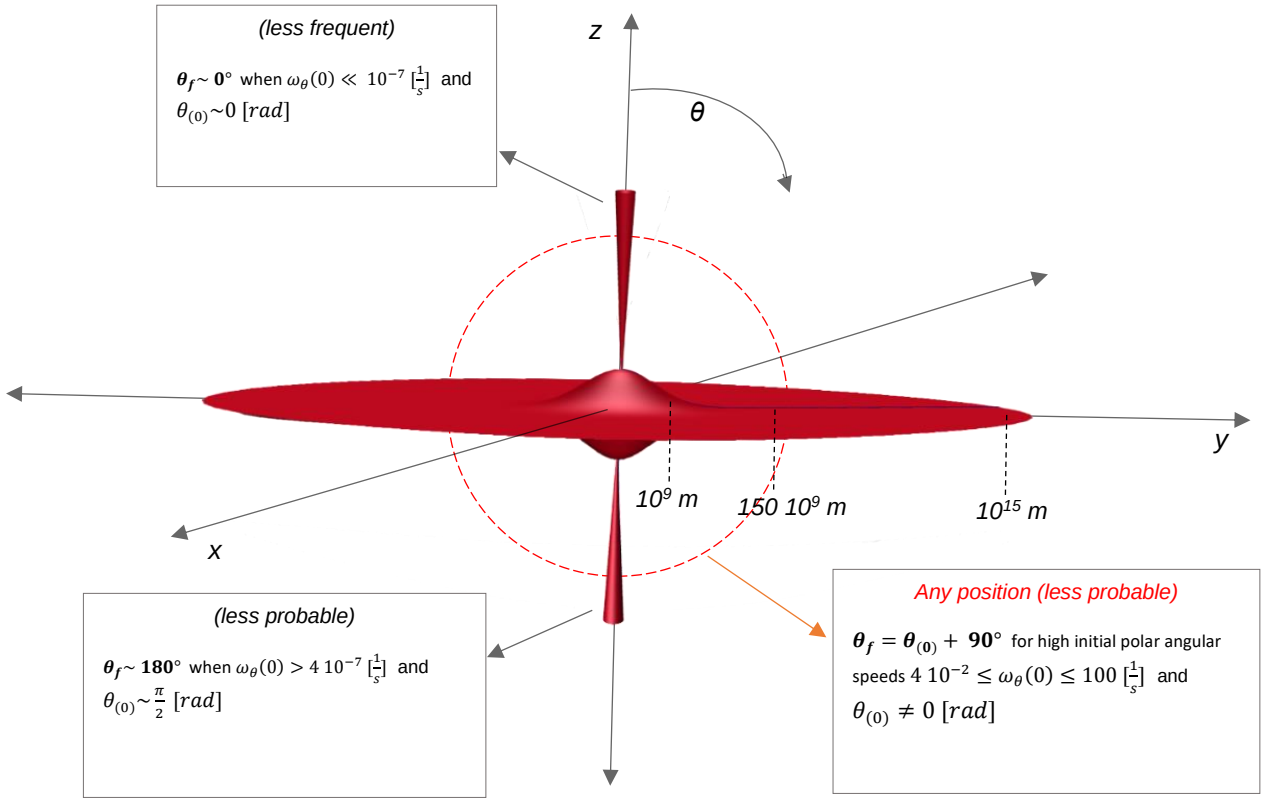
$\emptyset_{(0)} = 0\ [rad]$ *(azimuthal position of the center of mass **m** at t=0)*

Since there is a singularity for $\theta = 0$, the "zero" initial condition of the polar angle was taken close to zero as $\theta = \frac{\pi}{10^3}\ [rad]$.

$\dot{\phi}_{(t)} = \Omega$ *(is the relative angular velocity between the spin of the center of mass **M** and the translation of the center of mass **m**)*

Scheme of Positions of Mass m in Function of the Distance to Mass M, according to results

An approximate scheme (not in scale) based on the results of Table 1, Table 2, and Table 3, which illustrates most of the positions that mass ***m*** can adopt with respect to the distance (r) to mass ***M*** located at the origin of coordinates.



Comments About The Results

Even when the results of the present study were obtained with a limited set of initial conditions, they are in close agreement with observations of the universe, and what mother nature shows us.

Recall that the **polar component** of the gravity acceleration on mass *m* **points downwards** in the upper hemisphere, and **points upwards** in the lower hemisphere.

Then, it is somewhat logical to expect a pendulum-like motion in the polar direction, that may "force" mass ***m*** to stay approximately in the equatorial plane, or x-y plane. The universe shows that this is the general case.

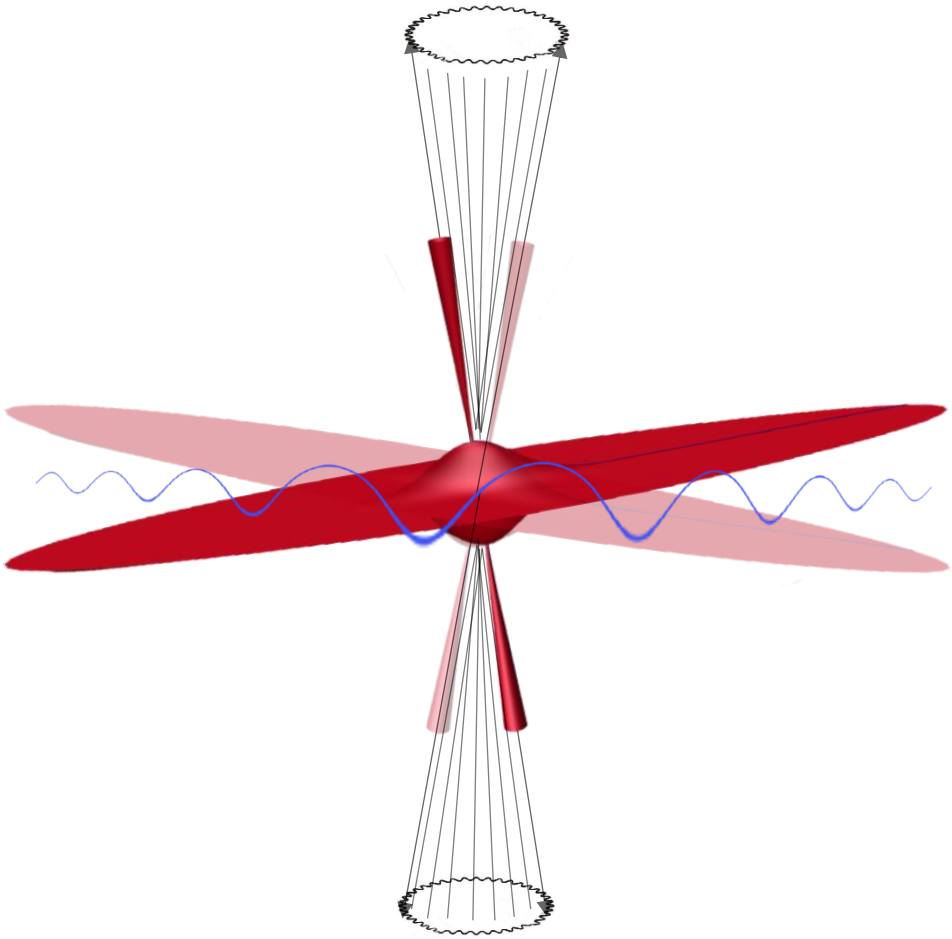
However, if the distance is short between the masses, then the secondary mass (***m***), depending on its initial angular position, may adopt any position around mass ***M***. In this region $\boldsymbol{\theta_f = \theta_{(0)}}$. The universe also shows that this is the general case.

According to what the universe shows us, one may classify some positions as "less frequent" and "less probable". In these cases, the initial conditions that gave such results could also be "less frequent" and "less probable" to happen in the universe.

In general, the center of mass ***m*** can reach its equilibrium or stable final position, in two ways: by approaching the final position asymptotically, or by oscillating. When oscillations are present, they are mostly *modulated in amplitude* by a lower frequency motion, which I believe could be due to the precession of mass ***M***. This lower frequency motion modulation is, in general, also of *variable frequency*. So, we have both: **amplitude and frequency modulation of the motion**.

There are also some unstable solutions of the motion (resonance), or because of some singularity, where amplitude increases steadily. There is no real equilibrium or stable final position in these cases.

Precession and Nutation – Not Always Present



Before the present study, I thought that Precession and Nutation were characteristic motions of any spinning body.

Surprisingly, some results show that the center of mass ***m*** *not always* reaches the equilibrium position with an oscillatory motion, but asymptotically. *Precession and Nutation are absent in such cases*.

For simplicity, Precession and Nutation motions are not shown in the scheme of the positions of mass ***m***, but they might be present in spinning bodies under certain conditions. In such cases, the center of mass ***m*** at equilibrium will follow those motions.

The z-axis will describe an oscillatory cone-like trajectory. In the x-y plane, the motion will be like a wobbling dish or coin on a table. Since gravitational acceleration is an EM wave [1], the motion will be subject to a delay in function of the distance between both centers of masses.

Conclusions

The present study demonstrates that in most of the cases, the secondary mass orbits in the equatorial plane of the main central mass. It is also shown the real orbit trajectories and the spiral nature of the motion. Additionally, it is also demonstrated that the motion modulation caused by precession and/or nutation is not always present. I don’t have access to high computation capacity clusters to validate some conclusions. The use of powerful computation machines is advisable for this study. However, the performed calculations and plots in this analysis are more than satisfactory to make conclusions.

Table 1 – General Results

The calculations were made for three initial positions of the vector *r(t)*:

$r_{(0)} = 1.186071579\ 10^{11}\ [m]$; $r_{(0)} = 100\ 10^{9}\ [m]$; $r_{(0)} = 150\ 10^{9}\ [m]$ *(distance Sun-Earth)*

	$\theta(0)$ [rad]	$\frac{\pi}{10^3}$ ($\theta \approx 0°$)	$\frac{\pi}{10^3}$	$\frac{\pi}{10^3}$	$\frac{\pi}{10^3}$	$\frac{\pi}{10^3}$	$\frac{\pi}{10^3}$	$\frac{\pi}{10^3}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$
	$\omega_\theta(0)$ $[\frac{1}{s}]$	0	10^{-15}	10^{-7}	10^{-2}	1	10	100	0	10^{-15}	10^{-7}	10^{-2}	1	10	100
$\Omega = 2\cdot10^{-7}$ $[\frac{1}{s}]$	θ_f [deg]	~0°-1°	~0°-0.85°	|90°| (+ 90°& -90°) (**) 90°	90°	90°	90°	90°	90°	90°	90° 109° (*)	150°	135°	180°	150°
$\Omega = -2.7\cdot10^{-6}$ $[\frac{1}{s}]$		~1.2°-3°	~1.2°-3°	90°	90°	90°	90°	90°	90°-92°	90°-91°	92°	150°	135°	180°	150°
$\Omega = 0$ $[\frac{1}{s}]$		$\theta(0)$	$\theta(0)$	80°-90°	90°	90°	90°	90°	$\theta(0)$	$\theta(0)$	88°-90° |90°| (+ 90°& -90°) (***)	150°	135°	180°	150°

(*): result for $r=150\cdot10^9$ *and* $\Omega = 2\cdot10^{-7}\left[\frac{1}{s}\right]$

(**): result for $r = 1.186071579\ 10^{11}$ *and* $\Omega = 2\cdot10^{-7}\left[\frac{1}{s}\right]$

(***): result for $r = 100\ 10^9$ *and* $\Omega = 0\left[\frac{1}{s}\right]$

Note:

1. Results are the same for the three initial position values of r(0), except for (*), (**) and (***).

2. In general, the final polar position is given by $\theta_f = \theta(0) + 90°$*, except for the colored table cells, where the final position is* $\theta_f = \theta(0)$*, when the initial value of the polar angular velocity* $\omega_\theta(0)$ *is very low or zero.*

See APPENDIX I for all the plots and the details of the results summarized in the table above.

Table 2 – Results only for initial polar positions θ(0) ≠ 0 of the center of mass m

The calculations were made for three initial positions of the vector *r(t)*:

$r_{(0)} = 1.186071579\ 10^{11}\ [m]$; $r_{(0)} = 100\ 10^{9}\ [m]$; $r_{(0)} = 150\ 10^{9}\ [m]$ *(distance Sun-Earth)*

	$\theta(0)$ [rad]	$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$						
	$\omega_\theta(0)$ $[\frac{1}{s}]$	0	10^{-15}	10^{-7}	10^{-2}	1	10	100
$\Omega = 2\cdot10^{-7}$ $[\frac{1}{s}]$	θ_f [deg]	82°-90°	87°-90°	89°-90° 125°(**) 109° (**)	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$
$\Omega = -2.7\cdot10^{-6}$ $[\frac{1}{s}]$		90°-92°	90°-92°	92°-94°	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$
$\Omega = 0$ $[\frac{1}{s}]$		$\theta(0)$	$\theta(0)$	82°-90° 357° (*) 329° (*)	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$	$\theta(0) + 90°$

There is a "critical" $\omega_\theta(0)$ for which the mass m reaches the equilibrium away from 90°. This value is around $3\cdot10^{-7} < \omega_\theta(0) < 4\cdot10^{-7}$

(*): the plot cannot be done beyond a certain point because of a probable singularity. The angle increases steadily, and it is impossible to see an equilibrium or a final stable angle.

(**): results for $r=150\cdot10^{9}$, $\theta(0)=\frac{\pi}{3}$ and $\theta(0)=\frac{\pi}{2}$ respectively

Note:

1. Results are quite the same for the four initial position values of $\theta(0)$ and the three initial position values of $r(0)$, except for (*) and (**).

2. In general, for initial values of the polar angular velocitiy $\omega_\theta(0) > 4\cdot10^{-7}$, the final polar position is given by $\theta_f = \theta(0) + 90°$. We also observe two cases on the colored table cells on the left, where the final position is $\theta_f = \theta(0)$ when the initial value of the polar angular velocity $\omega_\theta(0)$ is very low or zero.

See APPENDIX II for all the plots and the details of the results summarized in the table above.

Table 3 – Results for a short and long-distance (r) between the two centers of masses

The calculations were made for two initial positions of the vector *r(t)*:

$r1_{(0)} = 10^9\ [m]$ and $r2_{(0)} = 10^{15}\ [m]$

	$\theta(0)$ [rad]	$\frac{\pi}{10^3}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$						
	$\omega_\theta(0)$ $1\ [\frac{1}{s}]$	0	10^{-15}	10^{-7}	10^{-2}	1	10	100
$\Omega = 2\cdot10^{-7}$ $1\ [\frac{1}{s}]$	θ_f [deg]	*r1(0)* => $\theta(0)$ *r2(0)* => 90°	*r1(0)* => $\theta(0)$ *r2(0)* => 90°	*r1(0)* => $\theta(0)$ *r2(0)* => 90°-116°	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$
$\Omega = -2.7\cdot10^{-6}$ $1\ [\frac{1}{s}]$		*r1(0)* => $\theta(0)$ (*) *r2(0)* => 90°	*r1(0)* => $\theta(0)$ (*) *r2*(0) ⇒ 90°	*r1(0)* => $\theta(0)$ (*) *r2(0)* => 90°-92°	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$
$\Omega = 0$ $1\ [\frac{1}{s}]$		*r1(0)* => $\theta(0)$ (") *r2(0)* => $\theta(0)$ (")	*r1(0)* => $\theta(0)$ (") *r2(0)* => $\theta(0)$ (")	*r1(0)* => $\theta(0)$ (") *r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$	*r1(0) & r2(0)* => $\theta(0) + 90°$

("): there is some discontinuity ***for all*** *the initial positions of the polar angle. In some cases, it was taken the average value of the angle. In other cases, it was considered the maximum value of the angle before the discontinuty, since the angle increases steadily.*

(*): there is some discontinuity ***only*** *at the initial position* $\theta(0) = \frac{\pi}{10^3}$ *of the polar angle. It was taken the maximum value of the angle before the discontinuty, since the angle increases steadily.*

Note:

1. Results are the same for the five initial position values of $\theta(0)$ *and the two initial position values of r(0), when* $\omega_\theta(0) > 4\cdot10^{-7}$. *Results are also the same for* $\omega_\theta(0) < 4\cdot10^{-7}$ *and for r1(0), except for (") and (*), where the final polar angle is given by* $\theta_f = \theta(0)$ *when the initial value of the polar angular velocity* $\omega_\theta(0)$ *is very low or zero. However, for r2(0) we have three cases that differ of the rest, as shown in the highlighted values.*

2. In general, for initial values of the polar angular velocitiy $\omega_\theta(0) > 4\cdot10^{-7}$, *the final polar position is given by* $\theta_f = \theta(0) + 90°$.

See APPENDIX III for all the plots and the details of the results summarized in the table above.

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APPENDIX I

All plots and details of the results for r(0) = 1.186071579 10^{11} [m]

Analysis and plots of the motion for an initial distance of $r_{(0)} = 1.186071579\ 10^{11}\ \ [m]$, and the following initial conditions and parameters:

$G = 6.67408 * 10^{-11}\ \frac{Nm^2}{Kg^2}$ *(the universal gravitational constant)*

$M = 2\ 10^{30}\ Kg$ *(the mass of the Sun)*

$m = 5.9\ 10^{24}\ Kg$ *(the mass of the Earth)*

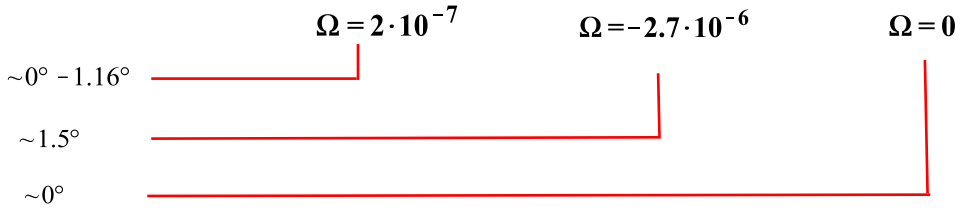
$\dot{r}_{(0)} = 0\ [m/s]$ *(linear velocity of the center of mass **m** at t=0)*

$\phi_{(0)} = 0\ [rad]$ *(azimuthal position of the center of mass **m** at t=0)*

Since there is a singularity for $\theta = 0$, the "zero" initial condition of the polar angle was taken close to zero as $\theta = \frac{\pi}{10^3}\ [rad]$.

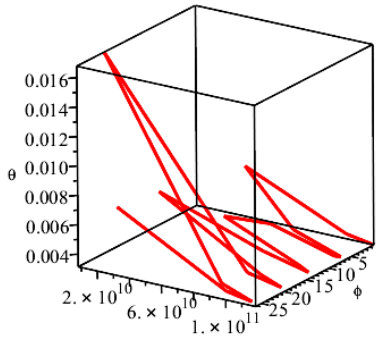
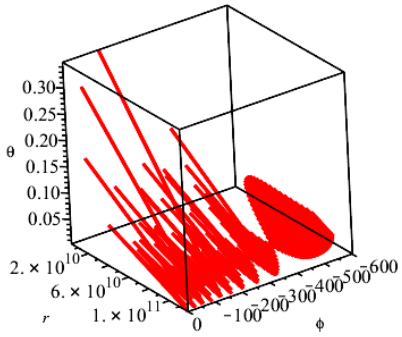
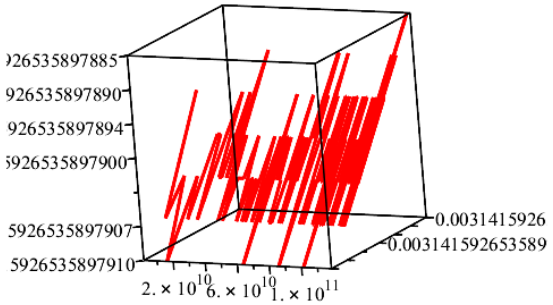
$\dot{\phi}_{(t)} = \Omega$ *(is the relative angular velocity between the spin of the center of mass **M** and the translation of the center of mass **m**)*

The polar angle of equilibrium (θ_f) is found for several values of the initial conditions of $\theta_{(0)}$, $\omega_{(0)}$, and Ω. The three vertical values of θ_f on each table cell, from top to bottom, are the results for each of the three values of Ω from left to right. For example:

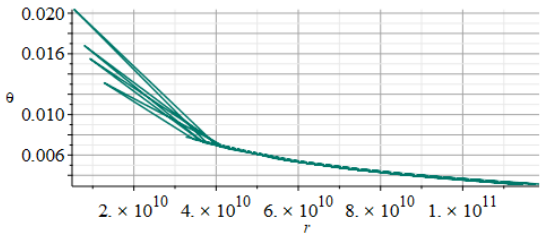


Obs.: the table is very long and continues over several pages.

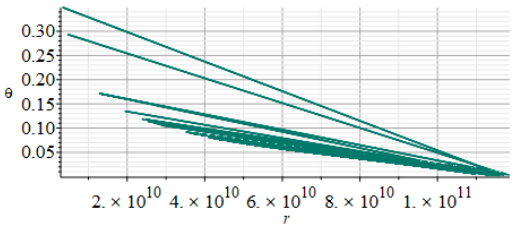
See Table 1 for a summary of the results obtained here.

$\theta(0)$ *[rad]*	$\omega_\theta(0)$ $[\frac{1}{s}]$	θ_f *[deg]*	$\Omega = 2\cdot 10^{-7}$ $[\frac{1}{s}]$ (makes the term in brackets = 0) $r(0) = 1.186071579\ 10^{11}, D(r)(0) = 0, \phi(0) = \frac{\pi}{10^3}, D(\phi)(0) = 2\cdot 10^{-7}$	$\Omega = -2.7\cdot 10^{-6}$ $[\frac{1}{s}]$ $r(0) = 1.186071579\ 10^{11}, D(r)(0) = 0, \phi(0) = \frac{\pi}{10^3}, D(\phi)(0) = -2.7\cdot 10^{-6}$	$\Omega = 0$ $[\frac{1}{s}]$ $r(0) = 1.186071579\ 10^{11}, D(r)(0) = 0, \phi(0) = \frac{\pi}{10^3}, D(\phi)(0) = 0$
$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	0	~0° - 1.16° ~1.5° ~0°	*The mass m reaches equilibrium by oscillating around its initial position from 0° to 1.16°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating around 1.5°, having a max. swing to ~27°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating around 0°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
			3D plot for $t = 0..5\cdot 10^8$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..6\cdot 10^6$* 

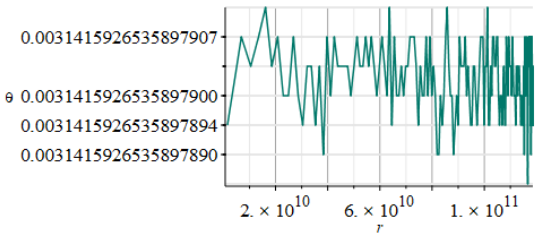
Plot of θ vs. r, for $t = 0..10^8$



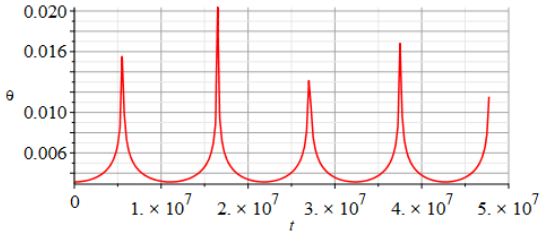
Plot of θ vs. r, for $t = 0..10^9$



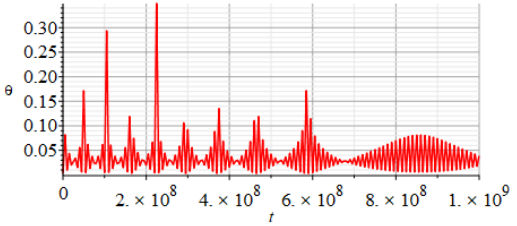
Plot of θ vs. r, for $t = 0..6 \cdot 10^6$



Plot of θ vs. t for $t = 0..5 \cdot 10^7$:



*Plot of θ vs. t for $t = 0..10^9$. For this time range, the period of the envelope increases from $T \approx 5.3 \cdot 10^7$ (~**1.7 years**) to $T \approx 1.6 \cdot 10^8$ (~**5 years**)*

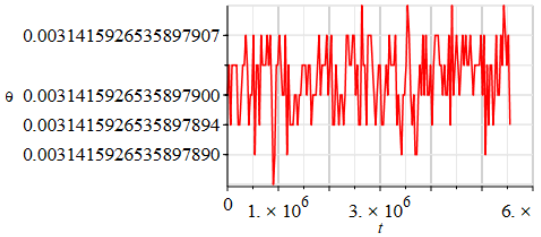


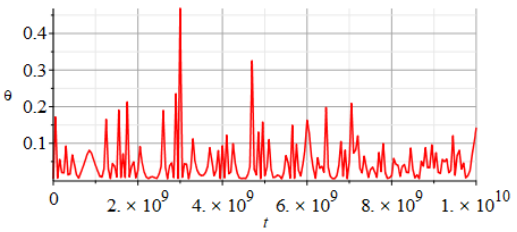
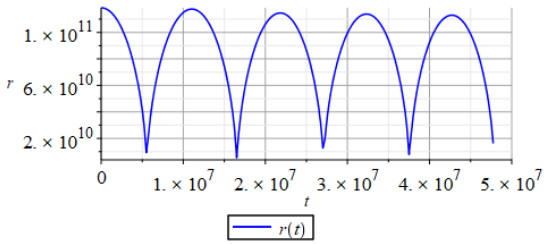
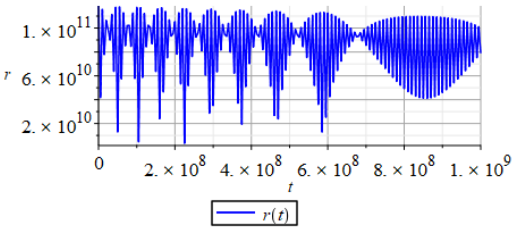
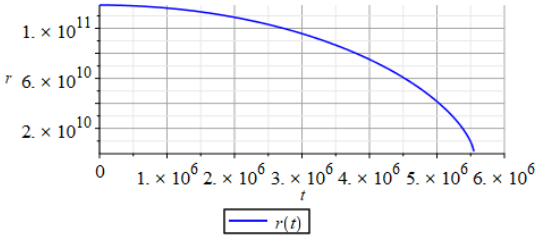
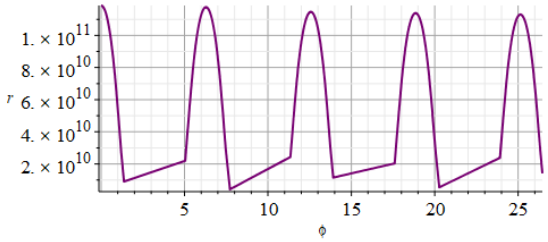
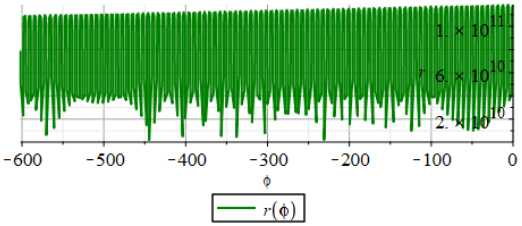
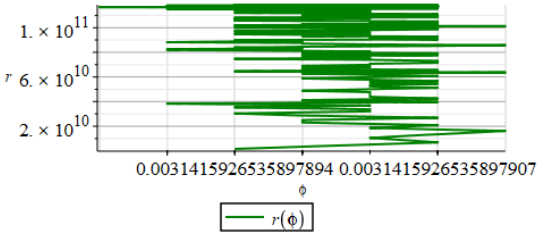
Precession and/or Nutation?

WARNING: *the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

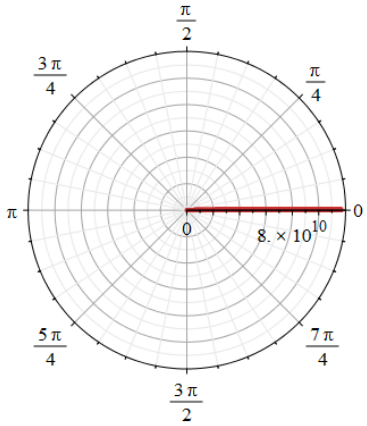
Plot of θ vs. t, for $t = 0..10^{10}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.*

Plot of θ vs. t for $t = 0..6 \cdot 10^6$.

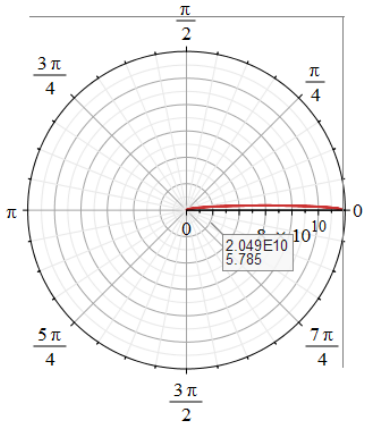


					
			Plot of r vs. t, for t = 0..5·10^7 	*Plot of r vs. t, for t = 0..10^9. For this time range, the period of the envelope increases from T≈ 5.3·10^7 (~**1.7 years**) to T≈ 1.6·10^8 (~**5 years**):* 	*Plot of r vs. t, for t = 0..6·10^6.* 
			Plot of r vs. ϕ, for t = 0..5·10^8 *It seems that r has a quasi-sinusoidal periodic variation with ϕ, with a very slightly decrease in amplitude.The changes in distance r w.r.t. the translation angle ϕ, may suggest an elliptic trajectory of the mass m.* 	*Plot of r vs. ϕ, for t = 0..10^9* 	*Plot of r vs. ϕ, for t = 0..6·10^6* 

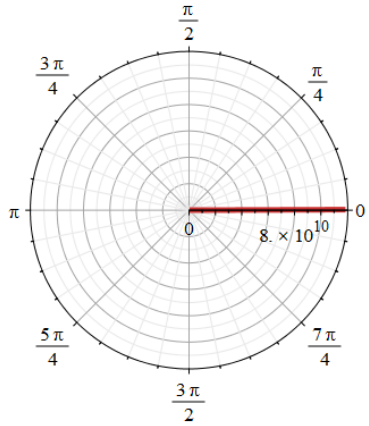
Polar plot of r vs θ, for $[R,\ Theta,\ 0..3\cdot 10^7]$



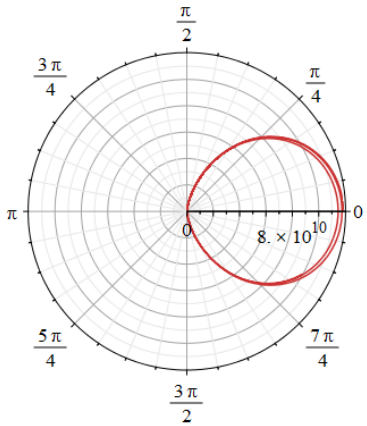
Polar plot of r vs θ, for $[R,\ Theta,\ 0..10^7]$



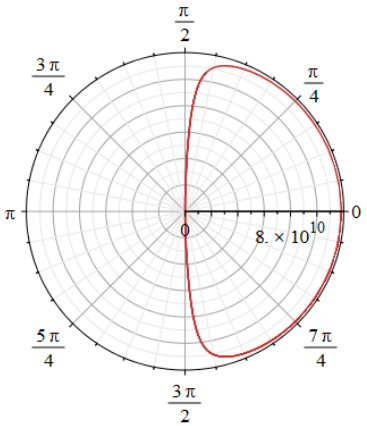
Polar plot of r vs θ, for $[R,\ Theta,\ 0..6\cdot 10^6]$



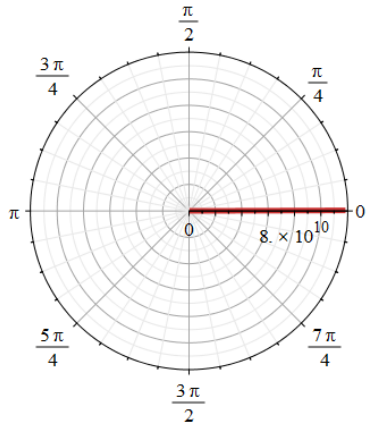
Polar plot of r vs ϕ, for $[R,\ Phi,\ 0..3\cdot 10^7]$
A slightly elliptic trajectory (orbit of mass m around central mass M)

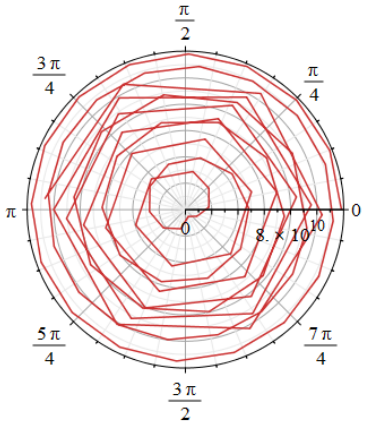
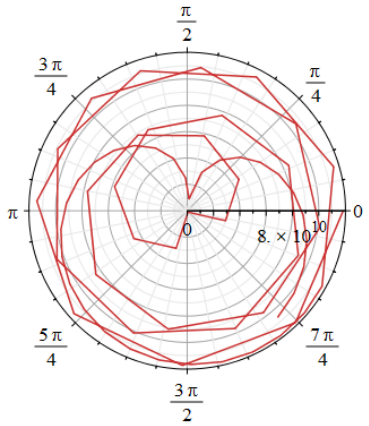
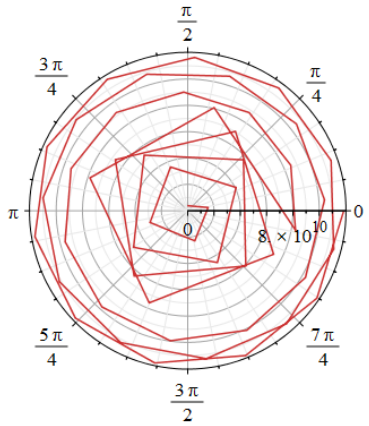


Polar plot of r vs ϕ, for $[R,\ Phi,\ 0..2\cdot 10^7]$
A quasi semi-circular trajectory (orbit of mass m around central mass M)



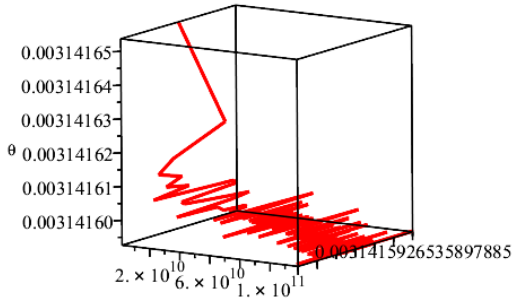
Polar plot of r vs ϕ, for $[R,\ Phi,\ 0..6\cdot 10^6]$



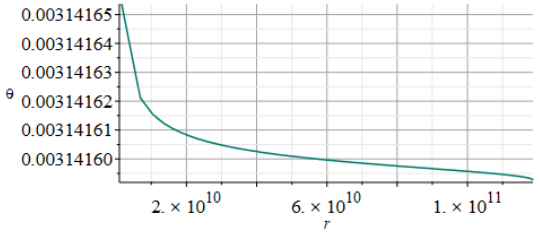
			Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10^{6}]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10^{7} \cdot 2]$, *numpoints = 3* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10^{6}]$, *numpoints = 3* *A spiral showing forward and backward motion* 

$\frac{\pi}{10^3}$ *(θ ≈ 0°)*	10^{-15}	~0° - 1.16° ~1.5° ~0°	*Same as above*	*Same as above*	*The mass m reaches equilibrium by oscillating around 0°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***

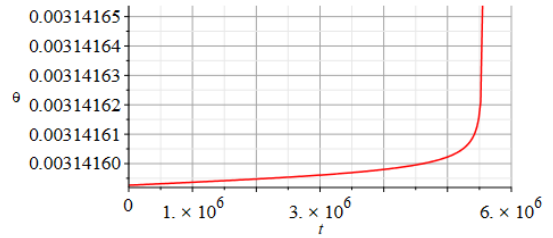
3D plot for $t = 0..6 \cdot 10^6$



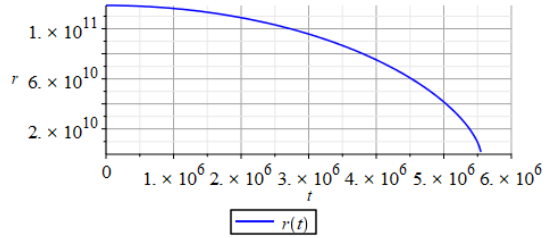
Plot of θ vs. r, for $t = 0..6 \cdot 10^6$



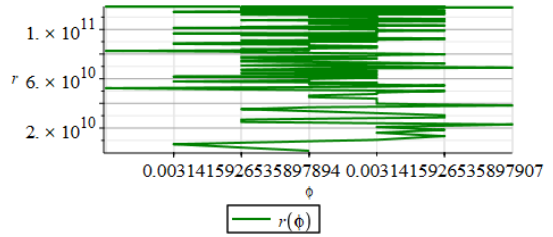
Plot of θ vs. t for t = 0..6·10^6.



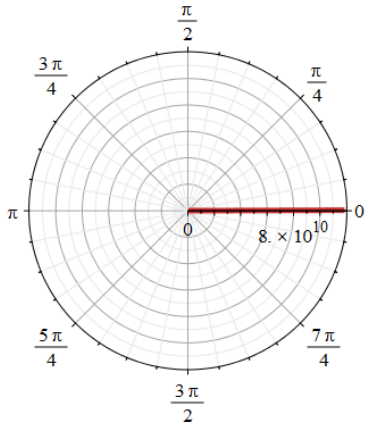
Plot of r vs. t, for t = 0..6·10^6.



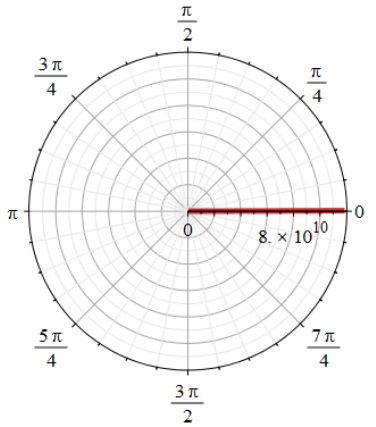
Plot of r vs. ϕ, for t = 0..6·10^6

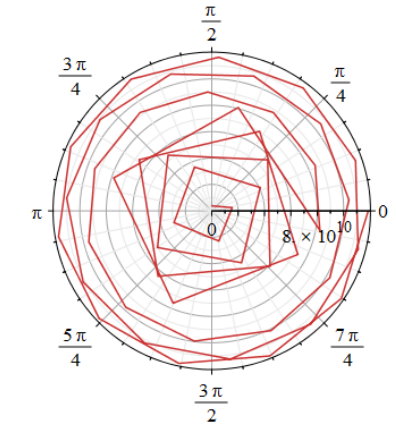


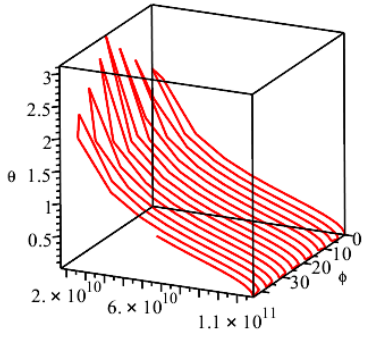
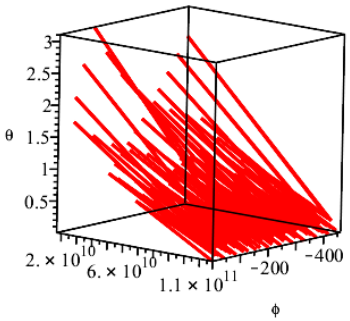
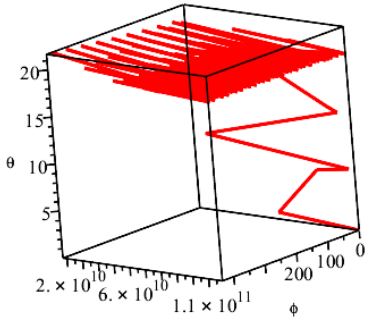
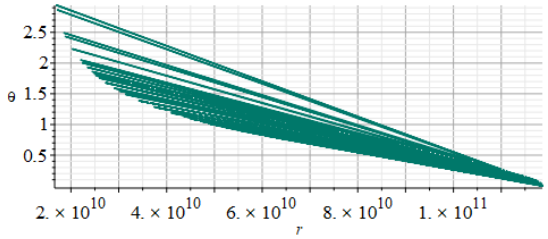
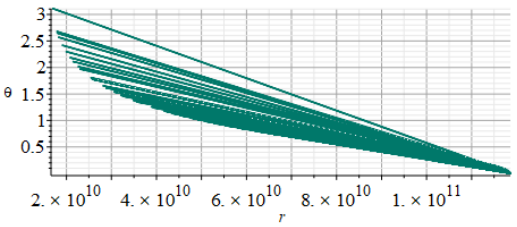
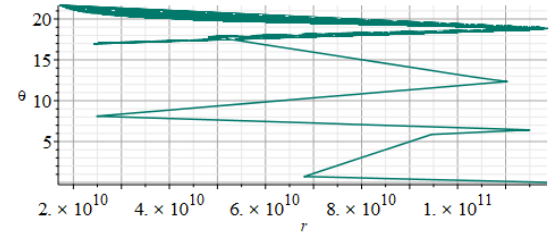
Polar plot of r vs θ, for $[R, Theta, 0..6 \cdot 10^{6}]$



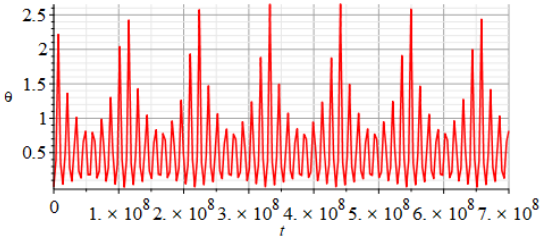
Polar plot of r vs ϕ, for $[R, Phi, 0..6 \cdot 10^{6}]$



					Polar plot of r vs t, for $[R(t), t, t = 0..7 \cdot 10^6]$, *numpoints = 3* *A spiral showing forward and backward motion* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10^{-7}	$\lvert 90° \rvert$ ($+90°$ & $-90°$) 90° ~80°	*The mass m oscillates from 0° to maximum ~ + 180°, and then swings to the opposite direction oscillating from 0° to maximum ~ -180°, having thus a mean oscillation angle of + 90° for some time, and then - 90° for a longer time. Therefore, 90° can be considered the (absolute) average equilibrium angle, with a* ***center line of oscillations around ~ ± 28°****.* *3D plot for t = 0..10^8:* 	*The mass m oscillates from 0° to maximum ~ 176°, having thus a mean oscillation angle of ~ 90°, with a* ***center line of oscillations around ~ 28°.*** *3D plot for t = 0..10^9:* 	*The mass m oscillates from 0° to maximum ~ 160°, having thus a mean oscillation angle of ~ 80°, with a* ***center line of oscillations around ~ 25°.*** *3D plot for t = 0..10^9:* 
			Plot of θ vs. r, for t = 0..10^9 	*Plot of θ vs. r, for t = 0..10^9* 	*Plot of θ vs. r, for t = 0..10^9* 

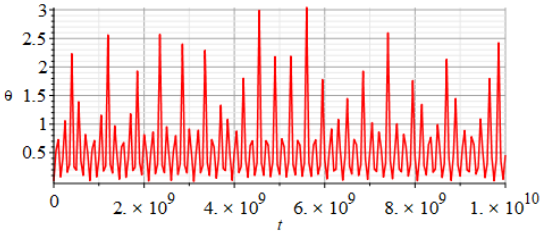
*Plot of θ vs. t, for $t = 0..7\cdot10^8$. By measuring the lower or upper envelope period, we get $T\approx1.1\cdot10^8$ (**~3.5 years**):*



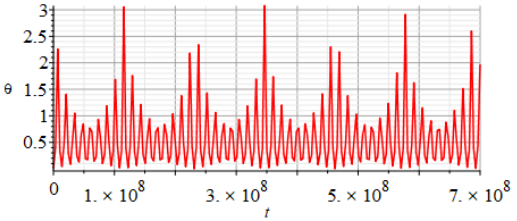
Precession and/or Nutation?

***WARNING**: the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

*Plot of θ vs. t, for $t = 0..10^{10}$. The period of the envelope increases to $T\approx650\cdot10^6$ (**~21 years**) and $T\approx800\cdot10^6$ (**~25 years**):*



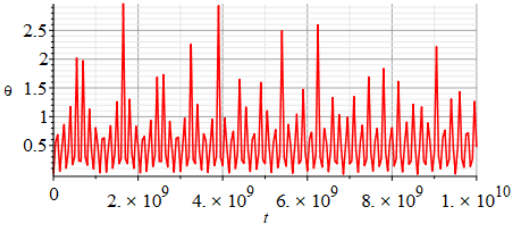
*Plot of θ vs. t, for $t = 0..7\cdot10^8$. By measuring the lower or upper envelope period, we get $T\approx1.1\cdot10^8$ (**~3.5 years**):*



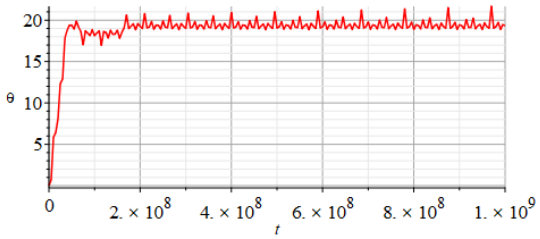
Precession and/or Nutation?

***WARNING**: the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

*Plot of θ vs. t, for $t = 0..10^{10}$. The period of the envelope increases to $T\approx0.4\cdot10^9$ (**~13 years**) and $T\approx1\cdot10^9$ (**~32 years**):*



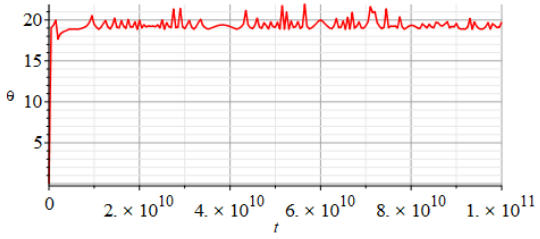
*Plot of θ vs. t, for $t = 0..10^9$. By measuring the envelope period of single bursts, we get $T\approx0.4\cdot10^8$ (**~15 months**), and the period between the highest peaks is $T\approx1\cdot10^8$ (**~3.2 years**) :*

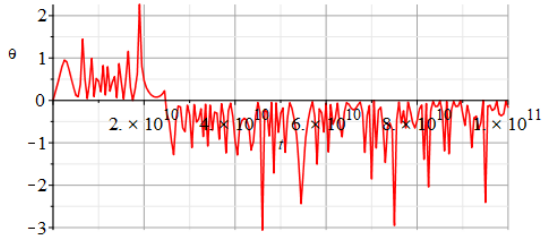
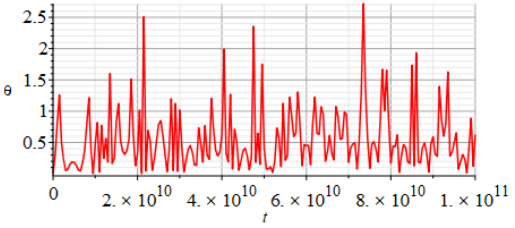
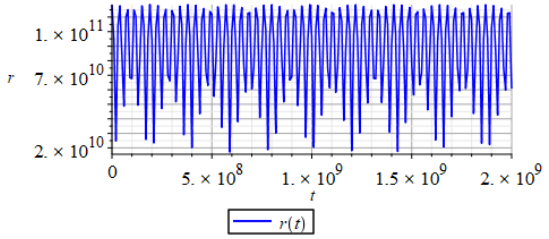
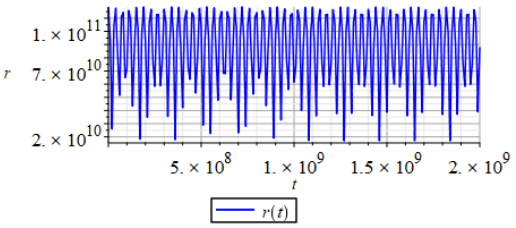
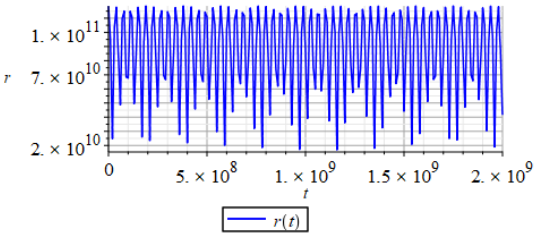
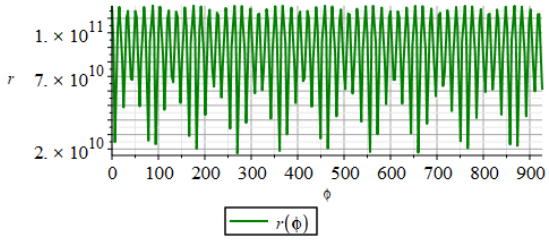
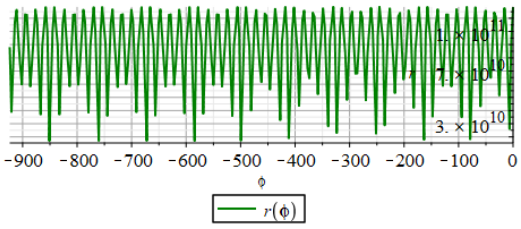
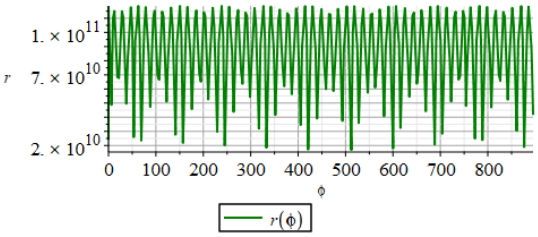


Precession and/or Nutation?

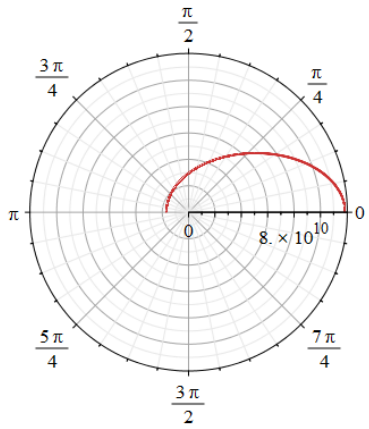
***WARNING**: the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

*Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: **amplitude and frequency modulation**. So, **the period of the mean value** oscillation (precession and/or nutation?) **is not constant**.*

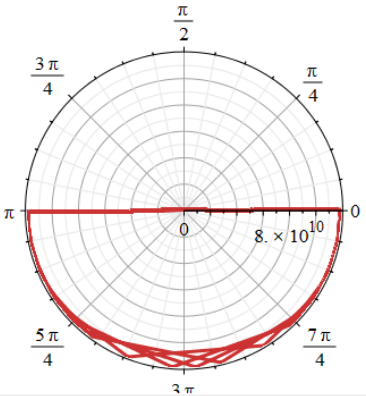


Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.* 	*Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both:* ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.* 	
*Plot of r vs. t, for $t = 0..2\cdot10^9$. For this scale, the period of the envelope is $T\approx0.2\cdot10^9$ (~**6.3 years**):* 	*Plot of r vs. t, for $t = 0..2\cdot10^9$. For this scale, the period of the envelope is $T\approx0.2\cdot10^9$ (~**6.3 years**) :* 	*Plot of r vs. t, for $t = 0..2\cdot10^9$. For this scale, the period of the envelope is $T\approx0.2\cdot10^9$ (~**6.3 years**) :* 
Plot of r vs. ϕ, for $t = 0..2\cdot10^9$ 	*Plot of r vs. ϕ, for $t = 0..2\cdot10^9$* 	*Plot of r vs. ϕ, for $t = 0..2\cdot10^9$* 

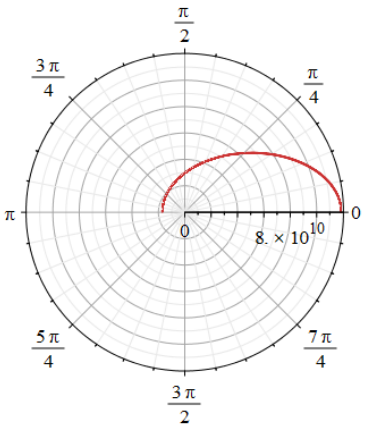
Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^7]$



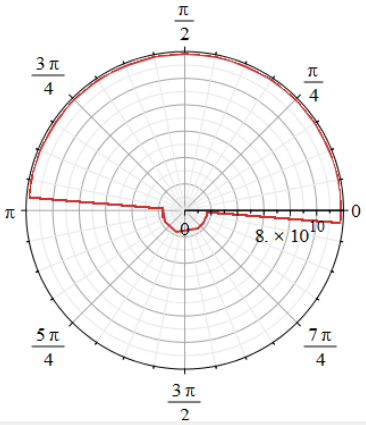
Polar plot of r vs ϕ, for
$[R, Phi, 0..6 \cdot 10^7]$, *numpoints = 10000*
A weird trajectory, back and forth mainly in the negative direction.



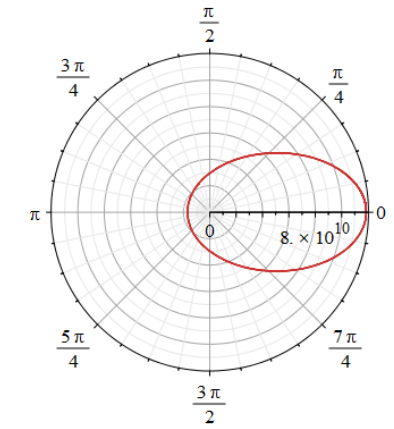
Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^7]$



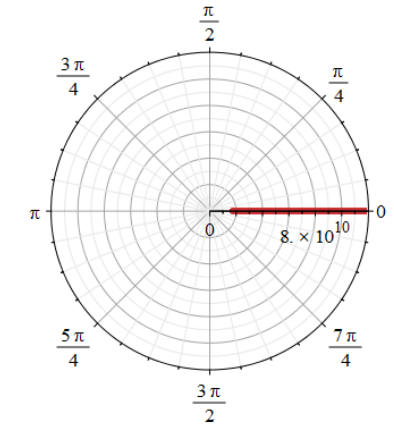
Polar plot of r vs ϕ, for
$[R, Phi, 0..2 \cdot 10^7]$, *numpoints = 10000*
A slightly rotated quasi semi-circular motion.

