Triangular A-Statistical Approximation by Double Sequences of Positive Linear Operators

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Abstract. In the present paper we introduce a new type of statistical convergence for double sequences called triangular A-statistical convergence and we show that triangular A-statistical convergence and A-statistical convergence overlap, neither contains the other. Also, we give a Korovkin-type approximation theorem using this new type of convergence. Finally we give some further developments.

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1. Introduction

One of the most important and basic results in approximation theory is the classical Bohman–Korovkin theorem (see for instance [1,2,16,19,28,31]). This theorem establishes the uniform convergence in the space $C([a,b])$ of all the continuous real functions defined on the interval $[a,b]$, for a sequence of positive linear operators (T_n) acting on $C([a,b])$, assuming the convergence only on the test functions 1, x, x^2. Also, other finite classes of test functions were considered, in both one- and multi-dimensional case. Several extensions were then considered, giving formulations of the Korovkin theorem to other functional spaces, like L^p, Orlicz or, more generally, modular spaces (see also [8,10,21,25,32,33,38,47]) and in other directions, for example fuzzy analysis (see also [3,5]) and quantitative versions (see also [41,42]). More recently, general versions of the Korovkin theorem were studied, in which a more general notion of convergence is used. In particular, the use of statistical convergence had a great impulse in recent years. Statistical convergence was first introduced independently by Fast and Steinhaus (see [27,44]), and then developed by many authors (see for instance [4,5,22,23,26,30,34–36,40]). Some Korovkin-type theorems in the setting of a general notion of convergence involving abstract filters were given in [6] and [24] (see also [7,12–14]).

For double sequences of positive linear operators, statistical convergence and some of its generalizations to convergence generated by summability matrix methods, were carried on by Demirci and Dirik in [18], [20], and the present article is a continuation of these researches. In particular, here we introduce a new kind of statistical convergence, called *triangular A-statistical convergence*, which is based on a new concept of *triangular A-density*. We show that this new kind of convergence is not comparable with the other ones, known in literature. For this new convergence we state a Korovkin type theorem, together with a study of suitable rates of convergence, in the space of the continuous functions defined on compact sets of $\mathbb{R}^2$. Our result applies to those double sequences which are neither A-statistical convergent in the usual sense nor in the Pringsheim sense, and we give some meaningful examples in this direction.

In the last section we will indicate another useful extension, which gives a "spectrum" of A-statistical convergences, depending on a suitable choice of a function $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, which determines a suitable set of matrix values $A = (a_{i,j})$.

2. Preliminaries

We begin with some definitions and notations which we will use in the sequel. Let $A = (a_{i,j})$ be a two-dimensional matrix transformation. For a double sequence $x = (x_{i,j})$ of real numbers, we put

$$(Ax)_i := \sum_{j=1}^{\infty} a_{i,j} x_{i,j},$$

if the series is convergent. We will say that A is *regular* if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for regular two-dimensional matrix transformations is known as the *Silverman–Toeplitz conditions* (see also [5,15,17,45]):

(i) $\|A\| = \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$,

(ii) $\lim_i a_{i,j} = 0$ for each $j \in \mathbb{N}$,

(iii) $\lim_i \sum_{j=1}^{\infty} a_{i,j} = 1$.

A double sequence $x = (x_{i,j})$ is said to be *convergent in the Pringsheim sense* if there exists a real number L such that for every $\varepsilon > 0$ there exists a positive integer N, with $|x_{i,j} - L| < \varepsilon$ whenever $i, j > N$. The number L is called the

Pringsheim limit of x and denoted by $P\text{-}\lim_{i,j} x_{i,j}$ (see [37]). More briefly, we will say that such an x is *P-convergent* to L. A double sequence is said to be *bounded* if there exists a positive number K such that $|x_{i,j}| < K$ for all $(i,j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that, in contrast to the case for single sequences, a convergent double sequence is not necessarily bounded. We denote the set of all convergent double sequences by c^2.

Let now $A = (a_{n,m,i,j})$ be a four-dimensional summability method. For a given double sequence $x = (x_{i,j})$, the *A-transform* of x, denoted by $Ax := ((Ax)_{n,m})$, is given by

$$(Ax)_{n,m} = \sum_{i,j=1,1}^{\infty,\infty} a_{n,m,i,j} x_{i,j},$$

provided the double series converges in the Pringsheim sense for $(n,m) \in \mathbb{N}^2$.