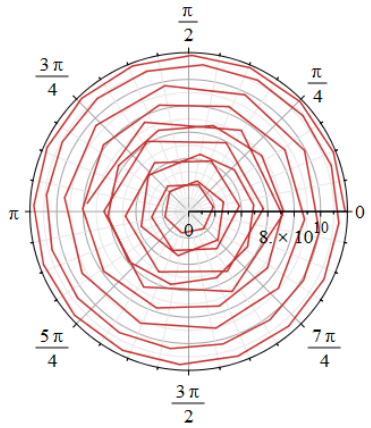
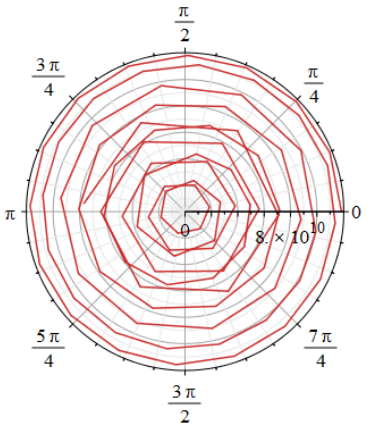
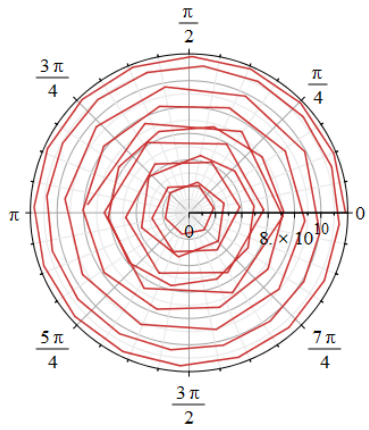


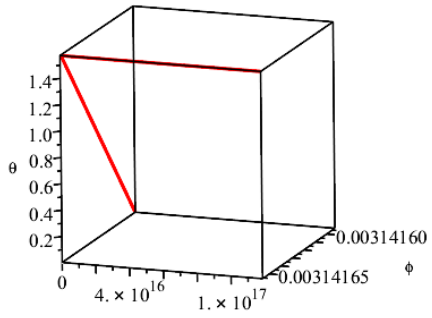
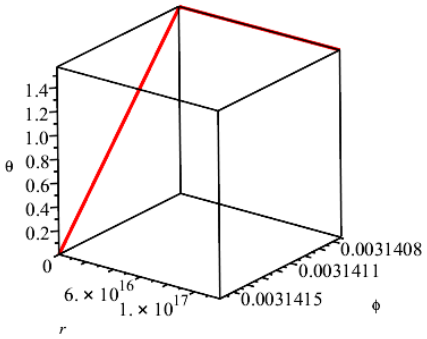
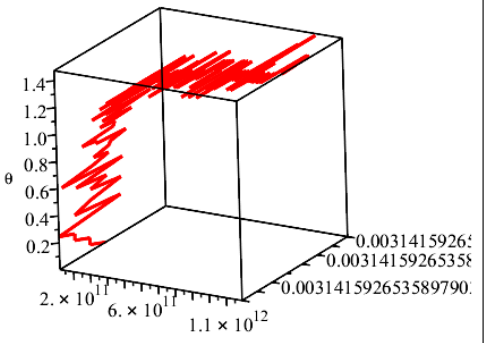
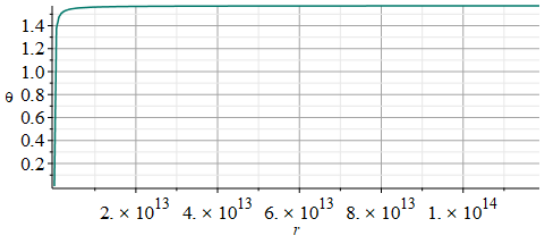
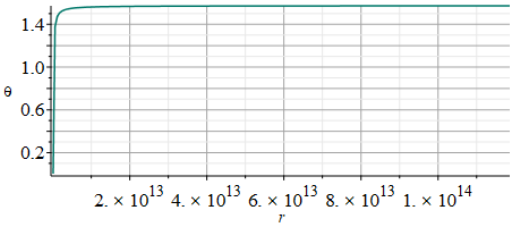
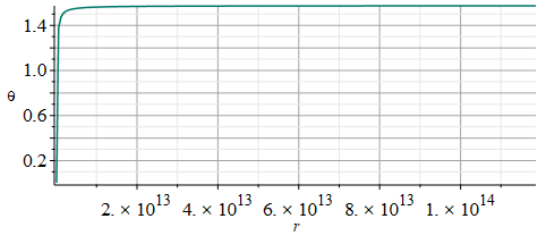
Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^7]$

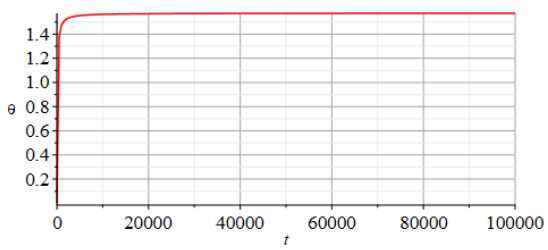
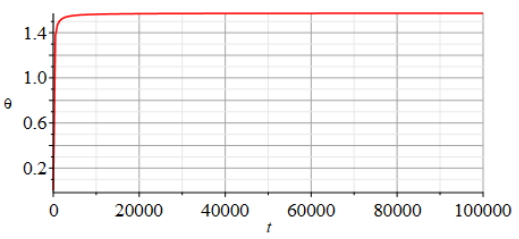
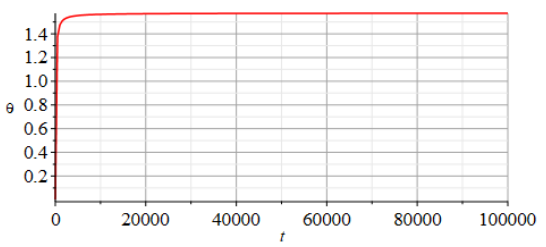
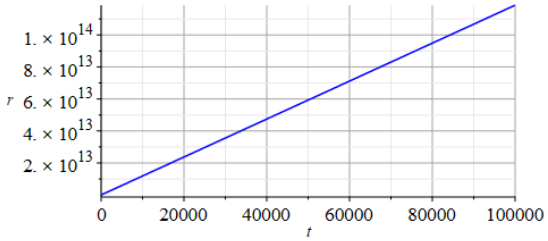
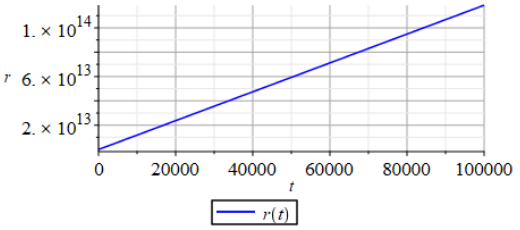
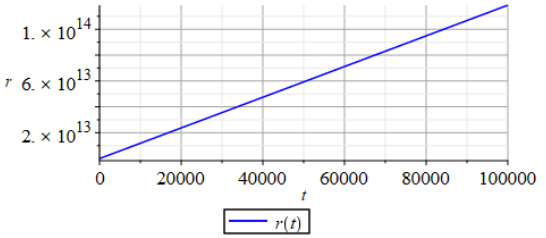


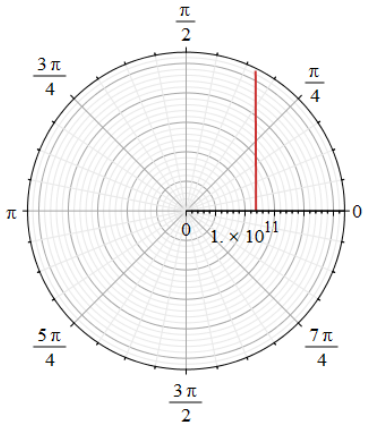
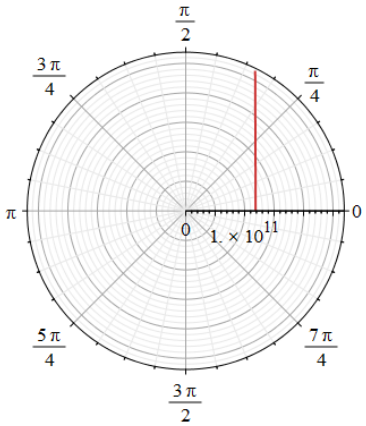
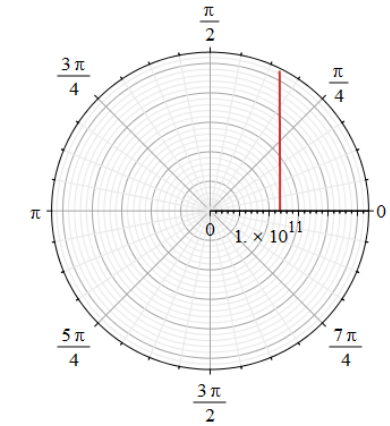
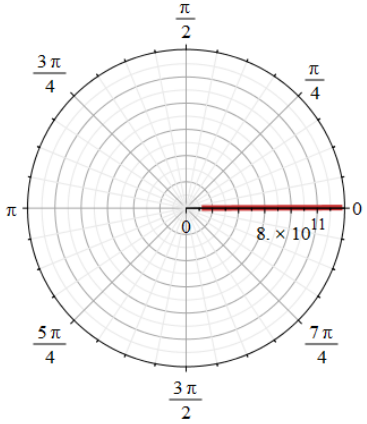
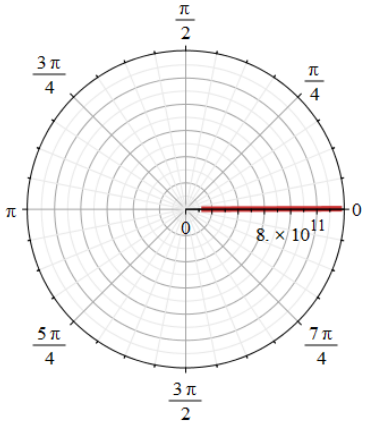
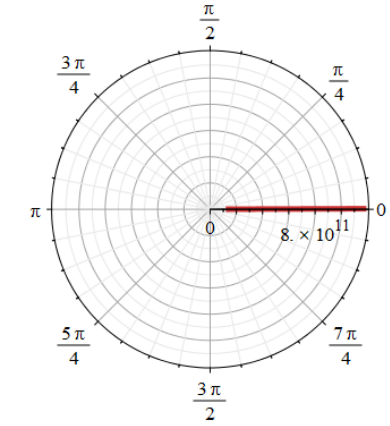
Polar plot of r vs ϕ, for $[R, Phi, 0..2 \cdot 10^7]$

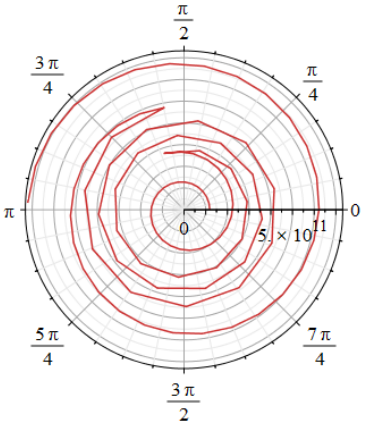
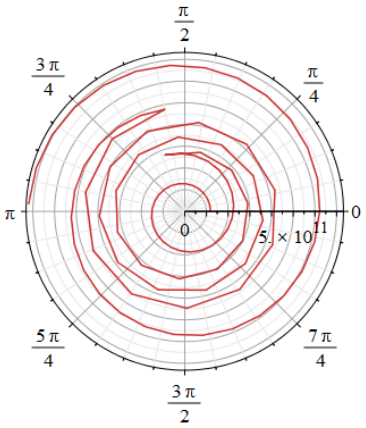
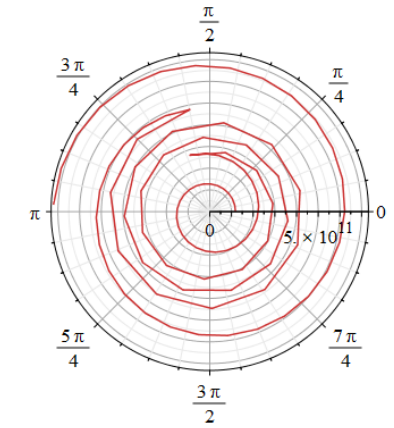


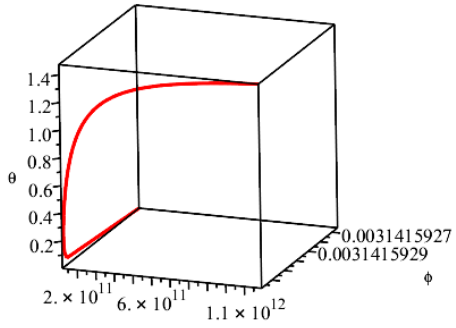
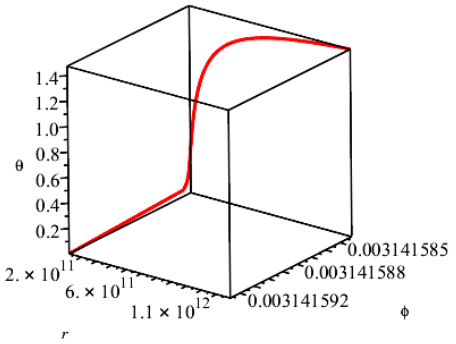
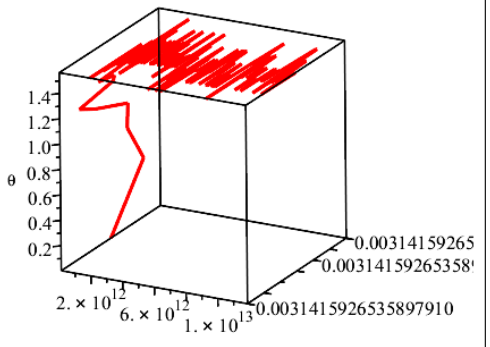
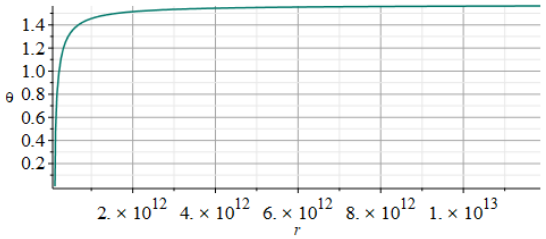
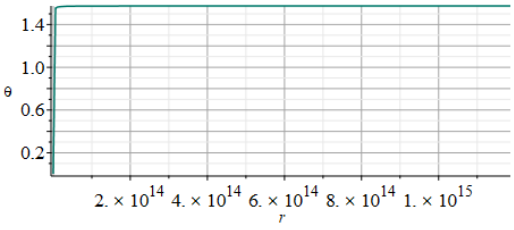
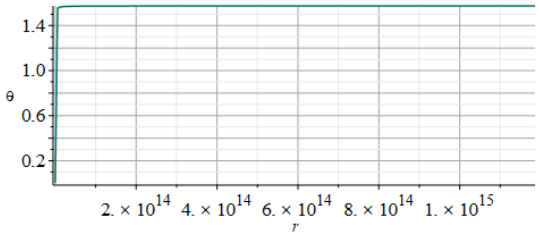
			Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10^6]$ *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A strange spiral...* 

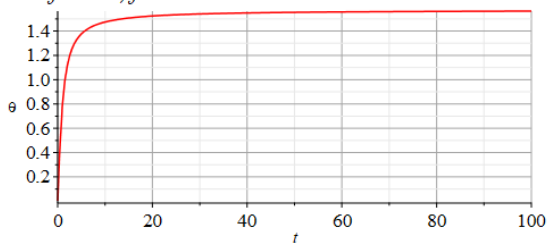
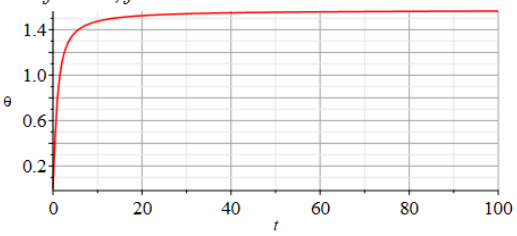
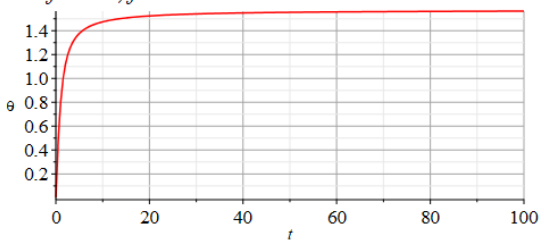
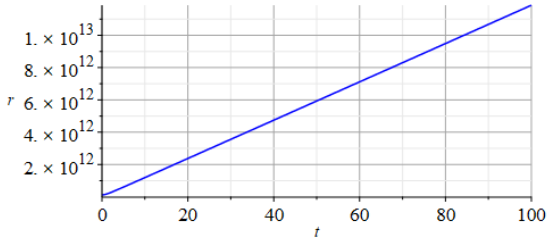
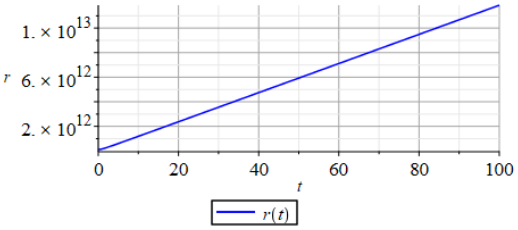
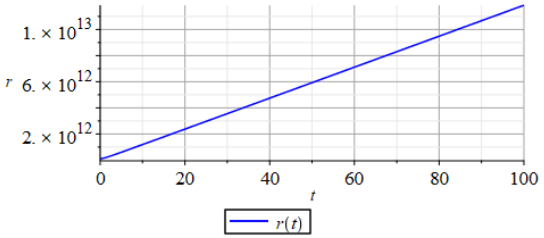
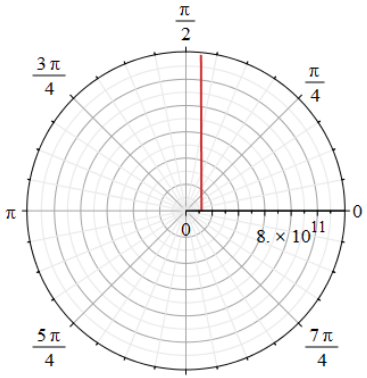
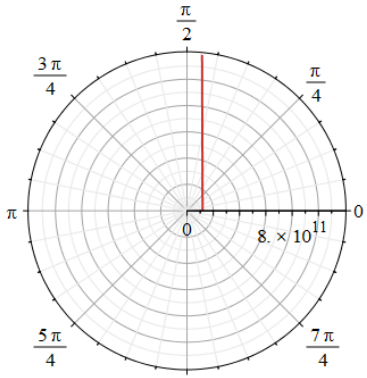
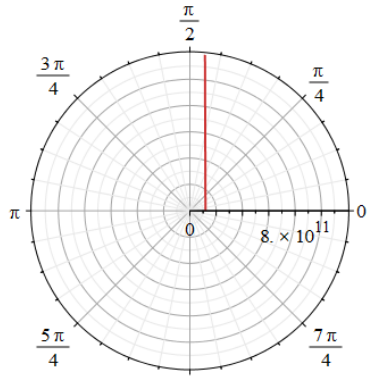
$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10^{-2}	90° 90° 90°	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^3$:* 
			Plot of θ vs. r, for $t = 0..10^5$ 	*Plot of θ vs. r, for $t = 0..10^5$* 	*Plot of θ vs. r, for $t = 0..10^5$* 

Plot of θ vs. t, for t = 0..10^5 	*Plot of θ vs. t, for t = 0..10^5* 	*Plot of θ vs. t, for t = 0..10^5* 
Plot of r vs. t, for t = 0..10^5 	*Plot of r vs. t, for t = 0..10^5* 	*Plot of r vs. t, for t = 0..10^5* 

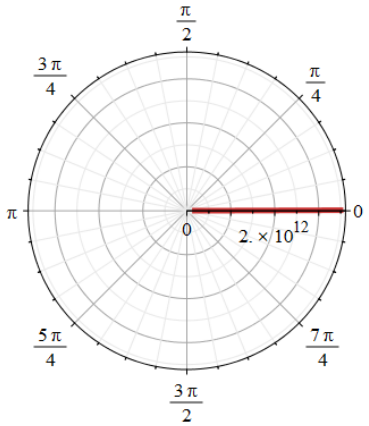
			Polar plot of r vs θ, for $[R, Theta, 0..2\cdot 10^2]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..2\cdot 10^2]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..2\cdot 10^2]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 

			Polar plot of r vs t, for $[R(t), t, t = 0..6 \cdot 10^2]$ *It seems to be a logaritmic spiral that evolves forwards and backwards...*  	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10^2]$ *It seems to be a logaritmic spiral that evolves forwards and backwards...*  	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10^2]$, *numpoints = 4* *It seems to be a logaritmic spiral that evolves forwards and backwards...* 

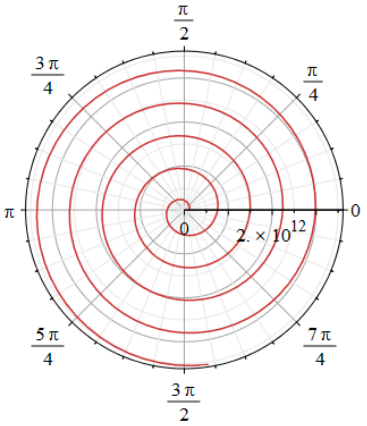
$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	1	90° 90° 90°	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10$:* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10$:* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^2$:* 
			Plot of θ vs. r, for $t = 0..10^4$ 	*Plot of θ vs. r, for $t = 0..10^4$* 	*Plot of θ vs. r, for $t = 0..10^4$* 

			Plot of θ vs. t, for t = 0..10² 	*Plot of θ vs. t, for t = 0..10²* 	*Plot of θ vs. t, for t = 0..10²* 
			Plot of r vs. t, for t = 0..10² 	*Plot of r vs. t, for t = 0..10²* 	*Plot of r vs. t, for t = 0..10²* 
			Polar plot of r vs θ, for [R, Theta, 0..10] 	*Polar plot of r vs θ, for* [R, Theta, 0..10] 	*Polar plot of r vs θ, for* [R, Theta, 0..10] 

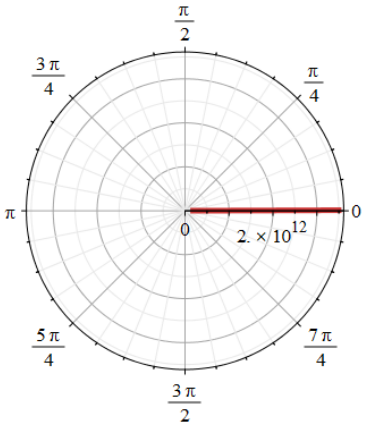
Polar plot of r vs ϕ, for $[R, Phi, 0..3 \cdot 10]$



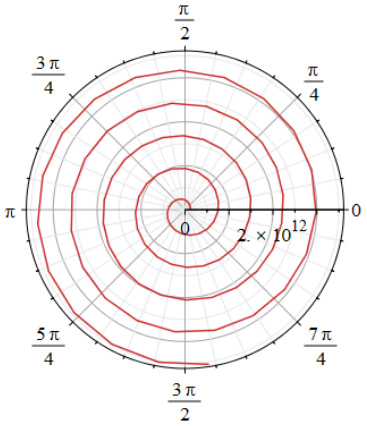
Polar plot of r vs t, for $[R(t), t, t = 0..3 \cdot 10]$
It seems to be a regular spiral



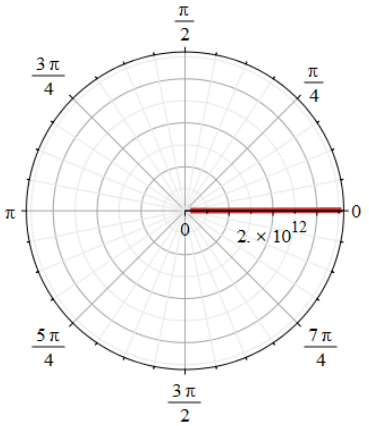
Polar plot of r vs ϕ, for $[R, Phi, 0..3 \cdot 10]$



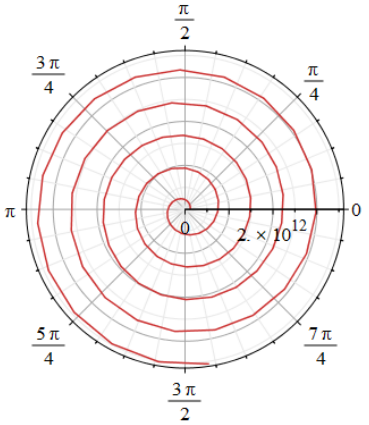
Polar plot of r vs t, for $[R(t), t, t = 0..3 \cdot 10]$*. It seems to be a regular spiral*

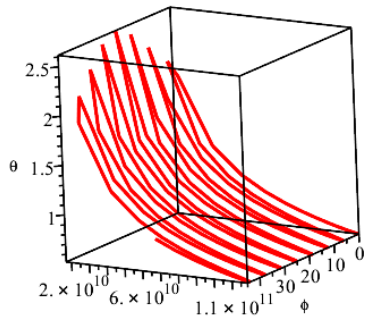
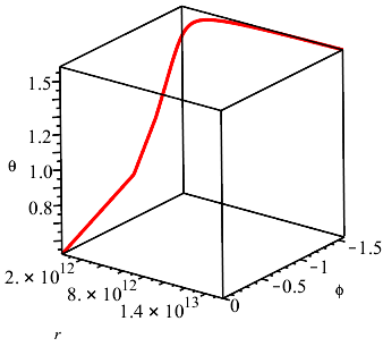
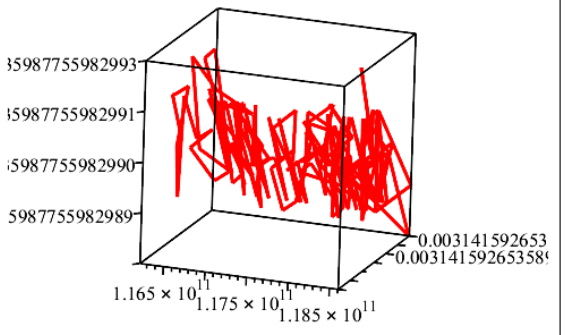


Polar plot of r vs ϕ, for $[R, Phi, 0..3 \cdot 10]$

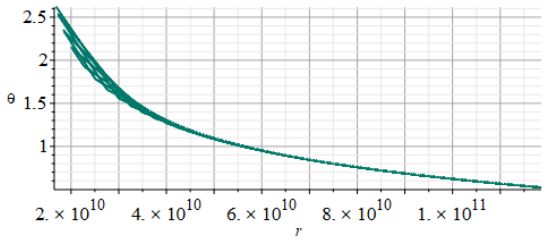


Polar plot of r vs t, for
$[R(t), t, t = 0..3 \cdot 10]$*, numpoints = 4. It seems to be a regular spiral*

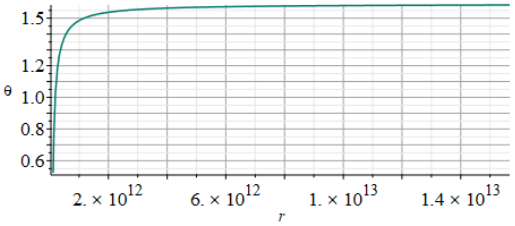


$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	100	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{6}$	0	90° 91° 30°	*The mass m oscillates from 30° to maximum 150°, having thus a mean oscillation angle of 90° (± 60°) which can be considered the average equilibrium angle, with smaller oscillations around ~45°.* *3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium at θ≈91°* *3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium oscillating around θ≈ 30°* *3D plot for $t = 0..10^6$:* 

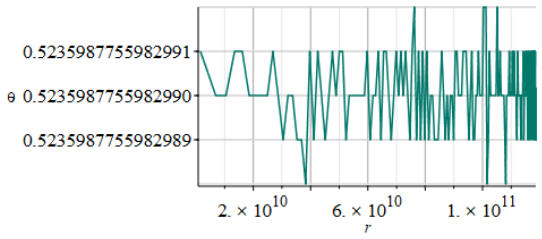
Plot of θ vs. r, for $t = 0..10^8$



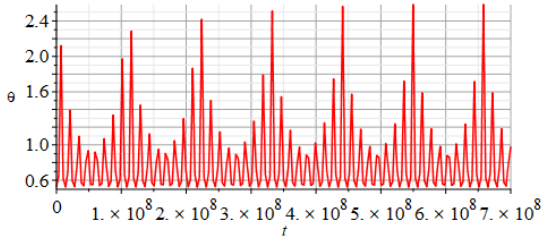
Plot of θ vs. r, for $t = 0..10^8$



Plot of θ vs. r, for $t = 0..6 \cdot 10^6$



*Plot of θ vs. t, for $t = 0..7 \cdot 10^8$. By measuring the lower or upper envelope period, we get $T \approx 1.1 \cdot 10^8$ (**~3.5 years**):*

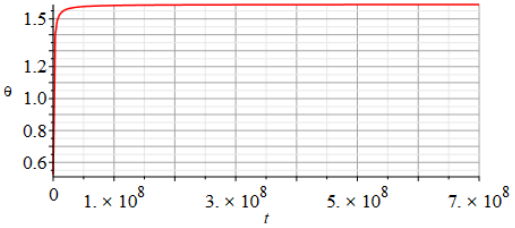


Precession and/or Nutation?

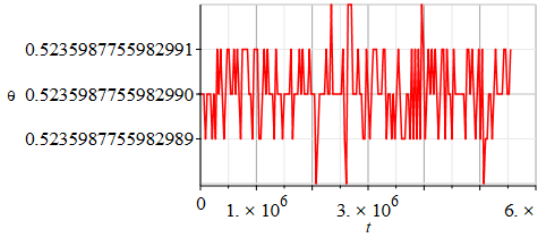
***WARNING**: the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

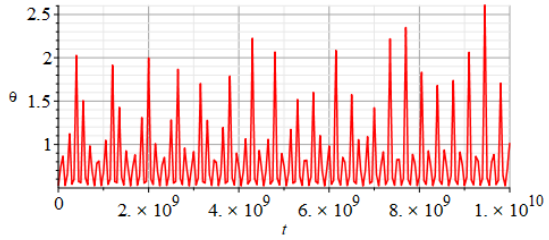
*Plot of θ vs. t, for $t = 0..10^{10}$. The period of the envelope increases to $T \approx 400 \cdot 10^6$ (**~13 years**) and $T \approx 800 \cdot 10^6$ (**~25 years**):*

Plot of θ vs. t, for $t = 0..7 \cdot 10^8$.

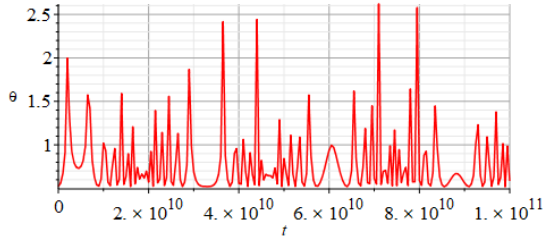


Plot of θ vs. t, for $t = 0..6 \cdot 10^6$.

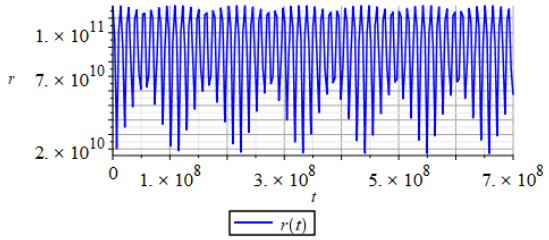




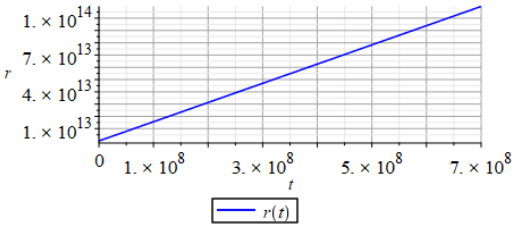
Plot of θ vs. t, for t = 0..10^11. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***



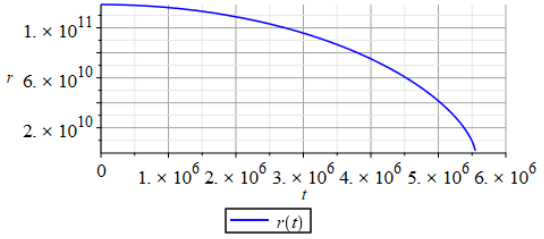
Plot of r vs. t, for t = 0..7·10^8 ' For this scale, the period of the envelope is T≈110·10^6 (~ ***3.5 years****).*

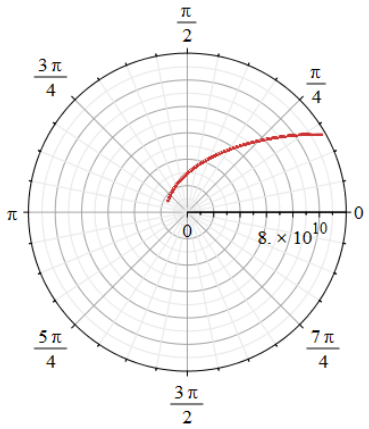
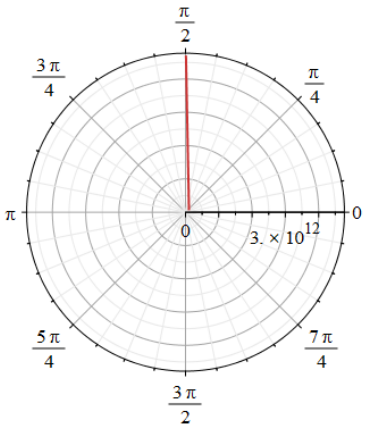
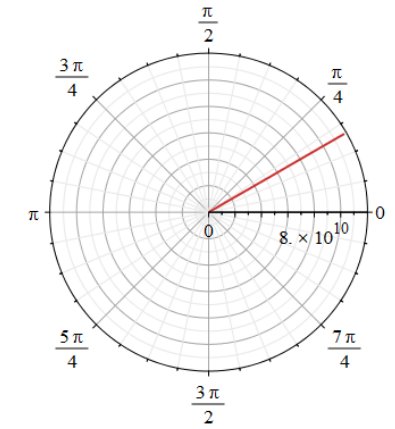
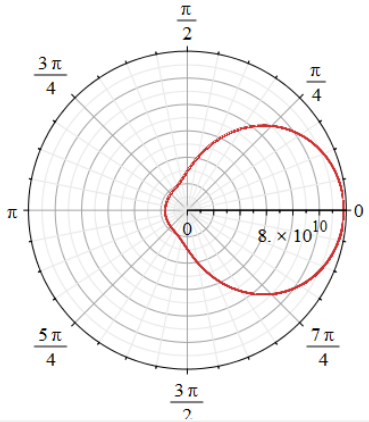
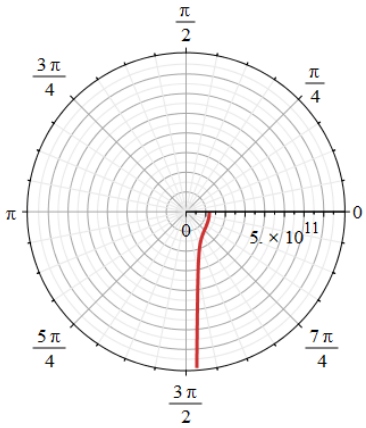
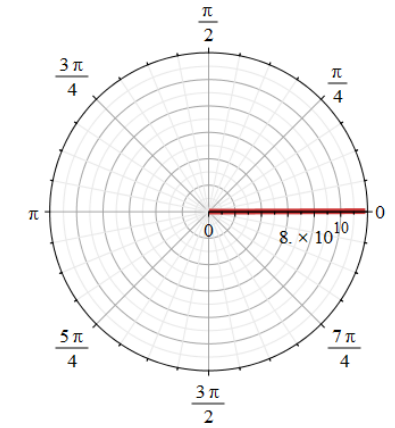


Plot of r vs. t, for t = 0..7·10^8.



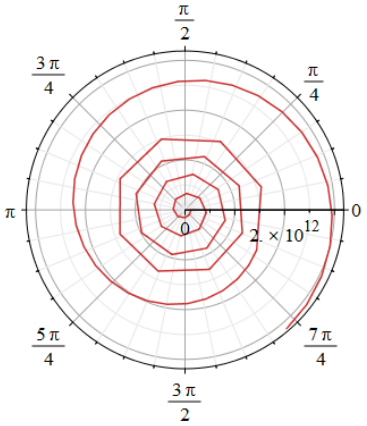
Plot of r vs. t, for t = 0..6·10^6.



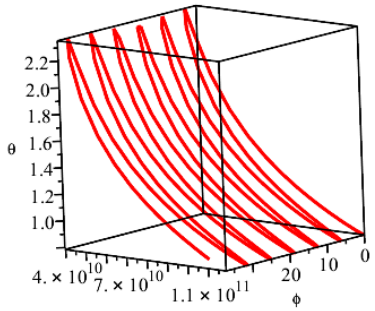
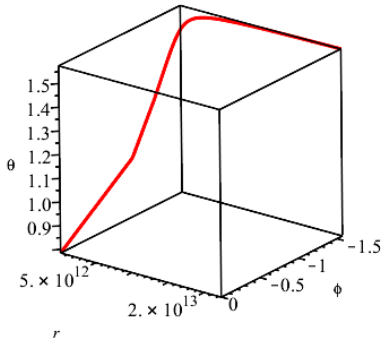
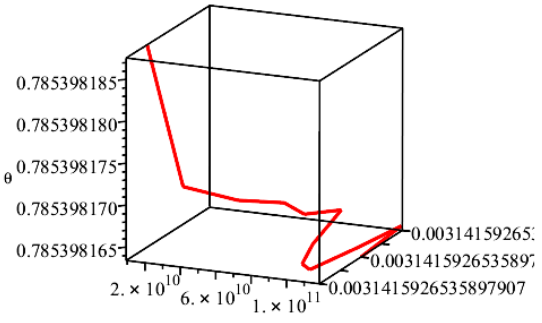
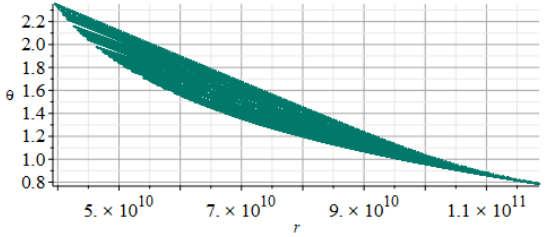
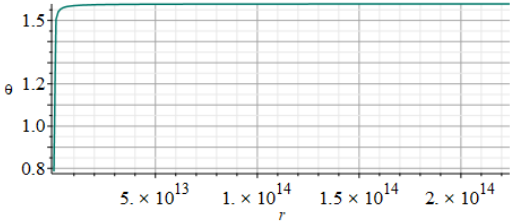
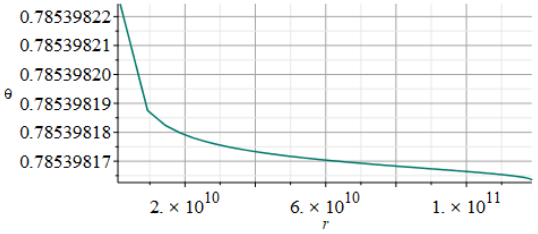
			Polar plot of r vs θ, for $[R, Theta, 0..3\cdot10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..3\cdot10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..6\cdot10^6]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..5\cdot10^7]$ *A slightly circular trajectory (orbit of mass m around central mass M)* 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..5\cdot10^6]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..6\cdot10^6]$ 

Polar plot of r vs t, for
$[R(t), t, t = 0..2 \cdot 10^7]$, *numpoints = 3*
A strange spiral...

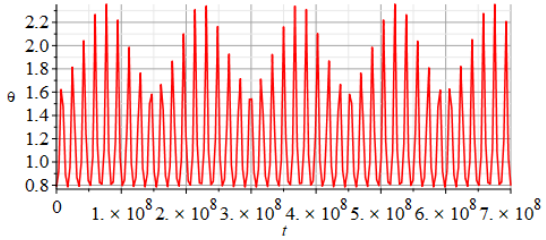
Polar plot of r vs t, for
$[R(t), t, t = 0..2 \cdot 10^7]$, *numpoints = 3*
A logaritmic spiral.



Polar plot of r vs t, for
$[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4*
A logaritmic spiral from outwards.

$\frac{\pi}{4}$	10^{-15}	90° 90° 45°	*The mass m swings from 45° to maximum 135°, so that it oscillates at an average angle of $\theta = 90° \pm 45°$. Therefore, $\theta = 90°$ can be considered the average equilibrium angle.* *3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^8$:* 	*The mass m reaches equilibrium at $\theta = 45°$. 3D plot for $t = 0..10^8$:* 
			Plot of θ vs. r, for $t = 0..10^9$ 	*Plot of θ vs. r, for $t = 0..10^9$* 	*Plot of θ vs. r, for $t = 0..10^7$* 

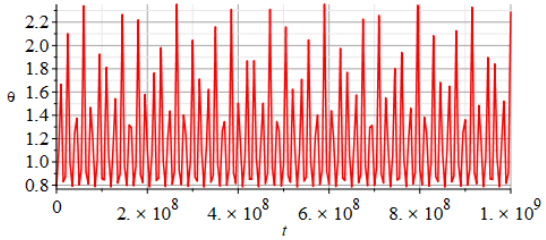
*Plot of θ vs. t, for $t = 0..7\cdot10^8$. By measuring the lower or upper envelope period, we get $T\approx1.5\cdot10^8$ (~**5 years**):*



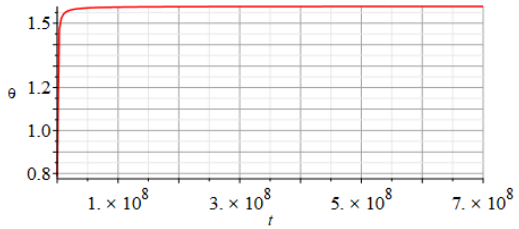
Precession and/or Nutation?

WARNING: *the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

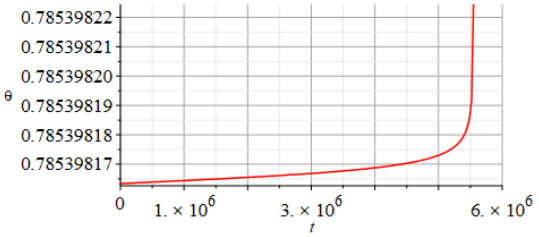
Plot of θ vs. t, for $t = 0..10^9$. It's very difficult to identify a clear envelope to measure its period:



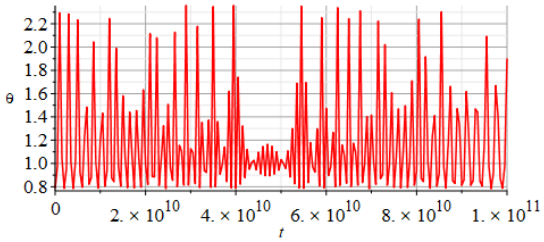
Plot of θ vs. t, for $t = 0..7\cdot10^8$:



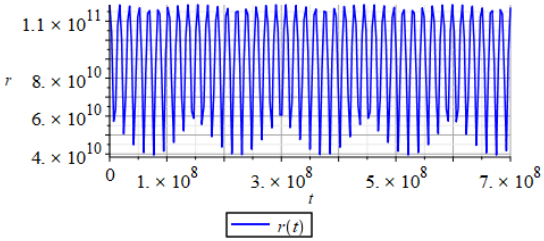
Plot of θ vs. t, for $t = 0..6\cdot10^6$:



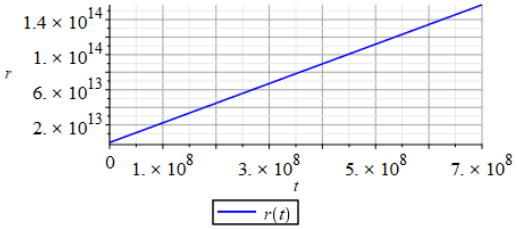
Plot of θ vs. t, for t = 0..10^{11}. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***



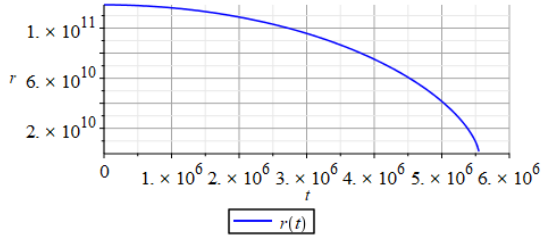
*Plot of r vs. t, for t = 0..7·10^8. For this scale, the period of the envelope is T≈1.5·10^8 (~**5 years**):*



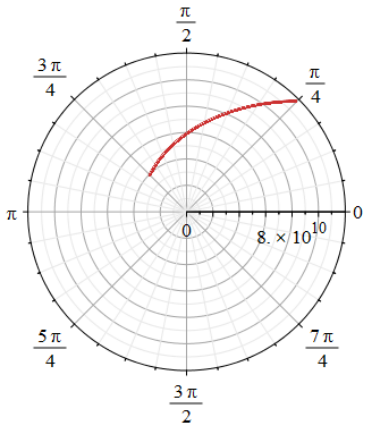
Plot of r vs. t, for t = 0..7·10^8:



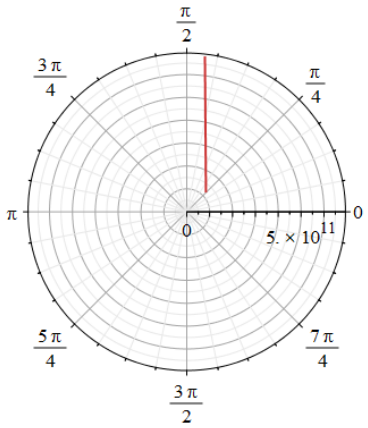
Plot of r vs. t, for t = 0..6·10^6:



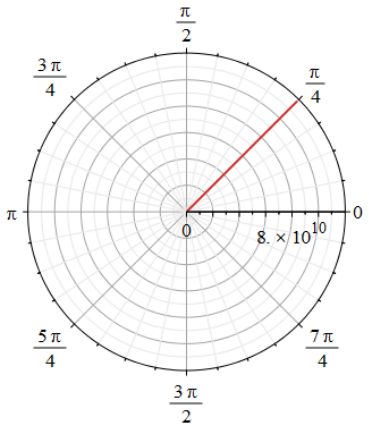
Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^7]$



Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^6]$

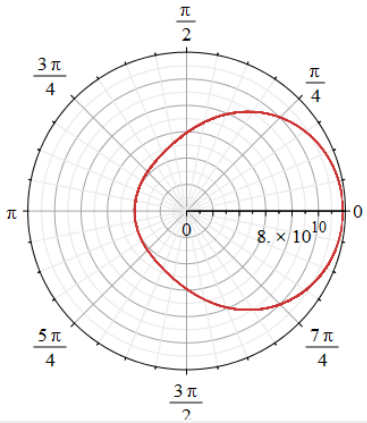


Polar plot of r vs θ, for $[R, Theta, 0..6 \cdot 10^6]$

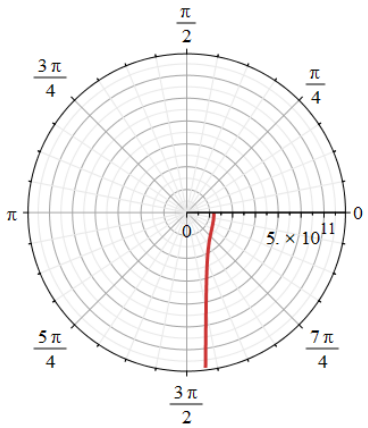


Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^7]$

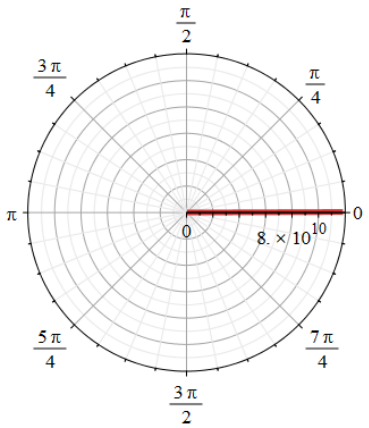
A slightly circular trajectory (orbit of mass m around central mass M)



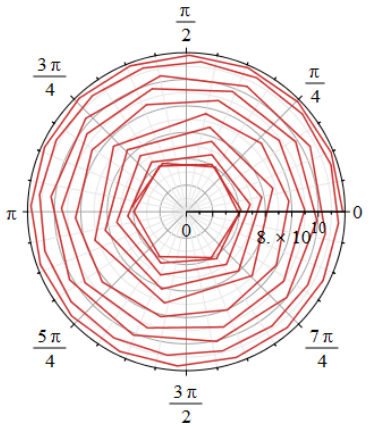
Polar plot of r vs ϕ, for $[R, Phi, 0..3 \cdot 10^6]$



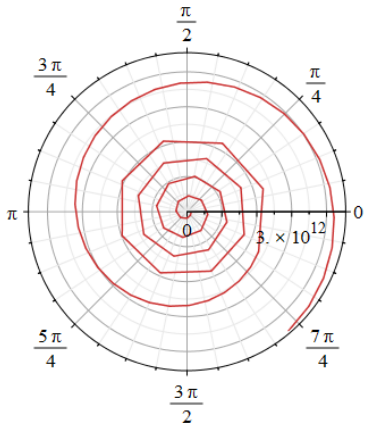
Polar plot of r vs ϕ, for $[R, Phi, 0..6 \cdot 10^6]$



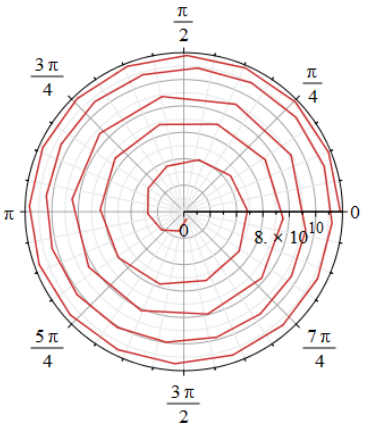
Polar plot of r vs t, for
$[R(t), t, t = 0..9 \cdot 10^{6}]$, *numpoints = 4*
Seems to be a regular spiral...

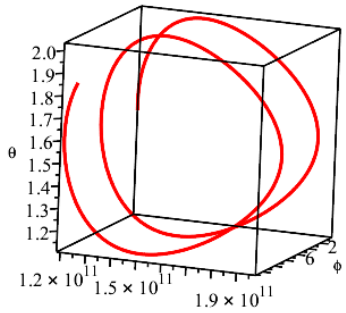
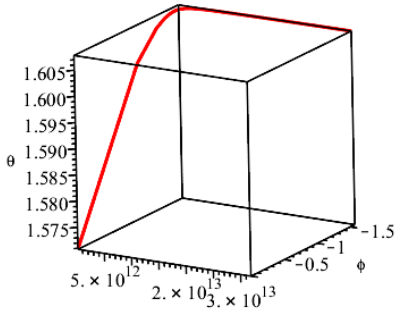
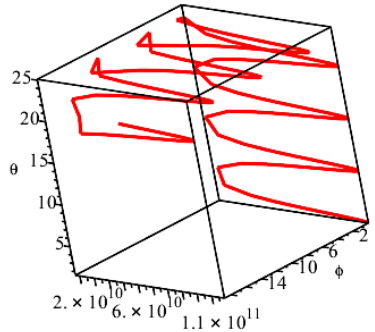
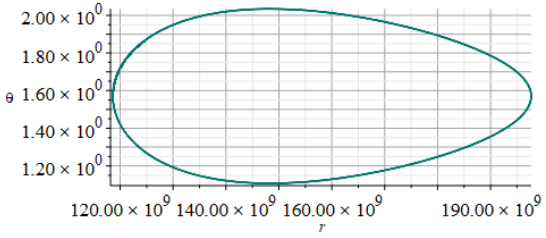
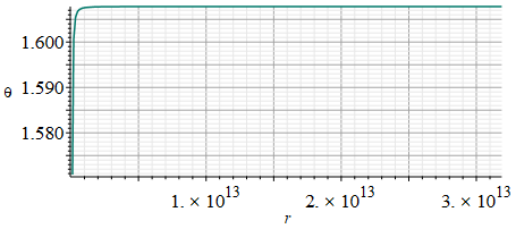
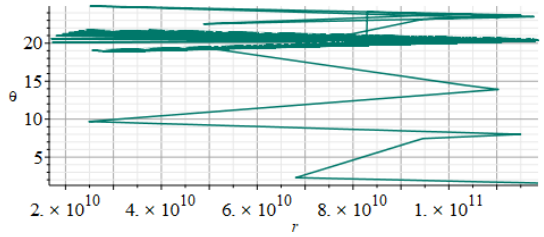


Polar plot of r vs t, for
$[R(t), t, t = 0..2 \cdot 10^{7}]$, *numpoints = 3*
A logaritmic spiral.

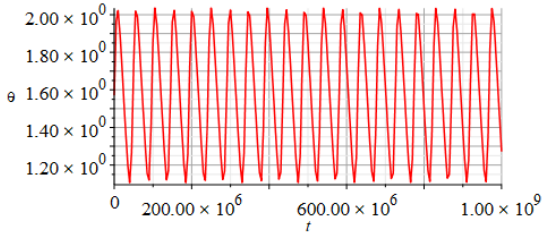


Polar plot of r vs t, for
$[R(t), t, t = 0..9 \cdot 10^{6}]$, *numpoints = 4*
A logaritmic spiral from outwards.



$\frac{\pi}{2}$	10^{-7}	90° 92° ~88°	*The mass m swings from 63° to maximum 116°, so that it oscillates at an average angle of* $\theta = 90° \pm 26.5°$. *Therefore,* $\theta = 90°$ *can be considered the average equilibrium angle.* *3D plot for* $t = 0..10^8$*:* 	*The mass m reaches equilibrium at* $\theta = 92°$. *3D plot for* $t = 0..10^8$*:* 	*The mass m reaches equilibrium oscillating around* $\theta \approx 88°$ ($\pm$ *83°). 3D plot for* $t = 0..10^8$*:* 
			Plot of θ *vs. r, for* $t = 0..10^8$ 	*Plot of* θ *vs. r, for* $t = 0..10^8$ 	*Plot of* θ *vs. r, for* $t = 0..10^9$ 

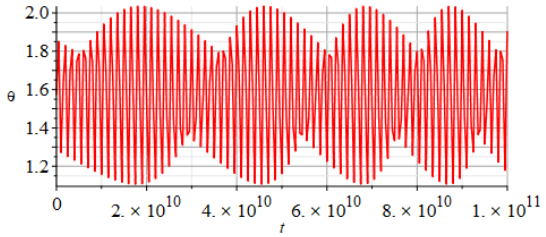
*Plot of θ vs. t, for $t = 0..10^9$. We can see a very smooth envelope with a period $T \approx 1.5 \cdot 10^8$ (~**5 years**):*



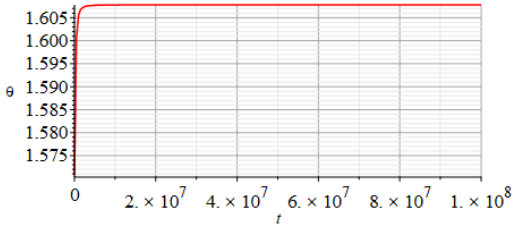
Precession and/or Nutation?

***WARNING**: the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

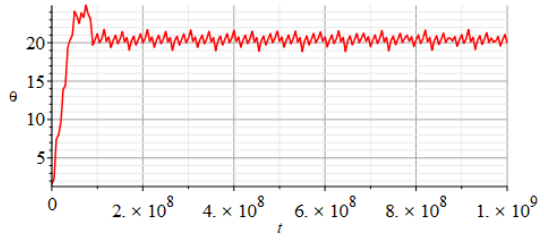
*Plot of θ vs. t, for $t = 0..10^{11}$. We can see both: **amplitude and frequency modulation**. So, **the period of the mean value** oscillation (precession and/or nutation?) **is not constant,** and for this time range it changes from $T \approx 1.8 \cdot 10^{10}$ (~ **571 years**), and $T \approx 3.12 \cdot 10^{10}$ (~ **989 years**):*



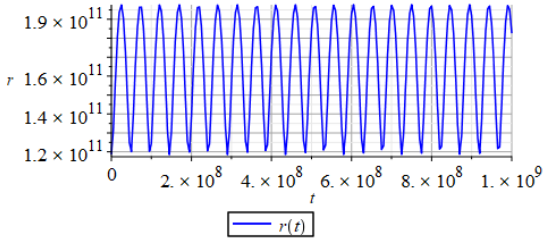
Plot of θ vs. t, for $t = 0..10^8$.