Recall that a four-dimensional matrix $A = (a_{n,m,i,j})$ is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison–Hamilton conditions (see also [29,39]) state that a four-dimensional matrix $A = (a_{n,m,i,j})$ is RH-regular if and only if

(i) $P\text{-}\lim_{n,m} a_{n,m,i,j} = 0$ for each i and j,

(ii) $P\text{-}\lim_{n,m} \sum_{i,j=1,1}^{\infty,\infty} a_{n,m,i,j} = 1$,

(iii) $P\text{-}\lim_{n,m} \sum_{i=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $j \in \mathbb{N}$,

(iv) $P\text{-}\lim_{n,m} \sum_{j=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $i \in \mathbb{N}$,

(v) $\sum_{i,j=1,1}^{\infty,\infty} |a_{n,m,i,j}|$ is P-convergent for every $(n,m) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that

$$\sum_{i,j>B} |a_{n,m,i,j}| < A$$

for every $(n,m) \in \mathbb{N}^2$.

Let $A = (a_{n,m,i,j})$ be a nonnegative RH-regular summability matrix. If $K \subset \mathbb{N}^2$, then the *A-density* of K is given by

$$\delta_A^2(K) := P\text{-}\lim_{n,m} \sum_{(i,j)\in K} a_{n,m,i,j},$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{i,j})$ is said to be *A-statistically convergent* to L and denoted by $st_A^2\text{-}\lim_{i,j} x_{i,j} = L$ if, for every $\varepsilon > 0$,

$$P\text{-}\lim_{n,m} \sum_{(i,j)\in K(\varepsilon)} a_{n,m,i,j} = 0,$$

where $K(\varepsilon) = \{(i,j) \in \mathbb{N}^2 : |x_{i,j} - L| \geq \varepsilon\}$ (see also [20,34]). If we take $A = C(1,1)$, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequences (see also [35,36]), where $C(1,1) = (c_{n,m,i,j})$ is the double Cesàro matrix, defined by $c_{n,m,i,j} = 1/nm$ if $1 \leq i \leq n$, $1 \leq j \leq m$, and $c_{n,m,i,j} = 0$ otherwise. We denote the set of all A-statistically convergent double sequences by st_A^2.

Our main aim in this paper is to present a new kind of statistical convergence for double sequences, called *triangular A-statistical convergence.* Later we give some examples, showing that triangular A-statistical convergence and A-statistical convergence overlap, neither contains the other, and we give a Korovkin-type approximation theorem for double sequences of positive linear operators.

3. Triangular Statistical Convergence

Let $x = (x_{i,j})$ be a double sequence, neither A-statistical convergent nor convergent in the Pringsheim sense. Then the question whether there is any kind of statistical convergence which is different from both A-statistical and Pringsheim convergence naturally arises. To answer this question we consider two-dimensional regular matrices for double sequences.

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix, $K \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $K_i = \{j \in \mathbb{N} : (i,j) \in K, j \leq i\}$. Let $|K_i|$ be the cardinality of K_i. We define the *triangular A-density* of K by

$$\delta_A^T(K) := \lim_i \sum_{j\in K_i} a_{i,j},$$

provided that the limit on the right-hand side exists in $\mathbb{R}$.

In a manner similar to the natural density, we can give some properties for the triangular A-density:

i) if $K_1 \subset K_2$, then $\delta_A^T(K_1) \leq \delta_A^T(K_2)$,
ii) if K has triangular A-density, then $\delta_A^T(\mathbb{N}^2 \setminus K) = 1 - \delta_A^T(K)$.

Definition 1. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. The double sequence $x = (x_{i,j})$ is *triangular A-statistically convergent* to L provided that for every $\varepsilon > 0$

$$\lim_i \sum_{j\in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} : j \leq i, \ |x_{i,j} - L| \geq \varepsilon\}$ and this is denoted by st_A^T-$\lim_i x_{i,j} = L$.