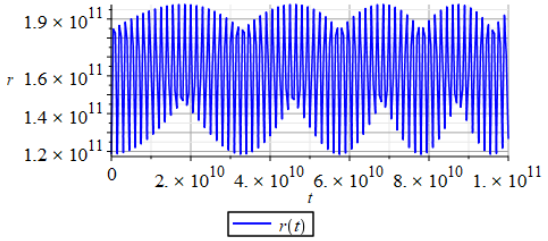
*Plot of θ vs. t, for $t = 0..10^9$. The period between the minima spikes (~ envelope) is $T \approx 1 \cdot 10^8$ (~ **3.2 years**).*



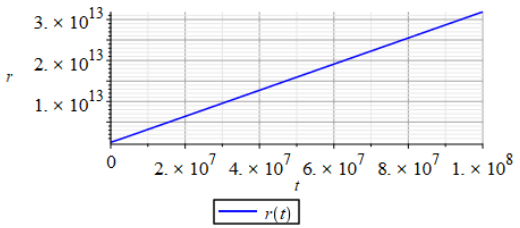
*Plot of r vs. t, for $t = 0..10^9$. We can see a very smooth envelope with a period $T \approx 1.5 \cdot 10^8$ (~**5 years**):*



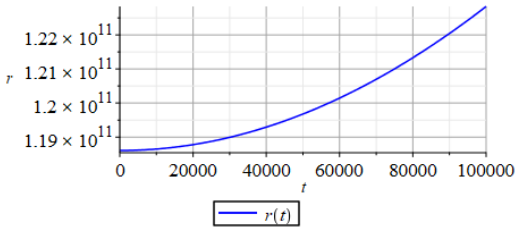
Plot of r vs. t, for $t = 0..10^{11}$. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession or nutation?)* ***is not constant,*** *and for this time range it changes from $T \approx 1.8 \cdot 10^{10}$ (~ **571 years**), and $T \approx 3.12 \cdot 10^{10}$ (~ **989 years**):*



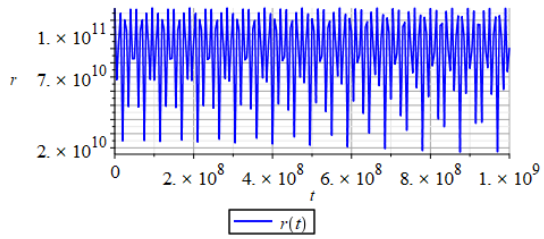
Plot of r vs. t, for $t = 0..10^8$:



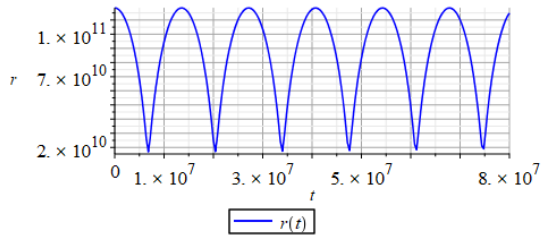
Details for $t = 0..10^5$:



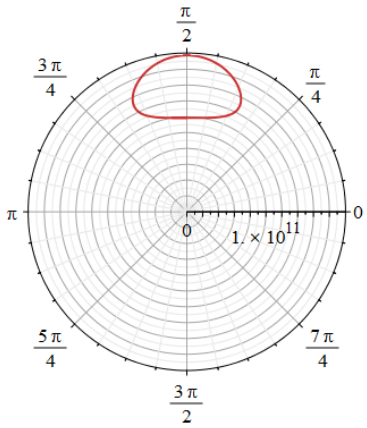
*Plot of r vs. t, for $t = 0..10^9$. The period between the minima spikes (~ envelope) is $T \approx 1 \cdot 10^8$ (~ **3.2 years**).*



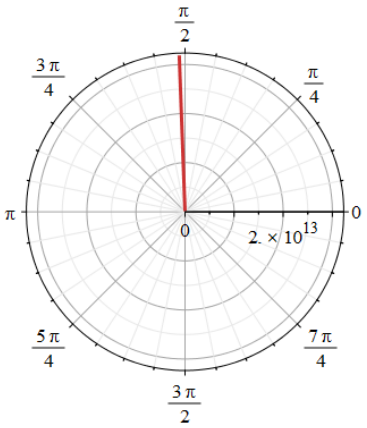
Details for $t = 0..8 \cdot 10^7$:



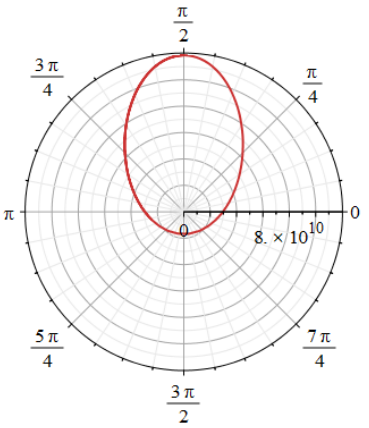
Polar plot of r vs θ, for $[R, Theta, 0..5 \cdot 10^7]$



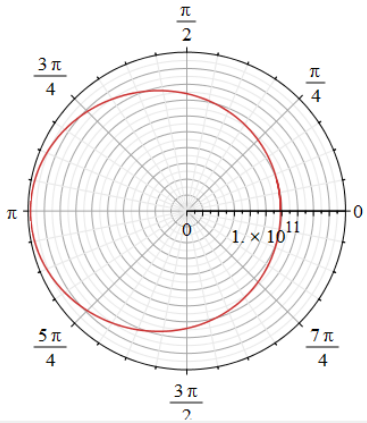
Polar plot of r vs θ, for $[R, Theta, 0..10^8]$:



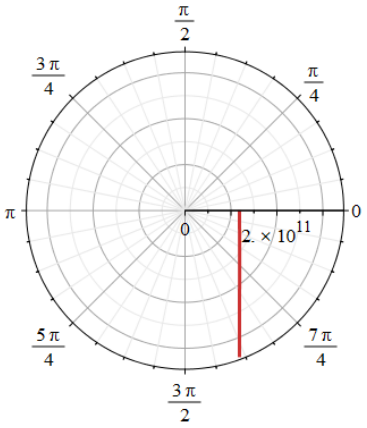
Polar plot of r vs θ, for $[R, Theta, 0..2 \cdot 10^7]$:



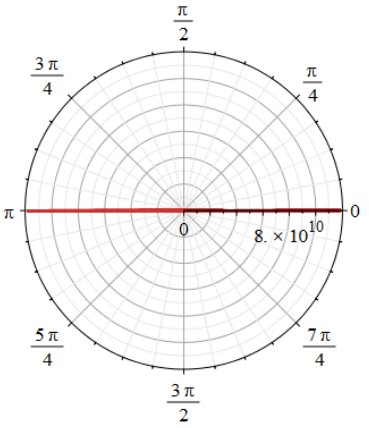
Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^7]$
Seems to be a circular trajectory (orbit of mass m around central mass M)



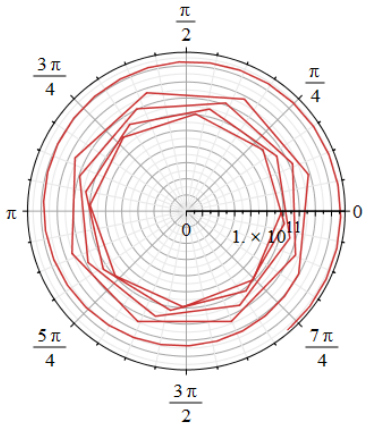
Polar plot of r vs ϕ, for $[R, Phi, 0..10^6]$:



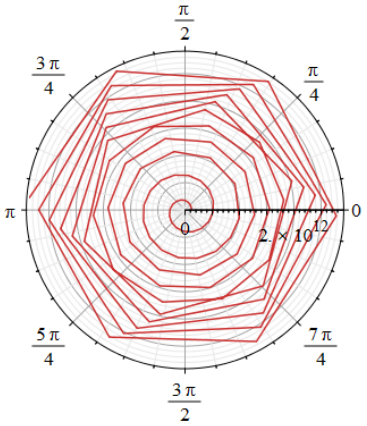
Polar plot of r vs ϕ, for $[R, Phi, 0..7 \cdot 10^7]$:



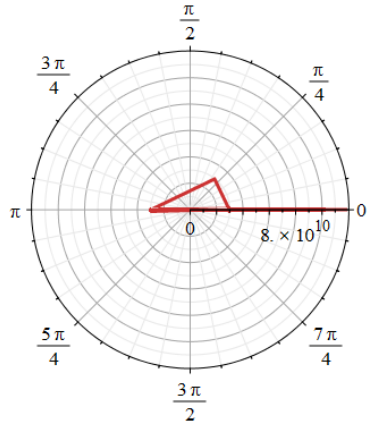
Polar plot of r vs t, for
$[R(t), t, t = 0..2\cdot 10^{7}]$, numpoints = 3
Seems to be a logaritmic spiral...



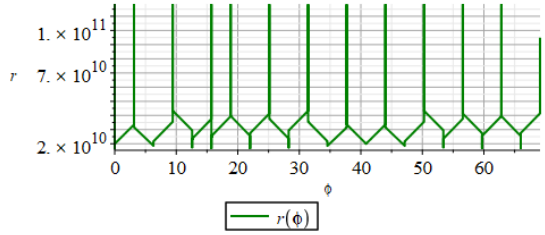
Polar plot of r vs t, for
$[R(t), t, t = 0..9\cdot 10^{6}]$, numpoints = 4
A logaritmic spiral from inwards...

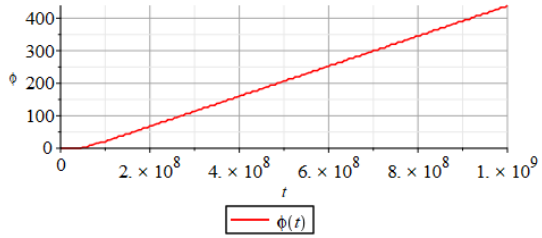


Polar plot of r vs ϕ, for
$[R, Phi, 0..6\cdot 10^{8}]$, numpoints = 10000

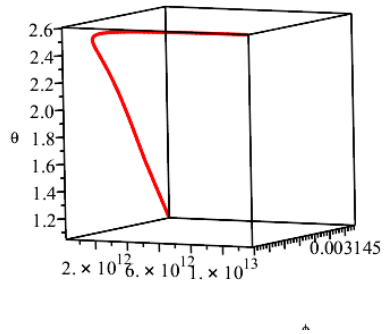
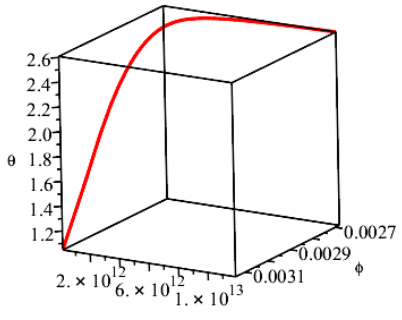
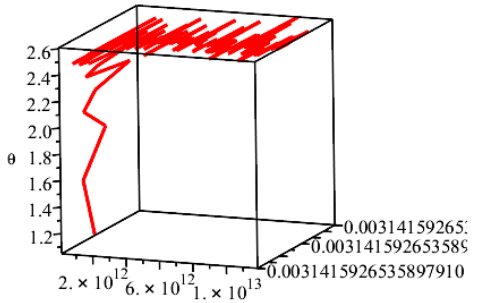
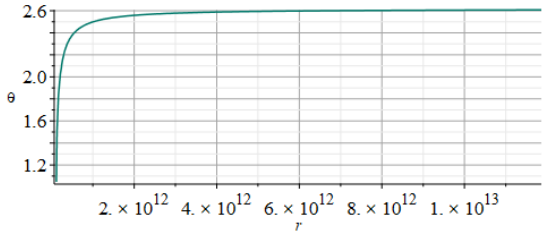
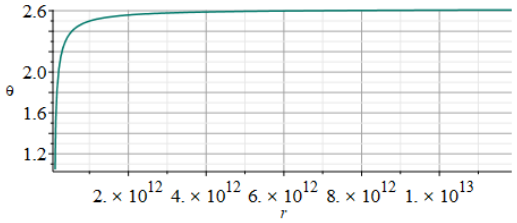
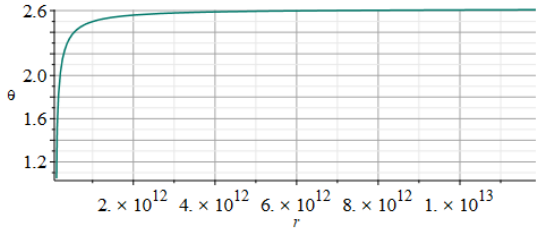


Something is wrong in both plots above. The cartesian plots of r vs ϕ for $t = 0..2\cdot 10^{8}$, and ϕ vs t for $t = 0..10^{9}$, show other behavior:

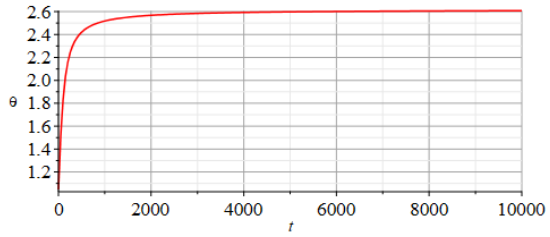




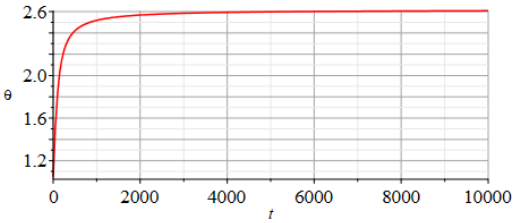
Polar plot of r vs t, for
[R(t), t, t = 0..9·10^6], numpoints = 4
A strange spiral...

$\frac{\pi}{3}$	10^{-2}	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10^4:* 	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10^4:* 	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10^4:* 
			Plot of θ vs. r, for t = 0..10^4 	*Plot of θ vs. r, for t = 0..10^4* 	*Plot of θ vs. r, for t = 0..10^4* 

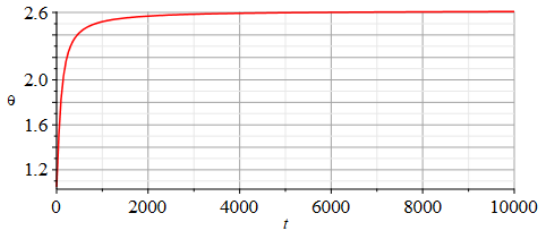
Plot of θ vs. t, for $t = 0..10^4$



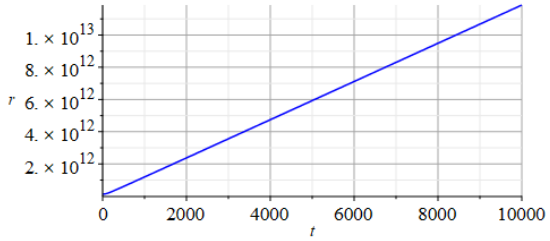
Plot of θ vs. t, for $t = 0..10^4$



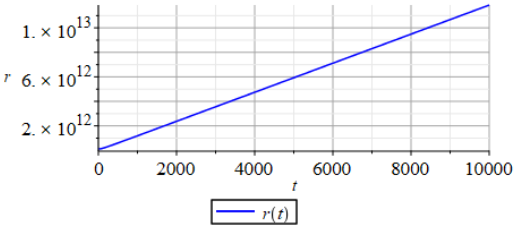
Plot of θ vs. t, for $t = 0..10^4$



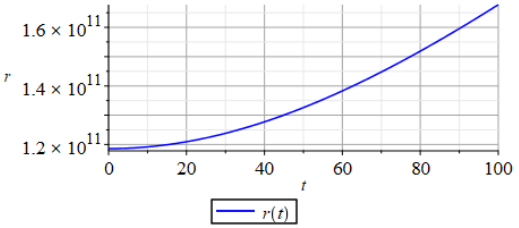
Plot of r vs. t, for $t = 0..10^4$



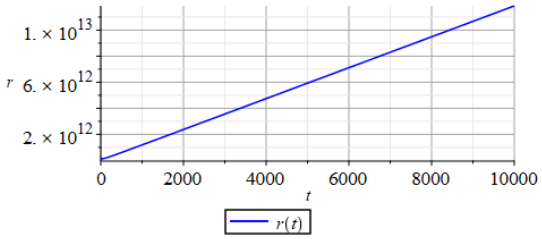
Plot of r vs. t, for $t = 0..10^4$:



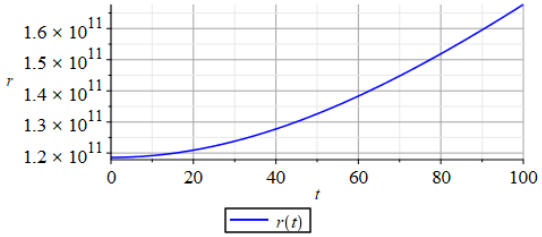
Details for $t = 0..10^2$:



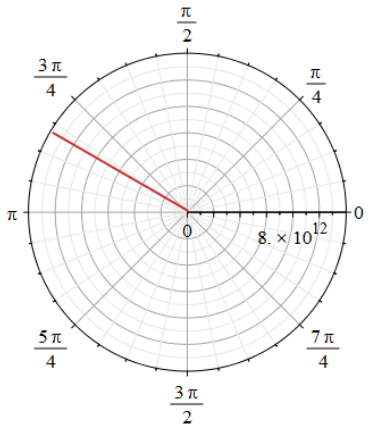
Plot of r vs. t, for $t = 0..10^4$:



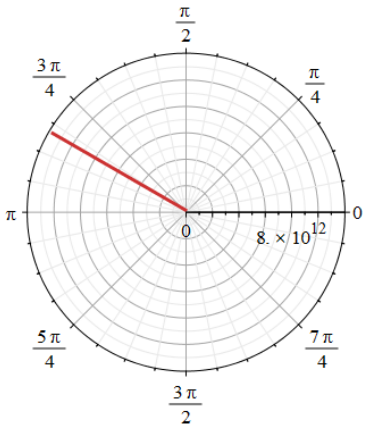
Details for $t = 0..10^2$:



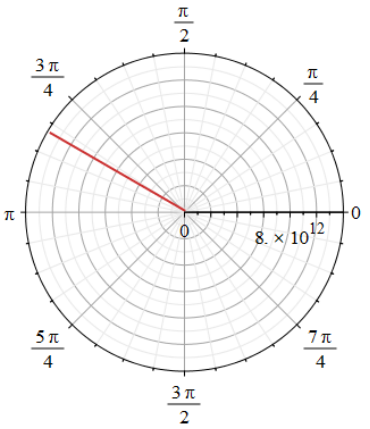
Polar plot of r vs θ, for $[R, Theta, 0..10^4]$



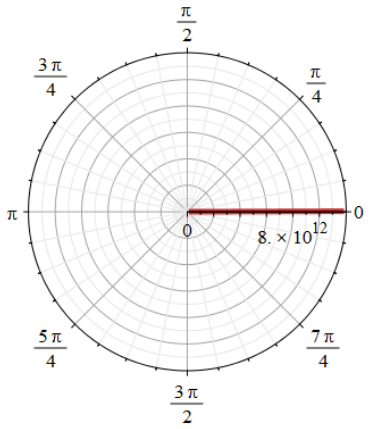
Polar plot of r vs θ, for $[R, Theta, 0..10^4]$:



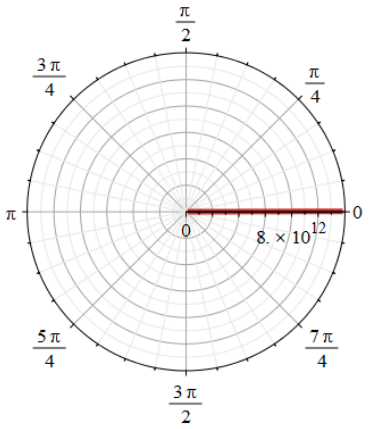
Polar plot of r vs θ, for $[R, Theta, 0..10^4]$:



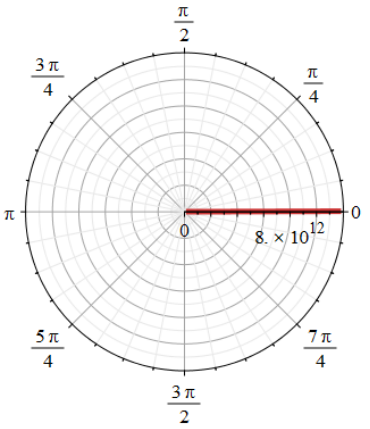
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$

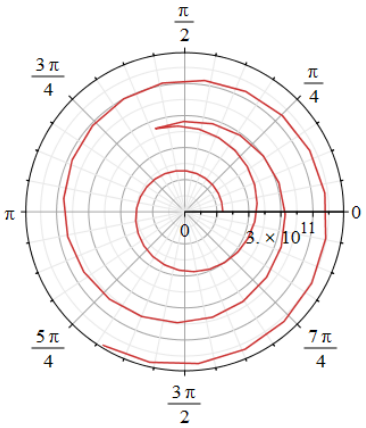
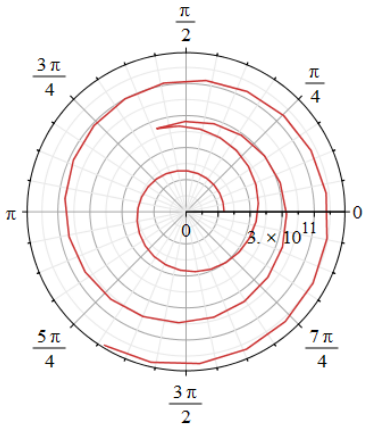
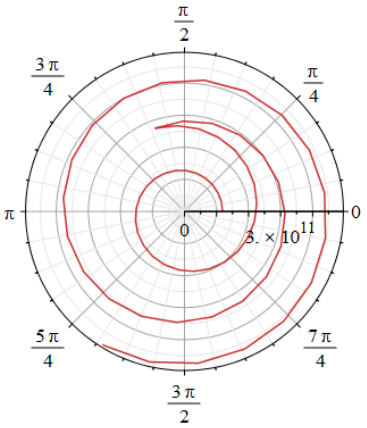


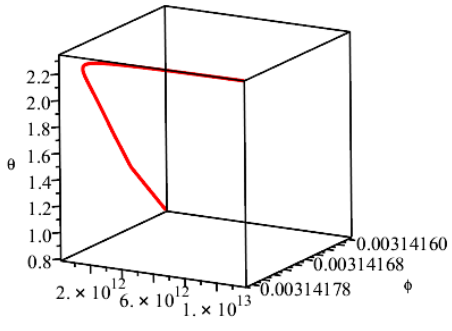
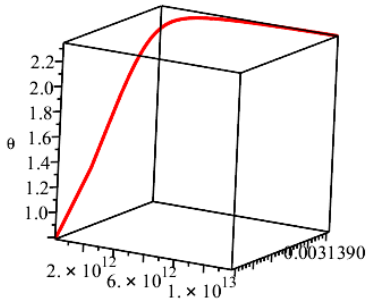
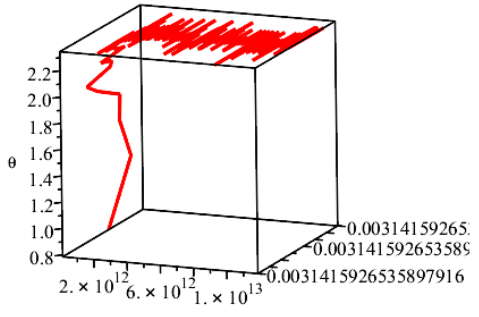
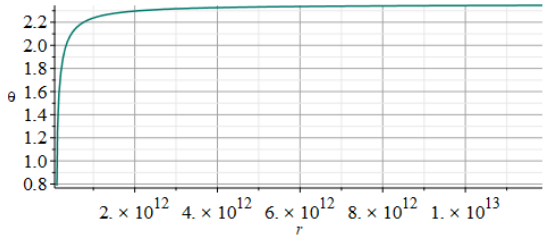
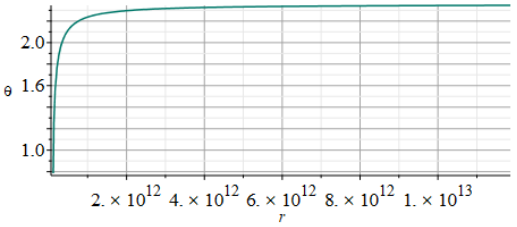
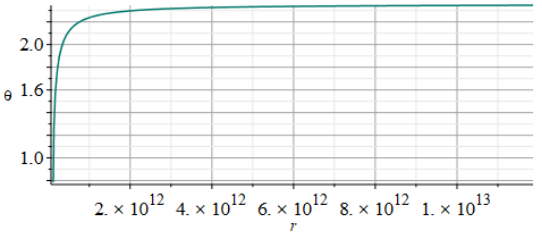
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$:



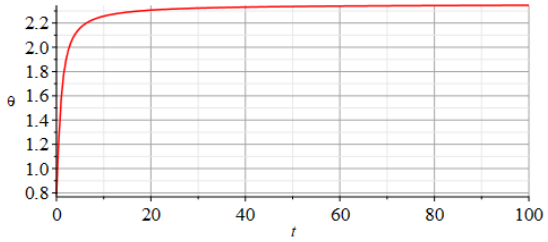
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$:



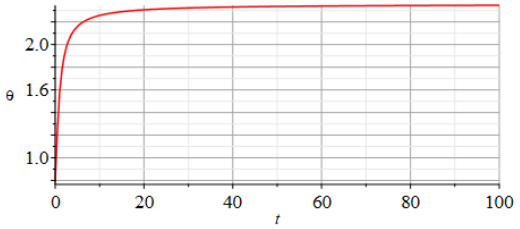
			Polar plot of r vs t, for $[R(t), t, t = 0..4 \cdot 10^2]$, *numpoints = 3* *A strange spiral going forward and backwards...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..4 \cdot 10^2]$, *numpoints = 3* *A strange spiral going forward and backwards...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..4 \cdot 10^2]$, *numpoints = 3* *A strange spiral going forward and backwards...* 

$\frac{\pi}{4}$	1	135° 135° 135°	*The mass m reaches equilibrium at θ = 135° (i.e., initial position of 45° + 90°)* *3D plot for $t = 0..10^2$:*  	*The mass m reaches equilibrium at θ = 135° (i.e., initial position of 45° + 90°)* *3D plot for $t = 0..10^2$:*  	*The mass m reaches equilibrium at θ = 135° (i.e., initial position of 45° + 90°)* *3D plot for $t = 0..10^2$:* 
			Plot of θ vs. r, for $t = 0..10^2$  	*Plot of θ vs. r, for $t = 0..10^2$:*  	*Plot of θ vs. r, for $t = 0..10^2$* 

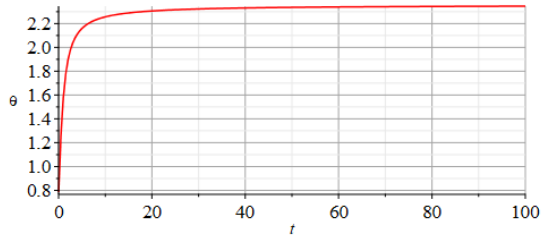
Plot of θ vs. t, for t = 0..10²



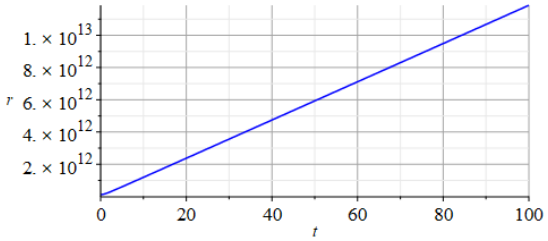
Plot of θ vs. t, for t = 0..10²:



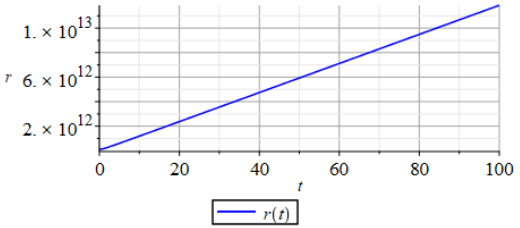
Plot of θ vs. t, for t = 0..10²



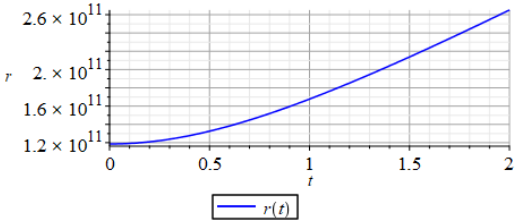
Plot of r vs. t, for t = 0..10²



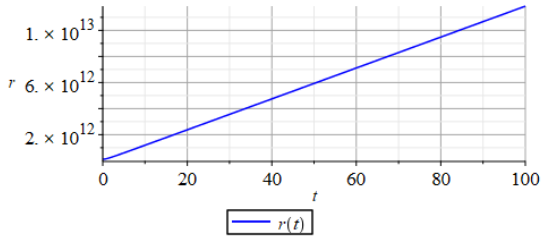
Plot of r vs. t, for t = 0..10²:



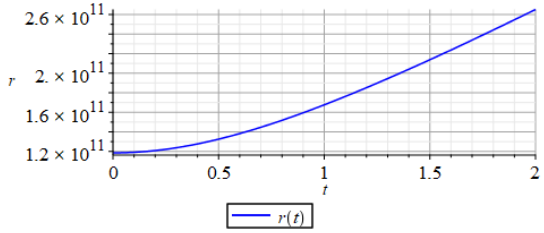
Details for t = 0..2:



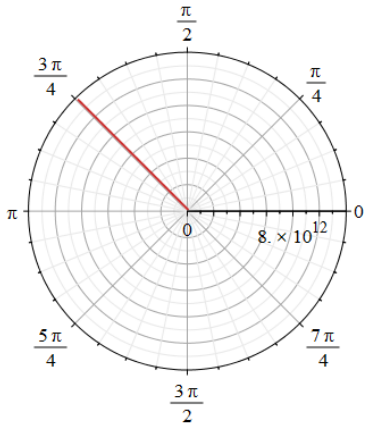
Plot of r vs. t, for t = 0..10²:



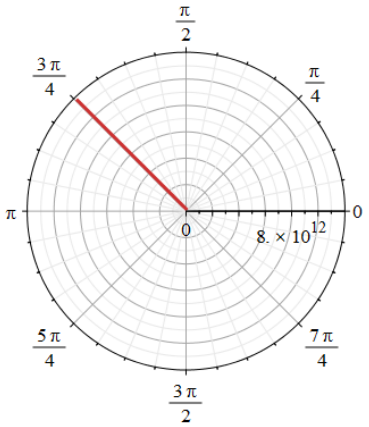
Details for t = 0..2:



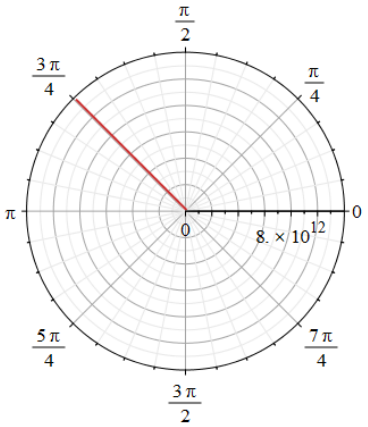
Polar plot of r vs θ, for $[R, Theta, 0..10^2]$



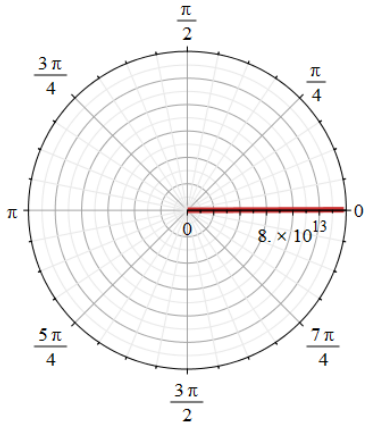
Polar plot of r vs θ, for $[R, Theta, 0..10^2]$:



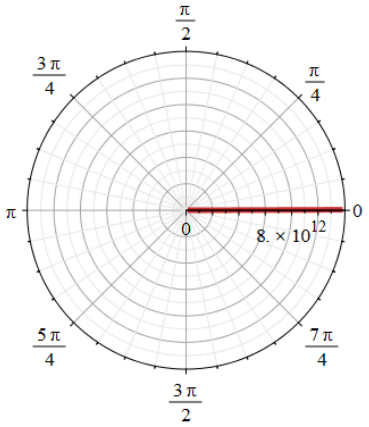
Polar plot of r vs θ, for $[R, Theta, 0..10^2]$:



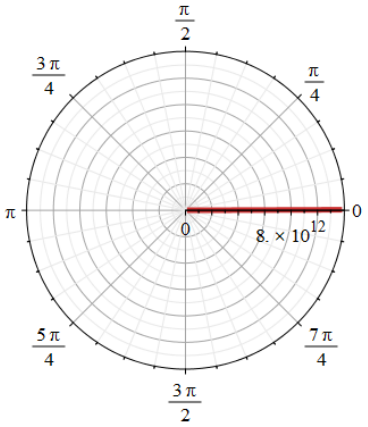
Polar plot of r vs ϕ, for $[R, Phi, 0..10^3]$



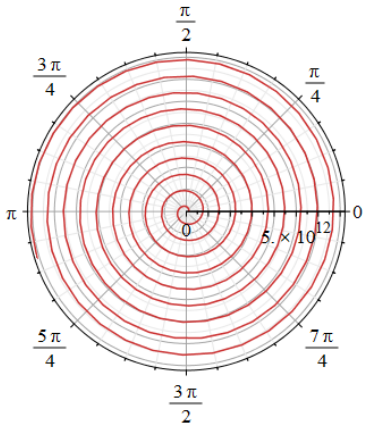
Polar plot of r vs ϕ, for $[R, Phi, 0..10^2]$



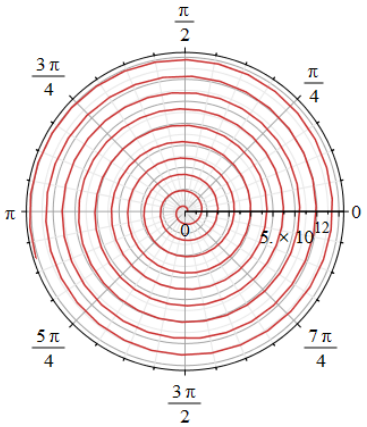
Polar plot of r vs ϕ, for $[R, Phi, 0..10^2]$:



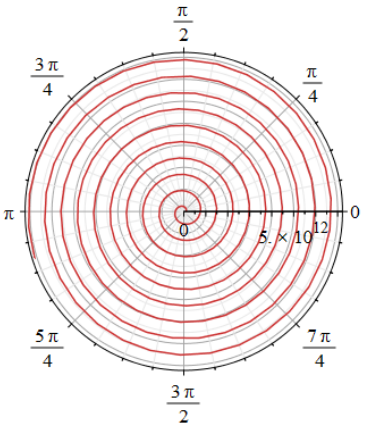
Polar plot of r vs t, for
$\left[R(t),\ t,\ t = 0..6\cdot 10\right]$, *numpoints = 10*
A regular spiral

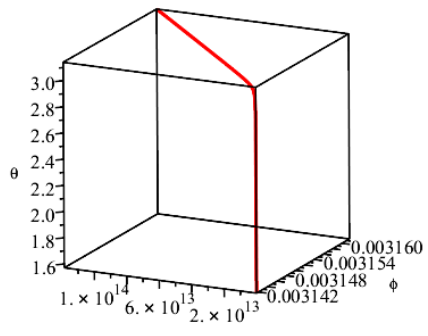
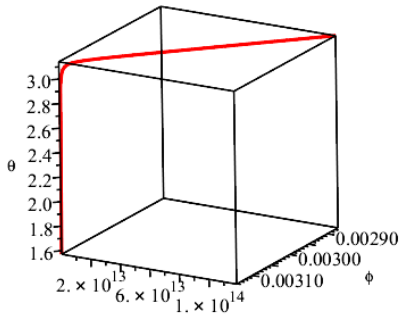
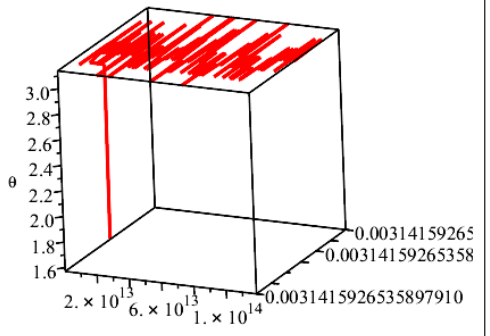
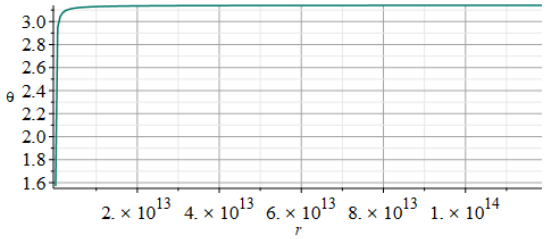
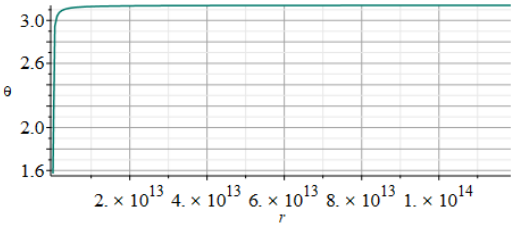
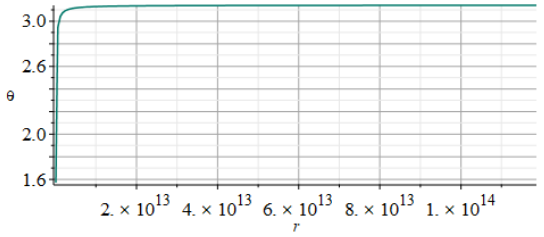


Polar plot of r vs t, for
$\left[R(t),\ t,\ t = 0..6\cdot 10\right]$, *numpoints = 10*
A regular spiral

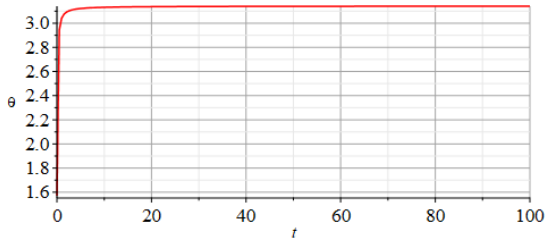


Polar plot of r vs t, for
$\left[R(t),\ t,\ t = 0..6\cdot 10\right]$, *numpoints = 10*
A regular spiral

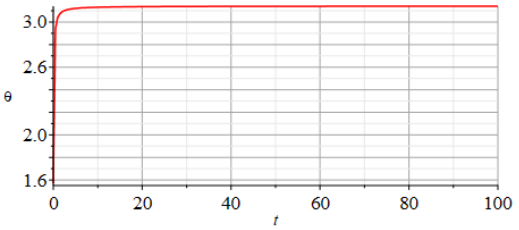


$\frac{\pi}{2}$	10	180° 180° 180°	*The mass m reaches equilibrium at θ = 180° (i.e., initial position of 90° + 90°)* *3D plot for $t = 0..10^2$:*  	*The mass m reaches equilibrium at θ = 180° (i.e., initial position of 90° + 90°)* *3D plot for $t = 0..10^2$:*  	*The mass m reaches equilibrium at θ = 180° (i.e., initial position of 90° + 90°)* *3D plot for $t = 0..10^2$:* 
			Plot of θ vs. r, for $t = 0..10^2$  	*Plot of θ vs. r, for $t = 0..10^2$*  	*Plot of θ vs. r, for $t = 0..10^2$* 

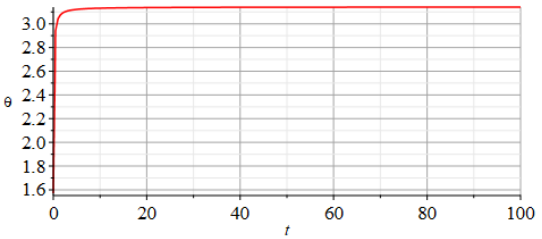
Plot of θ vs. t, for t = 0..10²



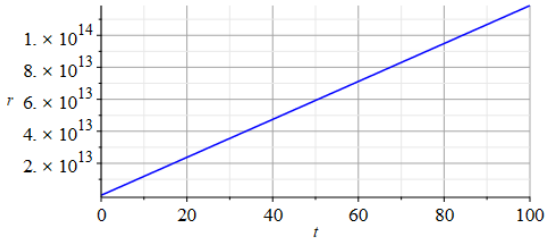
Plot of θ vs. t, for t = 0..10²



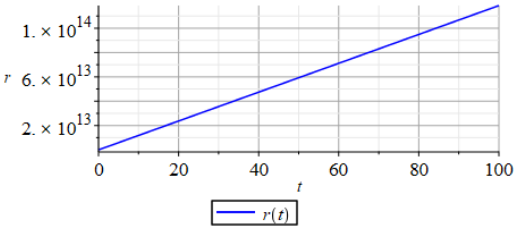
Plot of θ vs. t, for t = 0..10²



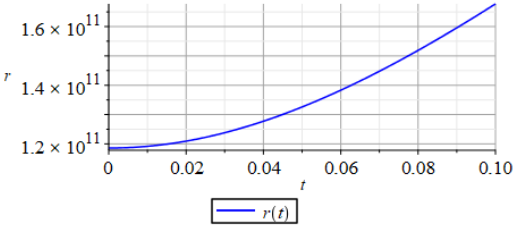
Plot of r vs. t, for t = 0..10²



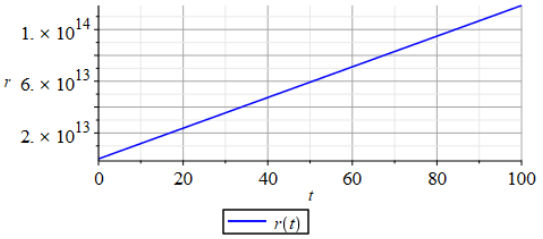
Plot of r vs. t, for t = 0..10²



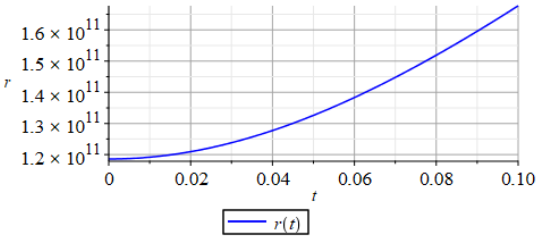
Details for t = 0..0.1:

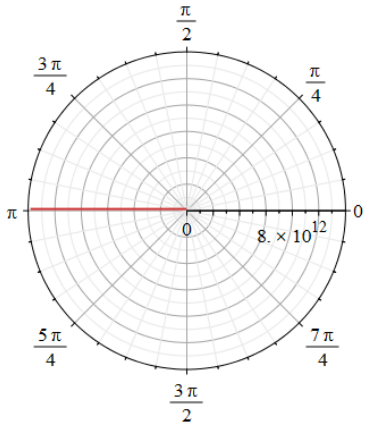
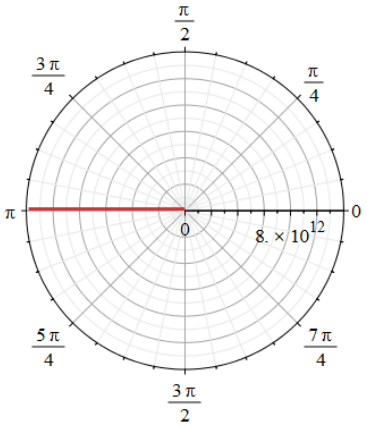
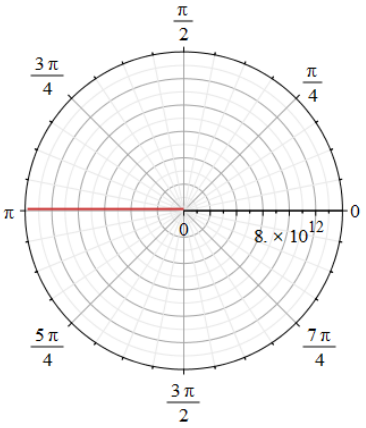
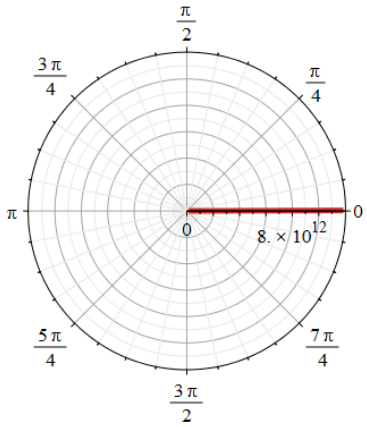
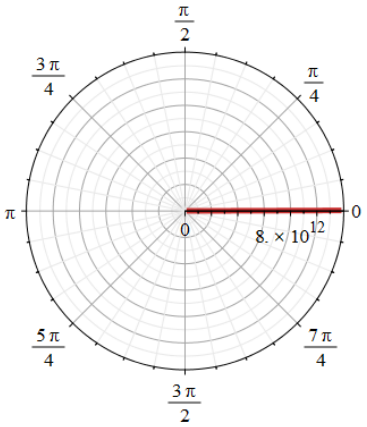
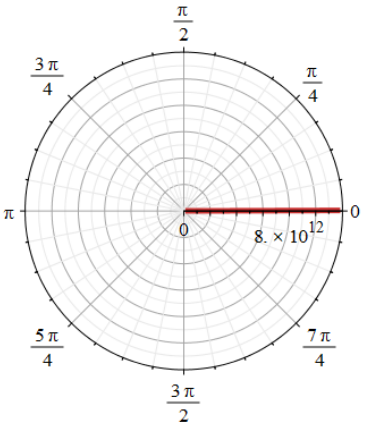


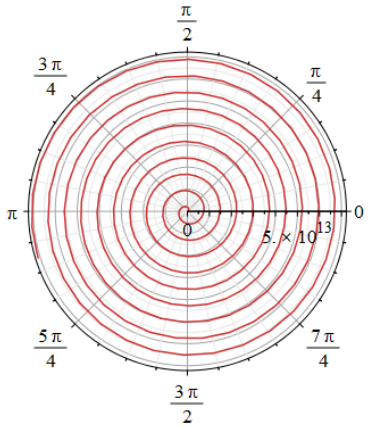
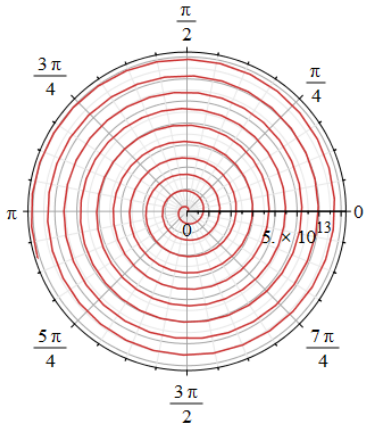
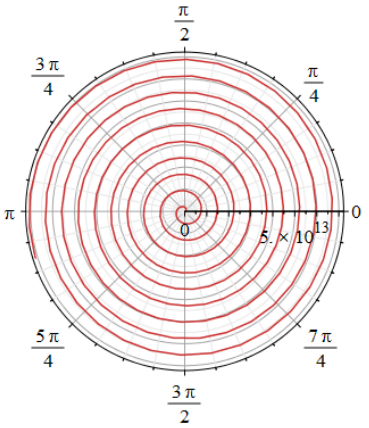
Plot of r vs. t, for t = 0..10²

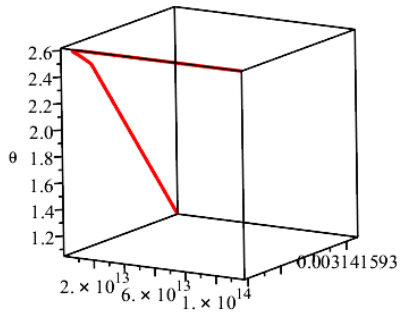
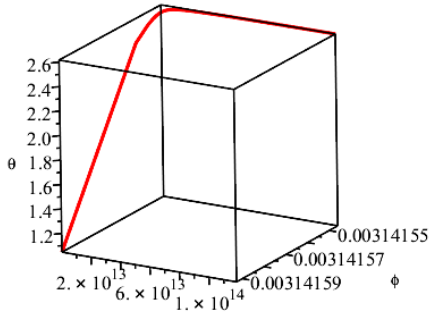
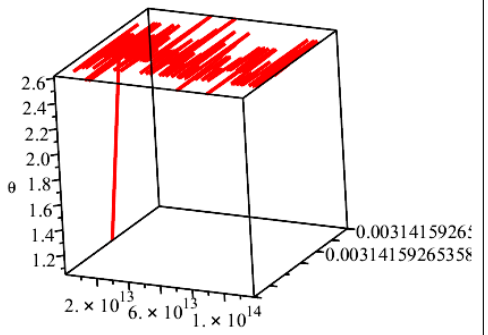
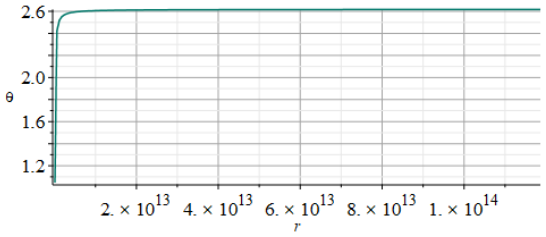
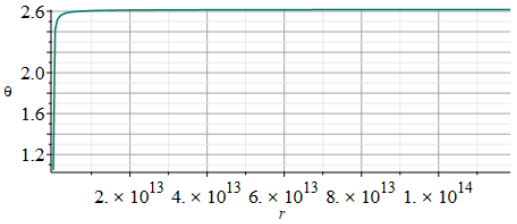
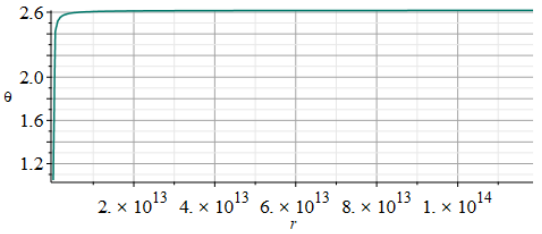


Details for t = 0..0.1:

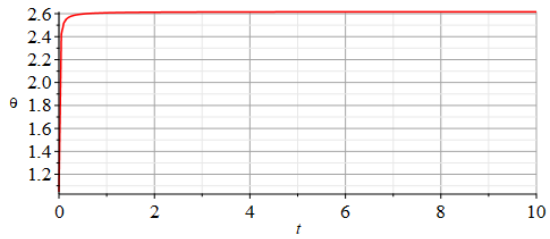


Polar plot of r vs θ, for [R, Theta, 0..10]	*Polar plot of r vs θ, for* [R, Theta, 0..10]	*Polar plot of r vs θ, for* [R, Theta, 0..10]
		
Polar plot of r vs ϕ, for [R, Phi, 0..10]	*Polar plot of r vs ϕ, for* [R, Phi, 0..10]	*Polar plot of r vs ϕ, for* [R, Phi, 0..10]
		

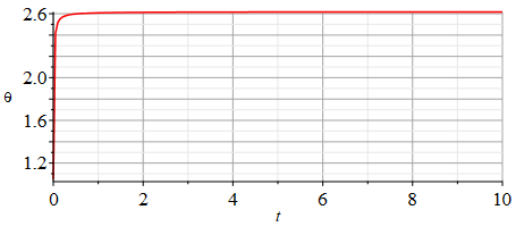
			Polar plot of r vs t, for $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 

$\frac{\pi}{3}$	100	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10:* 	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10:* 	*The mass m reaches equilibrium at θ = 150° (i.e., initial position of 60° + 90°)* *3D plot for t = 0..10:* 
			Plot of θ vs. r, for t = 0..10 	*Plot of θ vs. r, for t = 0..10* 	*Plot of θ vs. r, for t = 0..10* 

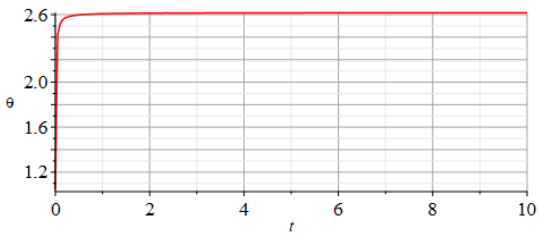
Plot of θ vs. t, for t = 0..10



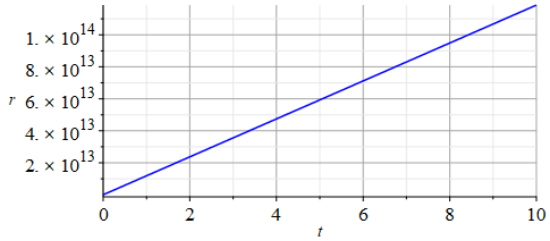
Plot of θ vs. t, for t = 0..10



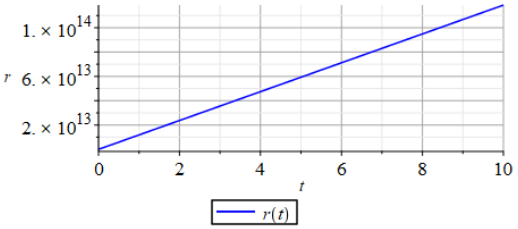
Plot of θ vs. t, for t = 0..10



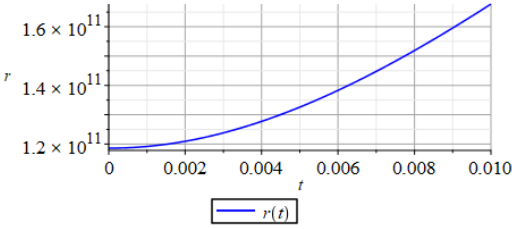
Plot of r vs. t, for t = 0..10



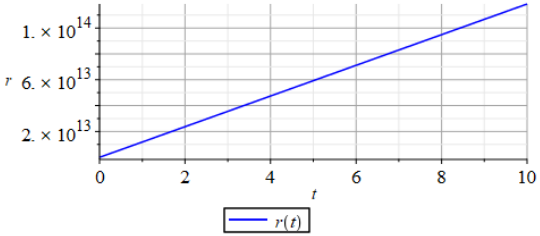
Plot of r vs. t, for t = 0..10



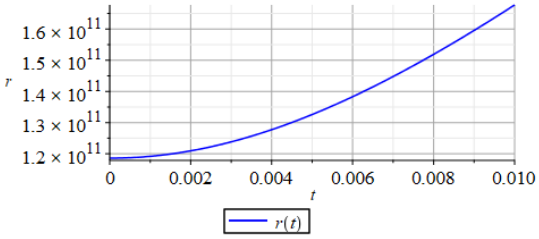
Details for t = 0..10^-2



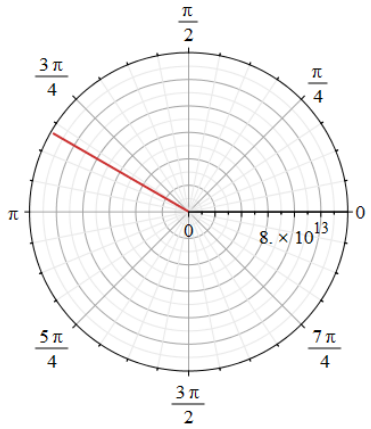
Plot of r vs. t, for t = 0..10



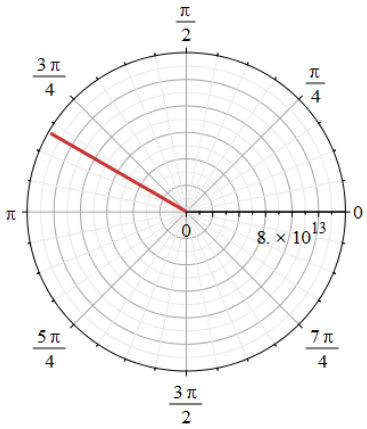
Details for t = 0..10^-2



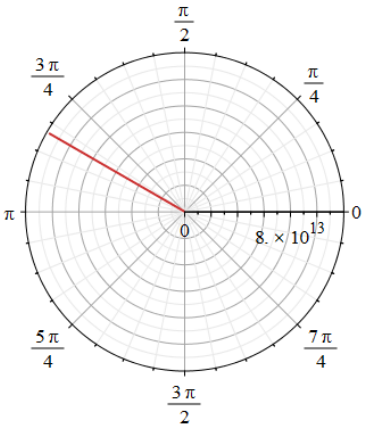
Polar plot of r vs θ, for $[R, Theta, 0..10]$



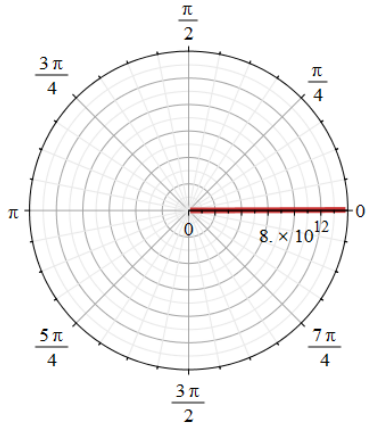
Polar plot of r vs θ, for $[R, Theta, 0..10]$



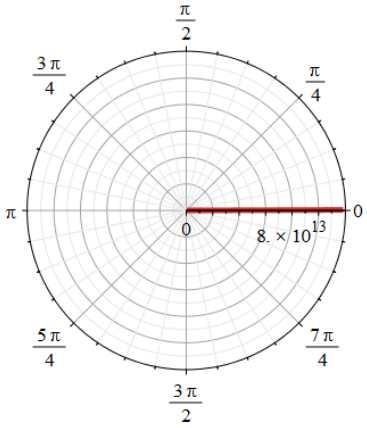
Polar plot of r vs θ, for $[R, Theta, 0..10]$



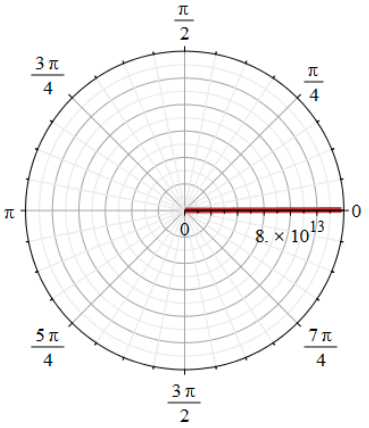
Polar plot of r vs ϕ, for $[R, Phi, 0..10]$

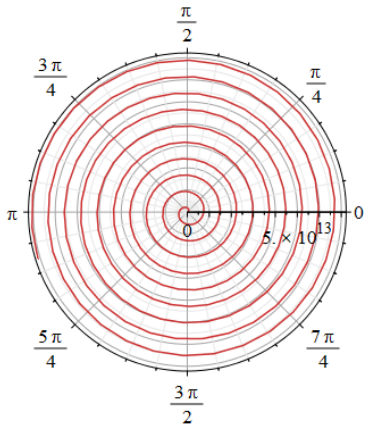
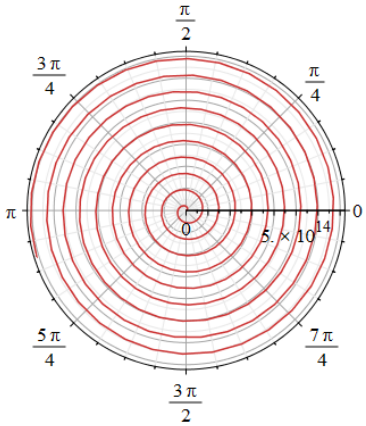
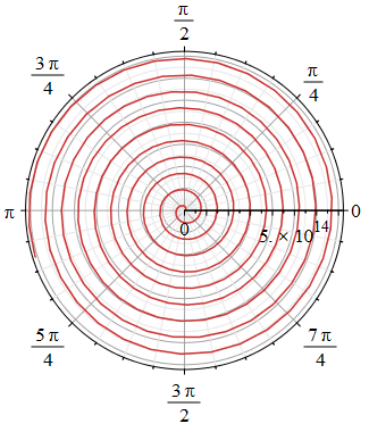


Polar plot of r vs ϕ, for $[R, Phi, 0..10]$



Polar plot of r vs ϕ, for $[R, Phi, 0..10]$



			Polar plot of r vs t, for $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10]$, *numpoints = 10* *A regular spiral* 

APPENDIX II

All plots and details of the results for r(0) = 100 10^9 [m]

Analysis and plots of the motion for an initial distance of $r_{(0)} = 100\ 10^9\ [m]$, and the following initial conditions and parameters:

$G = 6.67408 * 10^{-11}\ \frac{Nm^2}{Kg^2}$ *(the universal gravitational constant)*

$M = 2\ 10^{30}\ Kg$ *(the mass of the Sun)*

$m = 5.9\ 10^{24}\ Kg$ *(the mass of the Earth)*

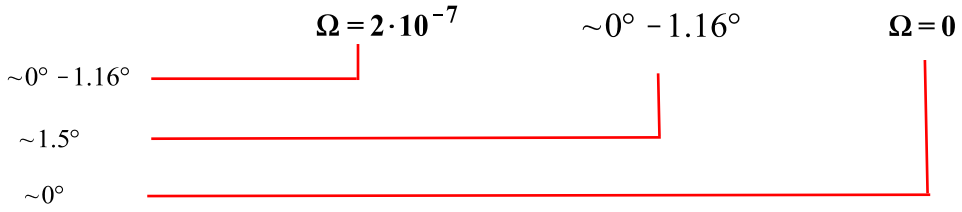
$\dot{r}_{(0)} = 0\ [m/s]$ *(linear velocity of the center of mass **m** at t=0)*

$\phi_{(0)} = 0\ [rad]$ *(azimuthal position of the center of mass **m** at t=0)*

Since there is a singularity for $\theta = 0$, the "zero" initial condition of the polar angle was taken close to zero as $\theta = \frac{\pi}{10^3}\ [rad]$.