We should note that if we take $A = C_1$, the Cesàro matrix, then the triangular C_1-statistical convergence coincides with the triangular statistical convergence.

The *triangular density* $\delta^T(K)$ is given by

$$\delta^T(K) = \lim_i \frac{1}{i}|K_i|,$$

or equivalently by

$$\delta^T(K) = \lim_i (C_1\chi_{K_i}(j))_i = \lim_i \sum_{j=1}^{\infty} c_{i,j}\chi_{K_i}(j),$$

if it exists (here and in the sequel, χ denotes the characteristic function of the set involved). The number sequence $x = (x_{i,j})$ is *triangular statistically convergent* to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) := \{(i,j) \in \mathbb{N}^2 : j \leq i,\ |x_{i,j} - L| \geq \varepsilon\}$ has triangular density zero; in that case we write st^T-$\lim_i x_{i,j} = L$.

We denote the set of all triangular A-statistically convergent sequences by st^T_A. Our preliminary result considers the problem of comparing triangular statistical convergence with statistical convergence.

Example 1. Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2 \\ \frac{k}{3(k+1)}, & i = 2k+1,\ j = 2k-1 \\ \frac{k}{2(k+1)}, & i = 2k,\ j = 2(k+1) \\ 0, & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N}.$$

For every $\varepsilon \in \left(0, \frac{1}{6}\right]$, we have

$$\frac{1}{i}|\{j \in \mathbb{N} : j \leq i,\ |x_{i,j}| \geq \varepsilon\}| = \begin{cases} 1, & i = 1 \\ \frac{2}{(2k+1)^2}, & i = (2k+1)^2 \\ \frac{1}{(2k)^2}, & i = (2k)^2 \\ \frac{1}{2k+1}, & i = 2k+1 \text{ and } i \text{ is not a square}, \\ 0, & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N}.$$

Clearly, we get

$$\lim_i \frac{1}{i}|\{j \in \mathbb{N} : j \leq i,\ |x_{i,j}| \geq \varepsilon\}| = 0.$$

So, we obtain $st^T_{C_1}$-$\lim_i x_{i,j} = 0$. Nevertheless, $x = (x_{i,j})$ is not Pringsheim and $C(1,1)$-statistically convergent.

Example 1 does not make possible to characterize the concept of triangular A-statistical convergence as given below:

A double sequence $x = (x_{i,j})$ is triangular A-statistically convergent to L if and only if there exists a set $K \subset \mathbb{N}^2$ such that the triangular A-density of K is 1 and

$$P\text{-}\lim_{i,j\to\infty \text{ and } (i,j)\in K} x_{i,j} = L \tag{1}$$

(see [36]). By (1) we mean that for every $\varepsilon > 0$ there exists an integer N such that

$$|x_{i,j} - L| \le \varepsilon \quad \text{if } i,j \ge N \text{ and } (i,j) \in K.$$

Example 2. Take $A = C(1,1)$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} \sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares} \\ \frac{1}{ij}, & \text{otherwise.} \end{cases}$$

It is easy to see that $st^2_{C(1,1)}\text{-}\lim_{i,j} x_{i,j} = 0$. Nevertheless, x is not Pringsheim and triangular statistically convergent.

Example 3. Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2 \\ 0, & \text{otherwise,} \end{cases} \quad k \in \mathbb{N}.$$

Similarly, $st^T_{C_1}\text{-}\lim_{i} x_{i,j} = 0$ and $st^2_{C(1,1)}\text{-}\lim_{i,j} x_{i,j} = 0$.

Example 4. Take $A = C_1$ and the double sequence $x = (x_{i,j})$ given by

$$x_{i,j} = \begin{cases} 1, & i = j = k^2 \\ \frac{k}{2k+1}, & i = 2k+1,\ j = 2k-1 \\ \frac{k}{4k+2}, & i = 2k,\ j = 2(k+1) \\ k, & i = k^2,\ j = k^2+1 \\ 0, & \text{otherwise,} \end{cases} \quad k \in \mathbb{N}.$$

For every $\varepsilon \in \left(0, \frac{1}{3}\right]$, we get

$$\frac{1}{i}|\{j \in \mathbb{N} : j \le i,\ |x_{i,j}| \ge \varepsilon\}| = \begin{cases} 1, & i = 1 \\ \frac{2}{(2k+1)^2}, & i = (2k+1)^2 \\ \frac{1}{(2k)^2}, & i = (2k)^2 \\ \frac{1}{2k+1}, & i = 2k+1 \text{ and } i \text{ is not a square} \\ 0, & \text{otherwise,} \end{cases}$$

$k \in \mathbb{N}$. Then we have

$$\lim_{i} \frac{1}{i}|\{j \in \mathbb{N} : j \le i,\ |x_{i,j}| \ge \varepsilon\}| = 0.$$

Thus we get $st^T_{C_1}\text{-}\lim_{i} x_{i,j} = 0$. So, neither $x = (x_{i,j})$ is Pringsheim and $C(1,1)$-statistically convergent nor bounded.

Remark 1. (i) Triangular statistical convergence and statistical convergence are not comparable; i.e., $st^T_A \not\subseteq st^2_A$ and $st^2_A \not\subseteq st^T_A$.

(ii) A P-convergent double sequence is A-statistically convergent and triangular A-statistically convergent to the same value, but the inverse implications are not true, i.e., $st_A^2 \nsubseteq c^2$ and $st_A^T \nsubseteq c^2$.

4. A Korovkin-Type Approximation Theorem

By $C(D)$ we denote the space of all continuous real valued functions on a fixed compact subset D of $\mathbb{R}^2$. This space is equipped with the supremum norm

$$\|f\|_{C(D)} = \max_{(x,y)\in D} |f(x,y)| \ (f \in C(D)).$$

For a function $f \in C(D)$ and $\delta > 0$, let us define the usual modulus of continuity ω by

$$\omega(f,\delta) := \max_{(x,y),(u,v)\in D, |(x,y)-(u,v)|\leq\delta} |f(x,y) - f(u,v)|,$$

where $|(x,y) - (u,v)| = \sqrt{(x-u)^2 + (y-v)^2}$. For the basic properties of the modulus of continuity see e.g. [19].

Let L be a linear operator from $C(D)$ into $C(D)$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $Lf \geq 0$. Also, we denote the value of Lf a point $(x,y) \in D$ by $L(f;x,y)$.

We now recall the following

Theorem 1. (see [46]) *Let $(L_{i,j})$ be a sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$P\text{-}\lim_{i,j} \|L_{i,j}f - f\|_{C(D)} = 0$$

if and only if

$$P\text{-}\lim_{i,j} \|L_{i,j}f_r - f_r\|_{C(D)} = 0 \, (r = 0,1,2,3),$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Theorem 2. (see [20]) *Let $A = (a_{n,m,i,j})$ be a nonnegative RH-regular summability matrix method. Let $(L_{i,j})$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f \in C(D)$,*

$$st_A^2\text{-}\lim_{i,j} \|L_{i,j}f - f\|_{C(D)} = 0 \tag{2}$$

if and only if

$$st_A^2\text{-}\lim_{i,j} \|L_{i,j}f_r - f_r\|_{C(D)} = 0 \, (r = 0,1,2,3). \tag{3}$$

We now turn to our main theorem.

Theorem 3. *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix and $(L_{i,j})$ be a double sequence of positive linear operators from $C(D)$ into $C(D)$. Then for every $f \in C(D)$ we have*

$$st_A^T\text{-}\lim_i \|L_{i,j}(f) - f\|_{C(D)} = 0 \tag{4}$$

if and only if

$$st_A^T\text{-}\lim_i \|L_{i,j}(f_r) - f_r\|_{C(D)} = 0 \quad \textit{for every } r = 0, 1, 2, 3. \tag{5}$$

Proof. Since each $f_r \in C(D)$ $(r = 0, 1, 2, 3)$, the implication (4)$\Rightarrow$(5) is obvious. Suppose now that (5) holds. By continuity of f on the compact set D, we can write

$$|f(x, y)| \leq M$$

where $M := \|f\|_{C(D)}$. Also, since f is continuous on D, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ for all $(u, v) \in D$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, we get

$$|f(u, v) - f(x, y)| < \varepsilon + \frac{2M}{\delta^2}\{(u - x)^2 + (v - y)^2\}. \tag{6}$$