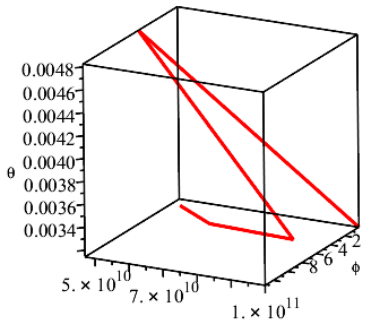
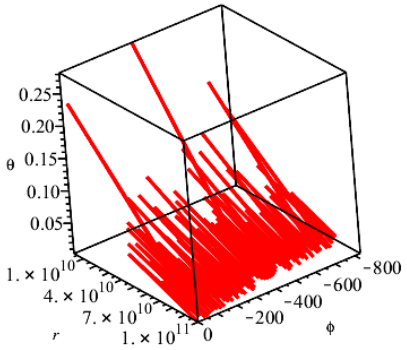
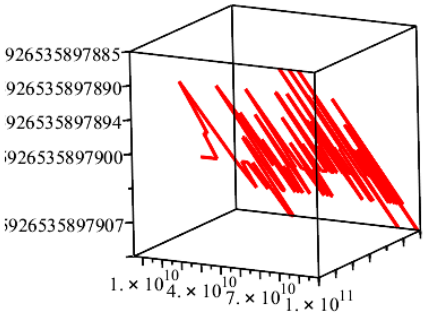
$\dot{\phi}_{(t)} = \Omega$ *(is the relative angular velocity between the spin of the center of mass **M** and the translation of the center of mass **m**)*

The polar angle of equilibrium (θ_f) is found for several values of the initial conditions of $\theta_{(0)}$, $\omega_{(0)}$, and Ω. The three vertical values of θ_f on each corresponding table cell, from top to bottom, are the results for each one of the three values of Ω from left to right. For example:

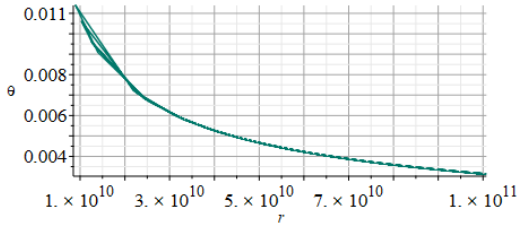


Obs.: the table is very long and continues over several pages.

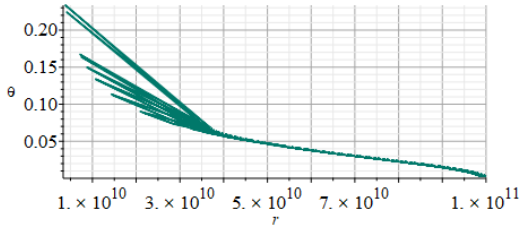
See Table 2 for a summary of the results obtained here.

$\theta(0)$ *[rad]*	$\omega_\theta(0)$ $[\frac{1}{s}]$	θ_f *[deg]*	$\Omega = 2\cdot 10^{-7}\ [\frac{1}{s}]$ $r(0)=100\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=2\cdot 10^{-7}$	$\Omega = -2.7\cdot 10^{-6}\ [\frac{1}{s}]$ $r(0)=100\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=-2.7\cdot 10^{-6}$	$\Omega = 0\ [\frac{1}{s}]$ $r(0)=100\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=0$
$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	0	~ 0° 1.2° ~0°	*The mass m reaches equilibrium by oscillating around its initial position from 0°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating around 1.2°, having a max. swing to ~15°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating at $\theta \approx 0°$.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
			3D plot for $t = 0..10^9$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..5\cdot 10^6$* 

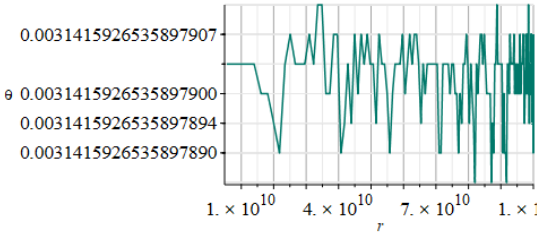
Plot of θ vs. r, for $t = 0..3\cdot 10^7$



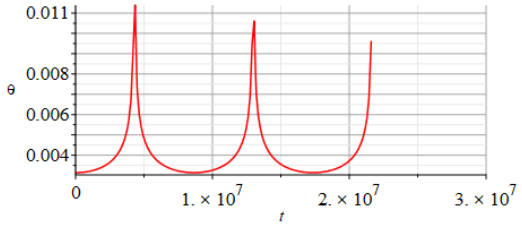
Plot of θ vs. r, for $t = 0..10^8$



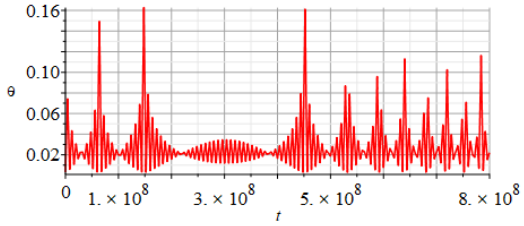
Plot of θ vs. r, for $t = 0..5\cdot 10^6$



Plot of θ vs. t, for $t = 0..3\cdot 10^7$



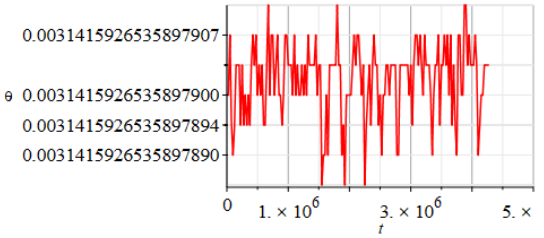
*Plot of θ vs. t, for $t = 0..8\cdot 10^8$. There are amplitude and frequency modulation of the envelope. For this time range, the max. period is $T\approx 1.7\cdot 10^8$ (~**5.4 years**), and the min. period is $T\approx 0.4\cdot 10^8$ (~**1.3 years**):*

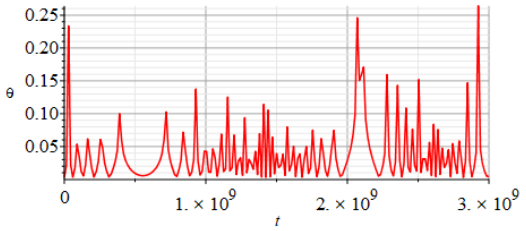
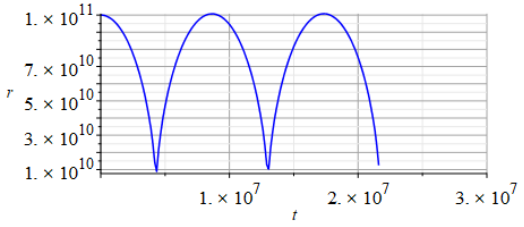
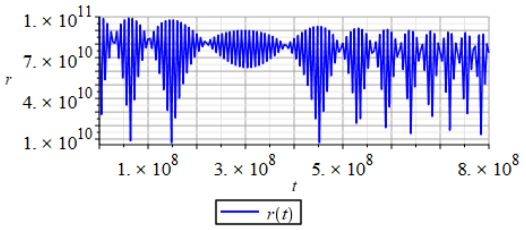
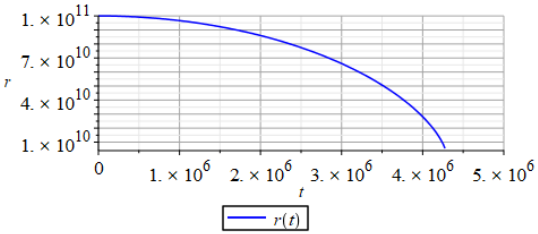


Precession and/or Nutation?

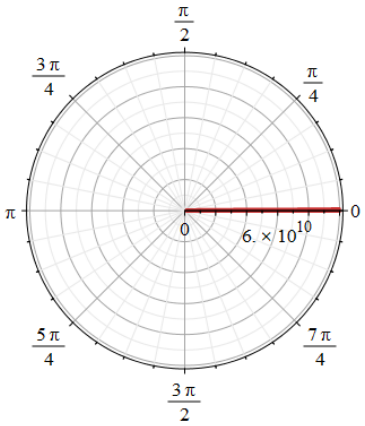
WARNING: *the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

Plot of θ vs. t, for $t = 0..5\cdot 10^6$.

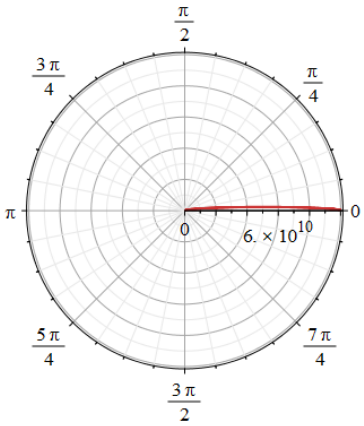


				Plot of θ vs. t, for t = 0..3·10^9. The pattern is not clear. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.* 	
			Plot of r vs. t, for t = 0..3·10^7 	*Plot of r vs. t, for t = 0..8·10^8. There are amplitude and frequency modulation of the envelope. For this time range, the max. period is T≈1.7·10^8 (~**5.4 years**), and the min. period is T≈0.4·10^8 (~**1.3 years**):* 	*Plot of r vs. t, for t = 0..5·10^6.* 

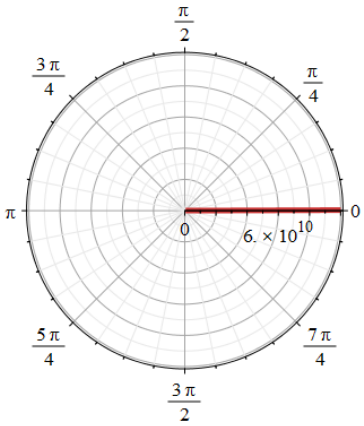
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$



Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

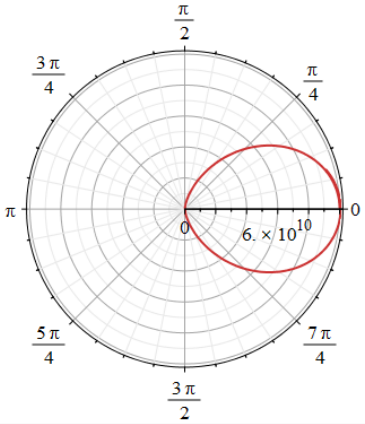


Polar plot of r vs θ, for $[R, Theta, 0..5 \cdot 10^6]$



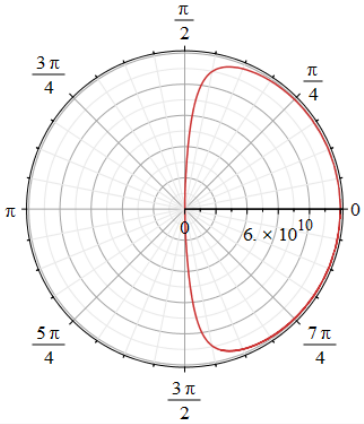
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$

Seems to be a slightly elliptic trajectory (orbit of mass m around central mass M)

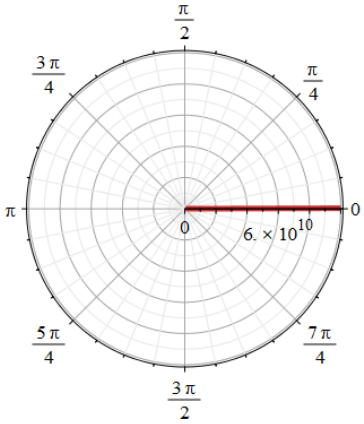


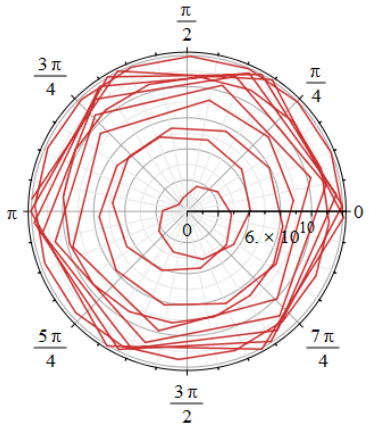
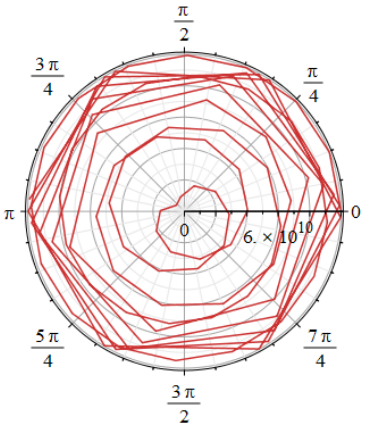
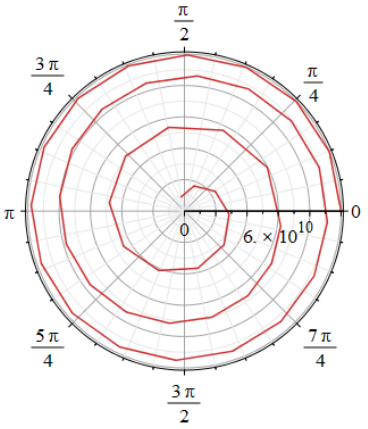
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$

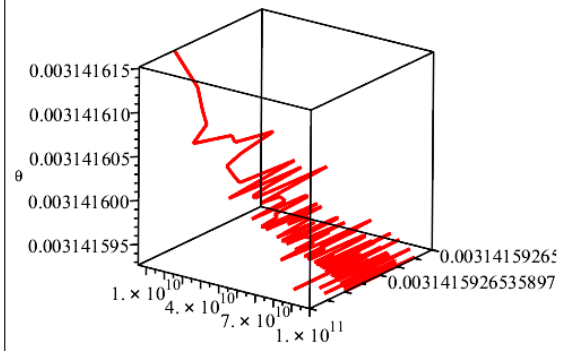
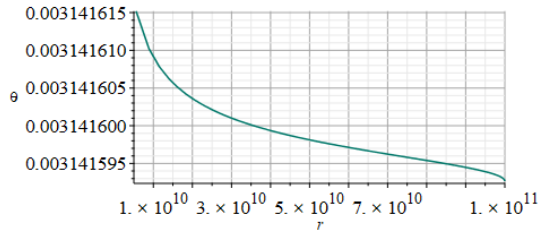
Seems to be a quasi semi-circular trajectory (orbit of mass m around central mass M)



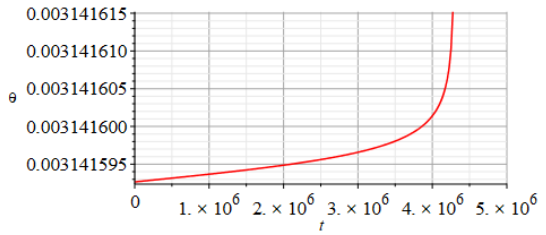
Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^6]$



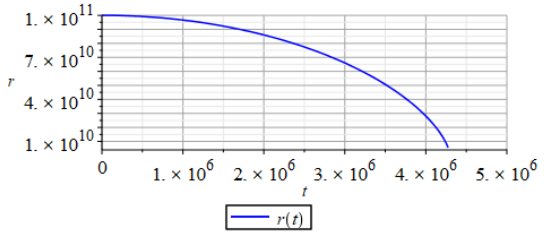
			Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A type of logaritmic spiral from outwards...* 

$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	10^{-15}	$\sim 0°$ 1.2° ~0°	*Same as above*	*Same as above*	*The mass m reaches equilibrium at $\theta \approx 0°$.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
					3D plot for $t = 0..5 \cdot 10^6$ 
					Plot of θ vs. r, for $t = 0..5 \cdot 10^6$ 

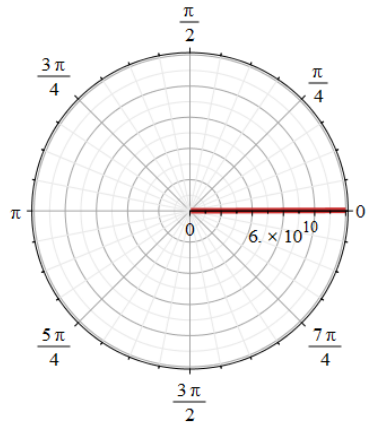
Plot of θ vs. t, for t = 0..5·10^6.



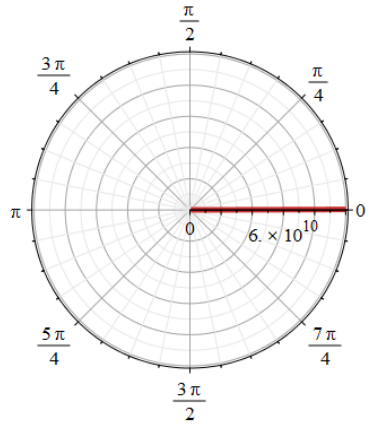
Plot of r vs. t, for t = 0..5·10^6.

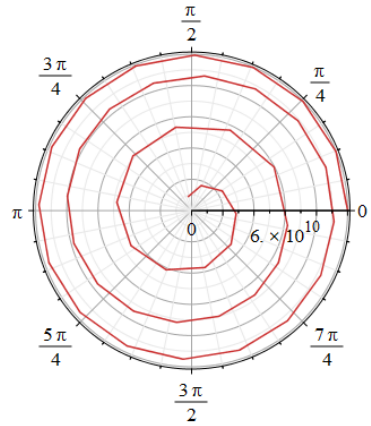


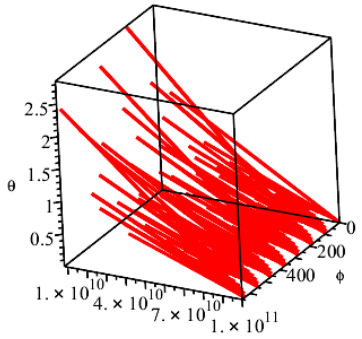
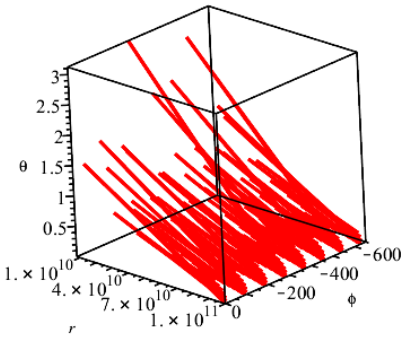
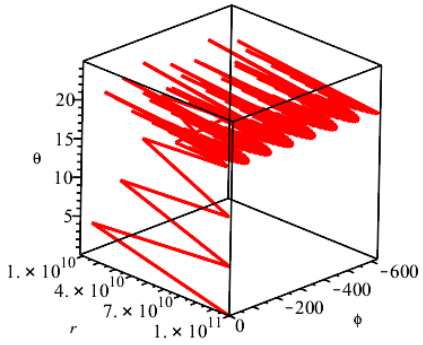
Polar plot of r vs θ, for $[R, Theta, 0..5 \cdot 10^6]$



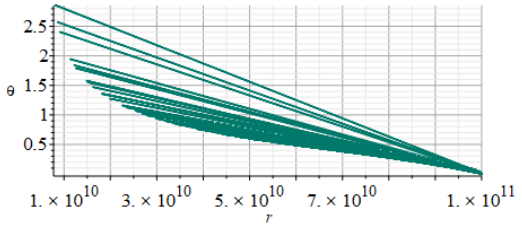
Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^6]$



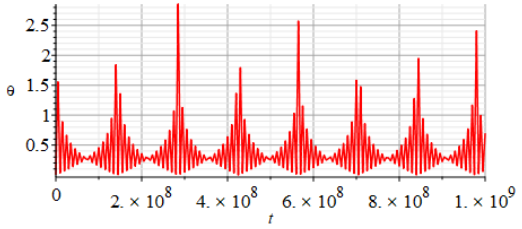
					Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10^{6}]$, *numpoints = 4* *A type of logaritmic spiral from outwards...* 

$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	10^{-7}	90° 90° ~85°	*The mass m oscillates from 0° to maximum ~180°, having thus a mean oscillation angle ~ 90° which can be considered the average equilibrium angle, with smaller oscillations around ~16°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days. **A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m oscillates from 0° to maximum ~180°, having thus a mean oscillation angle ~ 90° which can be considered the average equilibrium angle, with smaller oscillations around ~16°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days. **A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m oscillates from 0° to maximum ~171°, having thus a mean oscillation angle ~ 85° which can be considered the average equilibrium angle, with smaller oscillations around ~16°.*
			3D plot for $t = 0..10^9$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..10^9$* 

Plot of θ vs. r, for $t = 0..10^9$



*Plot of θ vs. t, for $t = 0..10^9$. The period of the envelope for this time range is $T \approx 1.4 \cdot 10^8$ (~ **4.4 years**):*

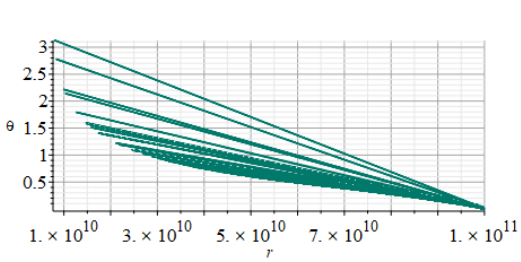


Precession and/or Nutation?

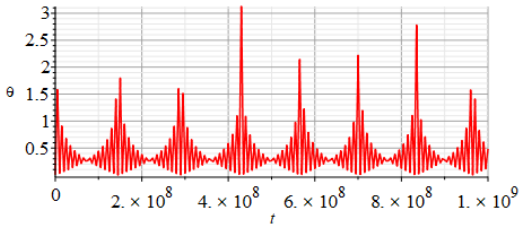
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***

Plot of θ vs. r, for $t = 0..10^9$



*Plot of θ vs. t, for $t = 0..10^9$. The period of the envelope for this time range is $T \approx 1.4 \cdot 10^8$ (~ **4.4 years**):*



Precession and/or Nutation?

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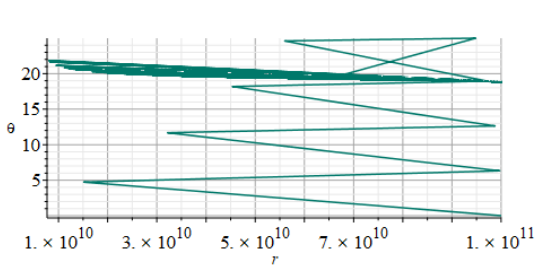
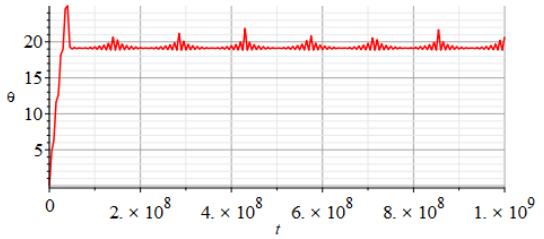
Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***

Plot of θ vs. r, for $t = 0..10^9$



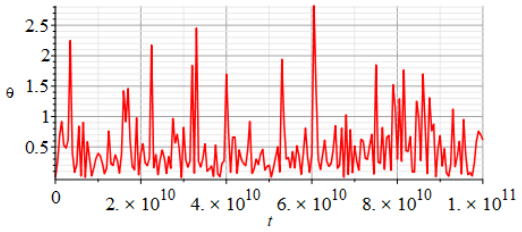
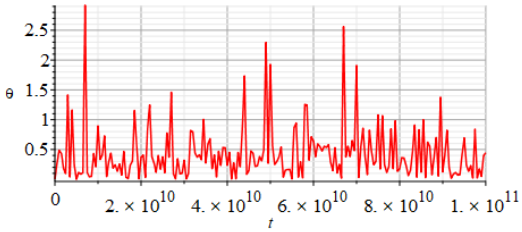
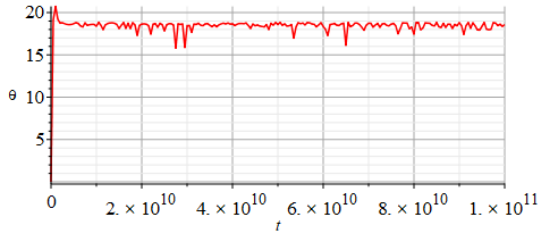
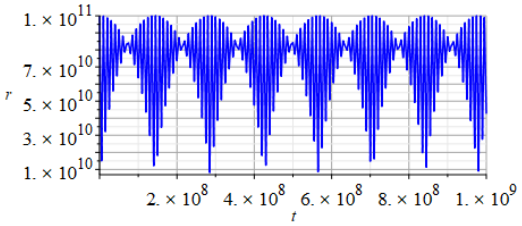
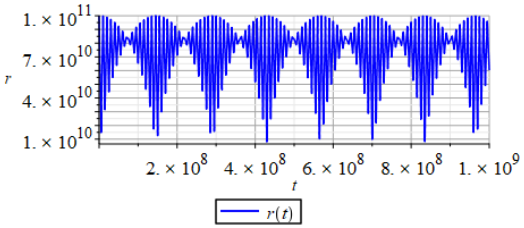
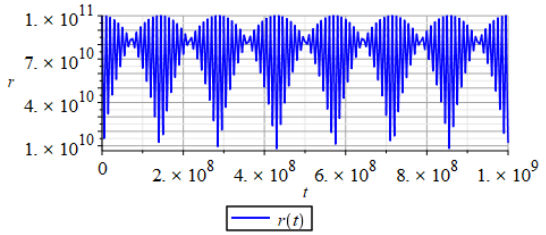
*Plot of θ vs. t, for $t = 0..10^9$. The period of the envelope for this time range is $T \approx 1.4 \cdot 10^8$ (~ **4.4 years**):*

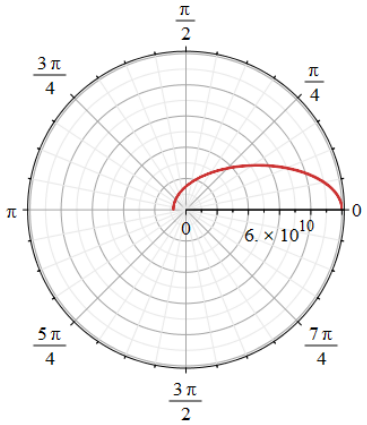
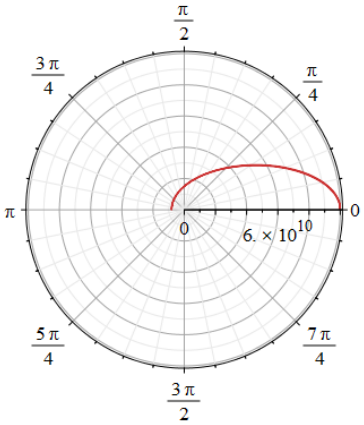
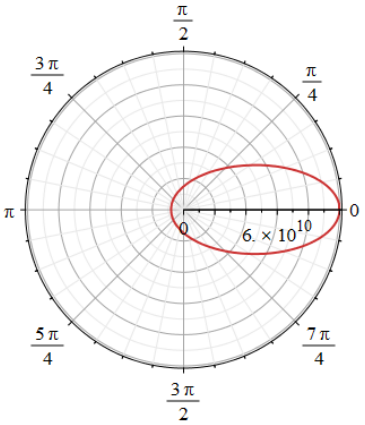
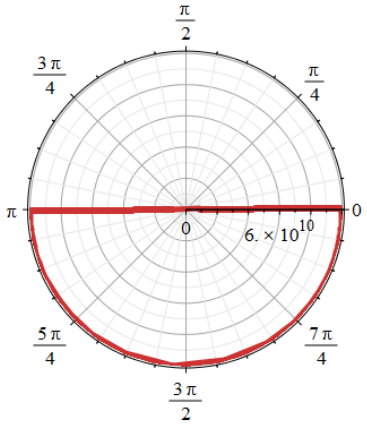
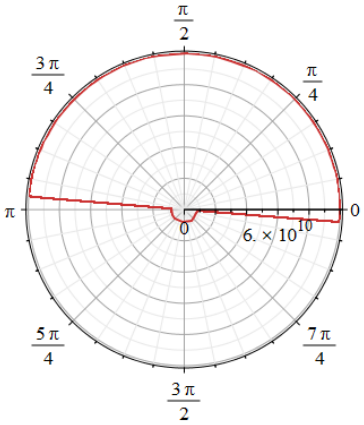
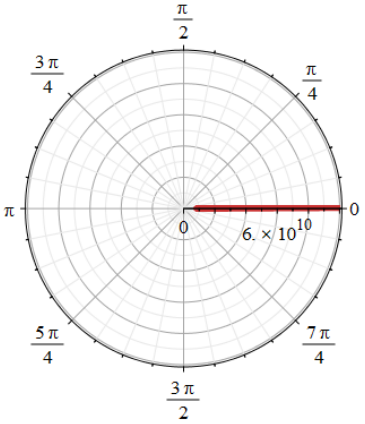


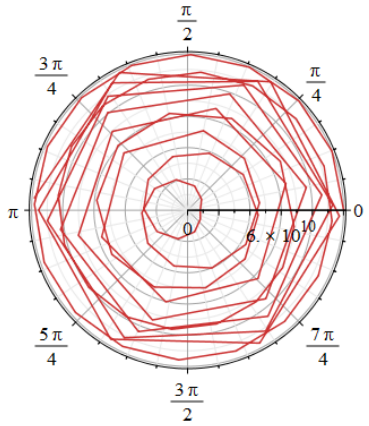
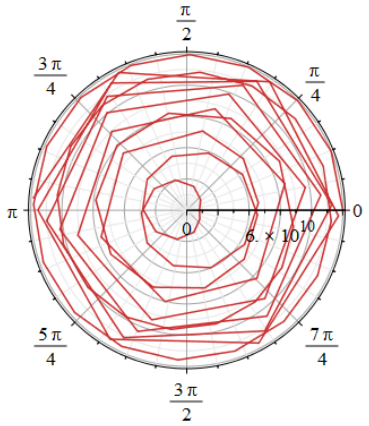
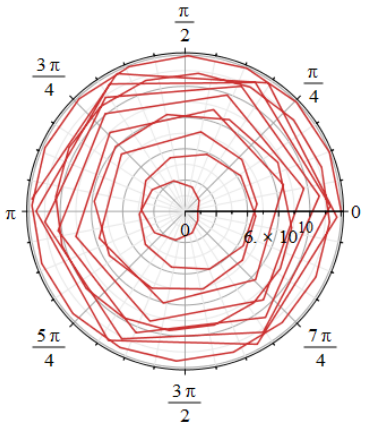
Precession and/or Nutation?

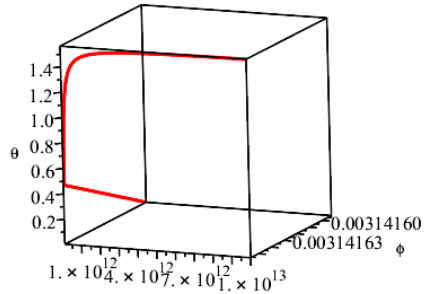
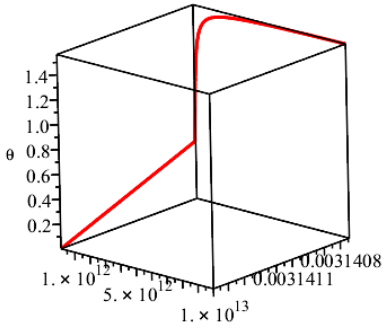
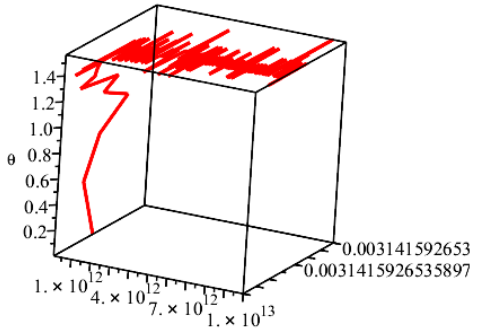
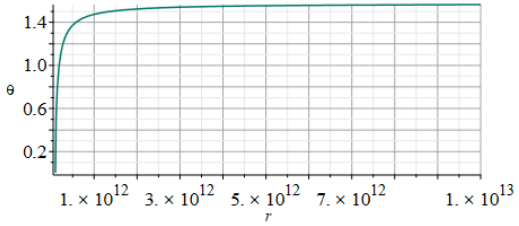
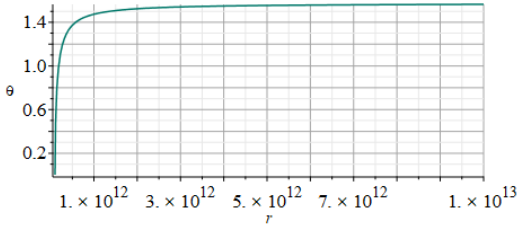
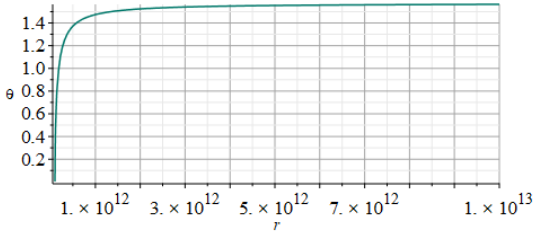
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***

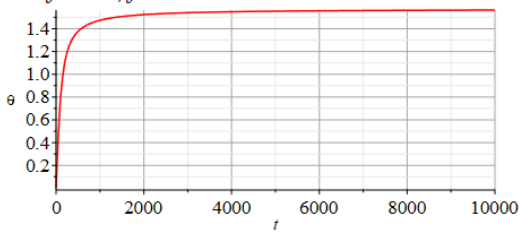
		
*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈1.4·10^8 (~ **4.4 years**):* 	*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈1.4·10^8 (~ **4.4 years**):* 	*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈1.4·10^8 (~ **4.4 years**):* 

			Polar plot of r vs θ, for $[R, Theta, 0..10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^7]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$, *numpoints = 5000* 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^7]$, *numpoints = 20000* 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^7]$ 

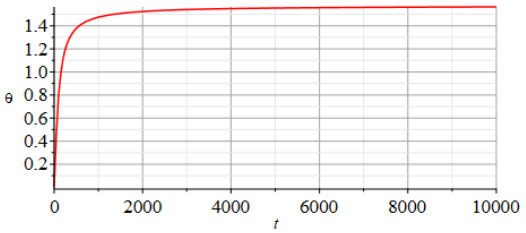
			Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10^6]$, *numpoints = 4* *A strange spiral...* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10^{-2}	90° 90° 90°	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^4$* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^4$* 	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^4$* 
			Plot of θ vs. r, for $t = 0..10^4$ 	*Plot of θ vs. r, for $t = 0..10^4$* 	*Plot of θ vs. r, for $t = 0..10^4$* 

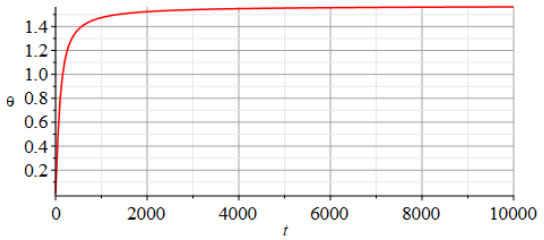
Plot of θ vs. t, for t = 0..10^4



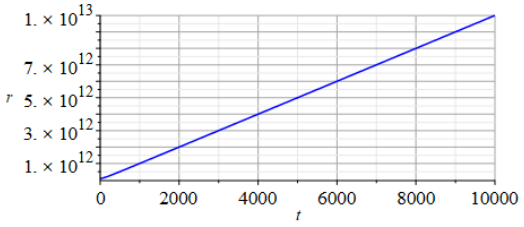
Plot of θ vs. t, for t = 0..10^4



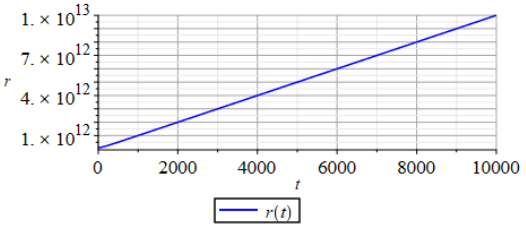
Plot of θ vs. t, for t = 0..10^4



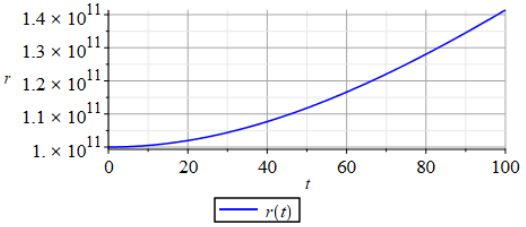
Plot of r vs. t, for t = 0..10^4



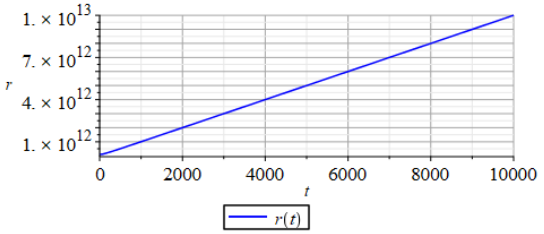
Plot of r vs. t, for t = 0..10^4



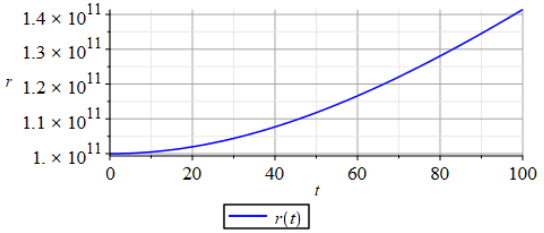
Details for t = 0..10^2



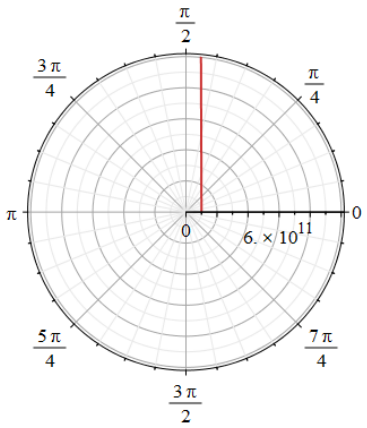
Plot of r vs. t, for t = 0..10^4



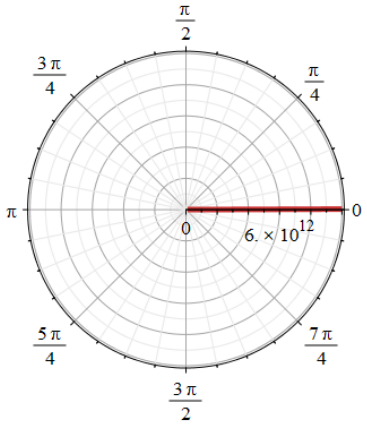
Details for t = 0..10^2



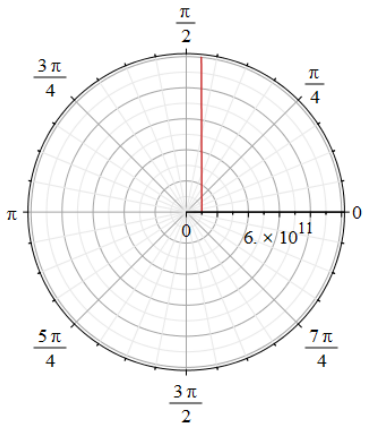
Polar plot of r vs θ, for $[R, Theta, 0..10^3]$



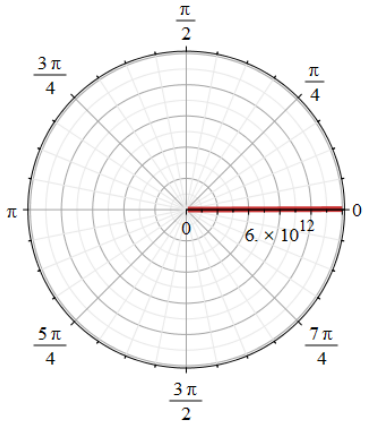
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$, *numpoints = 100*



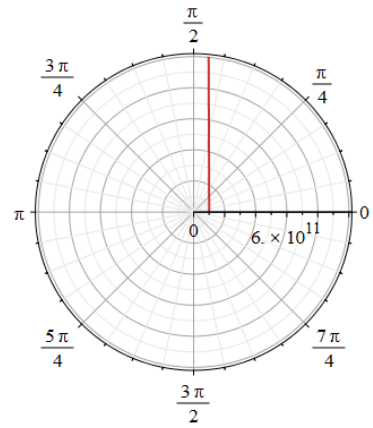
Polar plot of r vs θ, for $[R, Theta, 0..10^3]$



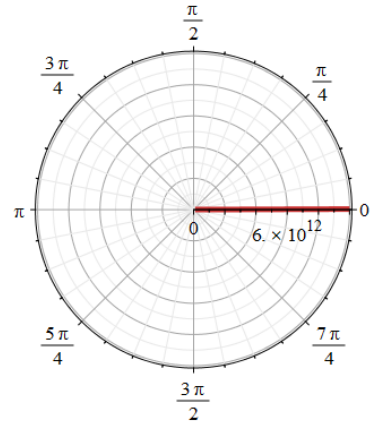
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$, *numpoints = 100*

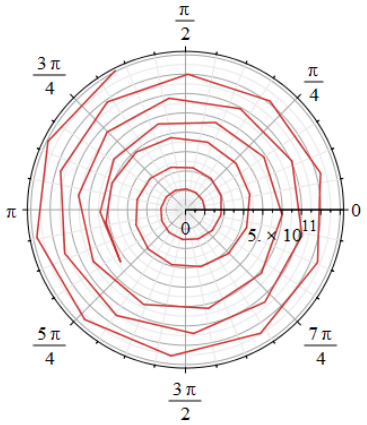
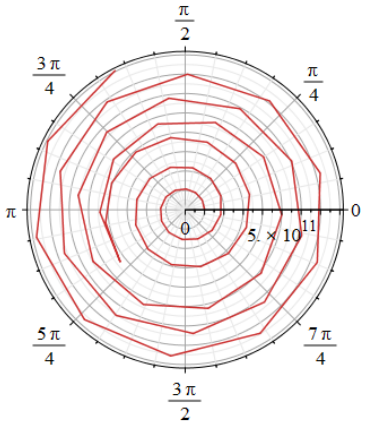
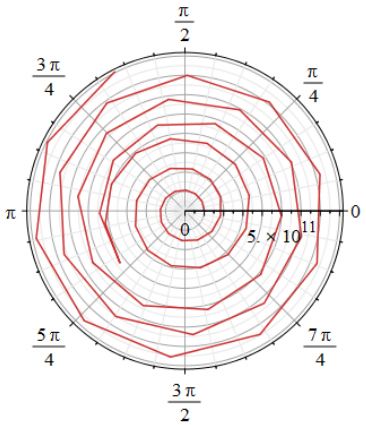


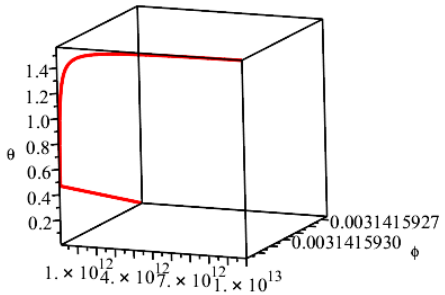
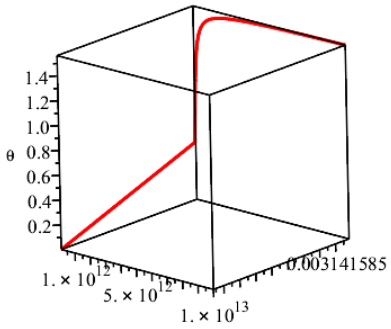
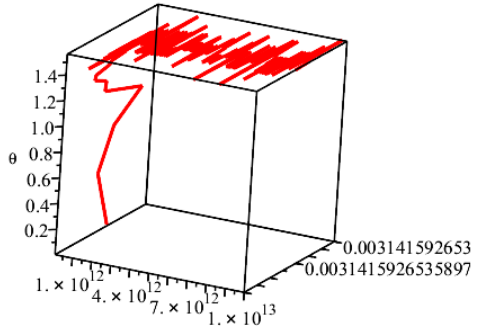
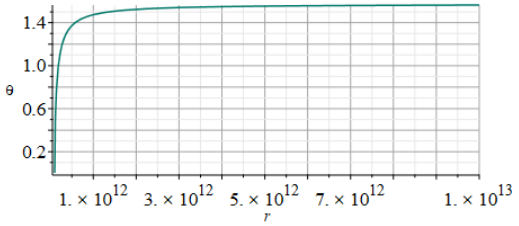
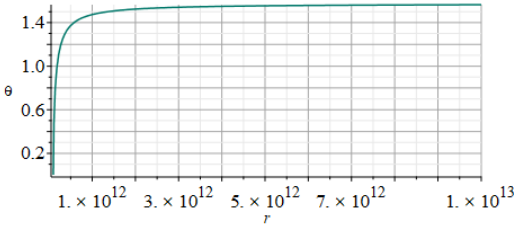
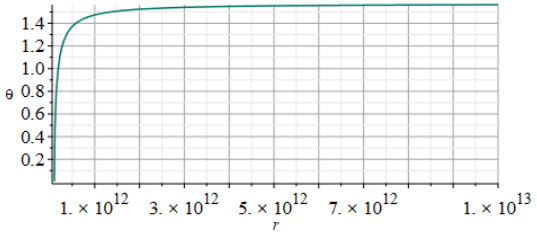
Polar plot of r vs θ, for $[R, Theta, 0..10^3]$



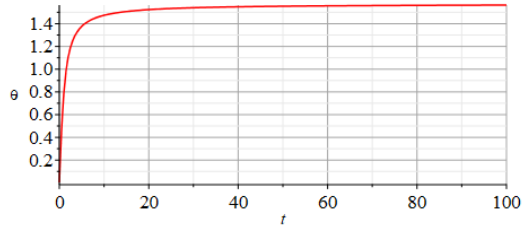
Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$



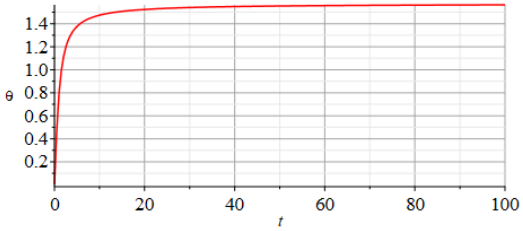
			Polar plot of r vs t, for $[R(t), t, t = 0..8 \cdot 10^2]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..8 \cdot 10^2]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..8 \cdot 10^2]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion...* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	1	90° 90° 90°	*The mass m reaches equilibrium at θ = 90°. 3D plot for t = 0..10²* 	*The mass m reaches equilibrium at θ = 90°. 3D plot for t = 0..10²* 	*The mass m reaches equilibrium at θ = 90°. 3D plot for t = 0..10²* 
			Plot of θ vs. r, for t = 0..10² 	*Plot of θ vs. r, for t = 0..10²* 	*Plot of θ vs. r, for t = 0..10²* 

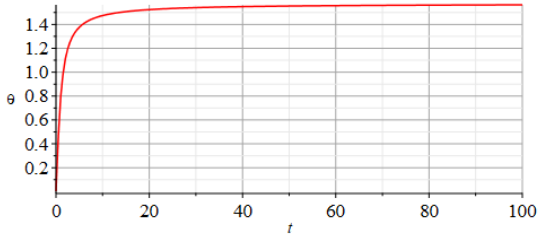
Plot of θ vs. t, for t = 0..10²



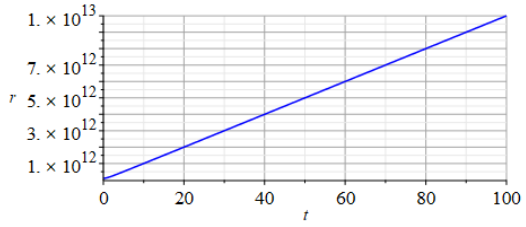
Plot of θ vs. t, for t = 0..10²



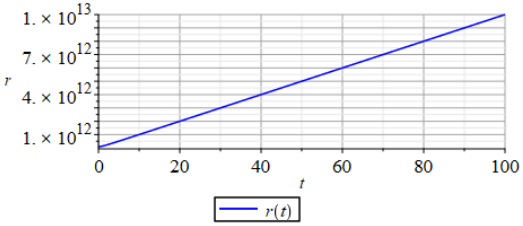
Plot of θ vs. t, for t = 0..10²



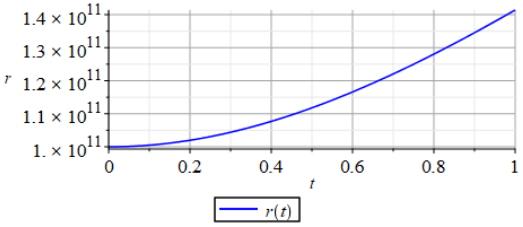
Plot of r vs. t, for t = 0..10²



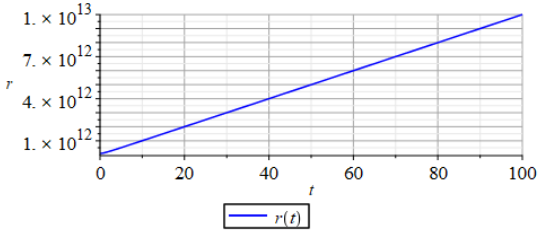
Plot of r vs. t, for t = 0..10²



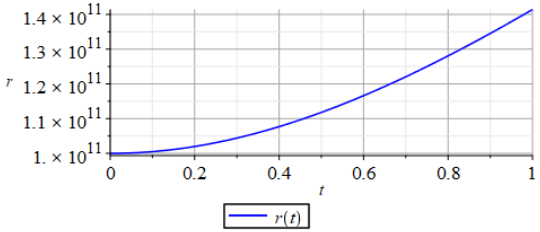
Details for t = 0..1



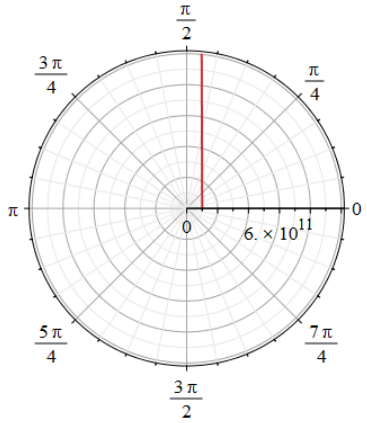
Plot of r vs. t, for t = 0..10²



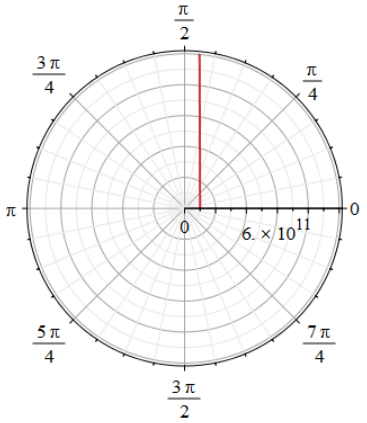
Details for t = 0..1



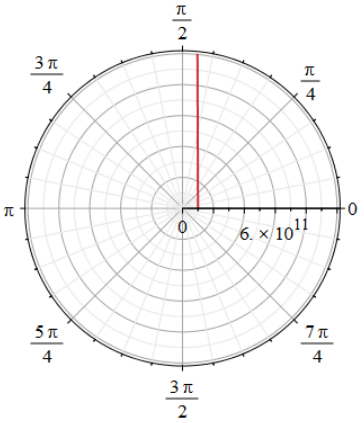
Polar plot of r vs θ, for $[R, Theta, 0..10]$



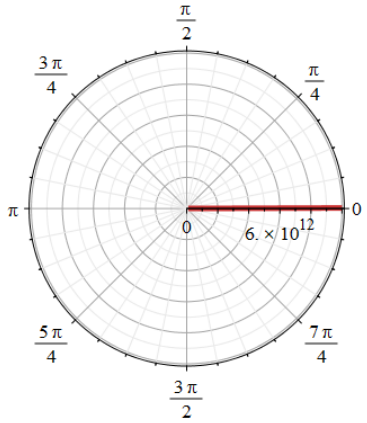
Polar plot of r vs θ, for $[R, Theta, 0..10]$



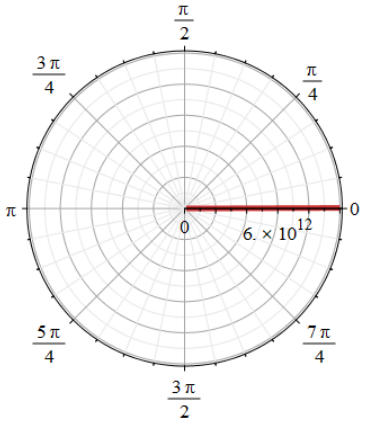
Polar plot of r vs θ, for $[R, Theta, 0..10]$



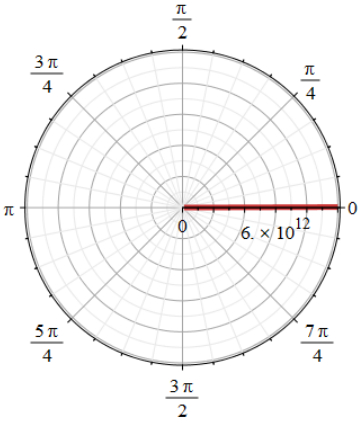
Polar plot of r vs ϕ, for
$[R, Phi, 0..10^2]$, *numpoints = 100*

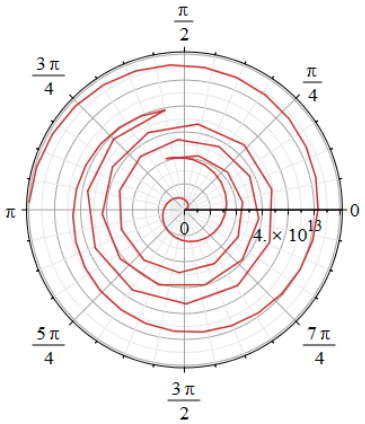
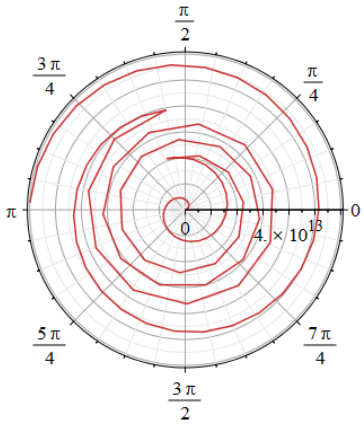
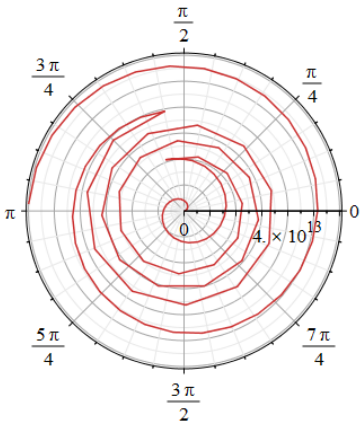


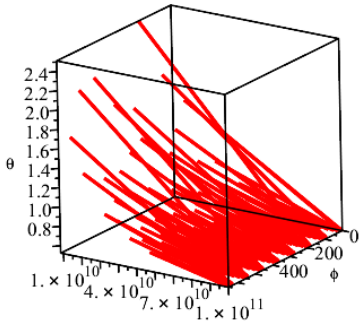
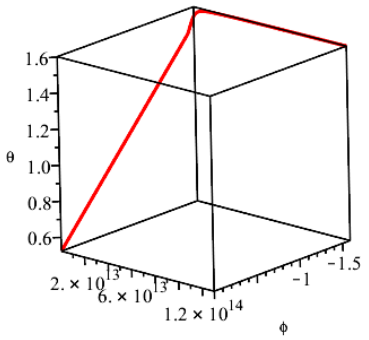
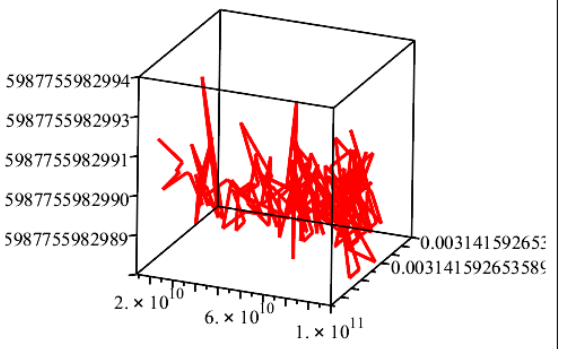
Polar plot of r vs ϕ, for
$[R, Phi, 0..10^2]$, *numpoints = 100*



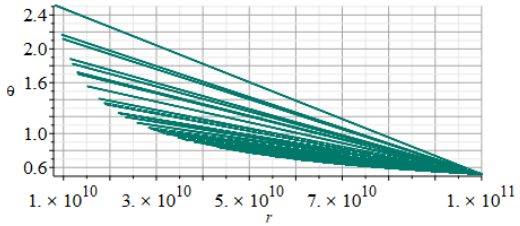
Polar plot of r vs ϕ, for $[R, Phi, 0..10^2]$



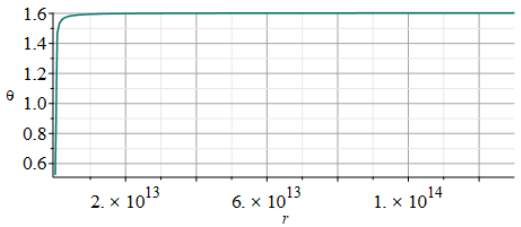
			Polar plot of r vs t, for $[R(t), t, t = 0..6 \cdot 10^2]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10^2]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..6 \cdot 10^2]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	100	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{6}$	0	~90° ~92° 30°	*The mass m oscillates from 30° to maximum ~143°, having thus a mean oscillation angle ~ 90° (± 56°) which can be considered the average equilibrium angle, with smaller oscillations around ~33°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days. **A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium at an angle ~ 92°.*	*The mass m reaches equilibrium at an angle $\theta = 30°$, with little oscillations around this angle.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days. **A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
			3D plot for $t = 0..10^9$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..4.2 \cdot 10^6$* 

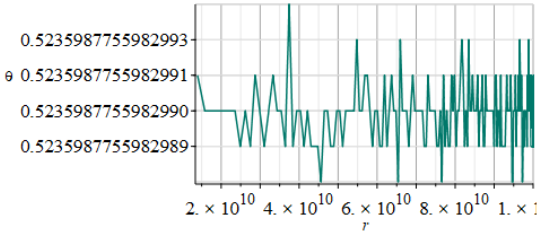
Plot of θ vs. r, for $t = 0..10^9$



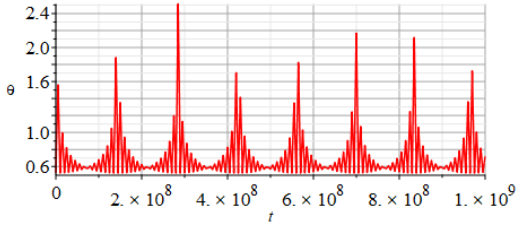
Plot of θ vs. r, for $t = 0..10^9$



Plot of θ vs. r, for $t = 0..4.2 \cdot 10^6$



*Plot of θ vs. t, for $t = 0..10^9$. The period of the envelope for this time range is $T \approx 1.4 \cdot 10^8$ (~ **4.4 years**):*

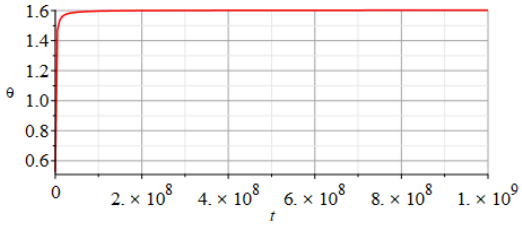


Precession and/or Nutation?

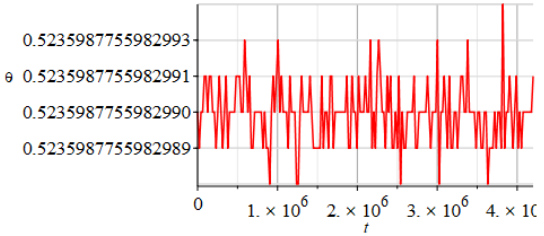
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

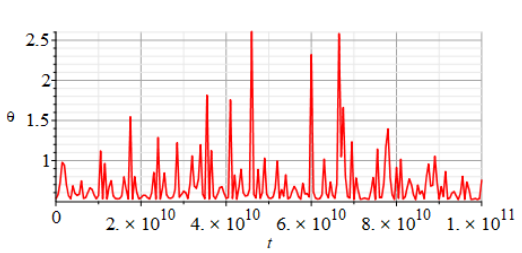
Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***

Plot of θ vs. t, for $t = 0..10^9$.

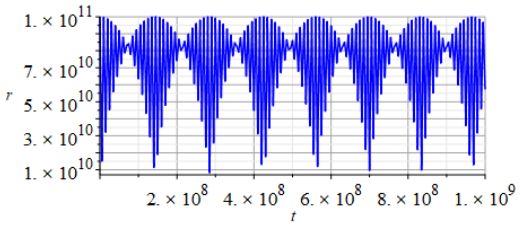


Plot of θ vs. t, for $t = 0..4.2 \cdot 10^6$.

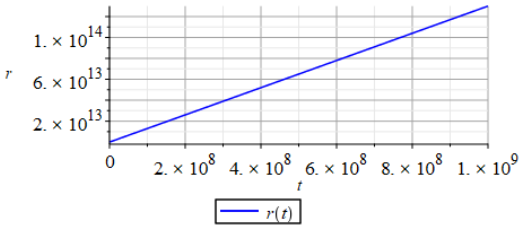




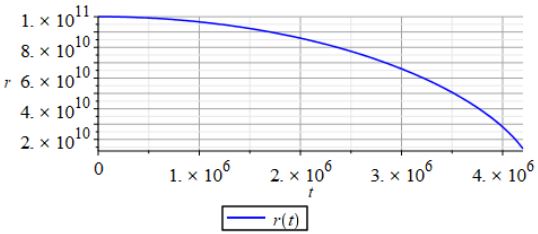
*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈$1.4 \cdot 10^8$ (~ **4.4 years**):*



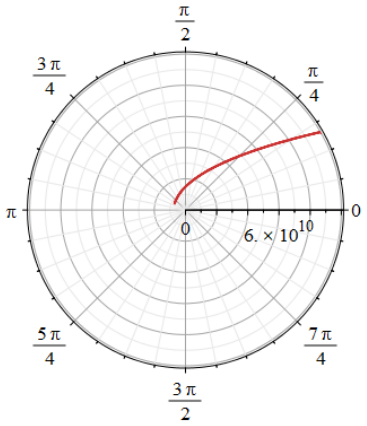
Plot of r vs. t, for t = 0..10^9.



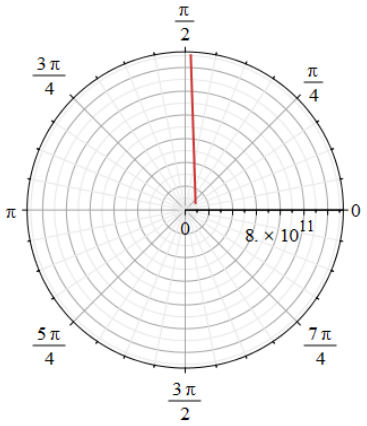
Plot of r vs. t, for t = 0..$4.2 \cdot 10^6$.



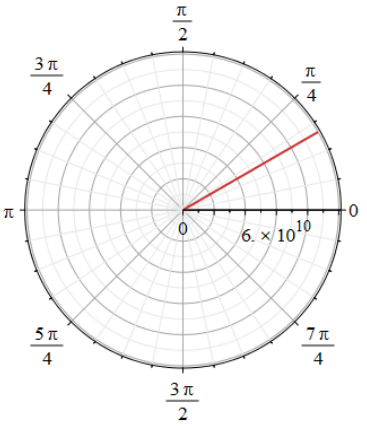
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$



Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

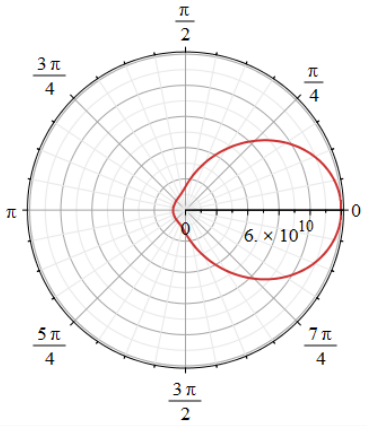


Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

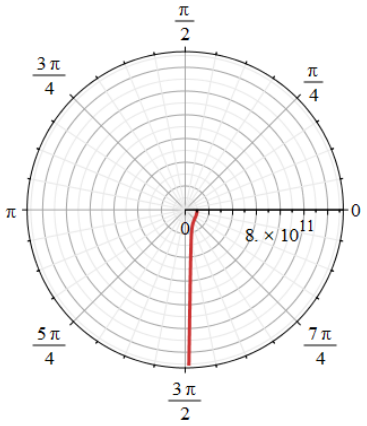


Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$

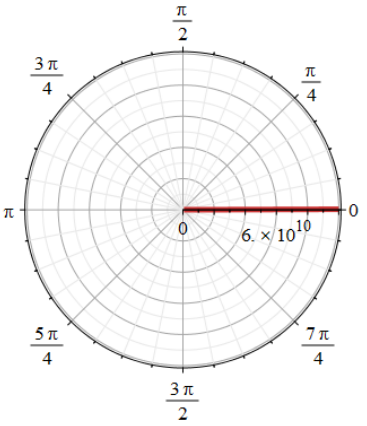
Seems to be a slightly elliptic trajectory (orbit of mass m around central mass M)

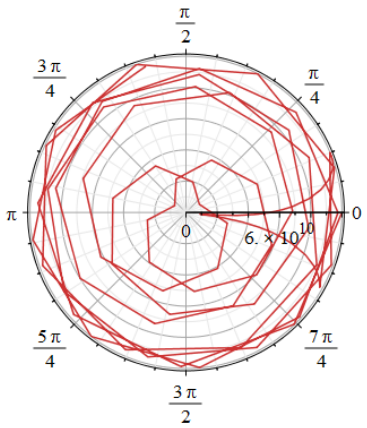
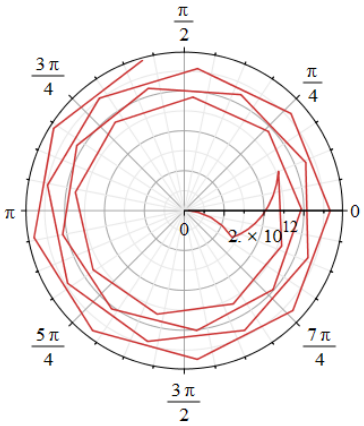
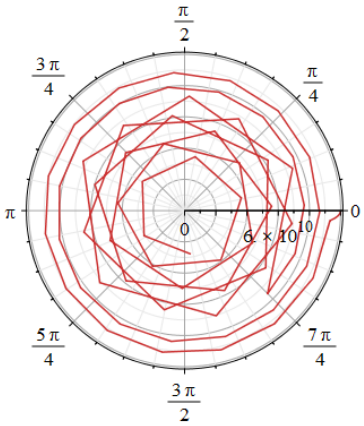


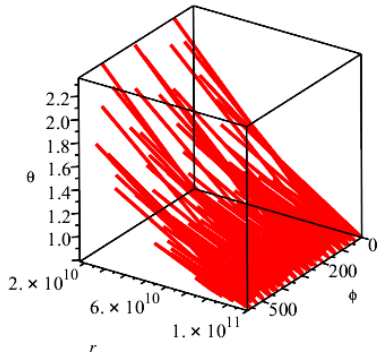
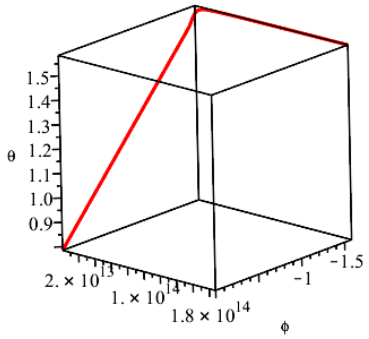
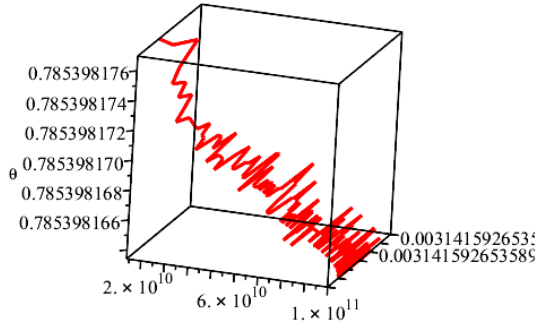
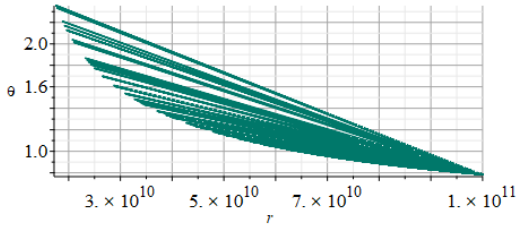
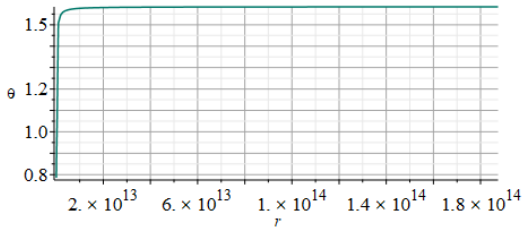
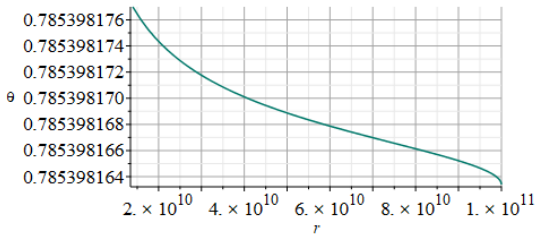
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



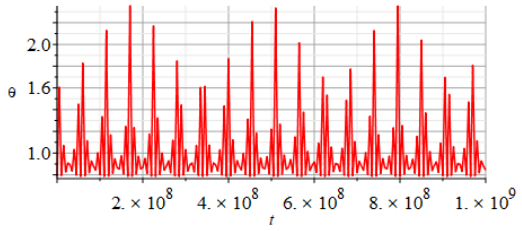
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



			Polar plot of r vs ϕ, for $[R(t), t, t = 0..3 \cdot 10^7]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs ϕ, for* $[R(t), t, t = 0..3 \cdot 10^7]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs ϕ, for* $[R(t), t, t = 0..4 \cdot 10^6]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 

$\frac{\pi}{4}$	10^{-15}	90° ~91° 45°	*The mass m has an **average oscillation angle of θ= 90°(±45°)**, though its center line of oscillation is around an angle of ~51°, swinging from 45° to ~135°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** *__**Note**__: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium at an angle θ≈ 91°*	*The mass m reaches equilibrium at an angle θ=45°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** *__**Note**__: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
			3D plot for $t = 0..10^9$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..4.2 \cdot 10^6$* 
			Plot of θ vs. r, for $t = 0..10^9$ 	*Plot of θ vs. r, for $t = 0..10^9$* 	*Plot of θ vs. r, for $t = 0..4.2 \cdot 10^6$* 

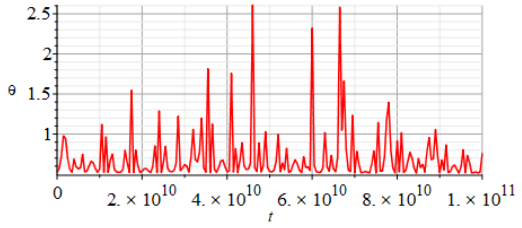
*Plot of θ vs. t, for t = 0..5·10^9. The period of the envelope for this time range is T≈0.6·10^8 (~ **1.9 years**):*



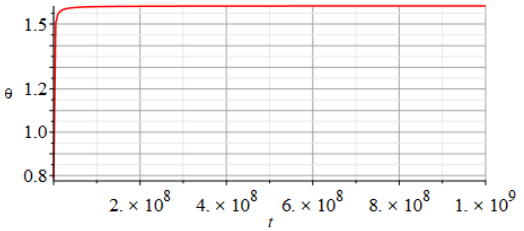
Precession and/or Nutation?

***WARNING**: the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

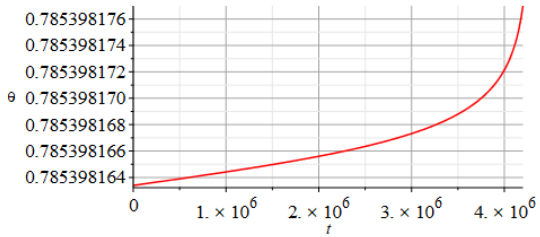
*Plot of θ vs. t, for t = 0..10^11. The pattern is not clear. We can see both: **amplitude and frequency modulation**.So, **the period of the mean value** oscillation (precession and/or nutation?) **is not constant:***



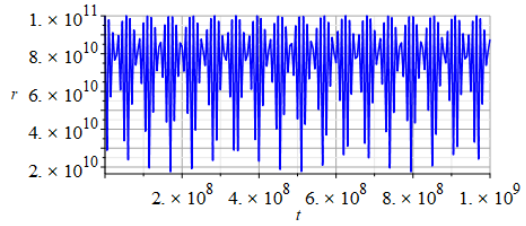
Plot of θ vs. t, for t = 0..10^9.



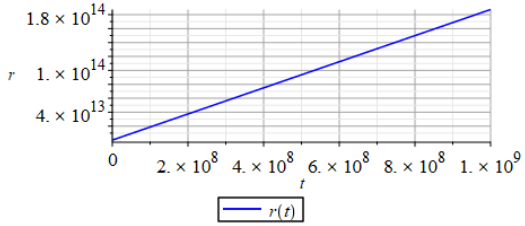
Plot of θ vs. t, for t = 0..4.2·10^6.



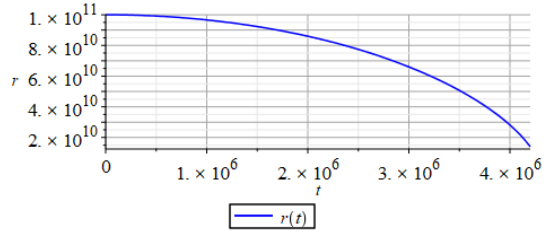
*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈0.6·10^8 (~ **1.9 years**):*



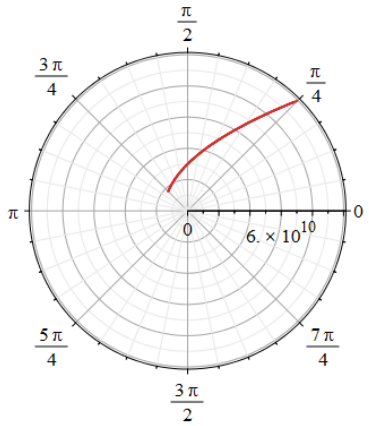
Plot of r vs. t, for t = 0..10^9.



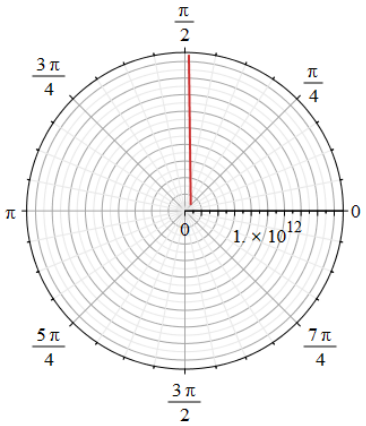
Plot of r vs. t, for t = 0..4.2·10^6.



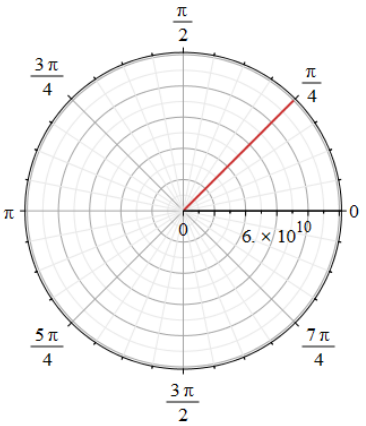
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$



Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

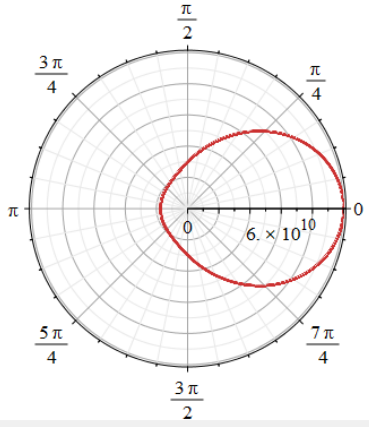


Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

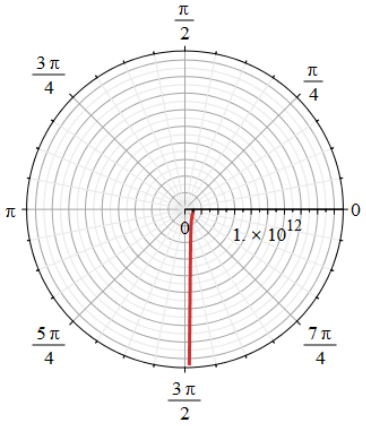


Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^7]$

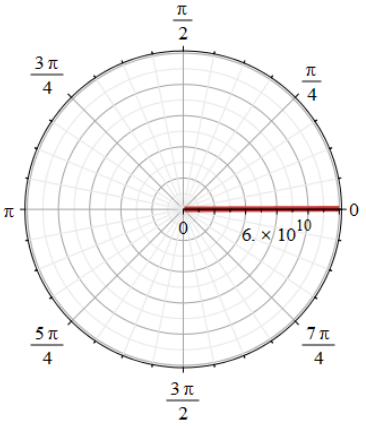
Seems to be a slightly elliptic trajectory (orbit of mass m around central mass M)

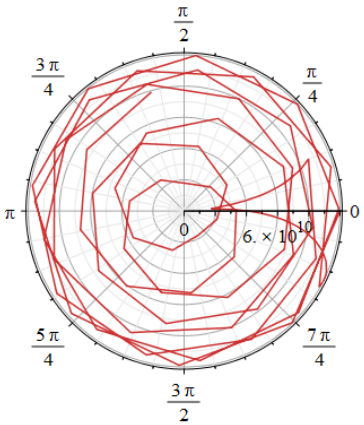
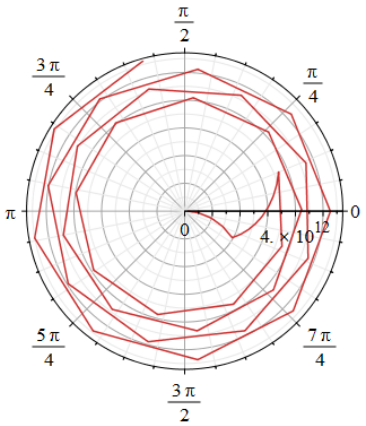
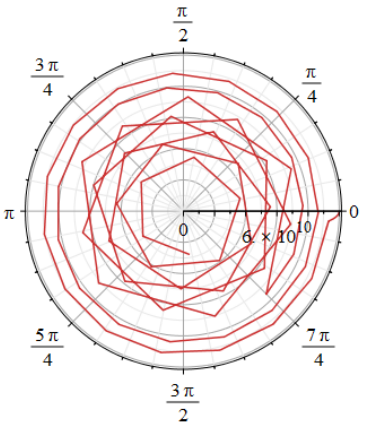


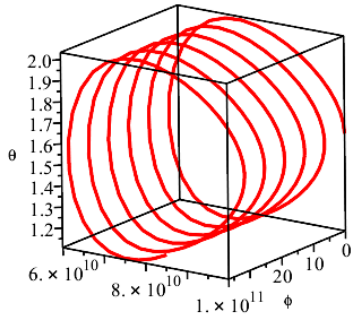
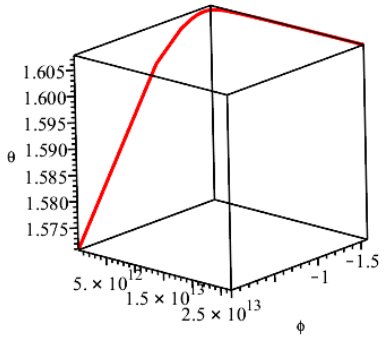
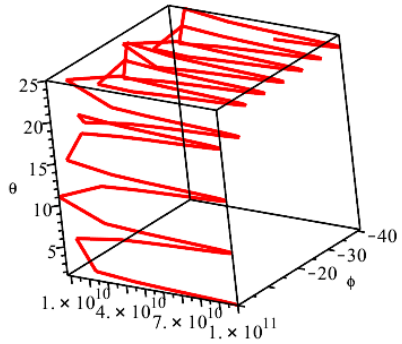
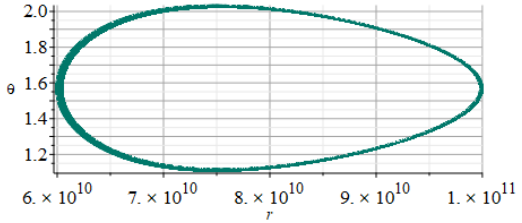
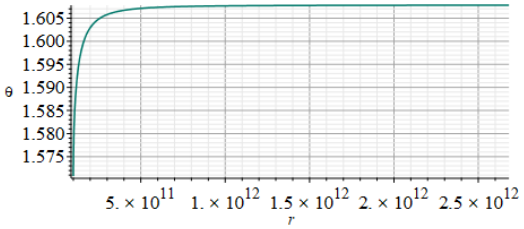
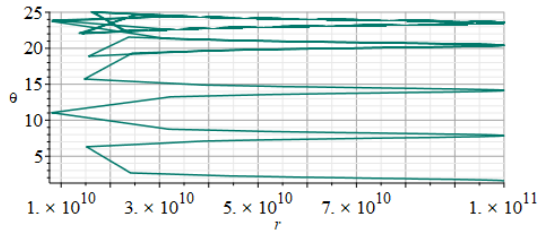
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



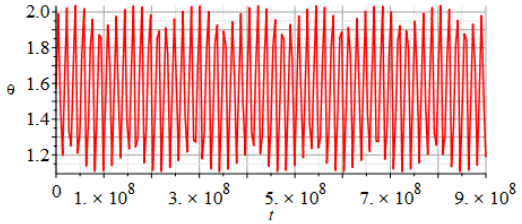
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



			Polar plot of r vs t, for $[R(t), t, t = 0..3 \cdot 10^7]$, *numpoints = 4* *A strange spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..3 \cdot 10^7]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..4 \cdot 10^6]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 

$\frac{\pi}{2}$	10^{-7}	90° ~92° 270° (-90°)	*The mass m oscillates around θ = 90° with an amplitud ~ of θ ± 26.5°.* *3D plot for t = 0..10^8* 	*The mass m reaches equilibrium at θ ≈ 92°. 3D plot for t = 0..10^8* 	*The mass m reaches equilibrium oscillating around θ ≈ 270°, with a swing ~ θ±85°. 3D plot for t = 0..10^8* 
			Plot of θ vs. r, for t = 0..10^9 	*Plot of θ vs. r, for t = 0..10^7* 	*Plot of θ vs. r, for t = 0..10^8* 

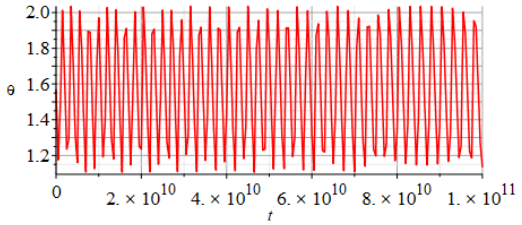
*Plot of θ vs. t, for $t = 0..9\cdot10^8$. The period of the envelope for this time range is $T\approx1.2\cdot10^8$ (~ **4 years**):*



Precession and/or Nutation?

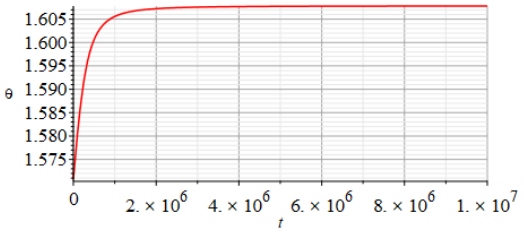
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

*Plot of θ vs. t, for $t = 0..10^{11}$. The pattern shows two clear envelopes at the end of this time range, the shorter with a period of $T\approx1.3\cdot10^{10}$ (~ **412 years**),and the longer with a period of $T\approx2.5\cdot10^{10}$ (~ **793 years**):*

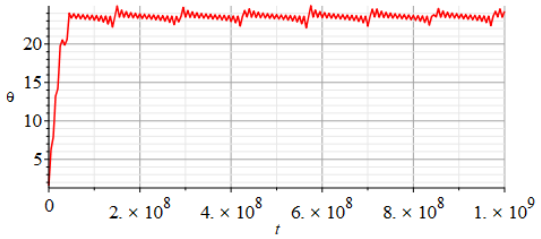


Plot of θ vs. t, for $t = 5\cdot10^{10}..5\cdot10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****.So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant:***

Plot of θ vs. t, for $t = 0..10^7$.



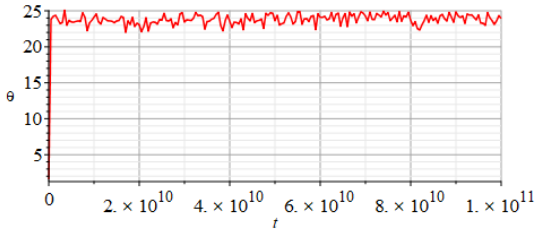
*Plot of θ vs. t, for $t = 0..10^9$. The period of the envelope for this time range is $T\approx1.4\cdot10^8$ (~ **4.4 years**):*

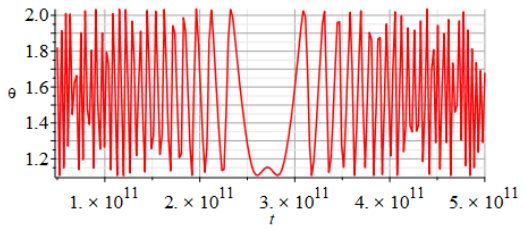


Precession and/or Nutation?

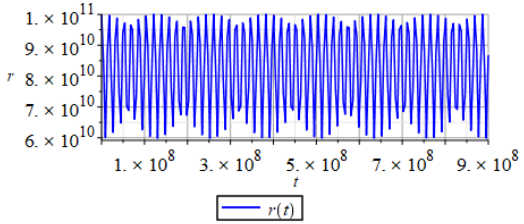
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear...

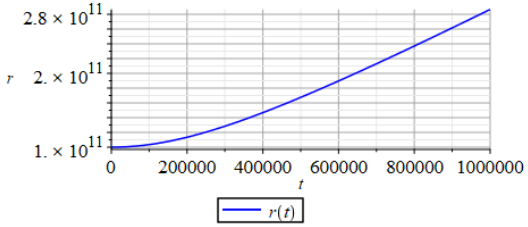




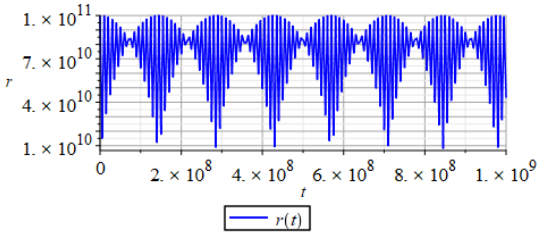
*Plot of r vs. t, for t = 0..9·10^8. The period of the envelope for this time range is T≈1.2·10^8 (~ **4 years**):*



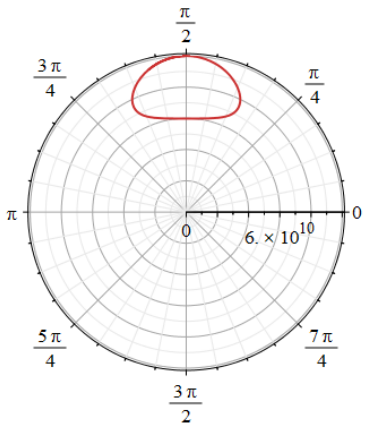
Plot of r vs. t, for t = 0..10^6.



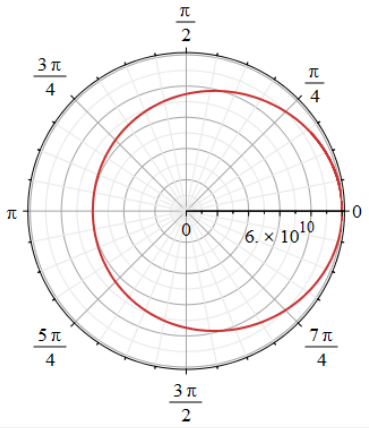
*Plot of r vs. t, for t = 0..10^9. The period of the envelope for this time range is T≈1.4·10^8 (~ **4.4 years**):*



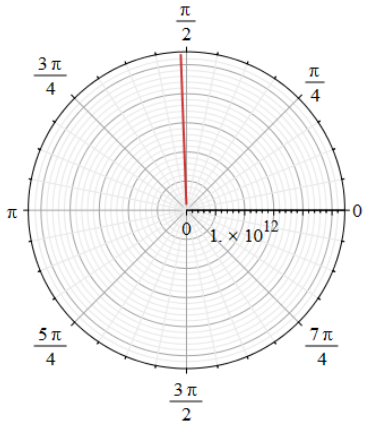
Polar plot of r vs θ, for $[R, Theta, 0..2 \cdot 10^7]$



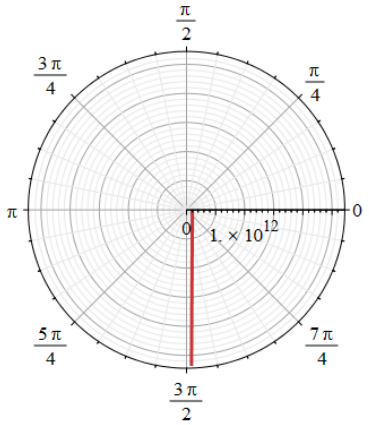
Polar plot of r vs ϕ, for $[R, Phi, 0..2 \cdot 10^7]$
Seems to be a circular trajectory (orbit of mass m around central mass M)



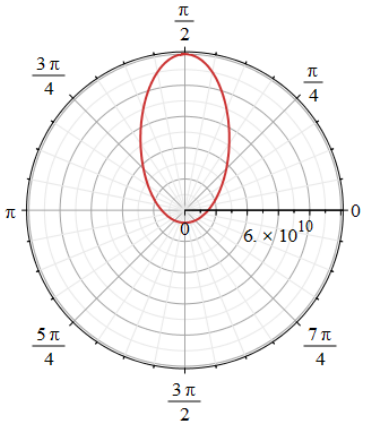
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$



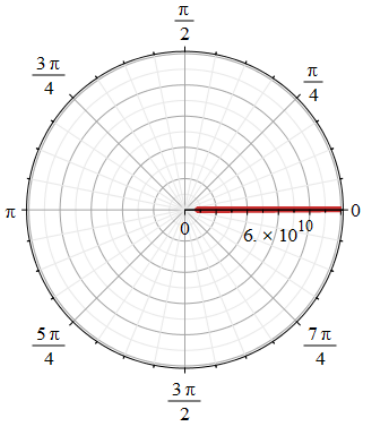
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$

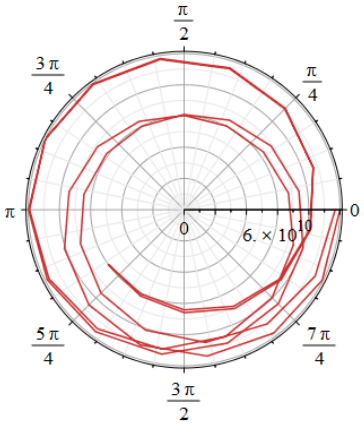
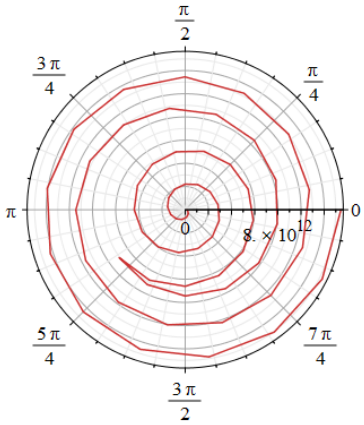
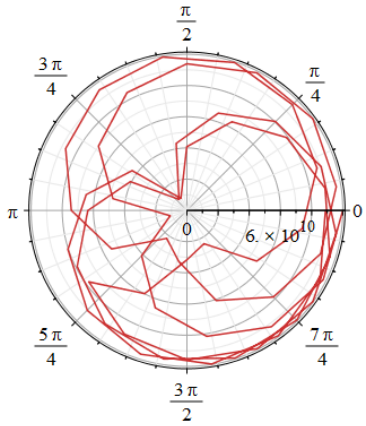


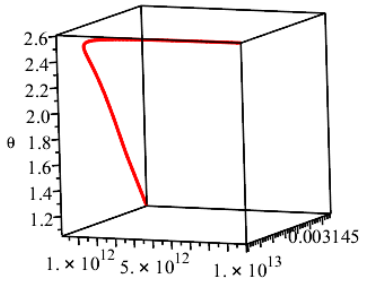
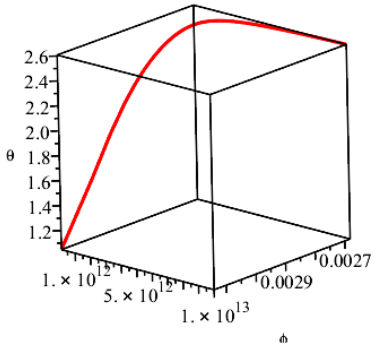
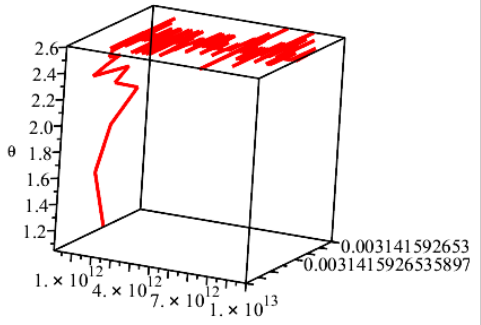
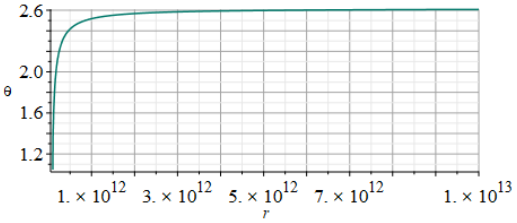
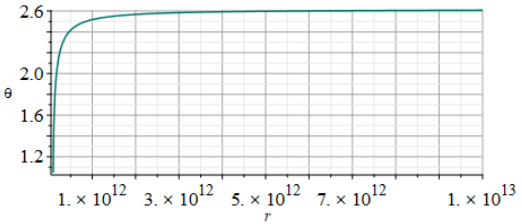
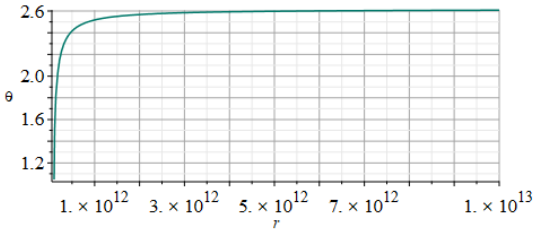
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$



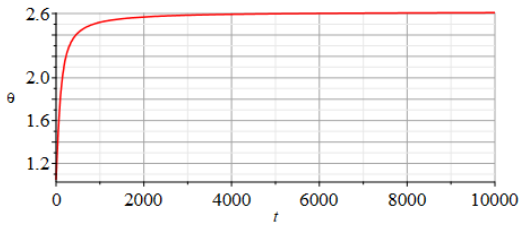
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



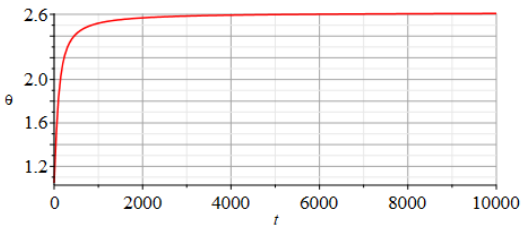
			Polar plot of r vs t, for $[R(t), t, t = 0..8 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..8 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..8 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* 

$\frac{\pi}{3}$	10^{-2}	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10^4* 	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10^4* 	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10^4* 
			Plot of θ vs. r, for t = 0..10^4 	*Plot of θ vs. r, for t = 0..10^4* 	*Plot of θ vs. r, for t = 0..10^4* 

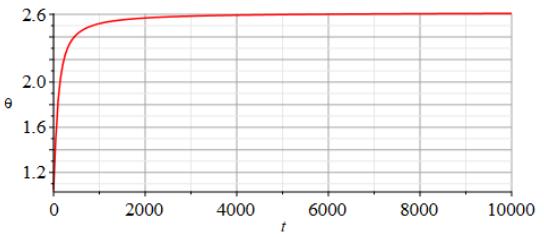
Plot of θ vs. t, for t = 0..10^4



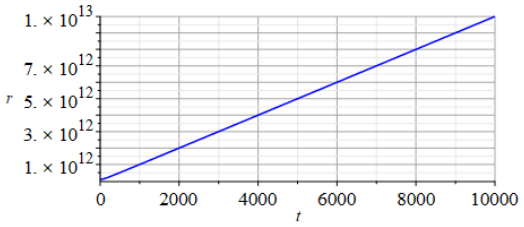
Plot of θ vs. t, for t = 0..10^4



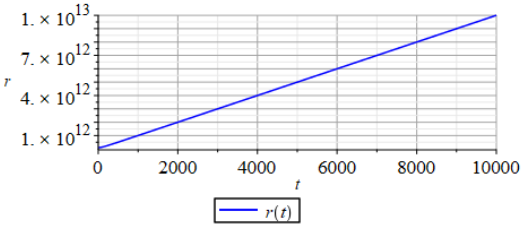
Plot of θ vs. t, for t = 0..10^4



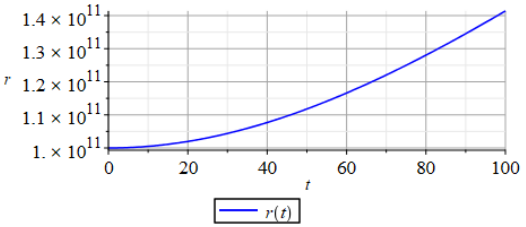
Plot of r vs. t, for t = 0..10^4



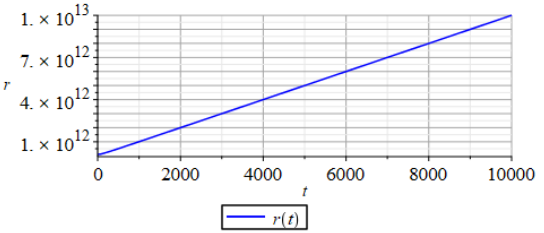
Plot of r vs. t, for t = 0..10^4



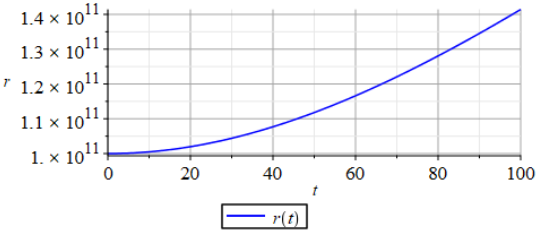
Details for t = 0..10^2

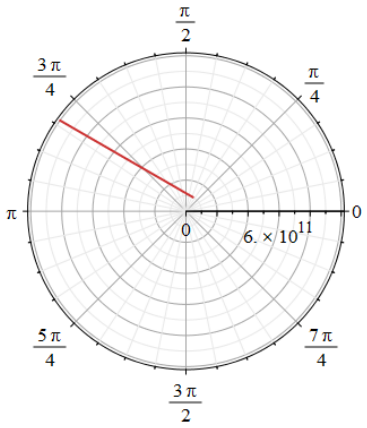
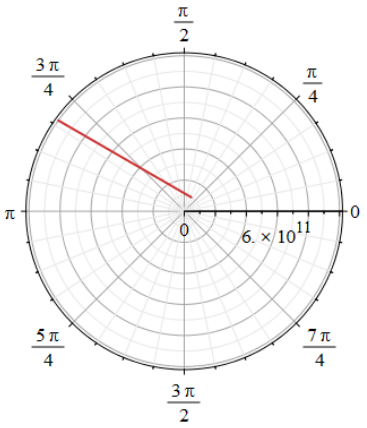
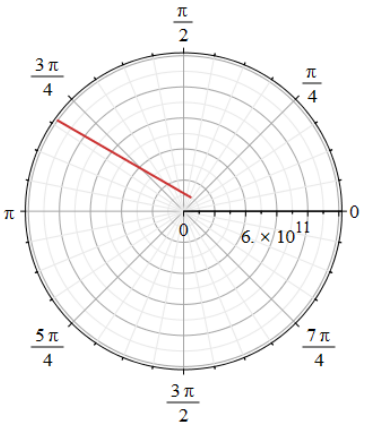
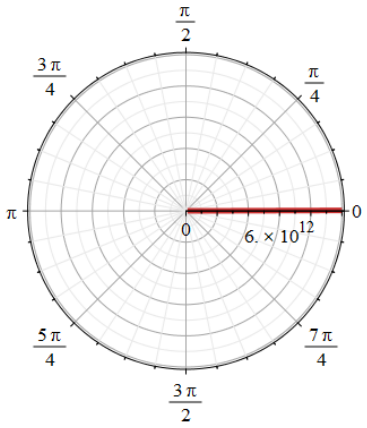
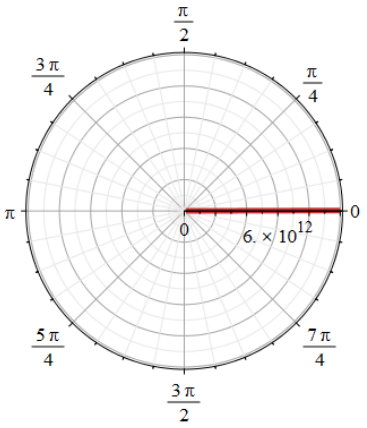
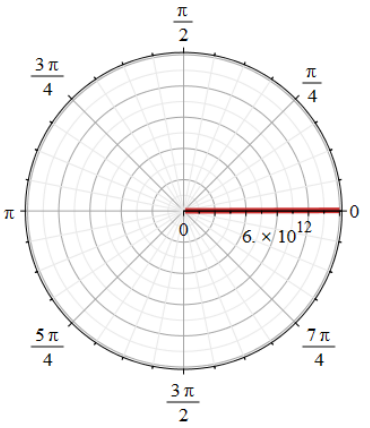


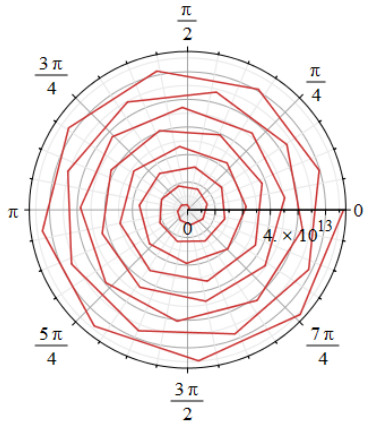
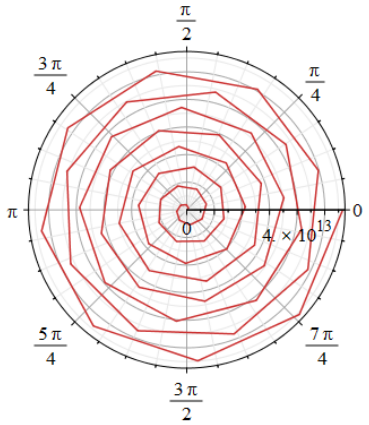
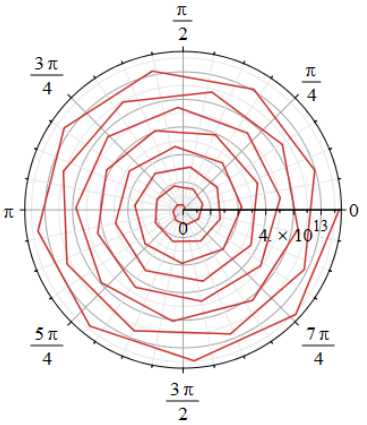
Plot of r vs. t, for t = 0..10^4

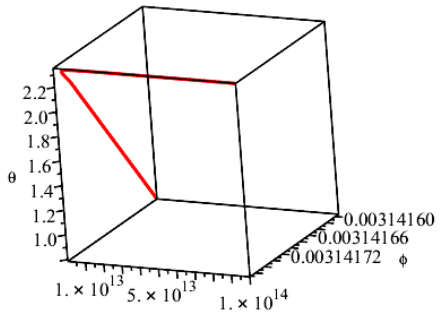
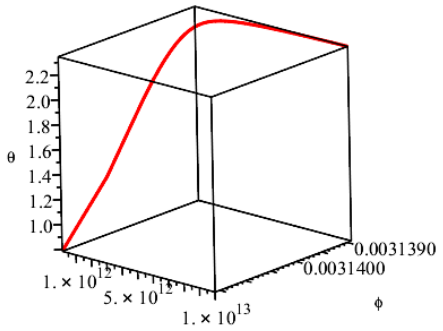
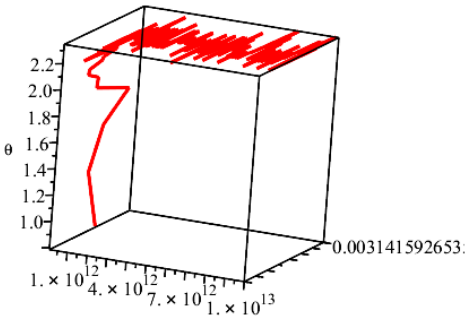
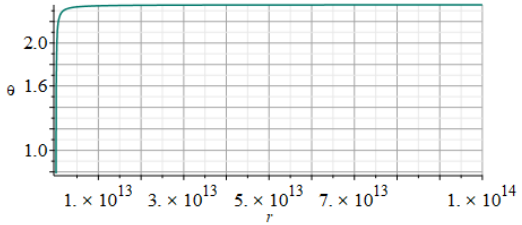
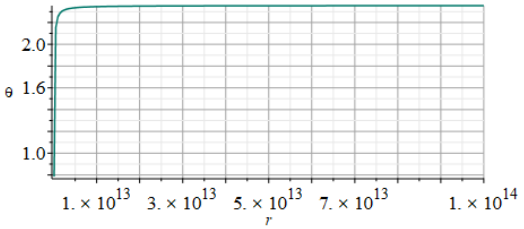
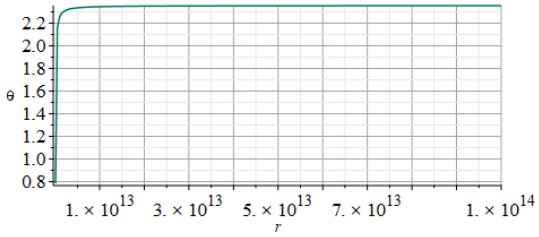