Since $L_{i,j}$ is linear and positive, we obtain

$$\begin{aligned}
|L_{i,j}(f; x, y) - f(x, y)| &= |L_{i,j}(f(u, v) - f(x, y); x, y) \\
&\quad - f(x, y)(L_{i,j}(f_0; x, y) - f_0(x, y))| \\
&\leq \left| L_{i,j}\left(\varepsilon + \frac{2M}{\delta^2}\{(u - x)^2 + (v - y)^2\}; x, y\right)\right| \\
&\quad + M|L_{i,j}(f_0; x, y) - f_0(x, y)| \\
&\leq \left(\varepsilon + M + \frac{2M}{\delta^2}(A^2 + B^2)\right)|L_{i,j}(f_0; x, y) - f_0(x, y)| \\
&\quad + \frac{4M}{\delta^2}A|L_{i,j}(f_1; x, y) - f_1(x, y)| \\
&\quad + \frac{4M}{\delta^2}B|L_{i,j}(f_2; x, y) - f_2(x, y)| \\
&\quad + \frac{2M}{\delta^2}|L_{i,j}(f_3; x, y) - f_3(x, y)| + \varepsilon
\end{aligned}$$

where $A := \max|x|$, $B := \max|y|$. Taking the supremum over $(x, y) \in D$ we get

$$\begin{aligned}
\|L_{i,j}f - f\|_{C(D)} \leq S\{&\|L_{i,j}(f_0) - f_0\|_{C(D)} + \|L_{i,j}(f_1) - f_1\|_{C(D)} \\
&+ \|L_{i,j}(f_2) - f_2\|_{C(D)} + \|L_{i,j}(f_3) - f_3\|_{C(D)}\} + \varepsilon,
\end{aligned} \tag{7}$$

where $S=\max\left\{\varepsilon+M+\frac{2M}{\delta^2}\left(A^2+B^2\right),\frac{4M}{\delta^2}A,\frac{4M}{\delta^2}B,\frac{2M}{\delta^2}\right\}$. Now, for a given $\varepsilon'>0$, choose $\varepsilon>0$ such that $\varepsilon<\varepsilon'$. Then, setting

$$D_i:=\left\{j\in\mathbb{N}:j\leq i,\ \|L_{i,j}(f)-f\|_{C(D)}\geq\varepsilon'\right\},$$

$$D_i^r:=\left\{j\in\mathbb{N}:j\leq i,\ \|L_{i,j}(f_r)-f_r\|_{C(D)}\geq\frac{\varepsilon'-\varepsilon}{4S}\right\},\qquad r=0,1,2,3.$$

Then it is easy to see that

$$D_i\subseteq\bigcup_{r=0}^{3}D_i^r$$

which gives, for all $i\in\mathbb{N}$,

$$\sum_{j\in D_i}a_{i,j}\leq\sum_{r=0}^{3}\sum_{j\in D_i^r}a_{i,j}.$$

Letting $i\to\infty$ and using (5), we obtain (4). The proof is complete. □

We now present two examples of sequence of positive linear operators. The first one shows that Theorems 2 and 1 do not work, but our approximation Theorem 3 works. The second one gives that Theorem 3 does not work but Theorem 2 works.

Example 5. Let us consider the following Bernstein operators (see also [43]) given by

$$B_{i,j}(f;x,y)=\sum_{k=0}^{i}\sum_{t=0}^{j}f\left(\frac{k}{i},\frac{t}{j}\right)\binom{i}{k}\binom{j}{t}x^k\left(1-x\right)^{i-k}y^t\left(1-y\right)^{j-t}\qquad(8)$$

where $(x,y)\in D=[0,1]\times[0,1]$; $f\in C(D)$. Also, observe that

$$\begin{aligned}B_{i,j}\left(f_0;x,y\right)&=f_0\left(x,y\right),\\ B_{i,j}\left(f_1;x,y\right)&=f_1\left(x,y\right),\\ B_{i,j}\left(f_2;x,y\right)&=f_2\left(x,y\right),\\ B_{i,j}\left(f_3;x,y\right)&=f_3\left(x,y\right)+\frac{x-x^2}{i}+\frac{y-y^2}{j},\end{aligned}$$