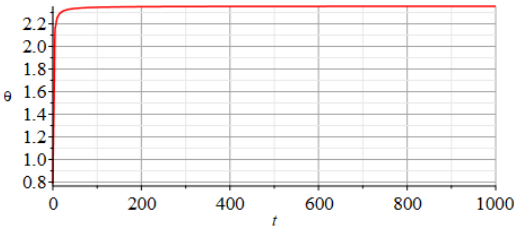
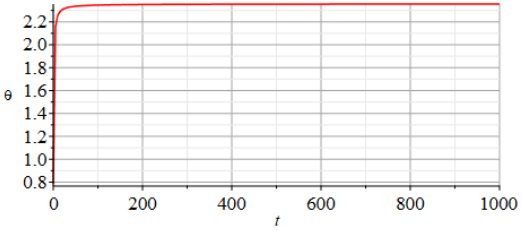
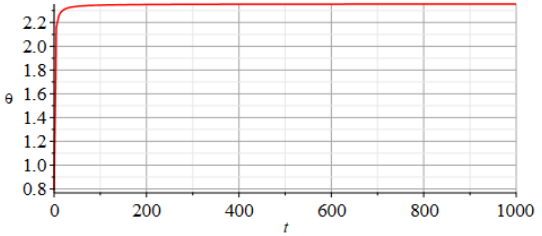
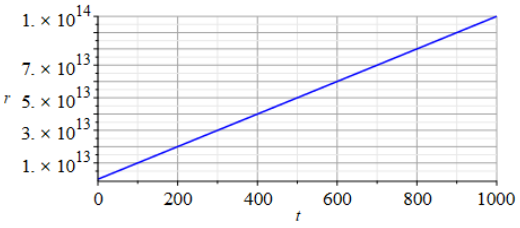
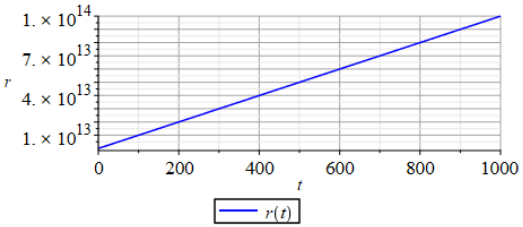
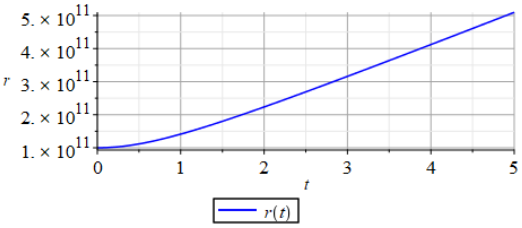
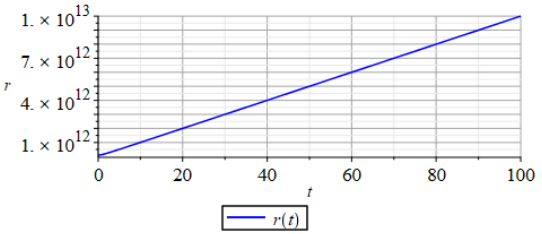
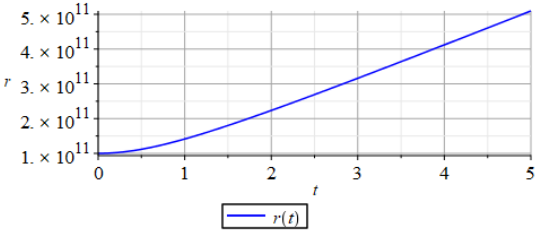
Details for t = 0..10^2

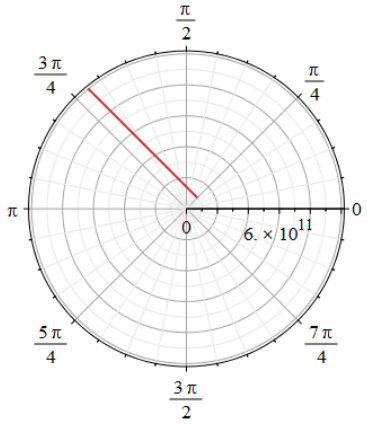
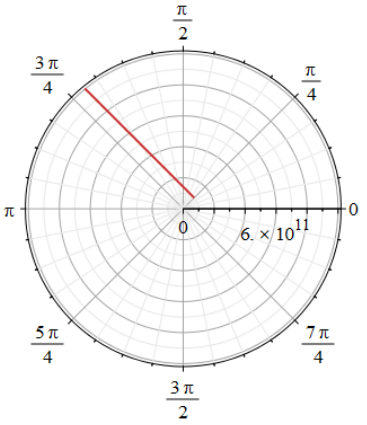
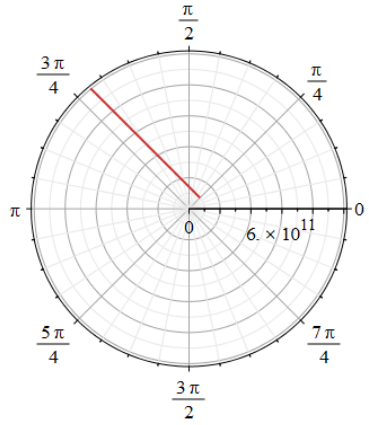
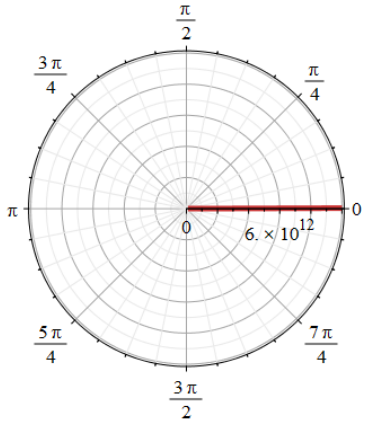
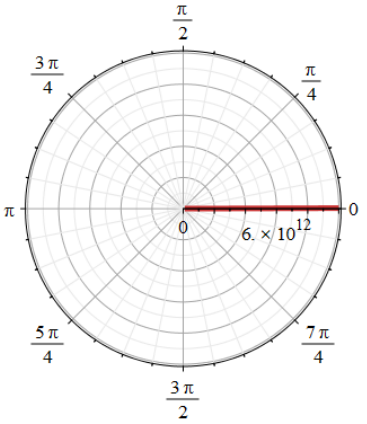
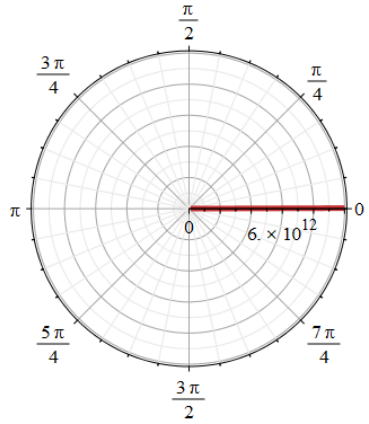


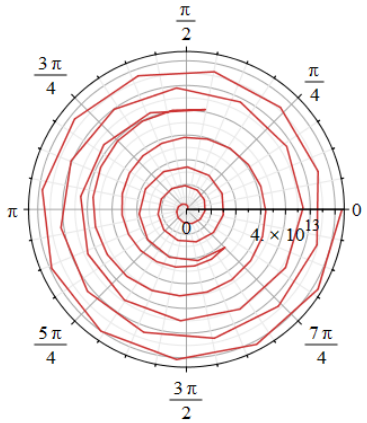
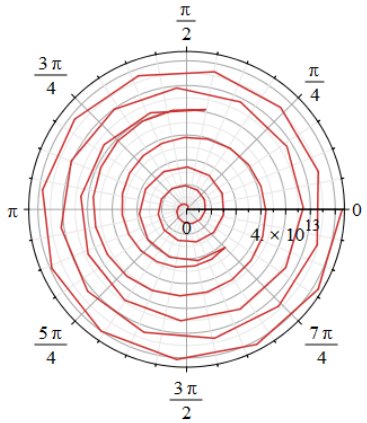
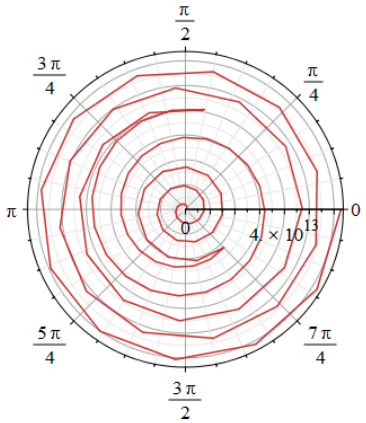
			Polar plot of r vs θ, for $[R, Theta, 0..10^3]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$ 

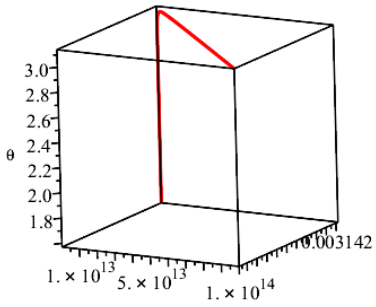
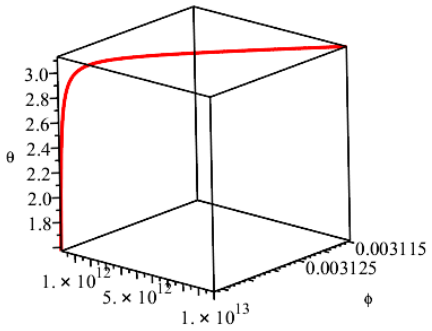
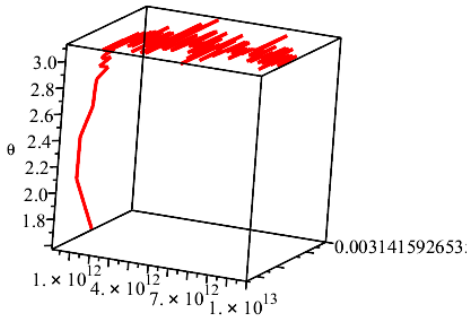
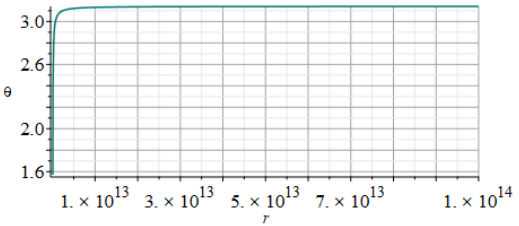
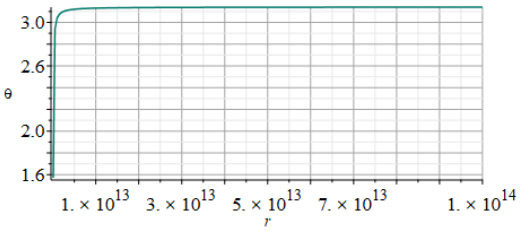
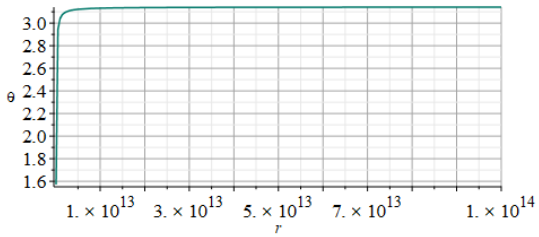
			Polar plot of r vs t, for $\left[R(t),\ t,\ t = 0..9\cdot 10^3\cdot 2\,\pi\right]$, *numpoints = 3* *A regular spiral.* 	*Polar plot of r vs t, for* $\left[R(t),\ t,\ t = 0..9\cdot 10^3\cdot 2\,\pi\right]$, *numpoints = 3* *A regular spiral.* 	*Polar plot of r vs t, for* $\left[R(t),\ t,\ t = 0..9\cdot 10^3\cdot 2\,\pi\right]$, *numpoints = 3* *A regular spiral.* 

$\frac{\pi}{4}$	1	135° 135° 135°	*The mass m reaches equilibrium at θ = 135° (initial position of θ=45° plus 90°)* *3D plot for t = 0..10^3* 	*The mass m reaches equilibrium at θ = 135° (initial position of θ=45° plus 90°)* *3D plot for t = 0..10^2* 	*The mass m reaches equilibrium at θ = 135° (initial position of θ=45° plus 90°)* *3D plot for t = 0..10^2* 
			Plot of θ vs. r, for t = 0..10^3 	*Plot of θ vs. r, for t = 0..10^3* 	*Plot of θ vs. r, for t = 0..10^3* 

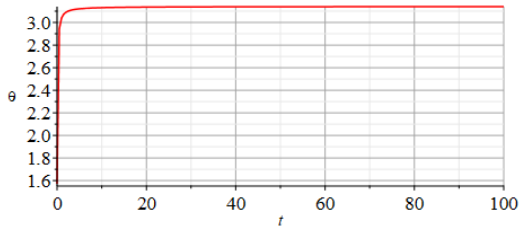
Plot of θ vs. t, for t = 0..10^3 	*Plot of θ vs. t, for t = 0..10^3* 	*Plot of θ vs. t, for t = 0..10^3* 
Plot of r vs. t, for t = 0..10^3 	*Plot of r vs. t, for t = 0..10^3*  *Details for t = 0..5* 	*Plot of r vs. t, for t = 0..10^2*  *Details for t = 0..5* 

Polar plot of r vs θ, for $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 
Polar plot of r vs ϕ, for $[R, Phi, 0..10^2]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^2]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^2]$ 

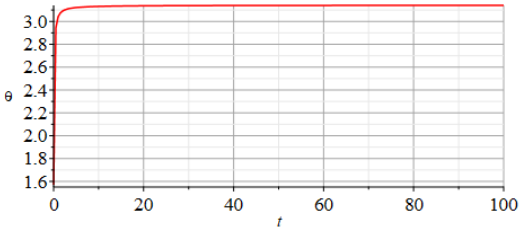
			Polar plot of r vs t, for $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 

$\frac{\pi}{2}$	10	180° 180° 180°	*The mass m reaches equilibrium at θ = 180° (initial position of θ=90° plus 90°)* *3D plot for t = 0..10²:* 	*The mass m reaches equilibrium at θ = 180° (initial position of θ=90° plus 90°)* *3D plot for t = 0..10:* 	*The mass m reaches equilibrium at θ = 180° (initial position of θ=90° plus 90°)* *3D plot for t = 0..10:* 
			Plot of θ vs. r, for t = 0..10² 	*Plot of θ vs. r, for t = 0..10²* 	*Plot of θ vs. r, for t = 0..10²* 

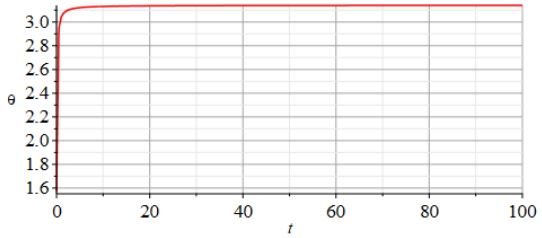
Plot of θ vs. t, for t = 0..10²



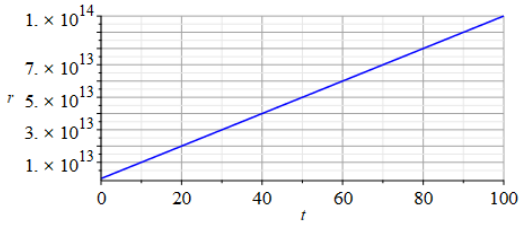
Plot of θ vs. t, for t = 0..10²



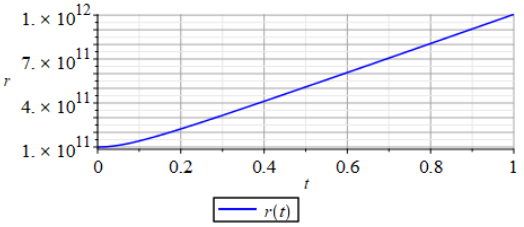
Plot of θ vs. t, for t = 0..10²



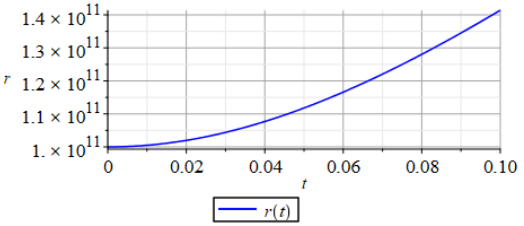
Plot of r vs. t, for t = 0..10²



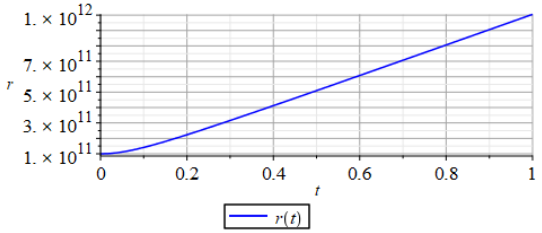
Plot of r vs. t, for t = 0..1



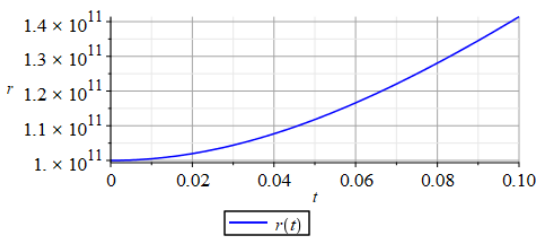
Details for t = 0..0.1



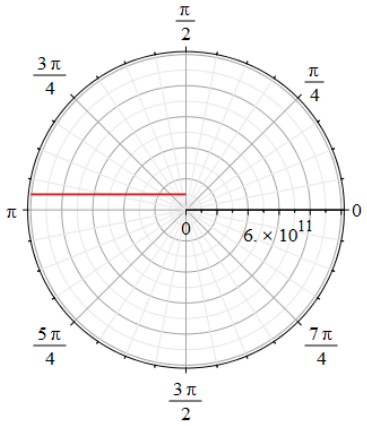
Plot of r vs. t, for t = 0..1



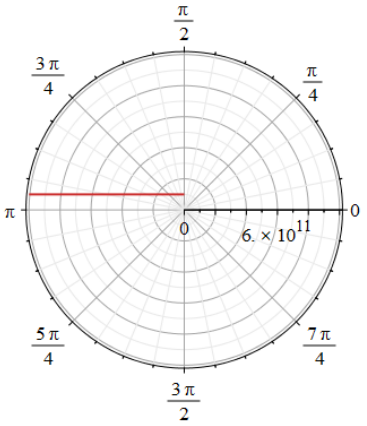
Details for t = 0..0.1



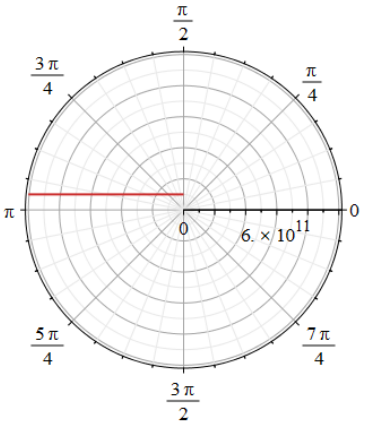
Polar plot of r vs θ, for [*R, Theta, 0..1*]



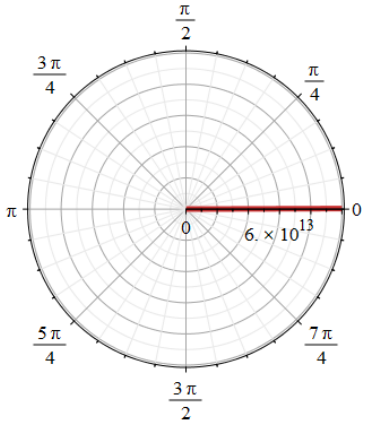
Polar plot of r vs θ, for [*R, Theta, 0..1*]



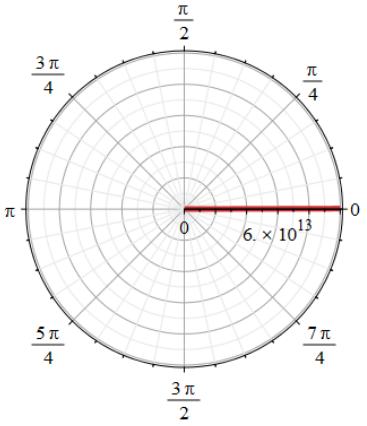
Polar plot of r vs θ, for [*R, Theta, 0..1*]



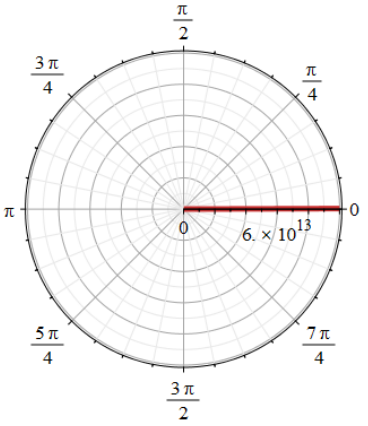
Polar plot of r vs ϕ, for [*R, Phi, 0..10²*]

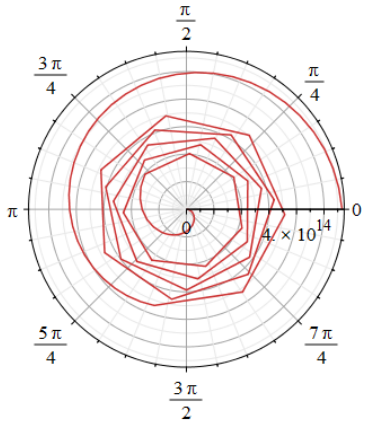
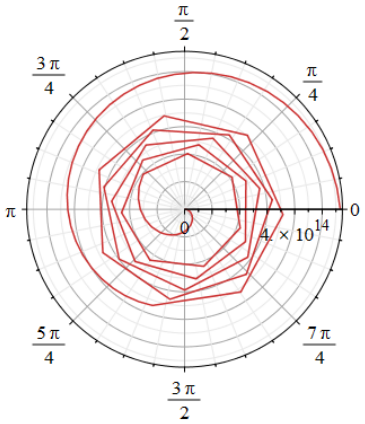
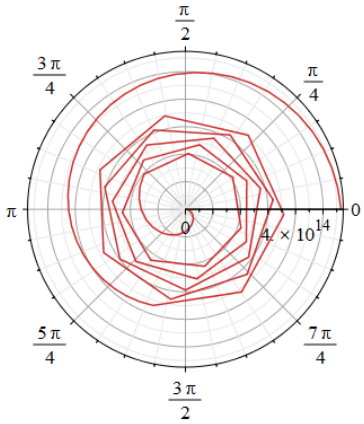


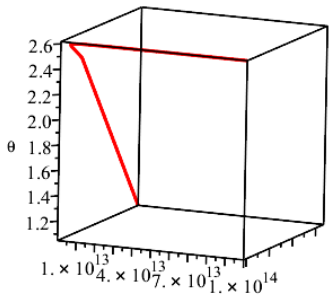
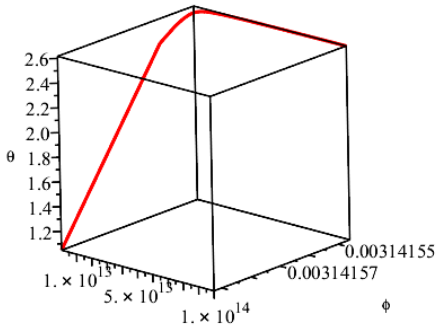
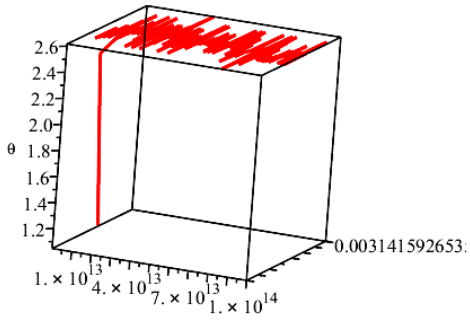
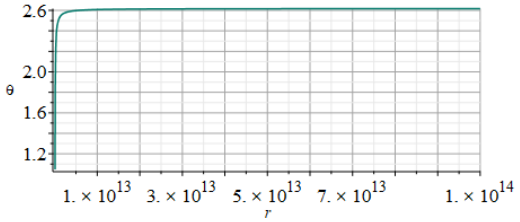
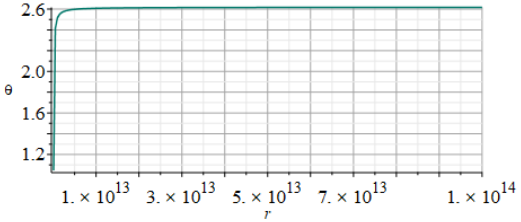
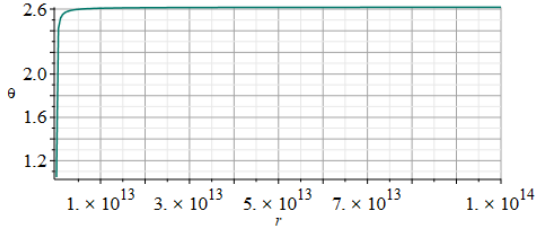
Polar plot of r vs ϕ, for [*R, Phi, 0..10²*]



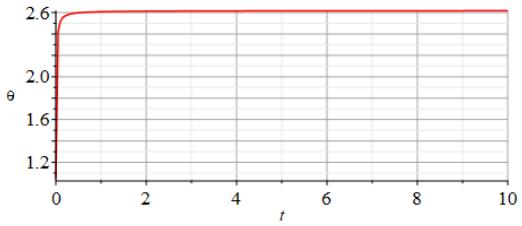
Polar plot of r vs ϕ, for [*R, Phi, 0..10²*]



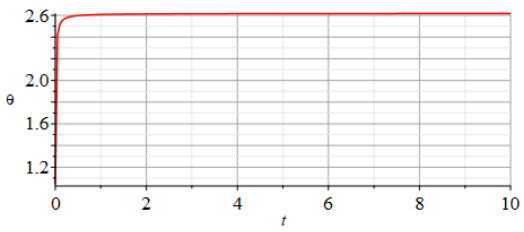
			Polar plot of r vs t, for $[R(t), t, t = 0..9 \cdot 10 \cdot 2\pi]$, *numpoints = 4* *A type of logaritmic spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10 \cdot 2\pi]$, *numpoints = 4* *A type of logaritmic spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..9 \cdot 10 \cdot 2\pi]$, *numpoints = 4* *A type of logaritmic spiral.* 

$\frac{\pi}{3}$	100	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10:*  	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10:*  	*The mass m reaches equilibrium at θ = 150° (initial position of θ=60° plus 90°)* *3D plot for t = 0..10:* 
			Plot of θ vs. r, for t = 0..10  	*Plot of θ vs. r, for t = 0..10*  	*Plot of θ vs. r, for t = 0..10* 

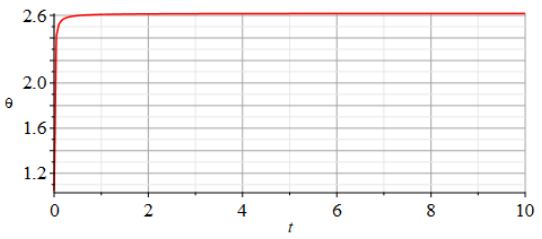
Plot of θ vs. t, for t = 0..10



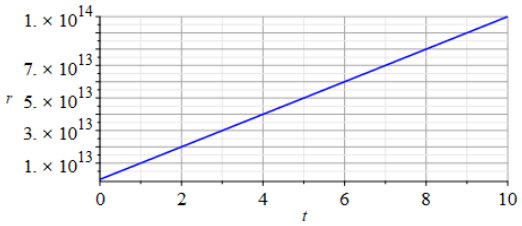
Plot of θ vs. t, for t = 0..10



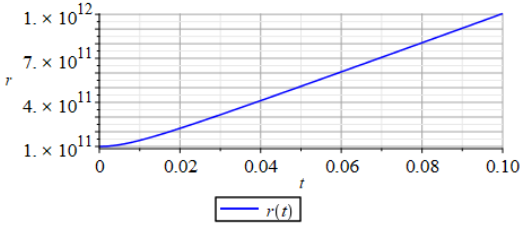
Plot of θ vs. t, for t = 0..10



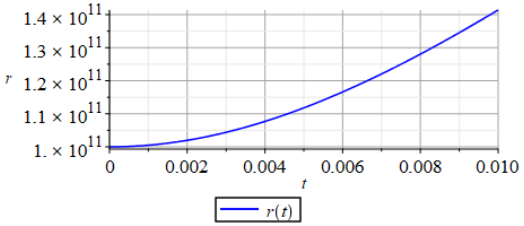
Plot of r vs. t, for t = 0..10



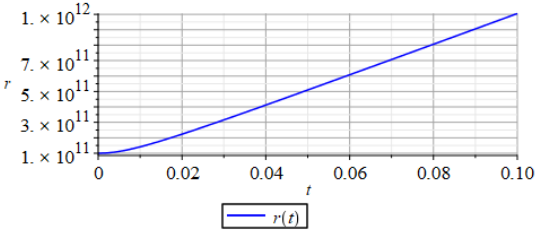
Plot of r vs. t, for t = 0..0.1



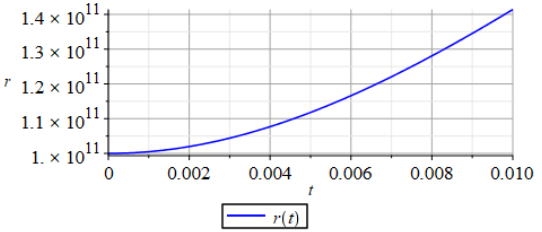
Details for t = 0..0.01

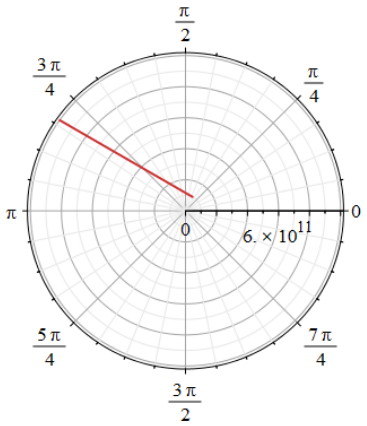
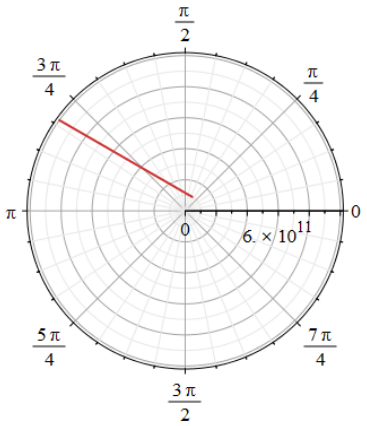
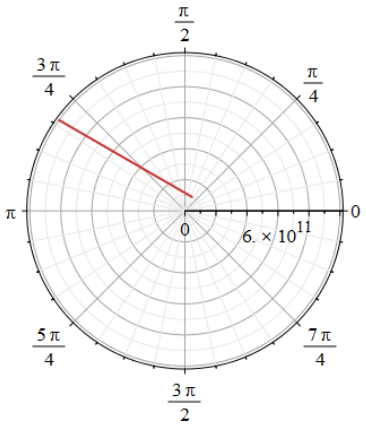
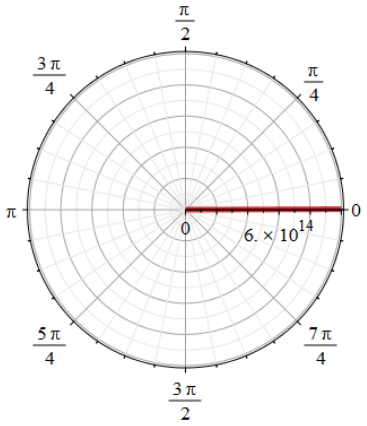
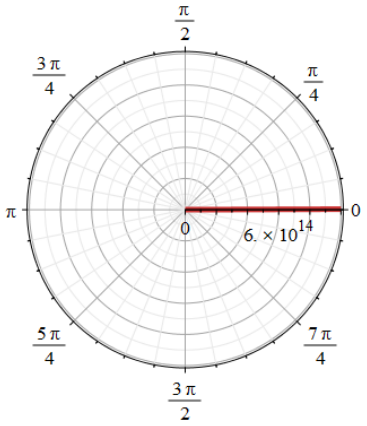
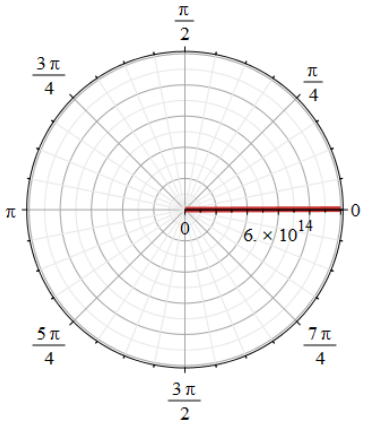


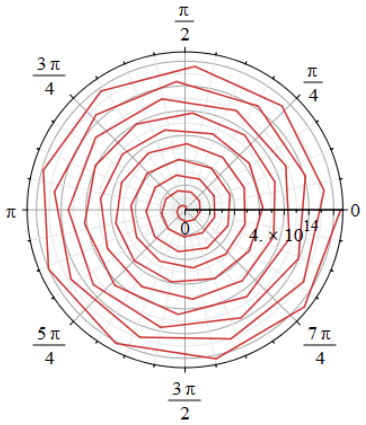
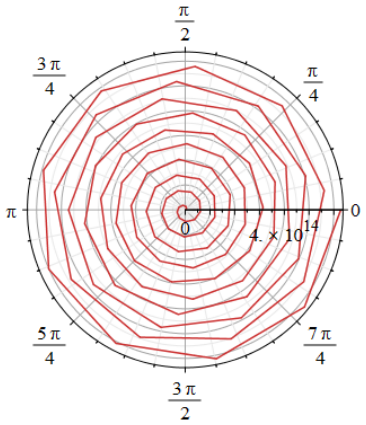
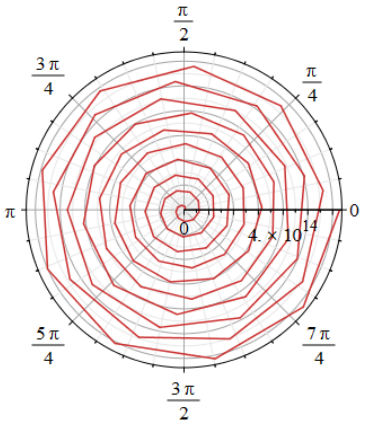
Plot of r vs. t, for t = 0..0.1



Details for t = 0..0.01



Polar plot of r vs θ, for [*R, Theta, 0..0.1*]	*Polar plot of r vs θ, for* [*R, Theta, 0..0.1*]	*Polar plot of r vs θ, for* [*R, Theta, 0..0.1*]
		
Polar plot of r vs ϕ, for [*R, Phi, 0..10^2*]	*Polar plot of r vs ϕ, for* [*R, Phi, 0..10^2*]	*Polar plot of r vs ϕ, for* [*R, Phi, 0..10^2*]
		

			Polar plot of r vs t, for $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 

APPENDIX III

All plots and details of the results for r(0) = 150 10^9 [m]

Analysis and plots of the motion for an initial distance of $r_{(0)} = 150\ 10^9\ [m]$, and the following initial conditions and parameters:

$G = 6.67408 * 10^{-11}\ \frac{Nm^2}{Kg^2}$ *(the universal gravitational constant)*

$M = 2\ 10^{30}\ Kg$ *(the mass of the Sun)*

$m = 5.9\ 10^{24}\ Kg$ *(the mass of the Earth)*

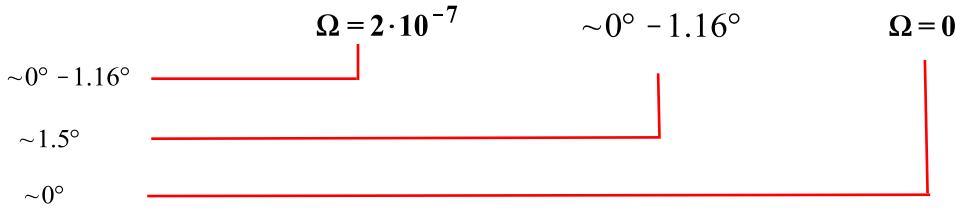
$\dot{r}_{(0)} = 0\ [m/s]$ *(linear velocity of the center of mass **m** at t=0)*

$\phi_{(0)} = 0\ [rad]$ *(azimuthal position of the center of mass **m** at t=0)*

Since there is a singularity for $\theta = 0$, the "zero" initial condition of the polar angle was taken close to zero as $\theta = \frac{\pi}{10^3}\ [rad]$.

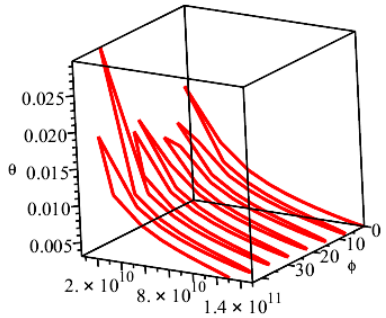
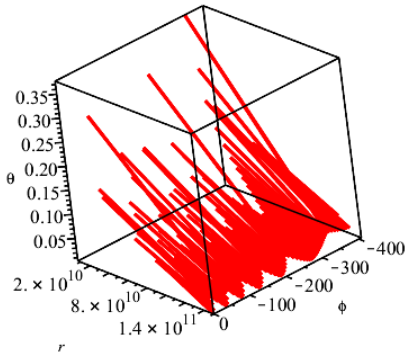
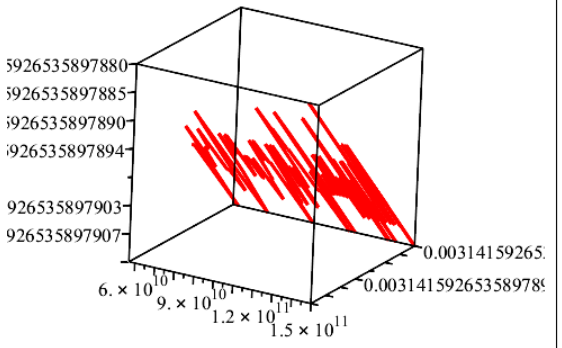
$\dot{\phi}_{(t)} = \Omega$ *(is the relative angular velocity between the spin of the center of mass **M** and the translation of the center of mass **m**)*

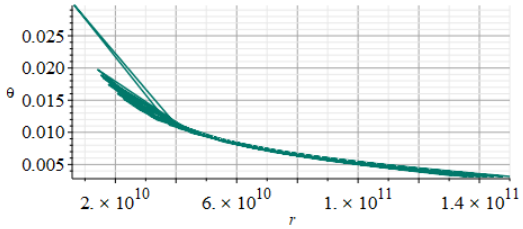
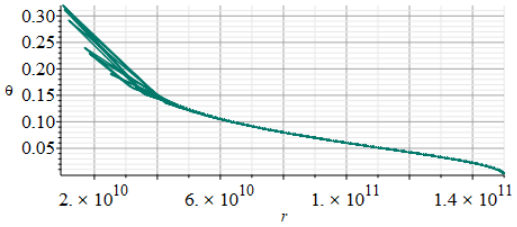
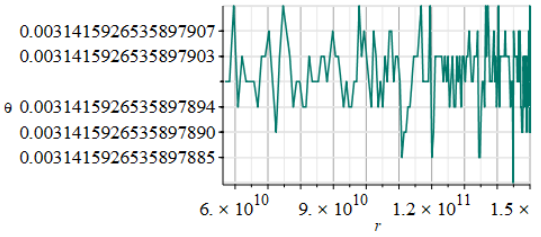
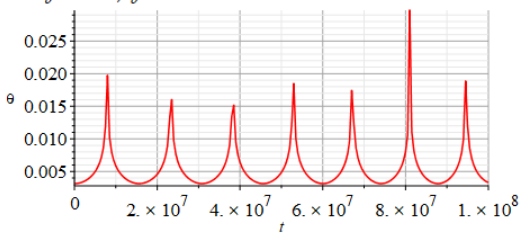
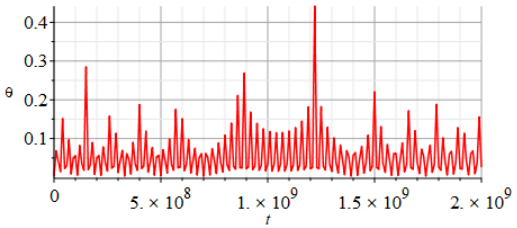
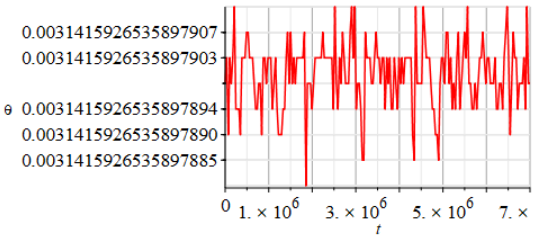
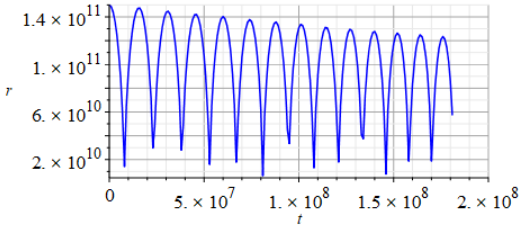
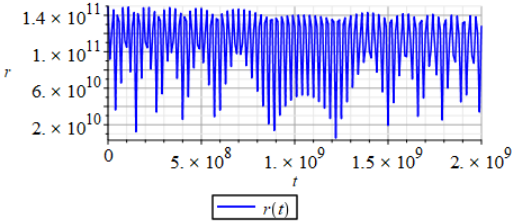
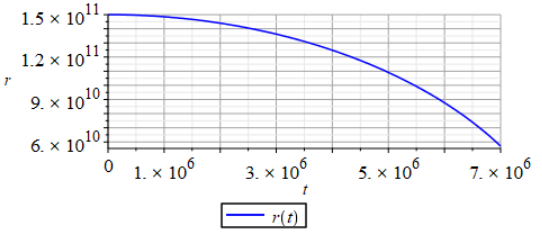
The polar angle of equilibrium (θ_f) is found for several values of the initial conditions of $\theta_{(0)}$, $\omega_{(0)}$, and Ω. The three vertical values of θ_f on each corresponding table cell, from top to bottom, are the results for each one of the three values of Ω from left to right. For example:

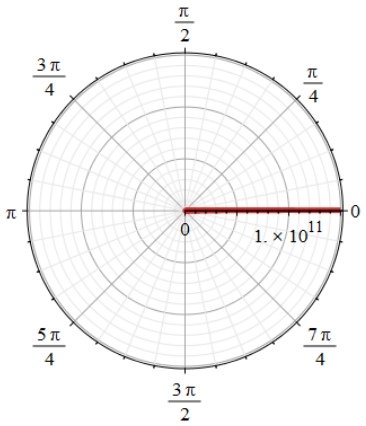
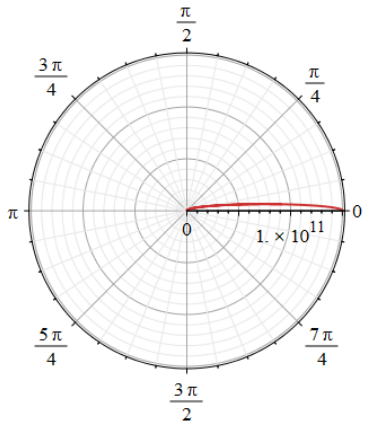
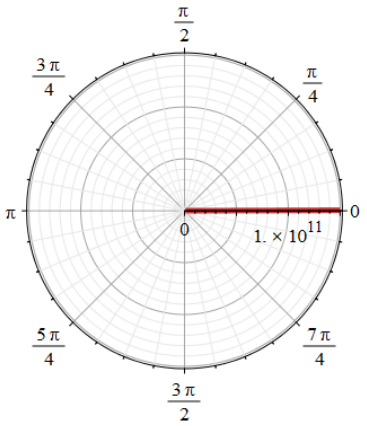
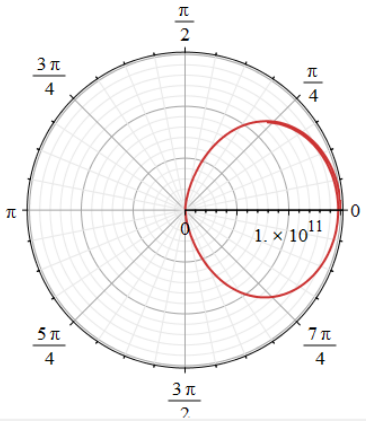
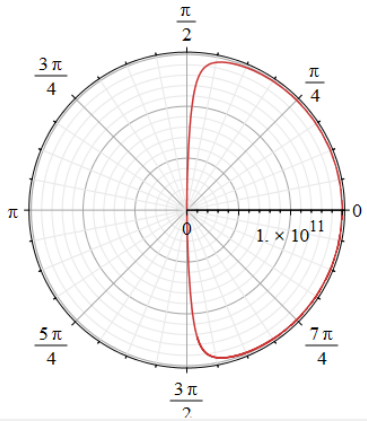
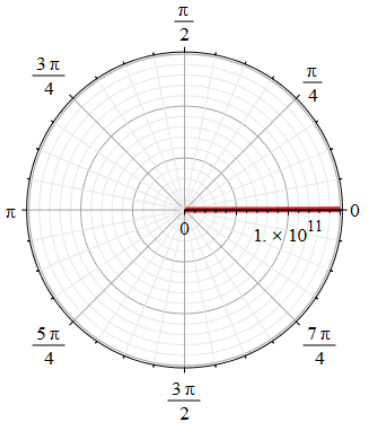


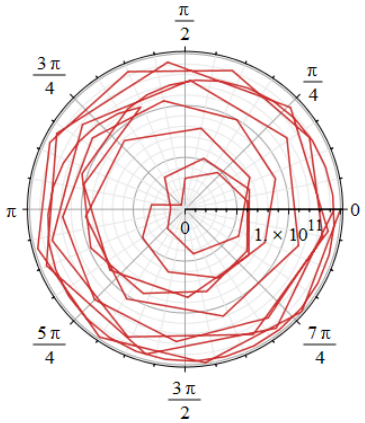
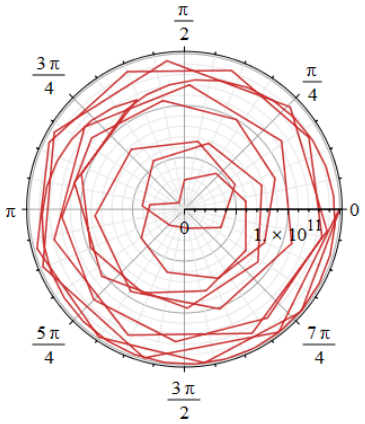
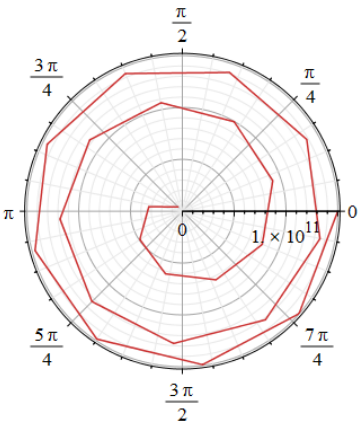
Obs.: the table is very long and continues over several pages.

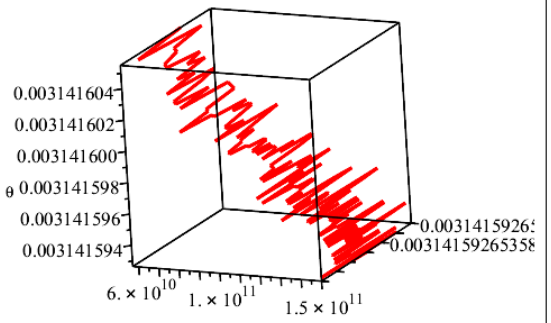
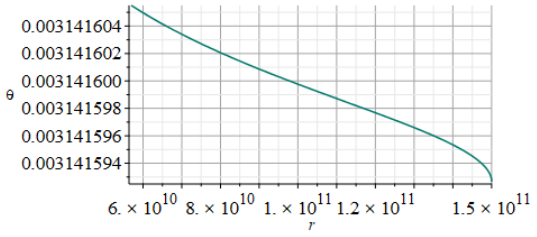
See Table 3 for a summary of the results obtained here.

$\theta(0)$ *[rad]*	$\omega_\theta(0)$ $[\frac{1}{s}]$	θ_f *[deg]*	$\Omega = 2\cdot 10^{-7}\,[\frac{1}{s}]$ $r(0)=150\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=2\cdot 10^{-7}$	$\Omega = -2.7\cdot 10^{-6}\,[\frac{1}{s}]$ $r(0)=150\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=-2.7\cdot 10^{-6}$	$\Omega = 0\,[\frac{1}{s}]$ $r(0)=150\ 10^9, D(r)(0)=0, \phi(0)=\frac{\pi}{10^3}, D(\phi)(0)=0$
$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	0	~0.85° ~3° ~0°	*The mass m reaches equilibrium by oscillating around its initial position from ~0°to ~ 1.7°, making thus a mean oscillation angle of ~ 0.85°.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating around $\theta \approx 3°$.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***	*The mass m reaches equilibrium by oscillating around the initial position $\theta \approx 0$.* ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note****: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
			3D plot for $t = 0..10^8$ 	*3D plot for $t = 0..10^9$* 	*3D plot for $t = 0..7\cdot 10^6$* 

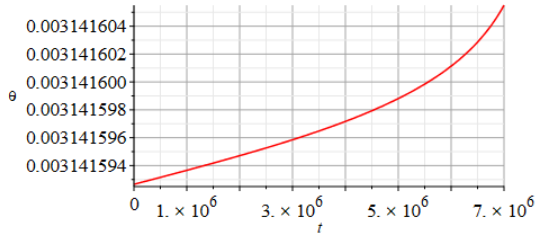
Plot of θ vs. r, for t = 0..10^8 	*Plot of θ vs. r, for t = 0..10^8* 	*Plot of θ vs. r, for t = 0..7·10^6* 
Plot of θ vs. t, for t = 0..10^8 	*Plot of θ vs. t, for t = 0..2·10^9. The envelope has a variable period in this time range, from T≈1.1·10^8 (~**3.5 years**) to T≈3.1·10^8 (~**9.8 years**).* 	*Plot of θ vs. t, for t = 0..7·10^6.* 
Plot of r vs. t, for t = 0..2·10^8 	*Plot of r vs. t, for t = 0..2·10^9. The envelope has a variable period in this time range, from T≈1.1·10^8 (~**3.5 years**) to T≈3.1·10^8 (~**9.8 years**).* 	*Plot of r vs. t, for t = 0..7·10^6.* 

			Polar plot of r vs θ, for $[R, Theta, 0..10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^7]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^7]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..2\cdot10^7]$ *A vertical elliptical trajectory (orbit of mass m around central mass M)* 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..2\cdot10^7]$ *A quasi semi-circular trajectory (orbit of mass m around central mass M)* 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..2\cdot10^7]$ 

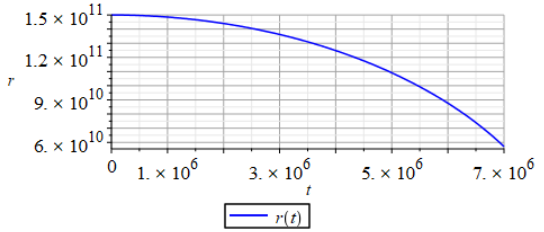
			Polar plot of r vs t, for $[R(t), t, t = 0..5 \cdot 10^6 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..5 \cdot 10^6 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..5 \cdot 10^6 \cdot 2\pi]$, *numpoints = 4* *A kind of logaritmic spiral from outwards* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10^{-15}	~0.85° ~3° ~0°	*Same as above for* $\omega_\theta(0) = 0$	*Same as above for* $\omega_\theta(0) = 0$	*The mass m reaches equilibrium by oscillating around the initial position* $\theta \approx 0$. ***This fact is not conclusive. Computation for longer times has to be made in order to check the behavior.*** ***Note**: I can't compute greater times for the plots. Whether it is not allowed or it might take days.* ***A much more powerful computer for numerical analysis is needed to arrive to a conclusion for this case.***
					3D plot for $t = 0..7 \cdot 10^6$ 
					Plot of θ *vs. r, for* $t = 0..7 \cdot 10^6$ 

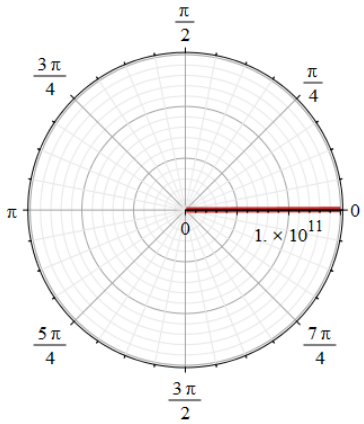
Plot of θ vs. t, for t = 0..7·10^6.



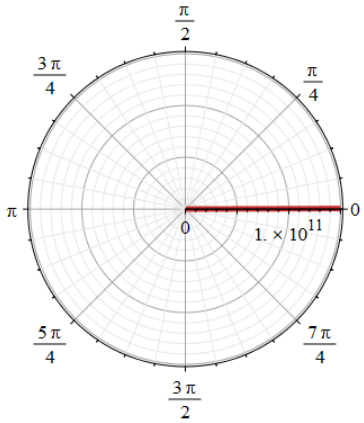
Plot of r vs. t, for t = 0..7·10^6.



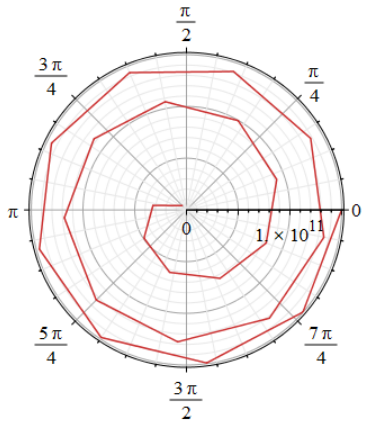
Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

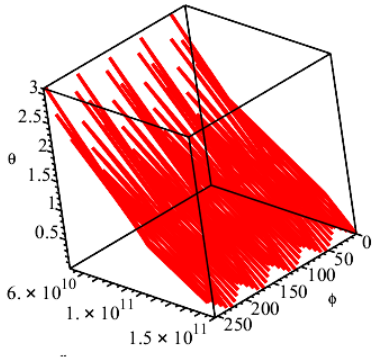
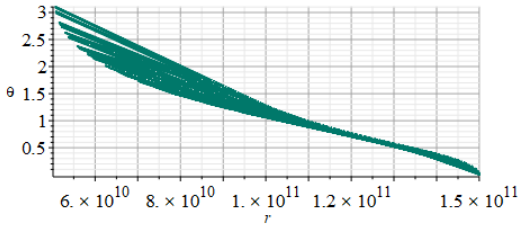
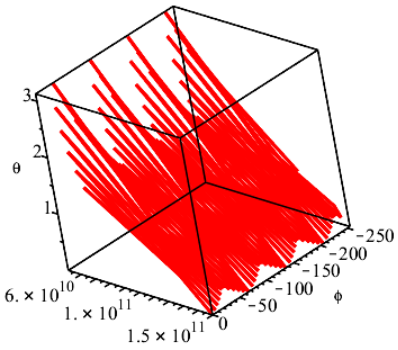
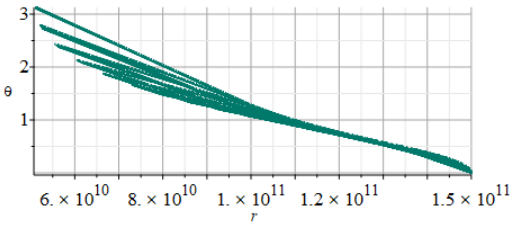
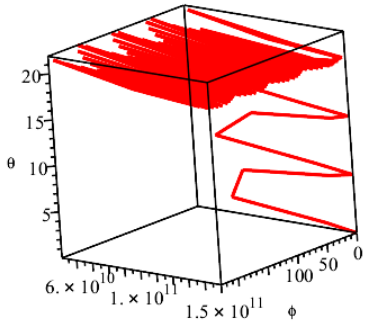
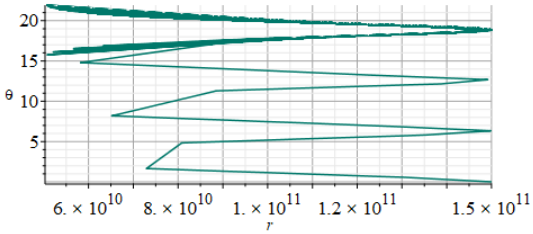


Polar plot of r vs ϕ, for $[R, Phi, 0..2 \cdot 10^7]$

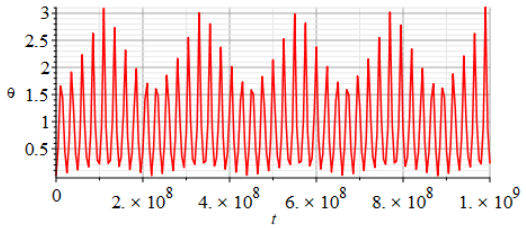


Polar plot of r vs t, for
$\left[R(t),\ t,\ t = 0..5 \cdot 10^{6} \cdot 2\,\pi\right]$, *numpoints = 4*
A kind of logaritmic spiral from outwards

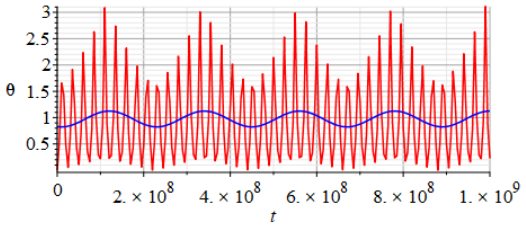


$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10^{-7}	~90° ~90° ~90°	*The mass m reaches equilibrium oscillating at a mean angle of $\theta \approx 90°$ ($\pm 90°$). It swings between $\theta \approx 0°$ and $\theta \approx 180°$* *3D plot for $t = 0..10^9$*  *Plot of θ vs. r, for $t = 0..10^9$* 	*The mass m oscillates from 0° to maximum ~ 180°, having thus a mean oscillation angle of ~ 90°, with a* ***center line of oscillations around*** *~ 57°. 3D plot for $t = 0..10^9$*  *Plot of θ vs. r, for $t = 0..10^9$* 	*The mass m oscillates from 0° to maximum ~ 180°, having thus a mean oscillation angle of ~ 90°, with a* ***center line of oscillations around*** *~ 90°. 3D plot for $t = 0..10^9$*  *Plot of θ vs. r, for $t = 0..10^9$* 

Plot of θ vs. t, for $t = 0..10^9$



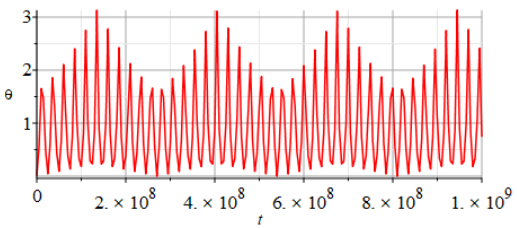
*Plot of θ vs. t, and the slower oscillation of the mean value, for $t = 0..10^9$. The mean approximate function (found manually) is: $0.825 + 0.3 \cdot \left(\sin\left(\frac{\omega \cdot t}{2} + 3 \right) \right)^2$, with $\omega := \frac{2\pi}{2.2 \cdot 10^8}$, and a period $T=2.2 \cdot 10^8$ (~**7 years**):*



Precession and/or Nutation?

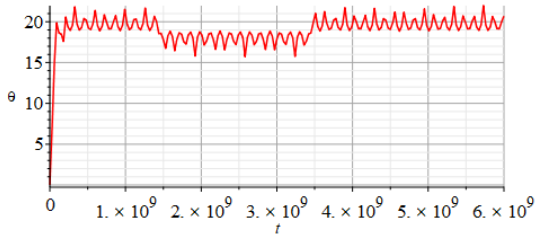
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

*Plot of θ vs. t, for $t = 0..10^9$. The envelope has a period in this time range, of $T \approx 2.7 \cdot 10^8$ (~**8.5 years**)*

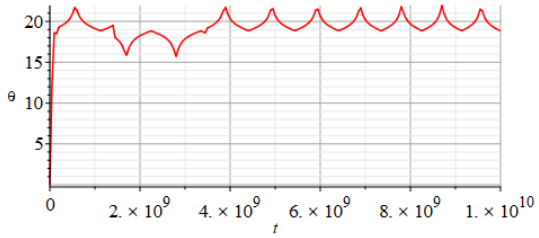


What is the mean value oscillation: Precession or Nutation?

Plot of θ vs. t, for $t = 0..6 \cdot 10^9$.



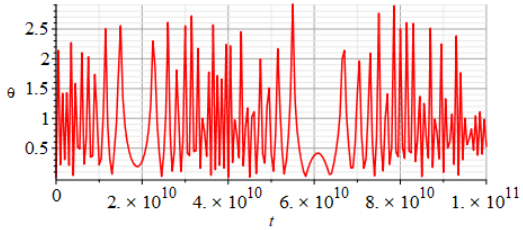
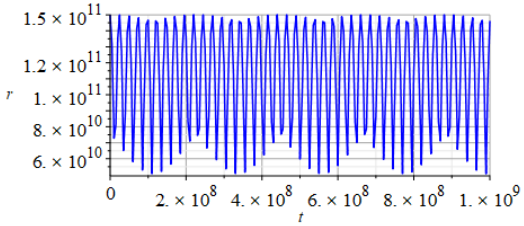
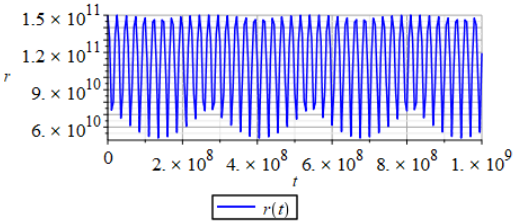
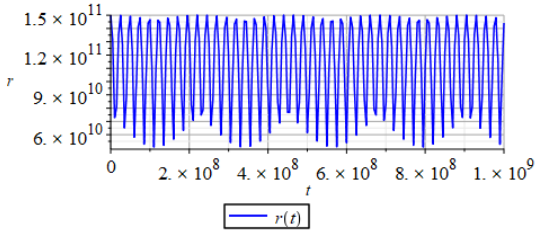
*Plot of θ vs. t, for $t = 0..10^{10}$. From $t = 4 \cdot 10^9$ the curve has a period in this time range, of $T \approx 1 \cdot 10^9$ (~**32 years**)*

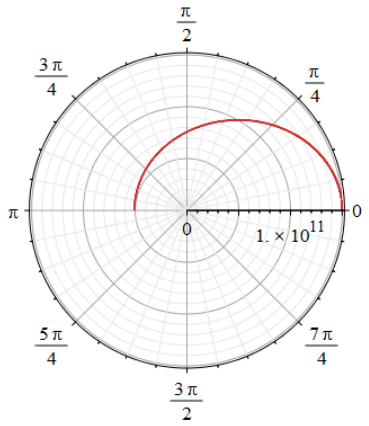
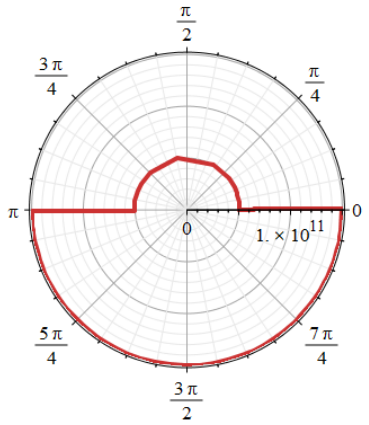
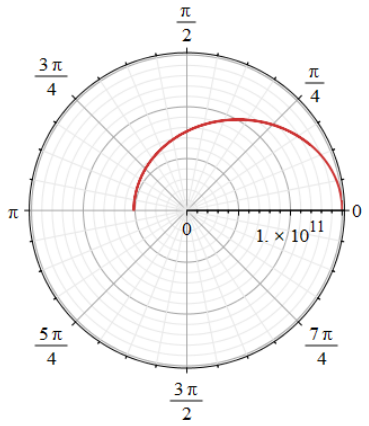
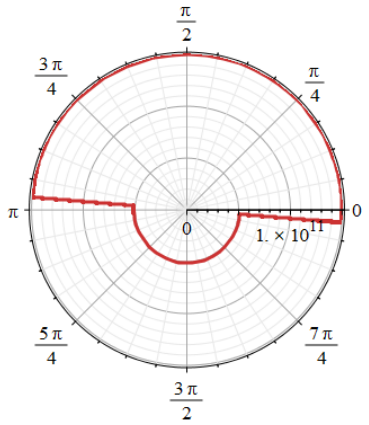
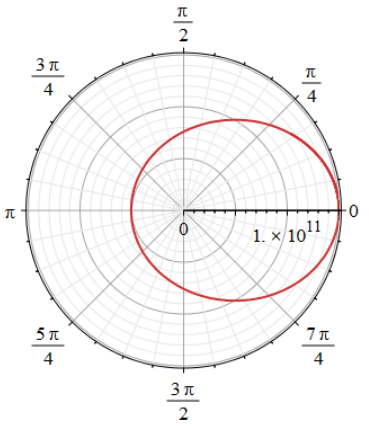
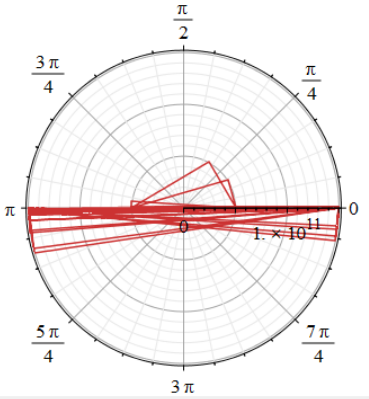


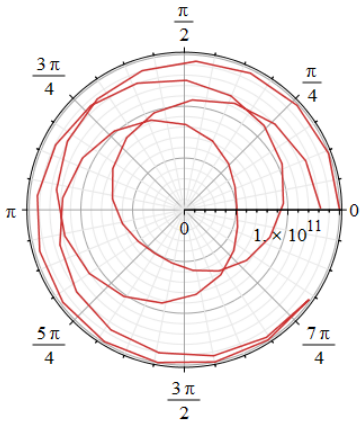
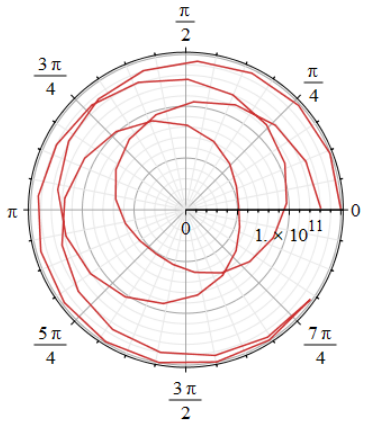
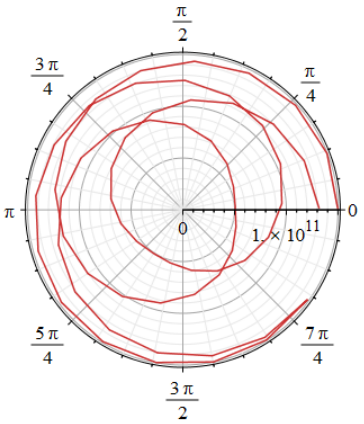
Precession and/or Nutation?

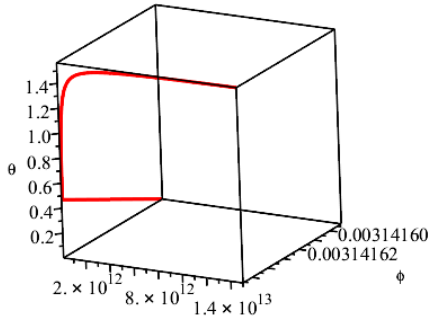
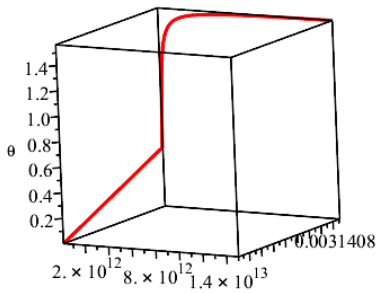
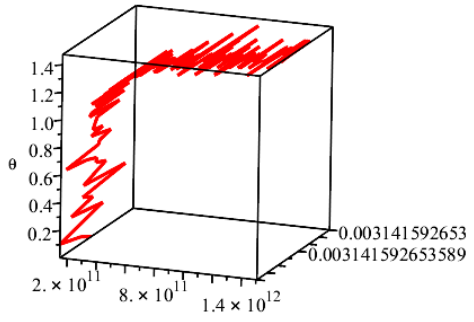
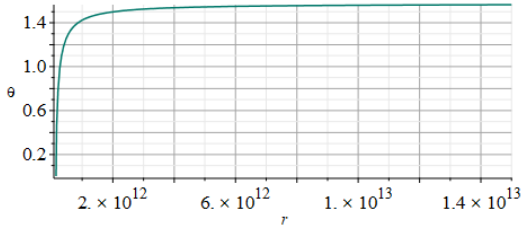
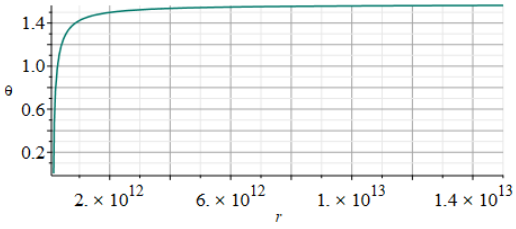
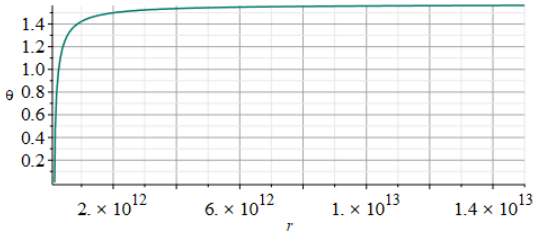
WARNING: *the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

Plot of θ vs. t, for $t = 0..10^{11}$: **not done because it takes too long.**

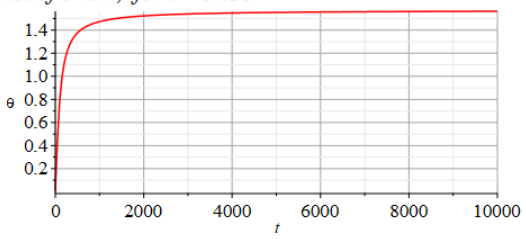
Plot of θ vs. t, for $t = 0..10^{11}$. The pattern is not clear. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.*  		
Plot of r vs. t, for $t = 0..10^9$  	*Plot of r vs. t, for $t = 0..10^9$. The envelope has a period in this time range, of $T \approx 2.7 \cdot 10^8$ (~**8.5 years**)*  	*Plot of r vs. t, for $t = 0..10^9$. The envelope has a period in this time range, of $T \approx 2 \cdot 10^8$ (~**6.3 years**)* 

			Polar plot of r vs θ, for $[R, Theta, 0..10^7]$  *Polar plot of r vs ϕ, for* $[R, Phi, 0..3\cdot10^7]$, *numpoints = 50000* 	*Polar plot of r vs θ, for* $[R, Theta, 0..2\cdot10^7]$  *Polar plot of r vs ϕ, for* $[R, Phi, 0..3\cdot10^7]$, *numpoints = 50000* 	*Polar plot of r vs θ, for* $[R, Theta, 0..3\cdot10^7]$  *Polar plot of r vs ϕ, for* $[R, Phi, 0..8\cdot10^8]$, *numpoints = 50000. The plot is not clear. I cannot calculate for longer times because it migt take days in PC...* 

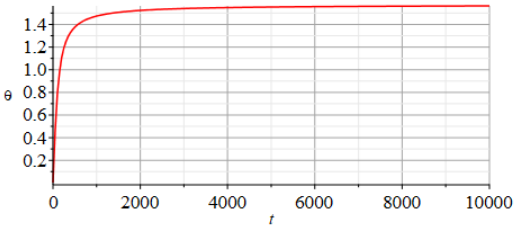
			Polar plot of r vs t, for $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 

$\frac{\pi}{10^3}$ *(=> $\theta \approx 0°$)*	10^{-2}	90° 90° 90°	*The mass m reaches equilibrium at $\theta = 90°$* *3D plot for $t = 0..10^4$*  	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^4$*  	*The mass m reaches equilibrium at $\theta = 90°$. 3D plot for $t = 0..10^3$* 
			Plot of θ vs. r, for $t = 0..10^4$  	*Plot of θ vs. r, for $t = 0..10^4$*  	*Plot of θ vs. r, for $t = 0..10^4$* 

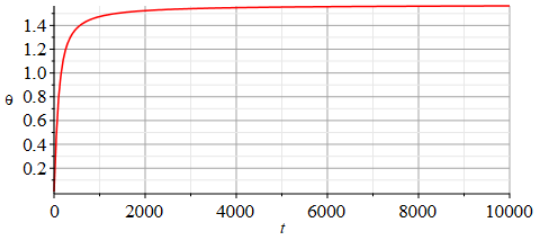
Plot of θ vs. t, for t = 0..10^4



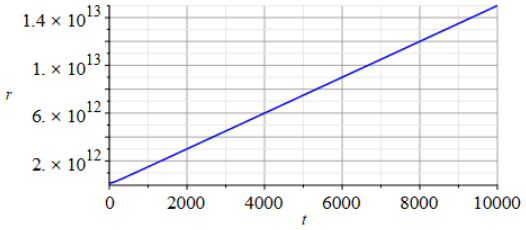
Plot of θ vs. t, for t = 0..10^4



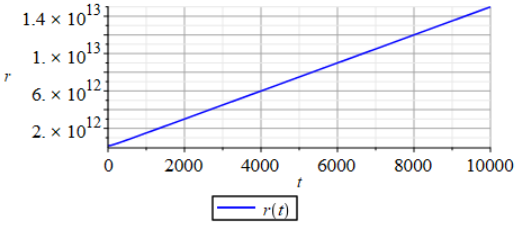
Plot of θ vs. t, for t = 0..10^4



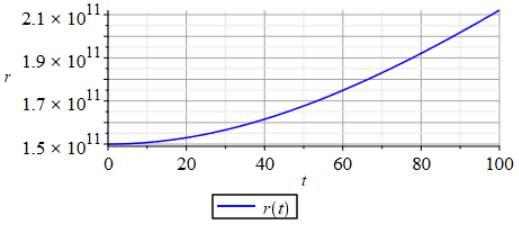
Plot of r vs. t, for t = 0..10^4



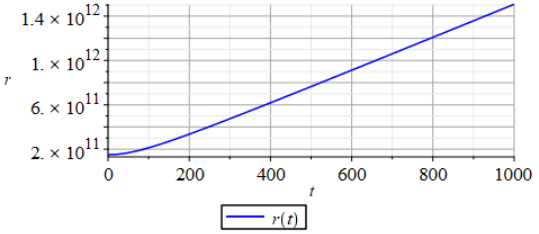
Plot of r vs. t, for t = 0..10^4



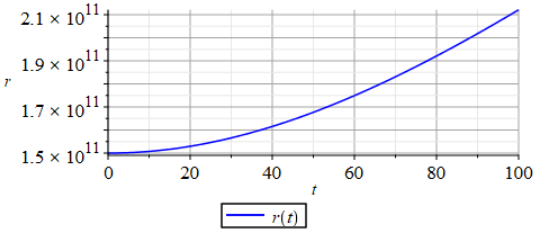
Details for t = 0..10^2

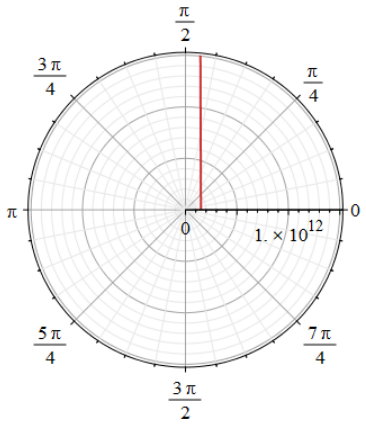
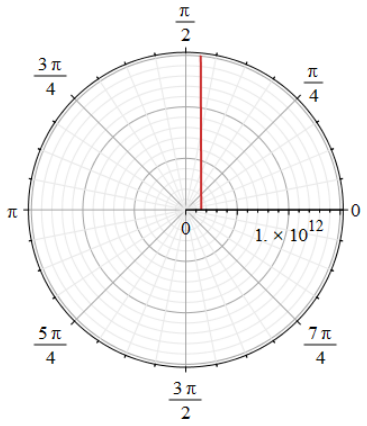
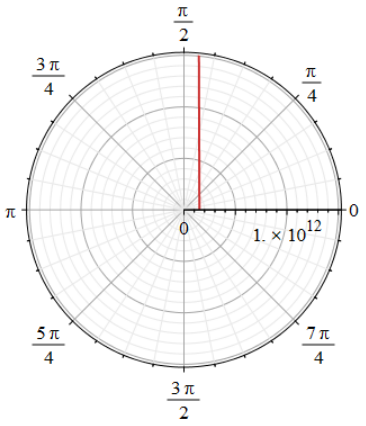
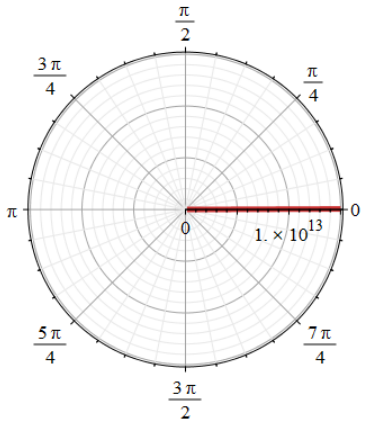
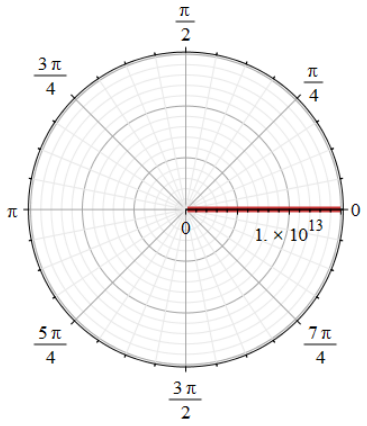
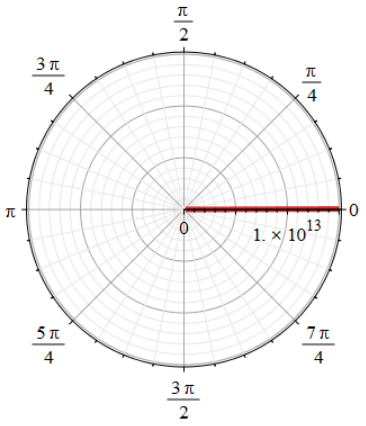


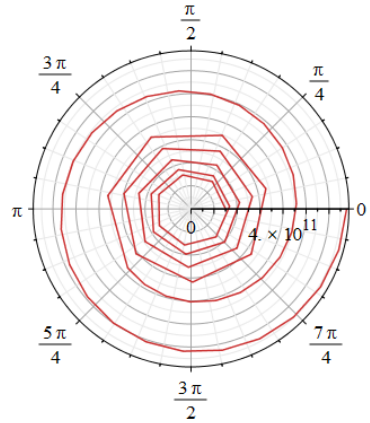
Plot of r vs. t, for t = 0..10^3

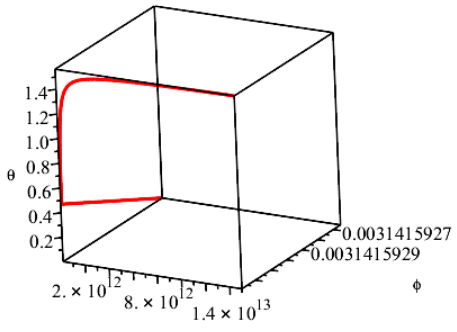
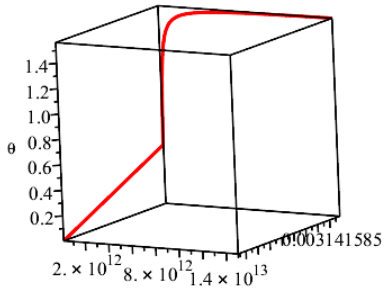
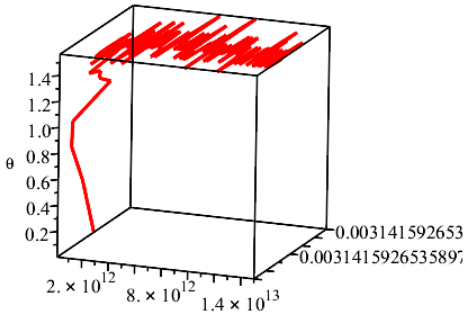
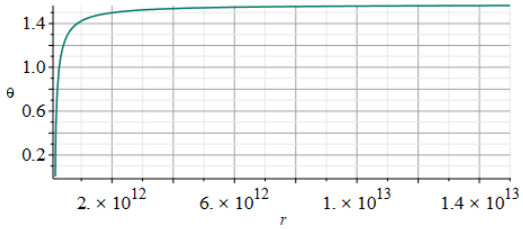
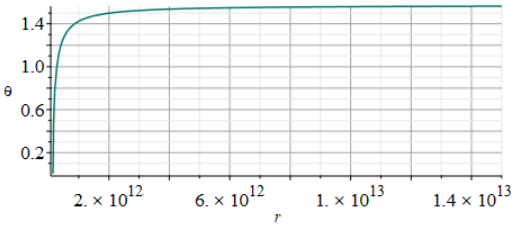
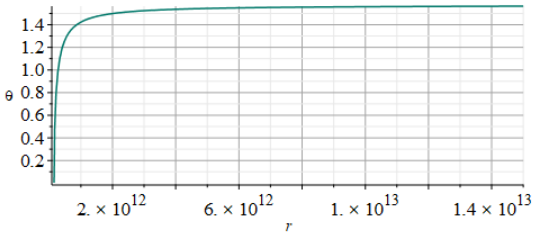


Details for t = 0..10^2

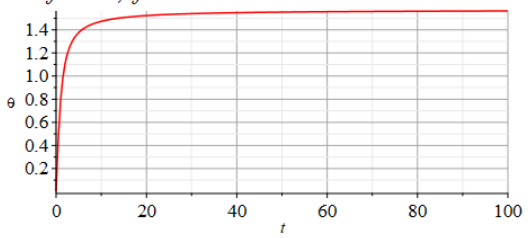


			Polar plot of r vs θ, for $[R, Theta, 0..10^3]$	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$
					
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$
					

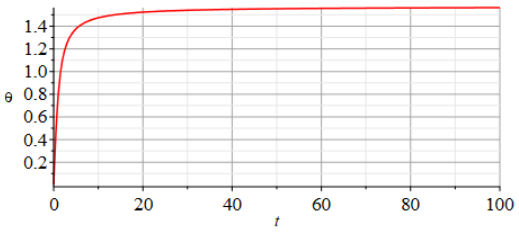
			Polar plot of r vs t, for $[R(t), t, t = 0..7 \cdot 10 \cdot 2\pi]$, *numpoints = 3* *A logaritmic spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10 \cdot 2\pi]$, *numpoints = 3* *A logaritmic spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10 \cdot 2\pi]$, *numpoints = 3* *A logaritmic spiral.* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	1	90° 90° 90°	*The mass m reaches equilibrium at* $\theta = 90°$ *3D plot for* $t = 0..10^2$ 	*The mass m reaches equilibrium at* $\theta = 90°$*. 3D plot for* $t = 0..10^2$ 	*The mass m reaches equilibrium at* $\theta = 90°$*. 3D plot for* $t = 0..10^2$ 
			Plot of θ *vs. r, for* $t = 0..10^2$ 	*Plot of* θ *vs. r, for* $t = 0..10^2$ 	*Plot of* θ *vs. r, for* $t = 0..10^2$ 

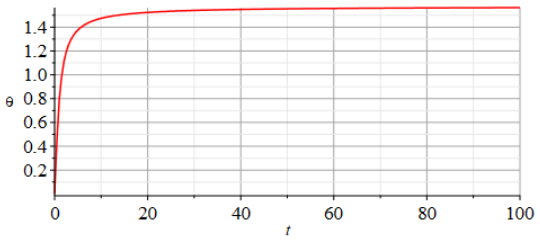
Plot of θ vs. t, for t = 0..10²



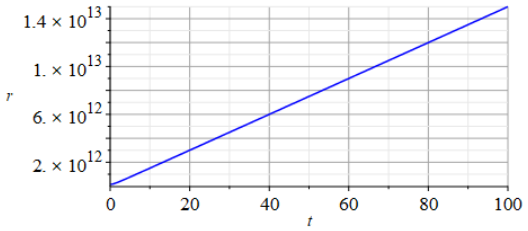
Plot of θ vs. t, for t = 0..10²



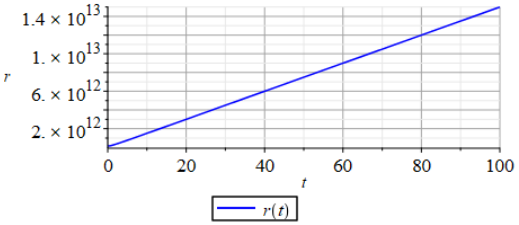
Plot of θ vs. t, for t = 0..10²



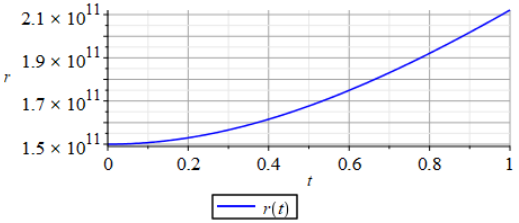
Plot of r vs. t, for t = 0..10²



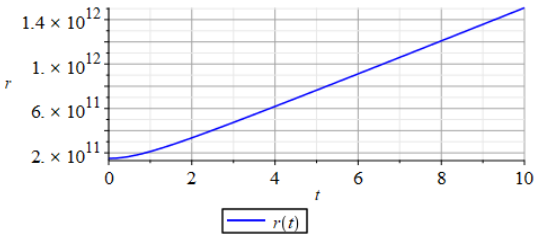
Plot of r vs. t, for t = 0..10²



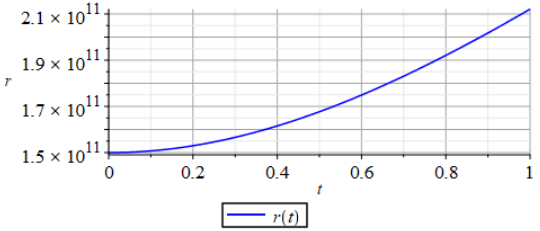
Details for t = 0..1

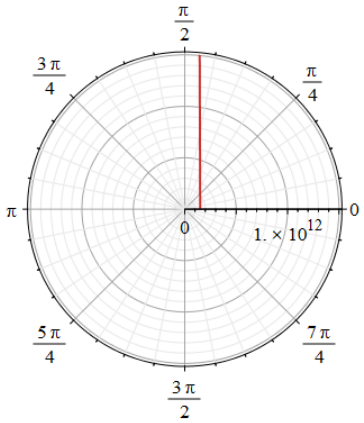
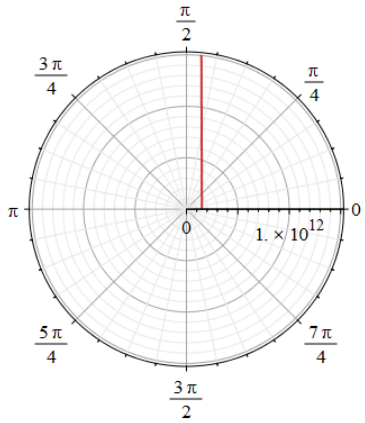
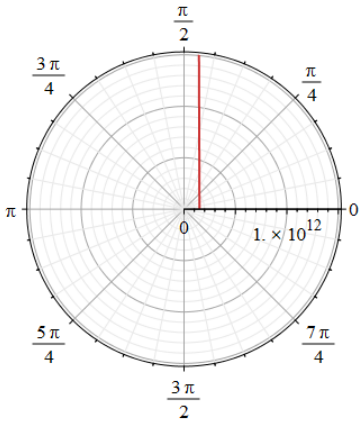
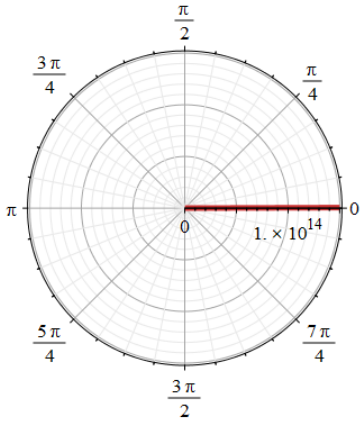
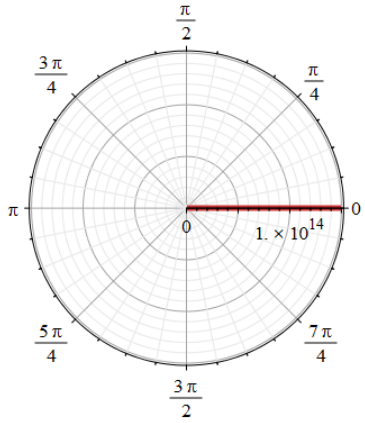
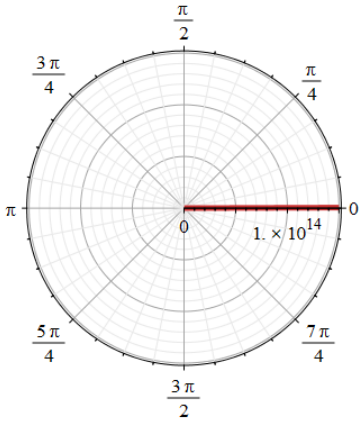


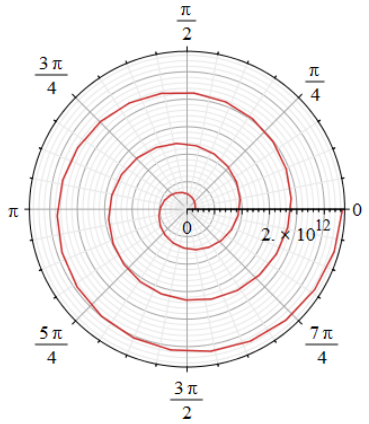
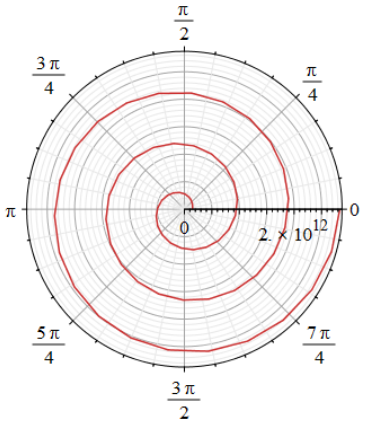
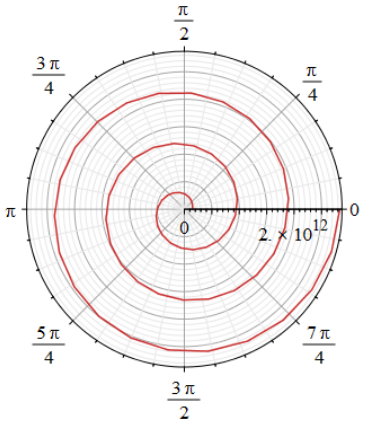
Plot of r vs. t, for t = 0..10

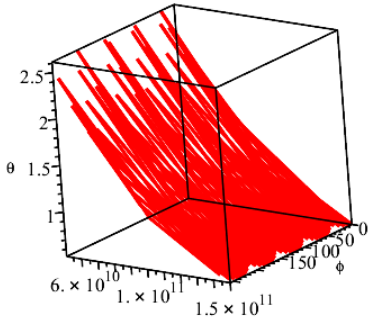
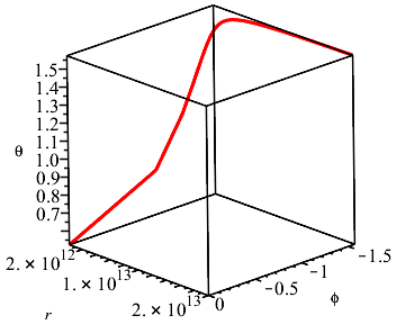
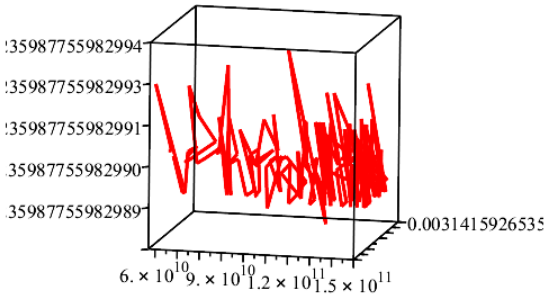


Details for t = 0..1

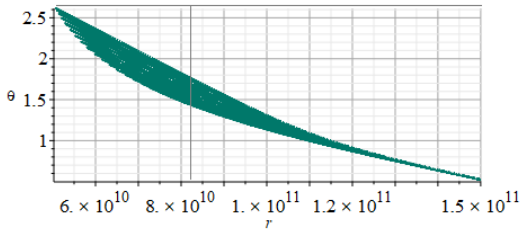


			Polar plot of r vs θ, for $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 

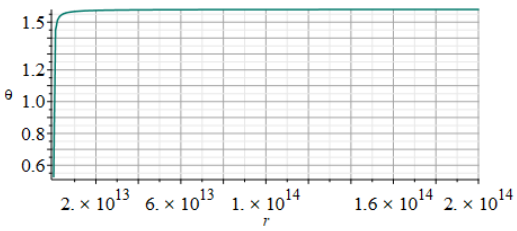
			Polar plot of r vs t, for $[R(t), t, t = 0..3\cdot 2\pi]$, *numpoints = 3* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..3\cdot 2\pi]$, *numpoints = 3* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..3\cdot 2\pi]$, *numpoints = 3* *A regular spiral.* 

$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	10	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{10^3}$ (=> $\theta \approx 0°$)	100	90° 90° 90°	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*	*Same as above for $\omega_\theta(0) = 1$*
$\frac{\pi}{6}$	0	90° 90° 30°	*The mass m reaches equilibrium oscillating at a mean angle of $\theta = 90°$ ($\pm 60°$). It swings between $\theta \approx 30°$ and $\theta \approx 150°$* *3D plot for $t = 0..10^9$* 	*The mass m reaches equilibrium at $\theta = 90°$.* *3D plot for $t = 0..10^9$* 	*The mass m reaches equilibrium oscillating around $\theta = 30°$ (the initial position).* *3D plot for $t = 0..7 \cdot 10^6$* 

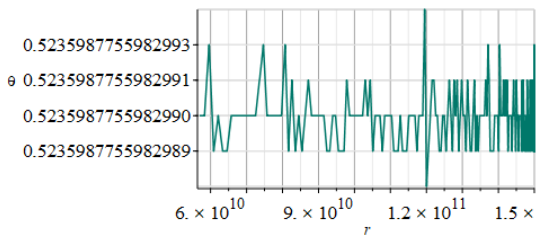
Plot of θ vs. r, for t = 0..10^9



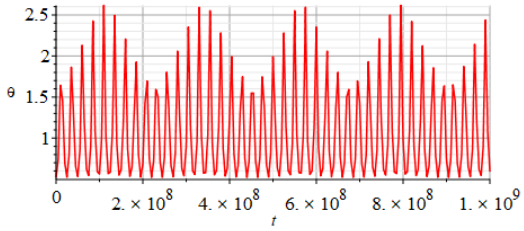
Plot of θ vs. r, for t = 0..10^9



Plot of θ vs. r, for t = 0..7·10^6

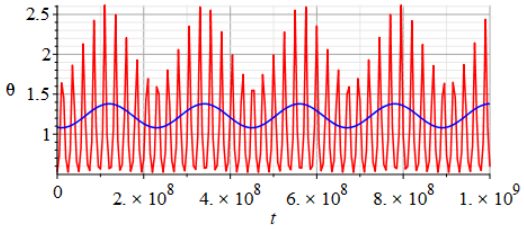


Plot of θ vs. t, for t = 0..10^9



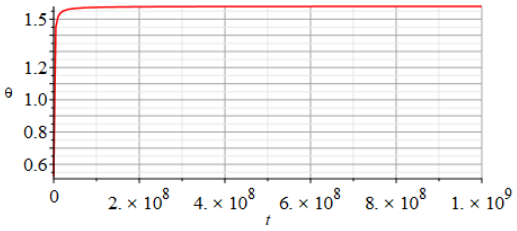
Plot of θ vs. t, and the slower oscillation of the mean value, for t = 0..10^9. The mean approximate function (found manually) is: $1.08 + 0.3 \cdot \left(\sin\left(\frac{\varpi \cdot t}{2} + 3 \right) \right)^2$, *with*

$\varpi := \frac{2\pi}{2.2 \cdot 10^8}$, *and a period T=2.2·10^8 (~**7 years**):*

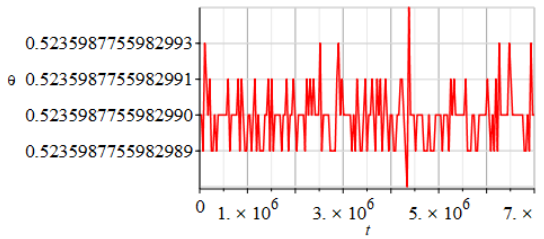


Precession and/or Nutation?

Plot of θ vs. t, for t = 0..10^9

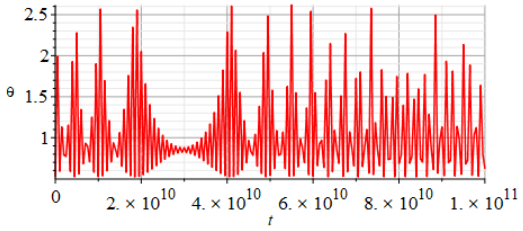


Plot of θ vs. t, for t = 0..7·10^6

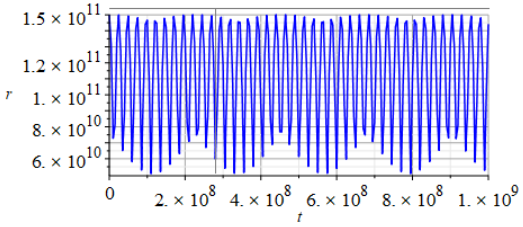


***WARNING**: the following calculations (plots) might take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

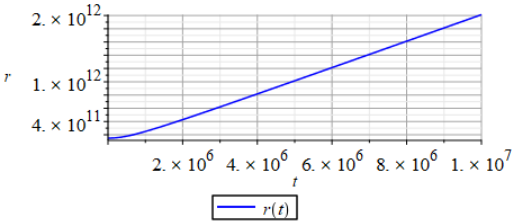
Plot of θ vs. t, for t = 0..10^11. The pattern is not clear. We can see both: ***amplitude and frequency modulation****. So,* ***the period of the mean value*** *oscillation (precession and/or nutation?)* ***is not constant****.*



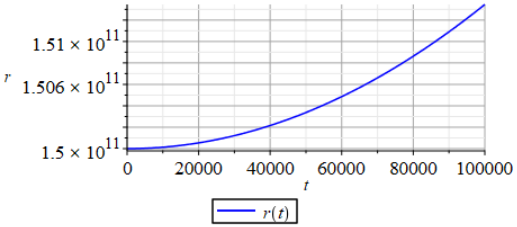
Plot of r vs. t, for t = 0..10^9



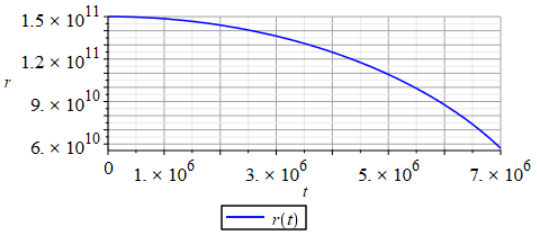
Plot of r vs. t, for t = 0..10^7



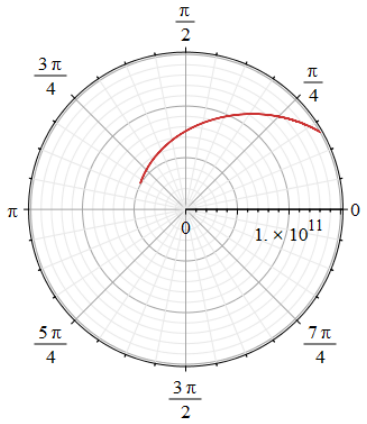
Details for t = 0..10^5



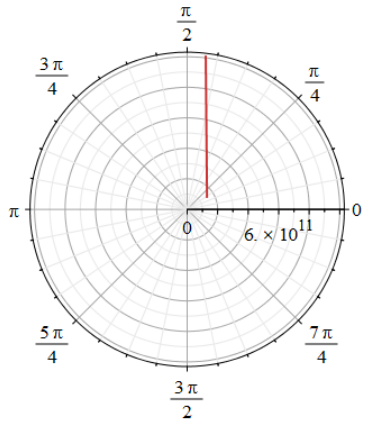
Plot of r vs. t, for t = 0..7·10^6



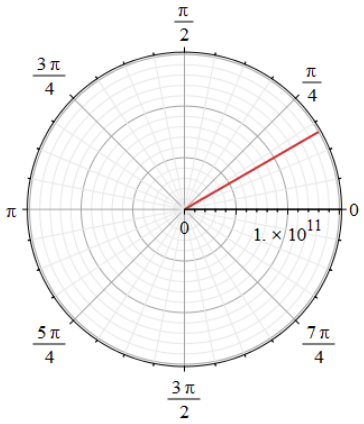
Polar plot of r vs θ, for $[R, Theta, 0..3 \cdot 10^7]$



Polar plot of r vs θ, for $[R, Theta, 0..5 \cdot 10^6]$

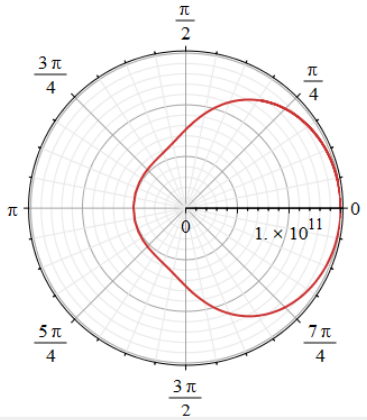


Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

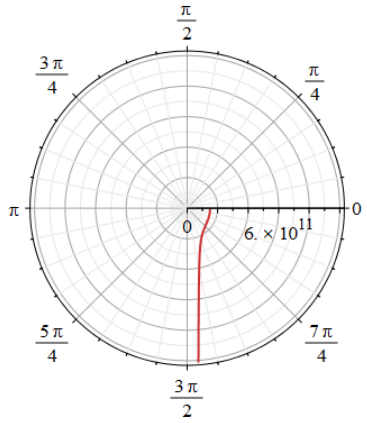


Polar plot of r vs ϕ, for $[R, Phi, 0..3 \cdot 10^7]$

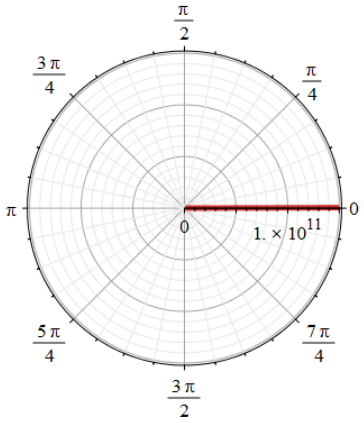
A vertical semi-elliptical trajectory (orbit of mass m around central mass M)

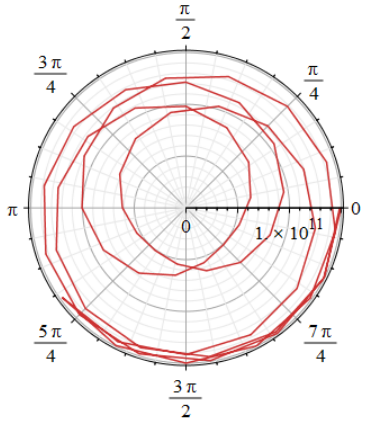
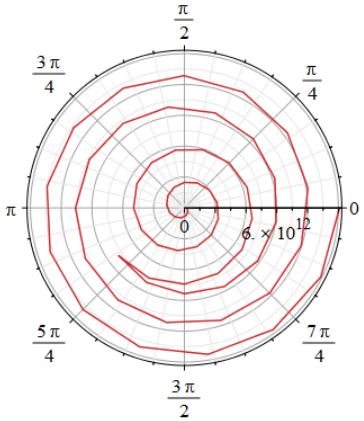
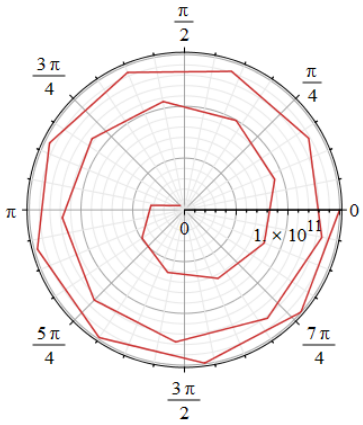


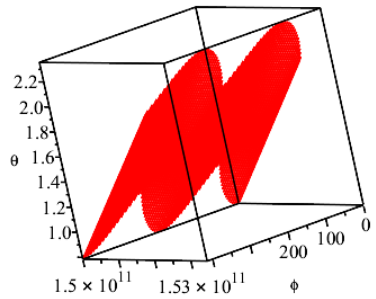
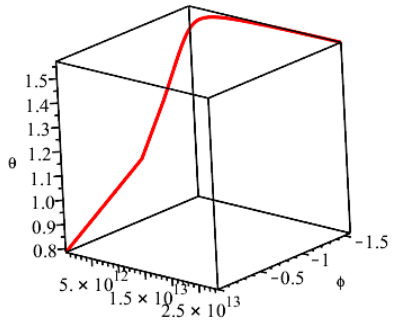
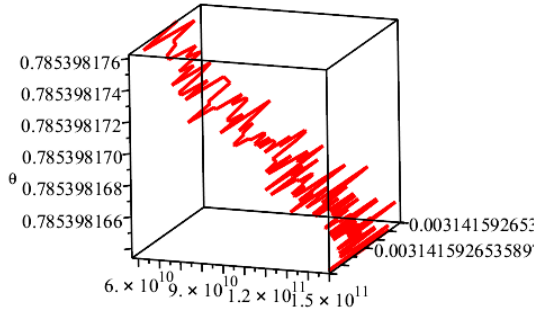
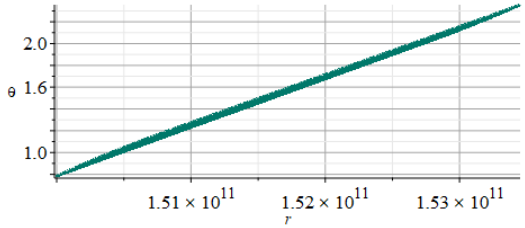
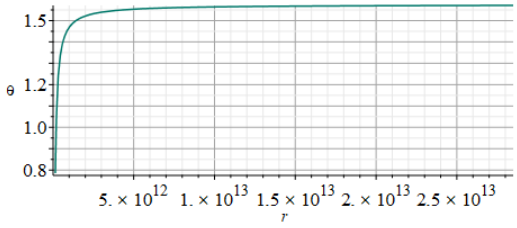
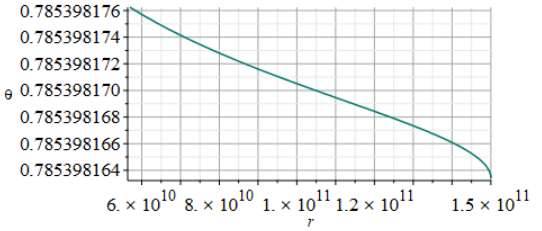
Polar plot of r vs ϕ, for $[R, Phi, 0..5 \cdot 10^6]$



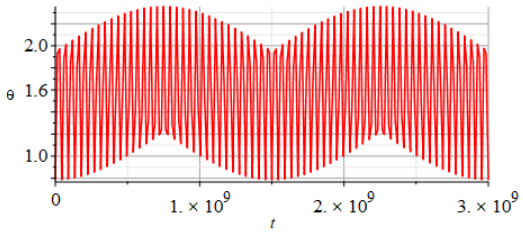
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



			Polar plot of r vs t, for $\left[R(t),\ t,\ t = 0..8\cdot 10^{6}\cdot 2\,\pi\right]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..*  	*Polar plot of r vs t, for* $\left[R(t),\ t,\ t = 0..8\cdot 10^{6}\cdot 2\,\pi\right]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..*  	*Polar plot of r vs t, for* $\left[R(t),\ t,\ t = 0..5\cdot 10^{6}\cdot 2\,\pi\right]$, *numpoints = 4* *A type of logaritmic spiral from outwards* 

$\frac{\pi}{4}$	10^{-15}	90° 90° 45°	*The mass m reaches equilibrium oscillating at a mean angle of $\theta = 90°$ ($\pm 45°$). It swings between $\theta = 45°$ and $\theta = 135°$* *3D plot for $t = 0..3 \cdot 10^9$* 	*The mass m reaches equilibrium at $\theta = 90°$.* *3D plot for $t = 0..10^8$* 	*The mass m reaches equilibrium oscillating around $\theta = 45°$(the initial position).* *3D plot for $t = 0..7 \cdot 10^6$* 
			Plot of θ vs. r, for $t = 0..3 \cdot 10^9$ 	*Plot of θ vs. r, for $t = 0..10^8$* 	*Plot of θ vs. r, for $t = 0..7 \cdot 10^6$* 

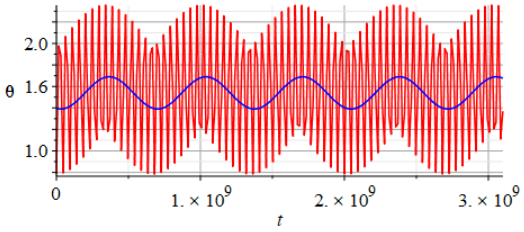
Plot of θ vs. t, for $t = 0..3\cdot10^9$



Plot of θ vs. t, and the slower oscillation of the mean value, for $t = 0..3.1\cdot10^9$. The mean approximate function (found manually) is:

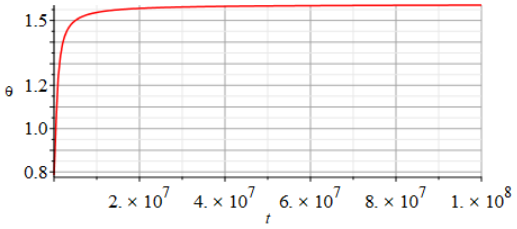
$$1.39 + 0.3\cdot\left(\sin\left(\frac{\varpi\cdot t}{2} + 3\right)\right)^2, \text{ with } \varpi := \frac{2\pi}{0.672\cdot10^9}$$

*, and a period $T=0.672\cdot10^9$ (**21.3 years**):*

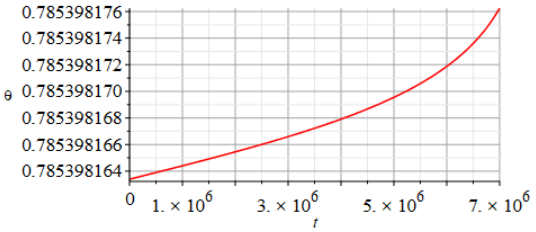


Precession and/or Nutation?