where $f_0(x,y)=1$, $f_1(x,y)=x$, $f_2(x,y)=y$ and $f_3(x,y)=x^2+y^2$. Now we take $A=C_1$ and define a double sequence $(\gamma_{i,j})$ by

$$\gamma_{i,j}=\begin{cases}1, & i=j=k^2\\ \frac{k}{3(k+1)}, & i=2k+1,\ j=2k-1\\ \frac{k}{2(k+1)}, & i=2k,\ j=2(k+1)\\ 0, & \text{otherwise,}\end{cases}\qquad k\in\mathbb{N}.\qquad(9)$$

It is clear that

$$st_{C_1}^{T}\text{-}\lim_i\gamma_{i,j}=0.\qquad(10)$$

Now, using (8) and (9), let us define the following positive linear operators on $C(D)$ as follows:

$$L_{i,j}(f;x,y) = (1+\gamma_{i,j})\, B_{i,j}\,(f;x,y)\,. \tag{11}$$

So, by Theorem 3 and (10), we see that

$$st_{C_1}^{T}\text{-}\lim_{i} \|L_{i,j}\,(f) - f\|_{C(D)} = 0.$$

Also, since $(\gamma_{i,j})$ is not P-convergent and statistical convergent, we can say that the Korovkin theorem in the Pringsheim and statistical sense does not work for our operators defined by (11).

Example 6. Let us consider the Bernstein operators as in Example 5, and define a double sequence $(\beta_{i,j})$ by

$$\beta_{i,j} = \begin{cases} \sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares,} \\ \frac{1}{ij} & \text{otherwise.} \end{cases} \tag{12}$$

It is clear that

$$st^2_{C(1,1)}\text{-}\lim_{i,j} \beta_{i,j} = 0. \tag{13}$$

Now, by using (8) and (12), we define the following positive linear operators on $C(D)$ as follows:

$$L_{i,j}(f;x,y) = (1+\beta_{i,j})\, B_{i,j}\,(f;x,y)\,, \quad i,j \in \mathbb{N}. \tag{14}$$

So, by Theorem 2 and (13), we see that

$$st^2_{C(1,1)}\text{-}\lim_{i,j} \|L_{i,j}\,(f) - f\|_{C(D)} = 0.$$

Also, since $(\beta_{i,j})$ is not P-convergent and triangular statistical convergent, we can say that the Korovkin theorem in the Pringsheim and triangular statistical sense does not work for operators defined by (14).

Example 7. The next example deals with bivariate moment-type operators (see also [9,11]).

For each $i,\, j \in \mathbb{N}$, set $A_{i,j} = \left[\frac{1}{i}, 1\right] \times \left[\frac{1}{j}, 1\right]$,

$$c_{i,j} = \int_0^1 \int_0^1 t_1\, t_2 (t_1^2 + t_2^2)^{i+j} \chi_{A_{i,j}}(t_1,t_2)\, dt_1\, dt_2, \quad d_{i,j} = \frac{1}{c_{i,j}}. \tag{15}$$

For every $i,\, j \in \mathbb{N}$, $t_1,\, t_2 \in [0,1]$, define $K_{i,j}(t_1,t_2) = d_{i,j} t_1\, t_2 (t_1^2 + t_2^2)^{i+j}$ $\chi_{A_{i,j}}(t_1,t_2)$. Moreover, let $\Gamma = \{(i,j) \in \mathbb{N}^2\colon \gamma_{i,j} = 0\}$ and C_1 be the Cesàro matrix. Finally, let us consider the double sequences of operators defined by

$$M_{i,j}(f;x_1,x_2) = \int_0^1 \int_0^1 K_{i,j}(t_1,t_2) f(t_1 x_1, t_2 x_2)\, dt_1\, dt_2, \quad f \in C([0,1]^2),$$

$$L_{i,j}(f;x_1,x_2) = (1+\gamma_{i,j})\, M_{i,j}\,(f;x_1,x_2)\,,$$

where $(\gamma_{i,j})$ is as in (9), $i,\, j \in \mathbb{N}$. We will prove that the $L_{i,j}$'s satisfy our approximation Theorem 3, though, by construction, they do not fulfil Theorems 1 and 2. We begin with the following

Lemma 1. *For every $i,\, j \geq 2$ we get*

$$d_{i,j} \leq \frac{(i+j+1)^2}{2^{i+j-3}}, \tag{16}$$

where $d_{i,j}$ is as in (15).