Plot of θ vs. t, for $t = 0..10^8$

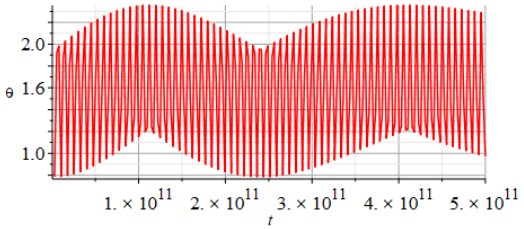


Plot of θ vs. t, for $t = 0..7\cdot10^6$

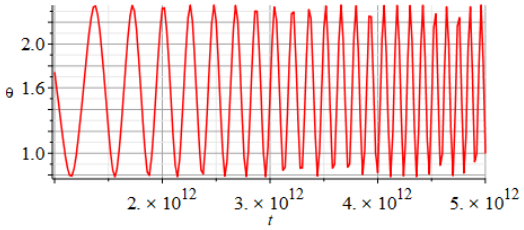


***WARNING**: the following calculations (plots) take from many hours to more than 1 day on a regular PC. To reach some conclusions, a much more powerful computer is necessary for plotting in this time range.*

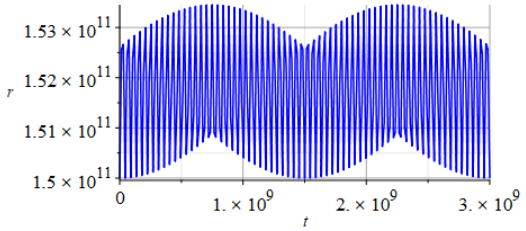
*Plot of θ vs. t, for $t = 3 \cdot 10^9 .. 5 \cdot 10^{11}$. It shows that the period of the mean value oscillation increases to $T \approx (4.1 \cdot 10^{11} - 1.1 \cdot 10^{11}) = 3 \cdot 10^{11}$ (**9513 years**):*



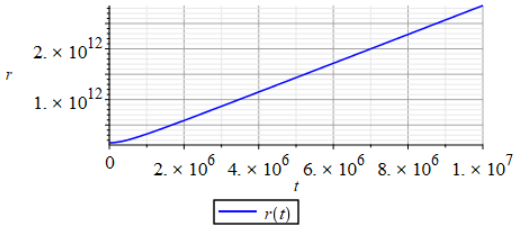
*Plot of θ vs. t, for $t = 10^{12} .. 5 \cdot 10^{12}$. The pattern is not clear. We can see both: **amplitude and frequency modulation**.So, **the period of the mean value** oscillation (precession and/or nutation?) **is not constant**.*



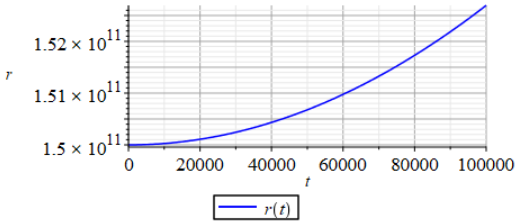
Plot of r vs. t, for $t = 0..3 \cdot 10^9$



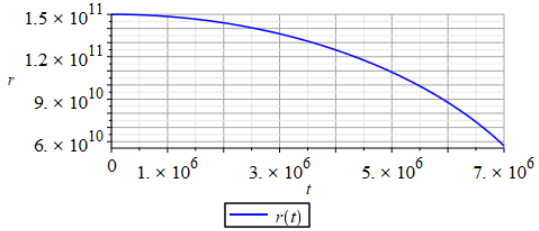
Plot of r vs. t, for $t = 0..10^7$



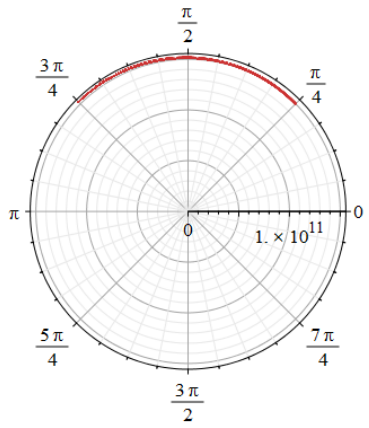
Details for $t = 0..10^5$



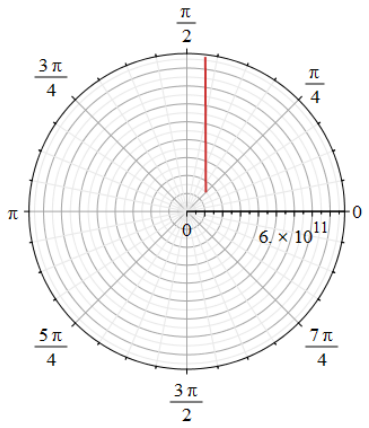
Plot of r vs. t, for $t = 0..7 \cdot 10^6$



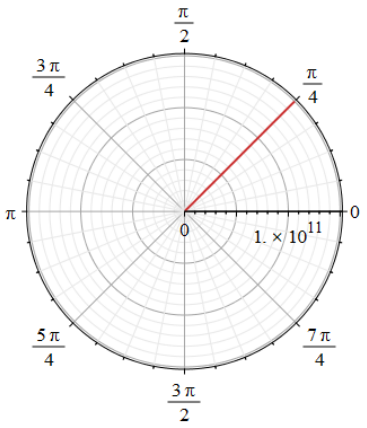
Polar plot of r vs θ, for $[R, Theta, 0..10^8]$



Polar plot of r vs θ, for $[R, Theta, 0..3\cdot10^6]$

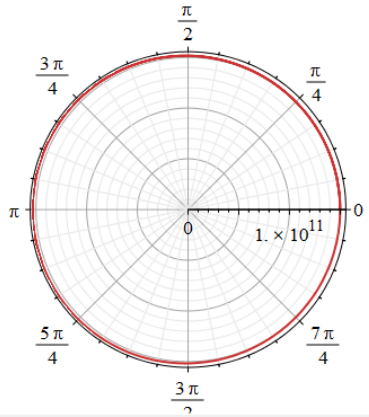


Polar plot of r vs θ, for $[R, Theta, 0..10^7]$

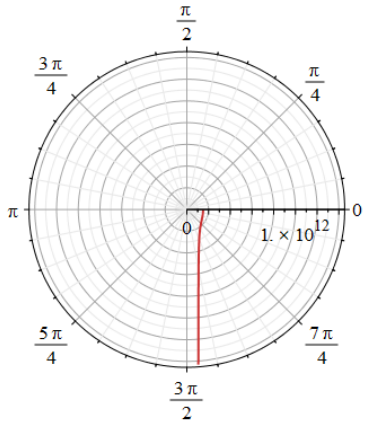


Polar plot of r vs ϕ, for $[R, Phi, 0..7\cdot10^7]$

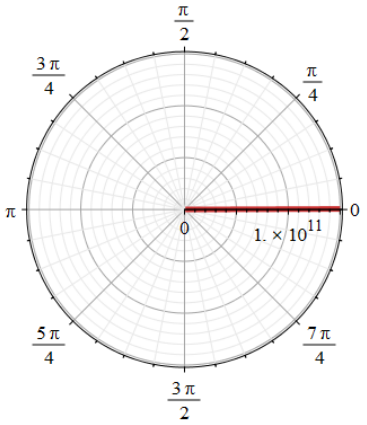
A slightly shifted circular trajectory (orbit of mass m around central mass M)

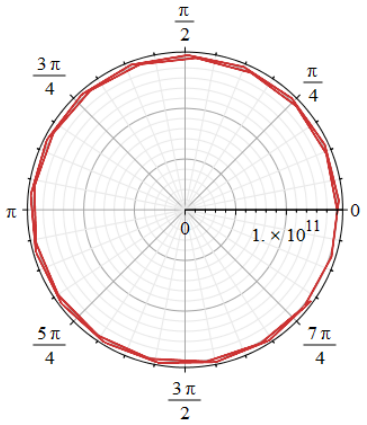
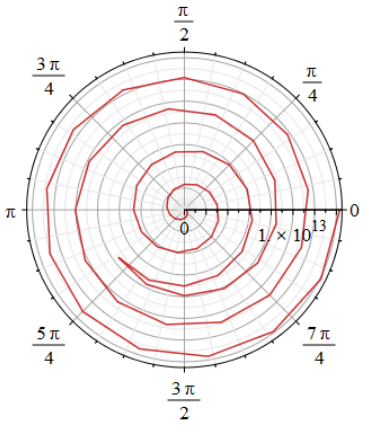
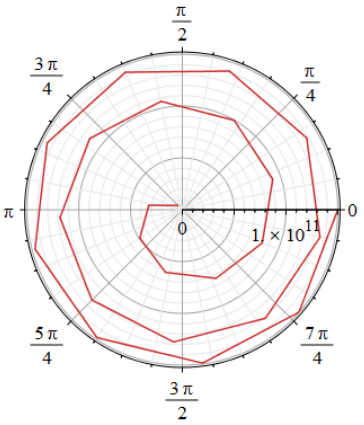


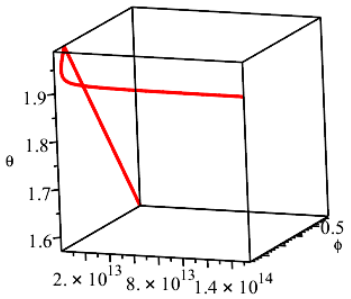
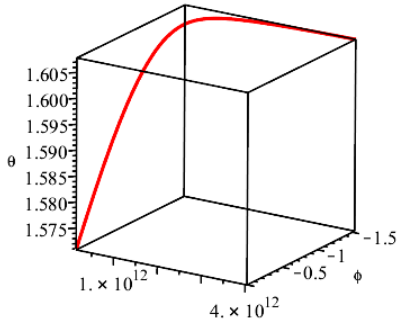
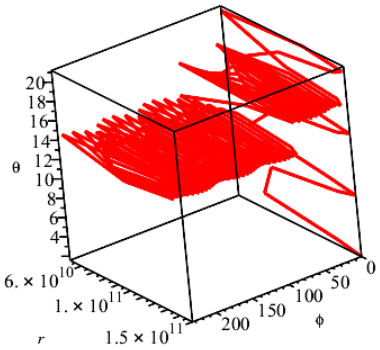
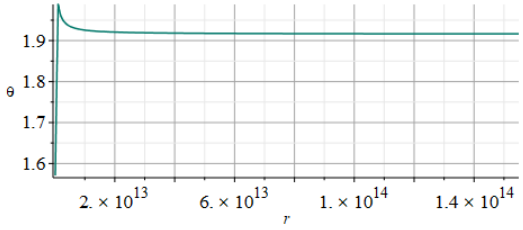
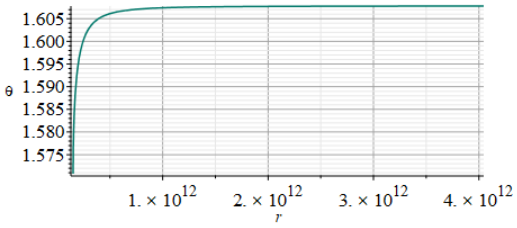
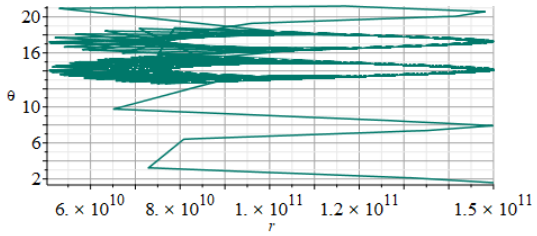
Polar plot of r vs ϕ, for $[R, Phi, 0..5\cdot10^6]$



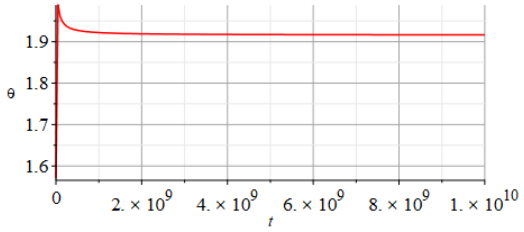
Polar plot of r vs ϕ, for $[R, Phi, 0..10^7]$



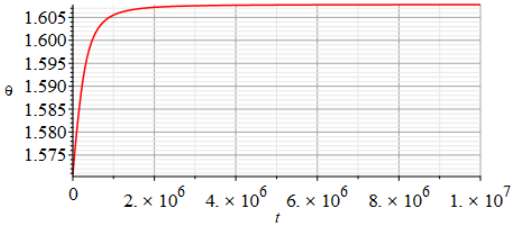
			Polar plot of r vs t, for $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of a tight spiral...* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..8 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..5 \cdot 10^6 \cdot 2\pi]$, *numpoints = 4* *A type of logaritmic spiral from outwards* 

$\frac{\pi}{2}$	10^{-7}	109° 92° 90°	*The mass m reaches equilibrium at θ = 109°* *3D plot for t = 0..10^10*  	*The mass m reaches equilibrium at θ ≈ 92°* *3D plot for t = 0..10^7*  	*The mass m oscillates with variable period around θ ≈ 90° (±90°).* *3D plot for t = 0..10^9* 
			Plot of θ vs. r, for t = 0..10^10  	*Plot of θ vs. r, for t = 0..10^7*  	*Plot of θ vs. r, for t = 0..10^9* 

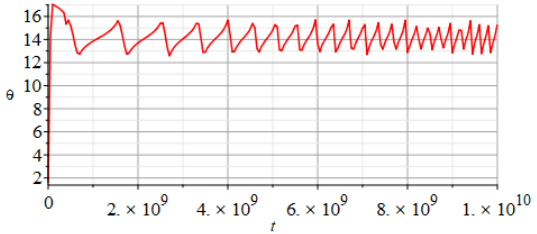
Plot of θ vs. t, for $t = 0..10^{10}$



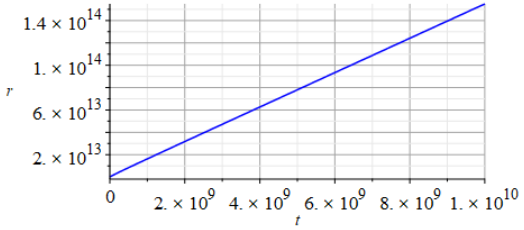
Plot of θ vs. t, for $t = 0..10^{7}$



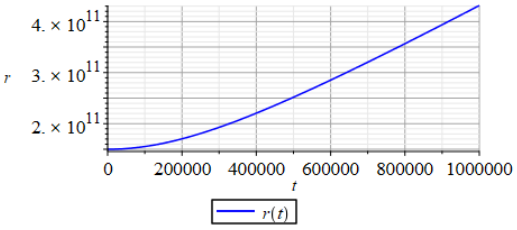
Plot of θ vs. t, for $t = 0..10^{10}$. The mass m oscillates with variable period around $\theta \approx 90°$ ($\pm 90°$).



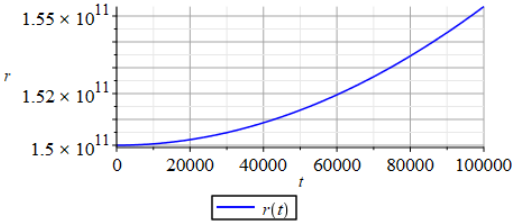
Plot of r vs. t, for $t = 0..10^{10}$



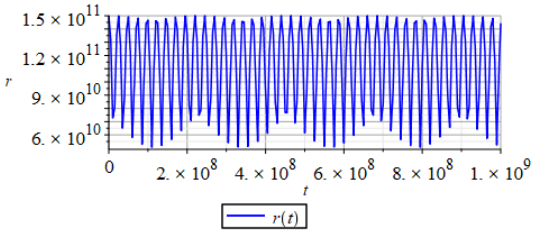
Plot of r vs. t, for $t = 0..10^{6}$



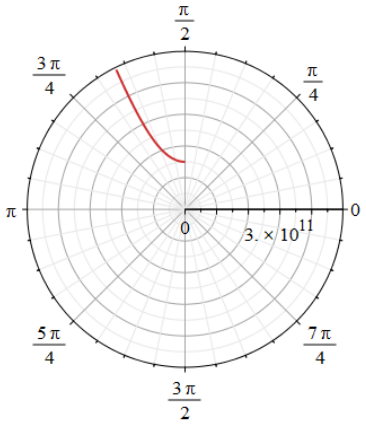
Details for $t = 0..10^{5}$



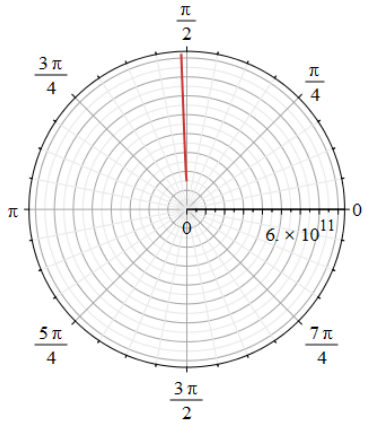
*Plot of r vs. t, for $t = 0..10^{9}$. The period of th envelope for this time range is $T \approx 2.2 \cdot 10^{8}$ (~**7 years**)*



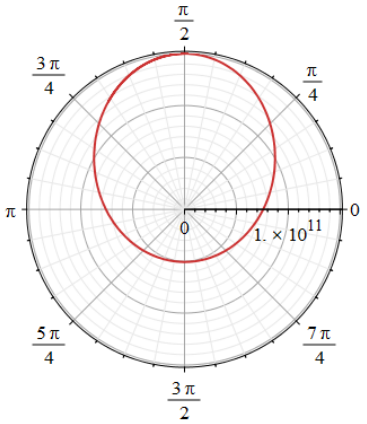
Polar plot of r vs θ, for $[R, Theta, 0..2\cdot10^{7}]$



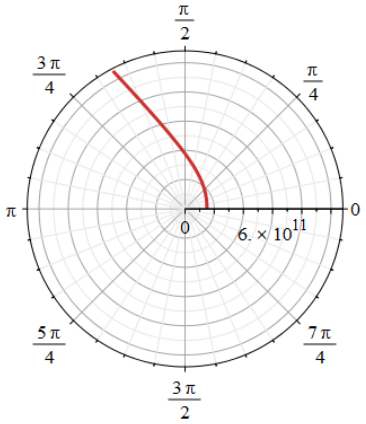
Polar plot of r vs θ, for $[R, Theta, 0..2\cdot10^{6}]$



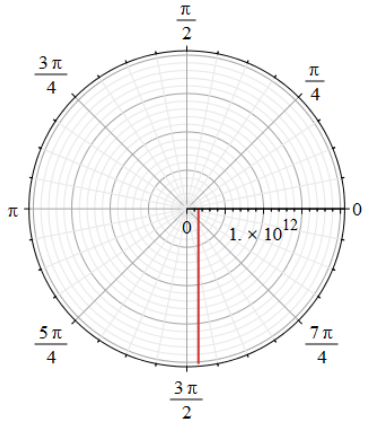
Polar plot of r vs θ, for $[R, Theta, 0..3\cdot10^{7}]$



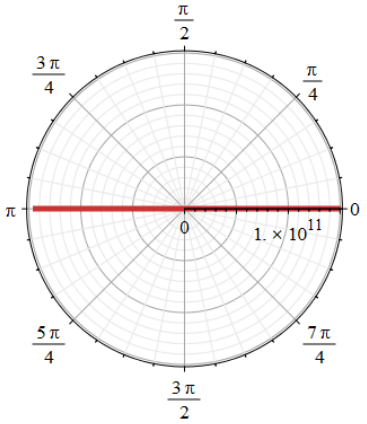
Polar plot of r vs ϕ, for $[R, Phi, 0..5\cdot10^{7}]$
The angle ϕ tends to 135° for longer times (can be verified with the plot r vs ϕ)

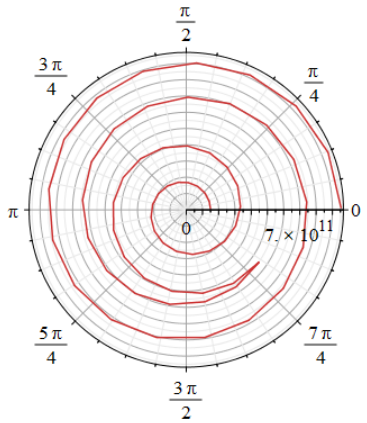
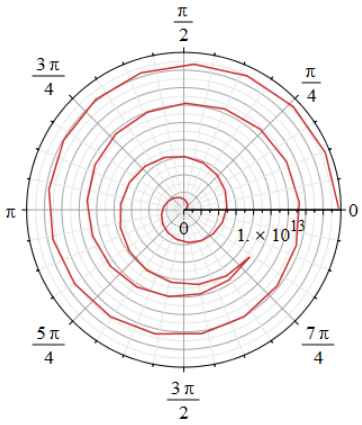
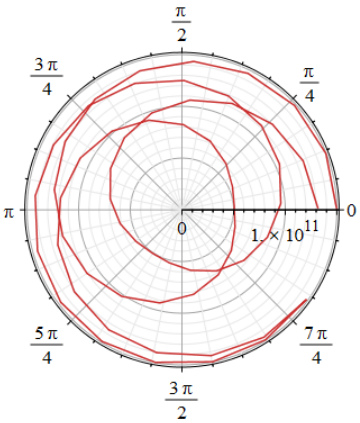


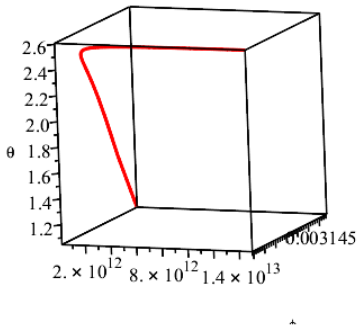
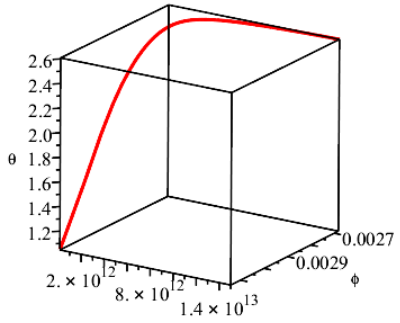
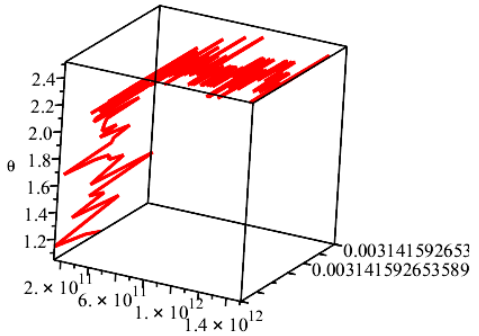
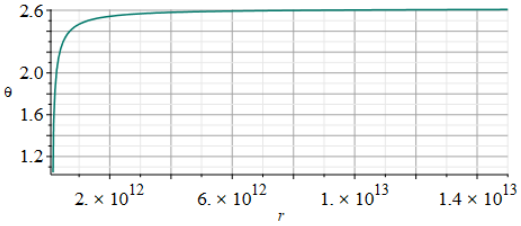
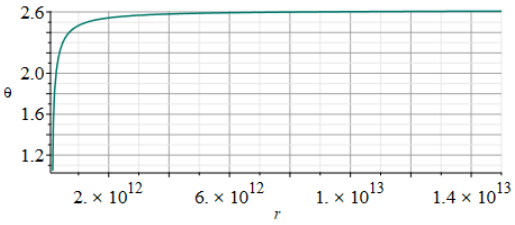
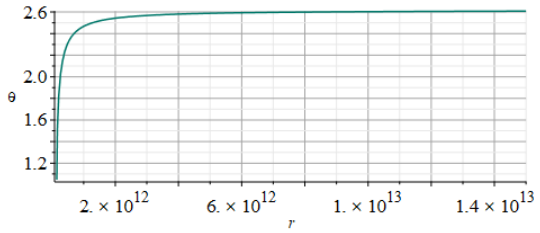
Polar plot of r vs ϕ, for $[R, Phi, 0..5\cdot10^{6}]$

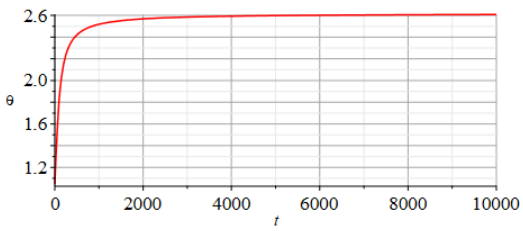
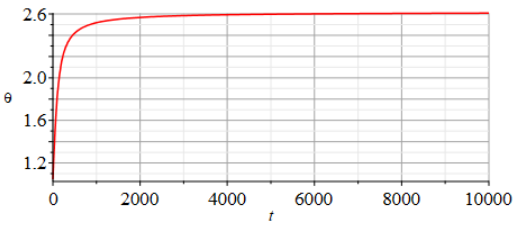
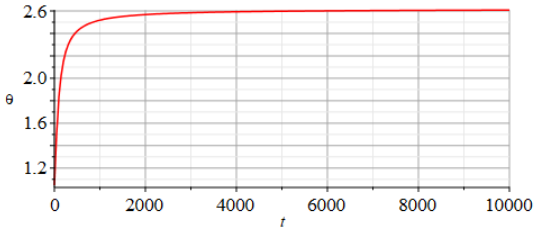
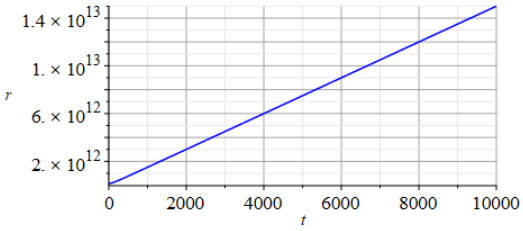
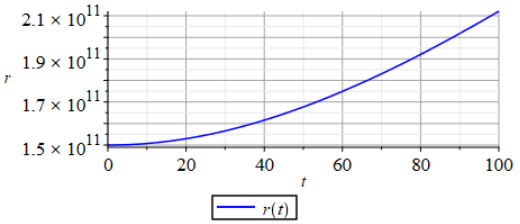
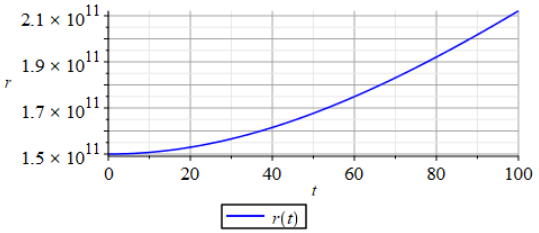


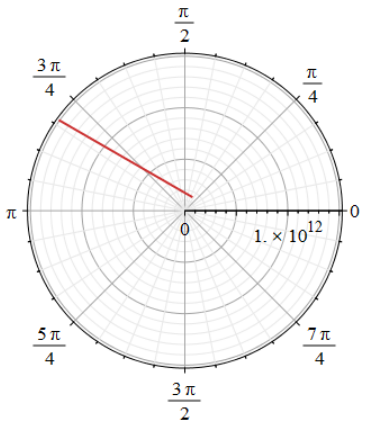
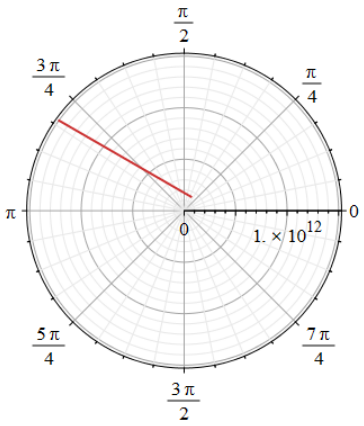
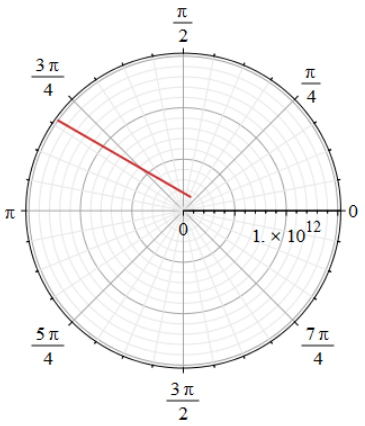
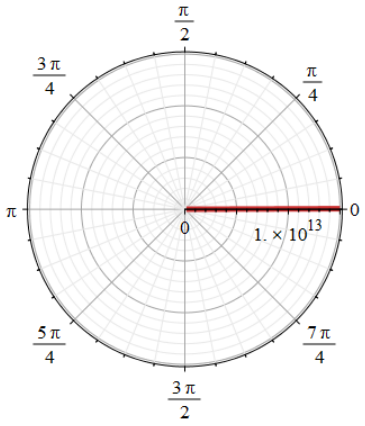
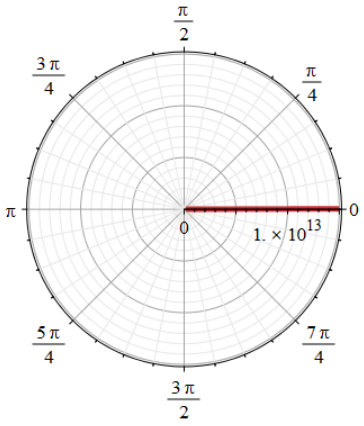
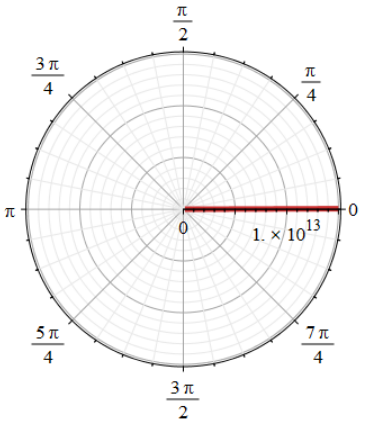
Polar plot of r vs ϕ, for $[R, Phi, 0..10^{8}]$

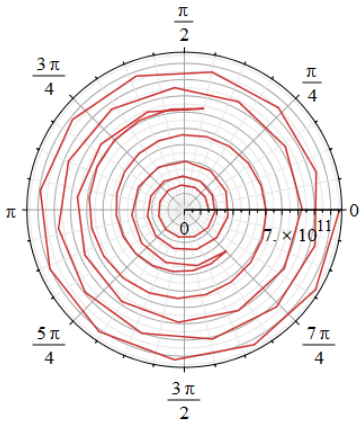
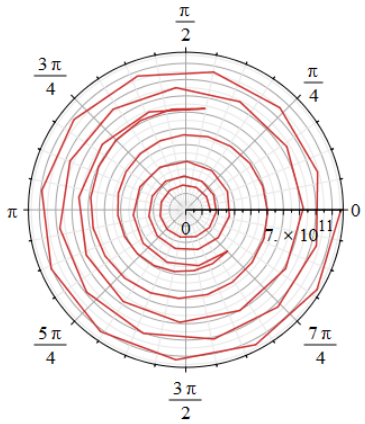
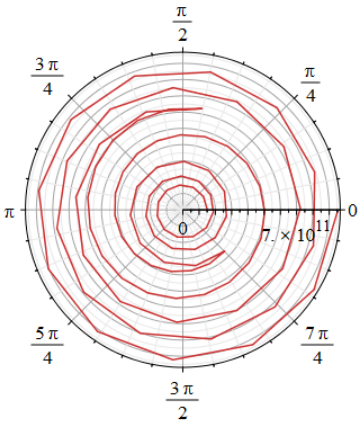


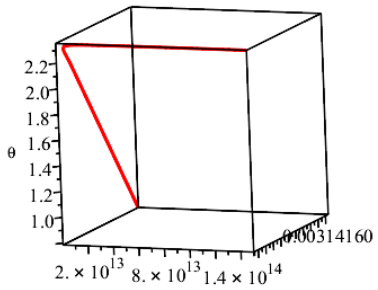
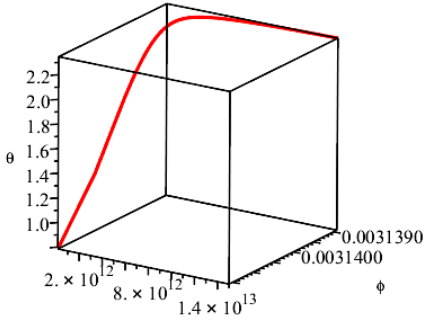
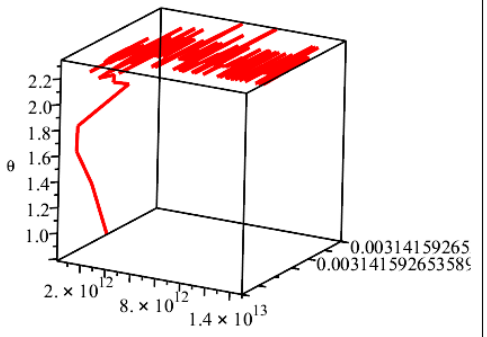
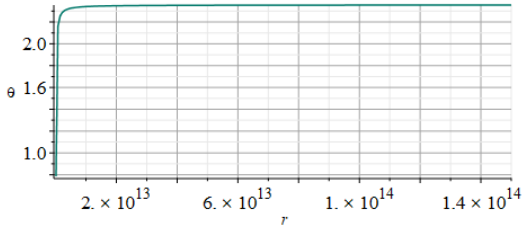
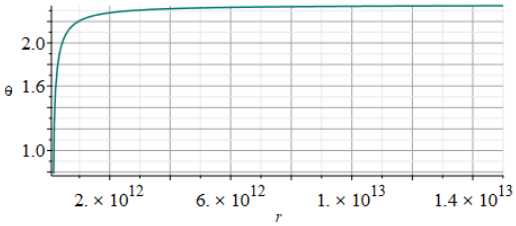
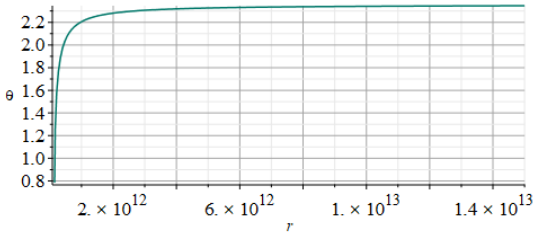
			Polar plot of r vs t, for $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..7 \cdot 10^6 \cdot 2\pi]$, *numpoints = 3* *A type of spiral that that evolves with forward and backward motion..* 

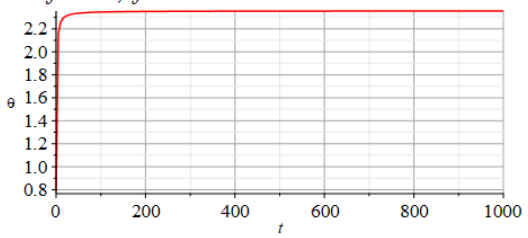
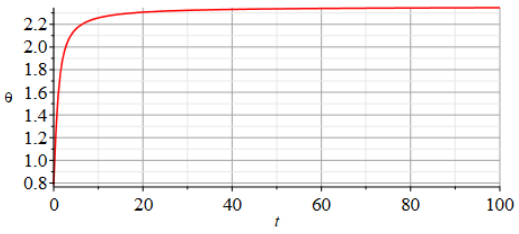
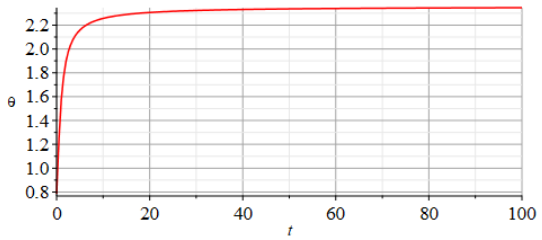
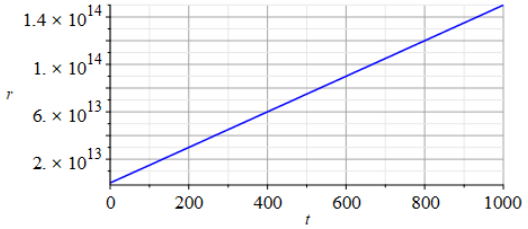
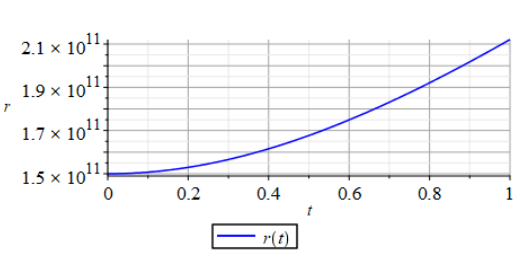
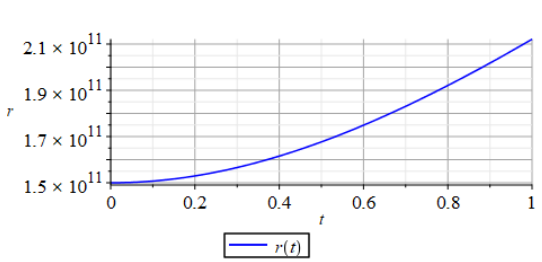
$\frac{\pi}{3}$	10^{-2}	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..10^4* 	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..10^4* 	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..10^3* 
			Plot of θ vs. r, for t = 0..10^4 	*Plot of θ vs. r, for t = 0..10^4* 	*Plot of θ vs. r, for t = 0..10^4* 

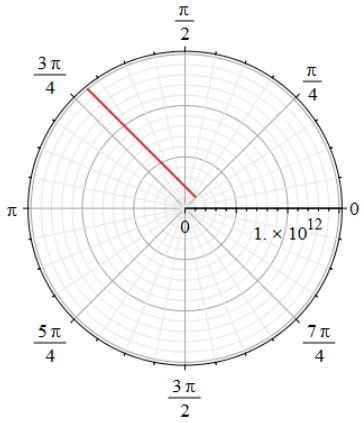
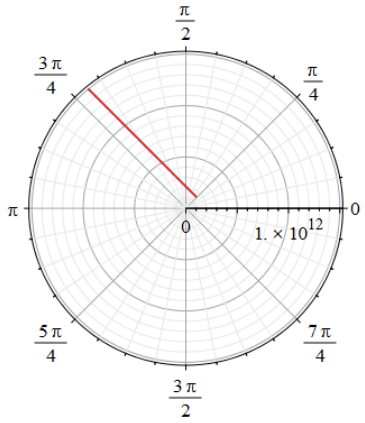
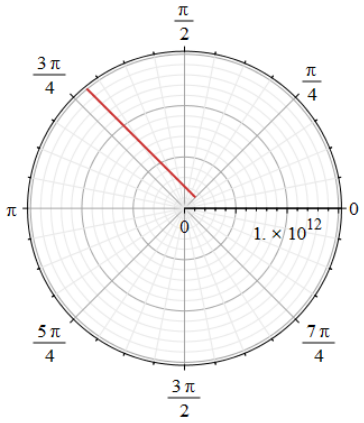
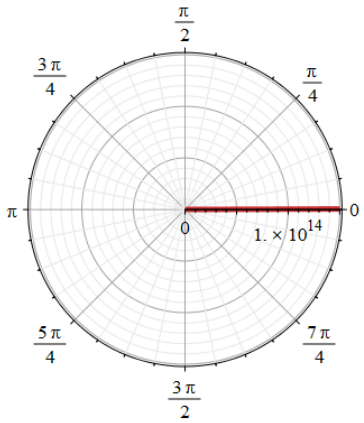
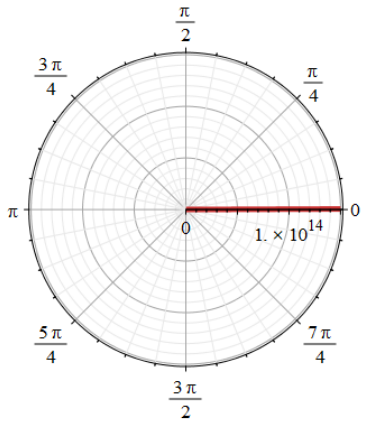
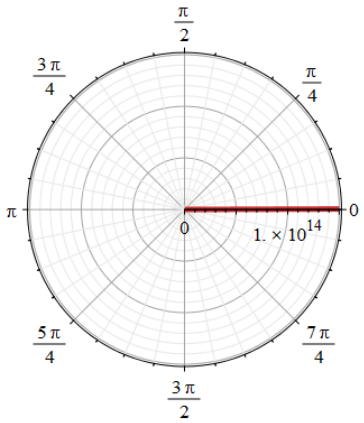
Plot of θ vs. t, for t = 0..10^4 	*Plot of θ vs. t, for t = 0..10^4* 	*Plot of θ vs. t, for t = 0..10^4* 
Plot of r vs. t, for t = 0..10^4 	*Plot of r vs. t, for t = 0..10^2* 	*Plot of r vs. t, for t = 0..10^2* 

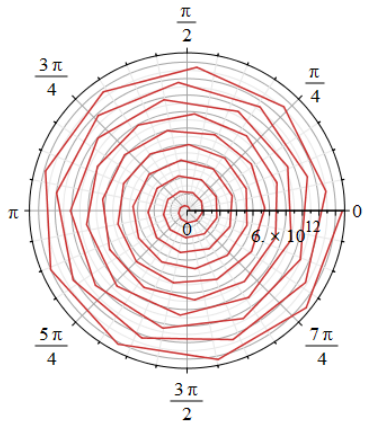
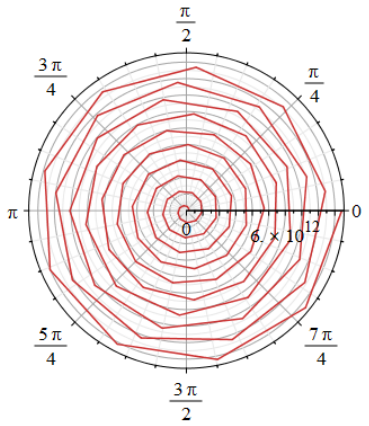
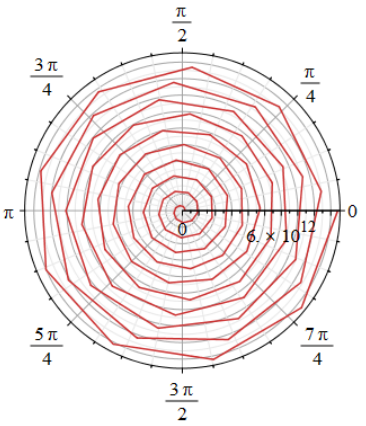
			Polar plot of r vs θ, for $[R, Theta, 0..10^3]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10^3]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^4]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^4]$ 

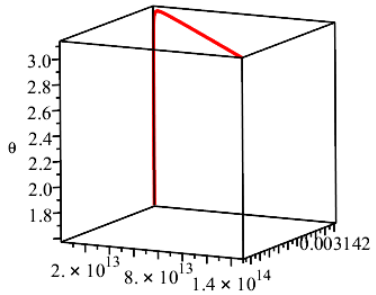
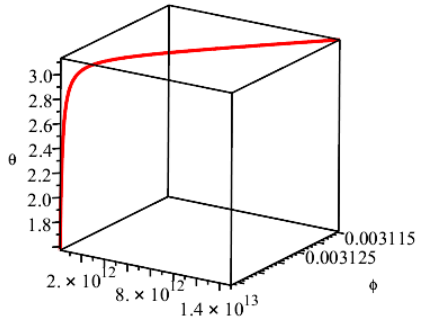
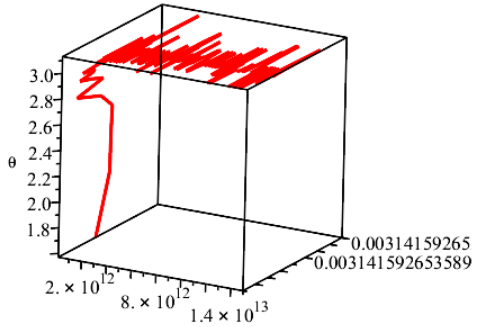
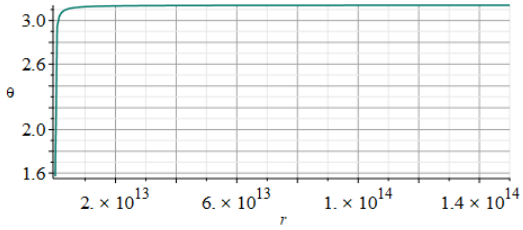
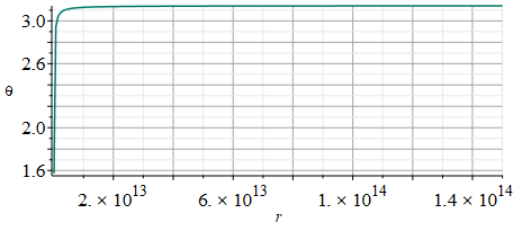
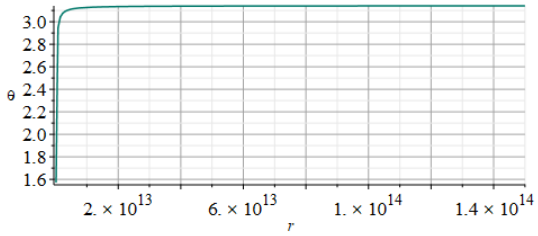
			Polar plot of r vs t, for $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion..* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10^2 \cdot 2\pi]$, *numpoints = 4* *A type of spiral that that evolves with forward and backward motion..* 

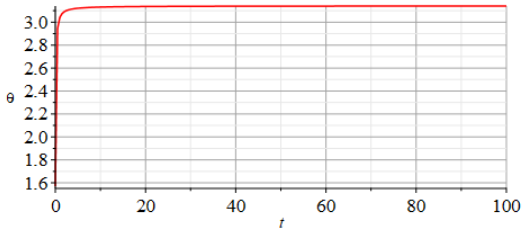
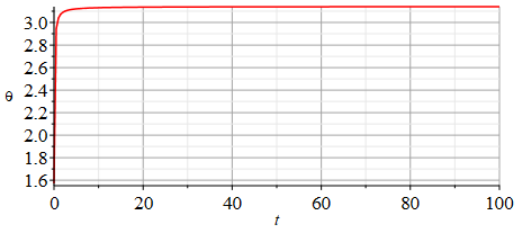
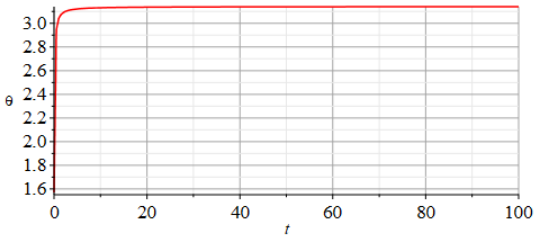
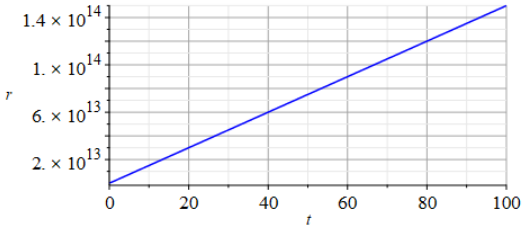
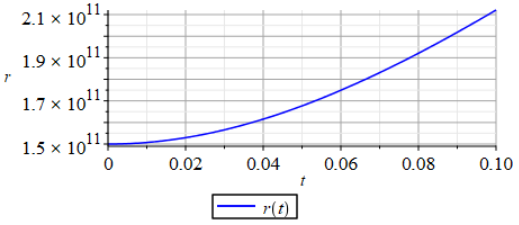
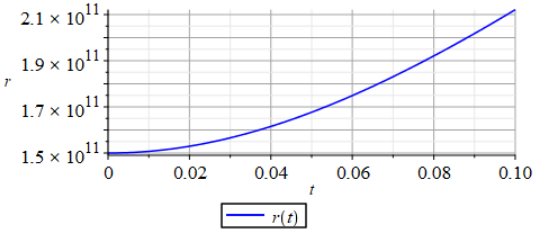
$\frac{\pi}{4}$	1	135° 135° 135°	*The mass m reaches equilibrium at $\theta = 135°$ (initial position of 45°, plus 90°).* *3D plot for $t = 0..10^3$* 	*The mass m reaches equilibrium at $\theta = 135°$ (initial position of 45°, plus 90°).* *3D plot for $t = 0..10^2$* 	*The mass m reaches equilibrium at $\theta = 135°$ (initial position of 45°, plus 90°).* *3D plot for $t = 0..10^2$* 
			Plot of θ vs. r, for $t = 0..10^3$ 	*Plot of θ vs. r, for $t = 0..10^2$* 	*Plot of θ vs. r, for $t = 0..10^2$* 

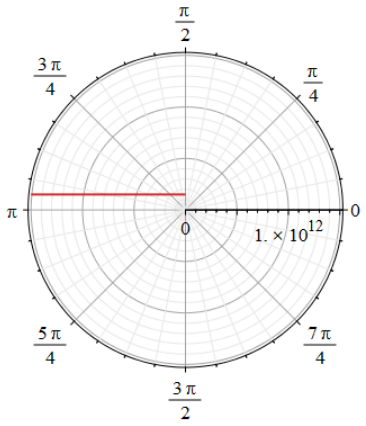
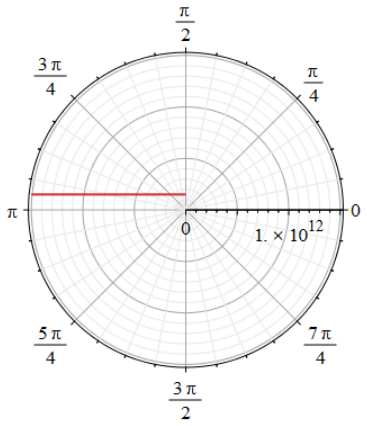
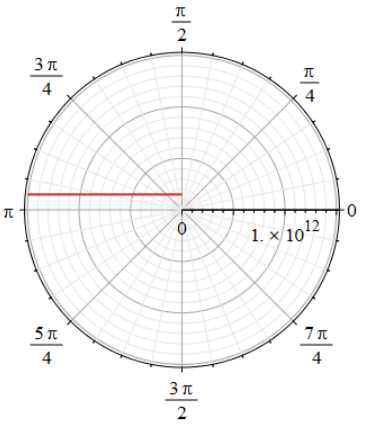
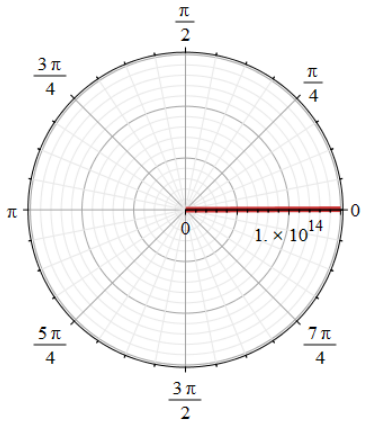
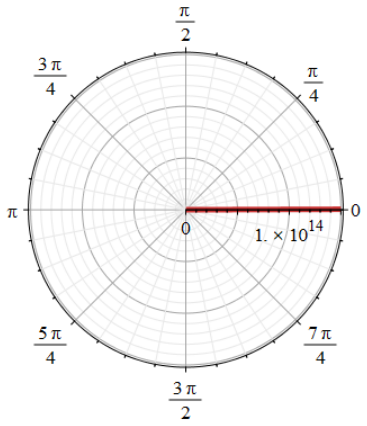
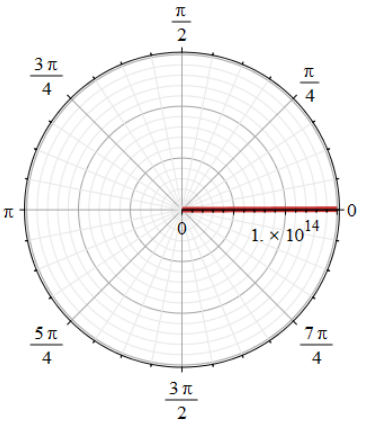
Plot of θ vs. t, for t = 0..10^3 	*Plot of θ vs. t, for t = 0..10^2* 	*Plot of θ vs. t, for t = 0..10^2* 
Plot of r vs. t, for t = 0..10^3 	*Plot of r vs. t, for t = 0..1* 	*Plot of r vs. t, for t = 0..1* 

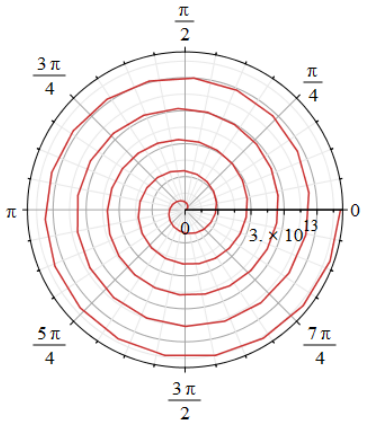
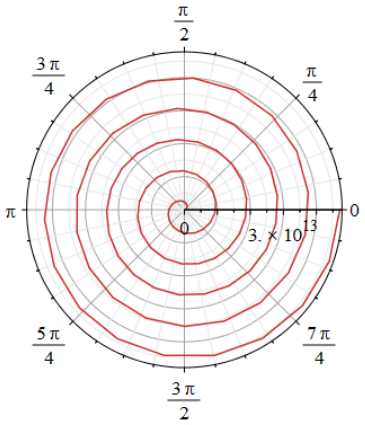
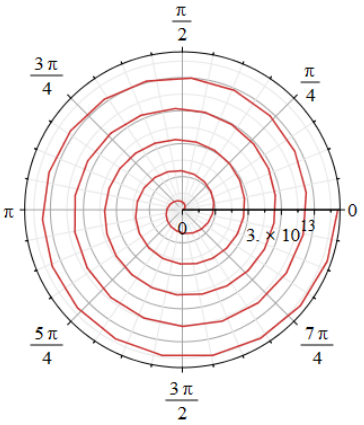
			Polar plot of r vs θ, for $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 	*Polar plot of r vs θ, for* $[R, Theta, 0..10]$ 
			Polar plot of r vs ϕ, for $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 	*Polar plot of r vs ϕ, for* $[R, Phi, 0..10^3]$ 

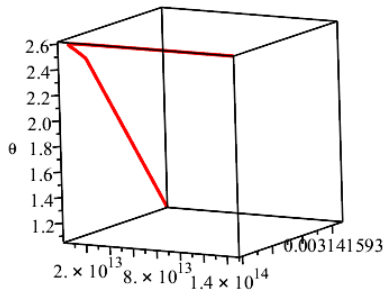
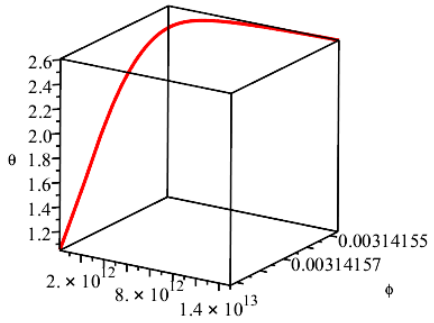
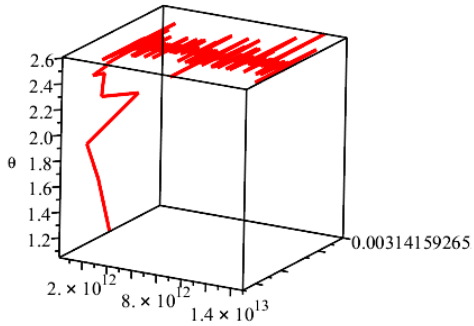
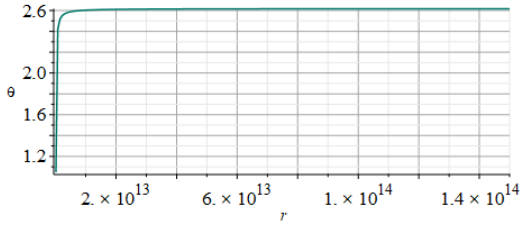
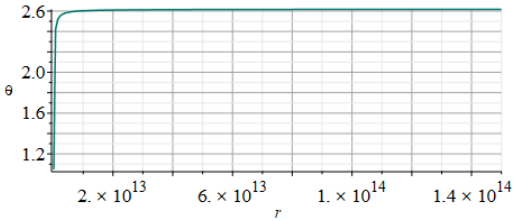
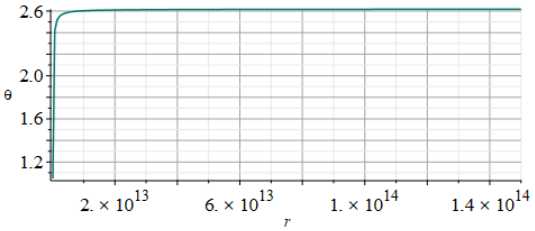
			Polar plot of r vs t, for $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..10 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 

$\frac{\pi}{2}$	10	180° 180° 180°	*The mass m reaches equilibrium at θ = 180° (initial position of 90°, plus 90°).* *3D plot for t = 0..10²:* 	*The mass m reaches equilibrium at θ = 180° (initial position of 90°, plus 90°).* *3D plot for t = 0..10:* 	*The mass m reaches equilibrium at θ = 180° (initial position of 90°, plus 90°).* *3D plot for t = 0..10:* 
			Plot of θ vs. r, for t = 0..10² 	*Plot of θ vs. r, for t = 0..10²* 	*Plot of θ vs. r, for t = 0..10²* 

Plot of θ vs. t, for t = 0..10² 	*Plot of θ vs. t, for t = 0..10²* 	*Plot of θ vs. t, for t = 0..10²* 
Plot of r vs. t, for t = 0..10² 	*Plot of r vs. t, for t = 0..0.1* 	*Plot of r vs. t, for t = 0..0.1* 

			Polar plot of r vs θ, for [*R, Theta, 0..1*] 	*Polar plot of r vs θ, for* [*R, Theta, 0..1*] 	*Polar plot of r vs θ, for* [*R, Theta, 0..1*] 
			Polar plot of r vs ϕ, for [*R, Phi, 0..10^2*] 	*Polar plot of r vs ϕ, for* [*R, Phi, 0..10^2*] 	*Polar plot of r vs ϕ, for* [*R, Phi, 0..10^2*] 

			Polar plot of r vs t, for $[R(t), t, t = 0..5 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..5 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 	*Polar plot of r vs t, for* $[R(t), t, t = 0..5 \cdot 2\pi]$, *numpoints = 4* *A regular spiral.* 

$\frac{\pi}{3}$	100	150° 150° 150°	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..10:* 	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..1:* 	*The mass m reaches equilibrium at θ = 150° (initial position of 60°, plus 90°).* *3D plot for t = 0..1:* 
			Plot of θ vs. r, for t = 0..10 	*Plot of θ vs. r, for t = 0..10* 	*Plot of θ vs. r, for t = 0..10*