Proof. For every $i,\, j \geq 2$ we get

$$\begin{aligned}
c_{i,j} &= \int_{1/i}^{1} t_1\, dt_1 \int_{1/j}^{1} t_2 (t_1^2 + t_2^2)^{i+j}\, dt_2 \\
&= \int_{1/i}^{1} t_1\, dt_1 \int_{1/j}^{1} t_2 \sum_{k=0}^{i+j} \binom{i+j}{k} t_1^{2(i+j-k)} t_2^{2k}\, dt_2 \\
&= \sum_{k=0}^{i+j} \binom{i+j}{k} \int_{1/i}^{1} t_1^{2(i+j-k)+1}\, dt_1 \int_{1/j}^{1} t_2^{2k+1}\, dt_2 \\
&= \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{1}{(2k+2)(2(i+j-k)+2)} \left(1 - \frac{1}{i^{2(i+j-k)+2}}\right)\left(1 - \frac{1}{j^{2k+2}}\right) \\
&\geq \frac{9}{16}\, \frac{2^{i+j}}{4\,(i+j+1)^2} \geq \frac{1}{8}\, \frac{2^{i+j}}{(i+j+1)^2} = \frac{2^{i+j-3}}{(i+j+1)^2}.
\end{aligned}$$

Therefore we get

$$d_{i,j} \leq \frac{(i+j+1)^2}{2^{i+j-3}} \quad \text{for each} \;\; i,\, j \geq 2,$$

that is the assertion. □

Lemma 2. *Under the same hypotheses and notations above, we get*

$$st^T_{C_1}\text{-}\lim_i \iint_{A_{i,j} \setminus U_\delta} K_{i,j}(t_1, t_2)\, dt_1\, dt_2 = 0 \tag{17}$$

for every $\delta \in (0,1)$, where U_δ is the ball centered in $(1,1)$ with radius δ.

Proof. Fix arbitrarily $\delta \in (0,1)$. First of all we observe that, if $(t_1,t_2) \in U_\delta$, then $\sqrt{t_1^2 + t_2^2} \leq a_\delta$, where $a_\delta = \sqrt{1 + (1-\delta)^2}$. Note that $0 < a_\delta < \sqrt{2}$. Taking into account (16), for every $(i,j) \in \Gamma$ with $i,\, j \geq 2$ we get:

$$\begin{aligned}
\iint_{A_{i,j} \setminus U_\delta} K_{i,j}(t_1,t_2)\, dt_1\, dt_2 &\leq d_{i,j} \iint_{A_{i,j} \setminus U_\delta} (t_1^2 + t_2^2)^{i+j}\, dt_1\, dt_2 \\
&\leq d_{i,j} \int_0^{\pi/2} d\theta \int_0^{a_\delta} \rho^{2(i+j)+1}\, d\rho \leq 8\pi (i+j+1) \frac{((a_\delta)^2)^{i+j+1}}{2^{i+j+1}}.
\end{aligned}$$

From this the assertion follows, since $0 < a_\delta < \sqrt{2}$. □

We now show that the hypotheses of Theorem 3 are satisfied. First of all note that, by construction, we get $L_{i,j}(f_0; x_1, x_2) = f_0(x_1, x_2) = 1$ for each $(i,j) \in \Gamma$ and x_1, $x_2 \in [0,1]$.

Fix now $\varepsilon \in \left(0, \frac{1}{4}\right)$ and let $U_\varepsilon \subset \mathbb{R}^2$ be the ball of center $(1,1)$ and radius ε. For $(i,j) \in \Gamma$, x_1, $x_2 \in [0,1]$ and $r = 1,2$ we get:

$$|L_{i,j}(f_r; x_1, x_2) - f_r(x_1, x_2)| \leq x_r \int_0^1 \int_0^1 K_{i,j}(t_1, t_2)(1 - t_r)\, dt_1\, dt_2$$

$$\leq \int_0^1 \int_0^1 K_{i,j}(t_1, t_2)(1 - t_r)\, dt_1\, dt_2 = \int\int_{A_{i,j} \cap U_\varepsilon} K_{i,j}(t_1, t_2)(1 - t_r)\, dt_1\, dt_2$$

$$+ \int\int_{A_{i,j} \setminus U_\varepsilon} K_{i,j}(t_1, t_2)(1 - t_r)\, dt_1\, dt_2$$

$$\leq 2 \int_{A_{i,j} \setminus U_\varepsilon} K_{i,j}(t_1, t_2)\, dt_1\, dt_2 + \frac{\pi}{2}\varepsilon^2 \leq 2 \int_{A_{i,j} \setminus U_\varepsilon} K_{i,j}(t_1, t_2)\, dt_1\, dt_2 + \frac{\varepsilon}{2}.$$

By (17) and arbitrariness of ε, it follows that $st^{T}_{C_1}\text{-}\lim_i \|L_{i,j}(f_r) - f_r\|_{C([0,1]^2)} = 0$, $r = 1, 2$.

Arguing analogously as above, considering for $r = 1,2$ the estimate

$$|L_{i,j}(f_r^2; x_1, x_2) - f_r^2(x_1, x_2)| \leq x_r^2 \int_0^1 \int_0^1 K_{i,j}(t_1, t_2)(1 - t_r^2)\, dt_1\, dt_2,$$

it is possible to check that $st^{T}_{C_1}\text{-}\lim_i \|L_{i,j}(f_r^2) - f_r^2\|_{C([0,1]^2)} = 0$, $r = 1, 2$, and so, by linearity, we get $st^{T}_{C_1}\text{-}\lim_i \|L_{i,j}(f_3) - f_3\|_{C([0,1]^2)} = 0$. So, it is possible to apply our Korovkin Theorem 3 and to deduce that $st^{T}_{C_1}\text{-}\lim_i \|L_{i,j}(f) - f\|_{C([0,1]^2)} = 0$ for every $f \in C([0,1]^2)$.

Example 8. We now consider a direct extension to the bivariate case of the classical one-dimensional moment kernel (see also [9,11]).

For every i, $j \in \mathbb{N}$ and t_1, $t_2 \in [0,1]$, let $K_{i,j}(t_1, t_2) = (i+1)(j+1)t_1^i t_2^j$, and for $f \in C([0,1]^2)$ and x_1, $x_2 \in [0,1]$ set

$$M^*_{i,j}(f; x_1, x_2) = \int_0^1 \int_0^1 K_{i,j}(t_1, t_2) f(t_1 x_1, t_2 x_2)\, dt_1\, dt_2,$$

and $L_{i,j}(f; x_1, x_2) = (1 + \gamma_{i,j}) M^*_{i,j}(f; x_1, x_2)$, where $(\gamma_{i,j})$ is as in (9). First of all, observe that

$$\int_0^1 \int_0^1 K_{i,j}(t_1, t_2) dt_1\, dt_2 = (i+1) \left(\int_0^1 t_1^i\, dt_1\right) (j+1) \left(\int_0^1 t_2^j\, dt_2\right) = 1,$$

and thus $L_{i,j}(f_0;x_1,x_2) = f_0(x_1,x_2) = 1$ for each $(i,j) \in \Gamma$ and $x_1,\ x_2 \in [0,1]$. Moreover we have

$$
\begin{aligned}
|L_{i,j}(f_1;x_1,x_2) - f_1(x_1,x_2)| &\leq \int_0^1 \int_0^1 K_{i,j}(t_1,t_2)(1-t_1)\,dt_1\,dt_2 \\
&= (i+1)(j+1)\int_0^1 t_1^i(1-t_1)\,dt_1 \int_0^1 t_2^j\,dt_2 \\
&= (i+1)\int_0^1 t_1^i\,dt_1 - (i+1)\int_0^1 t_1^{i+1}\,dt_1 = 1 - \frac{i+1}{i+2} = \frac{1}{i+2}
\end{aligned}
$$

for $(i,j) \in \Gamma$ and $x_1,\ x_2 \in [0,1]$. Thus, $st_{C_1}^T\text{-}\lim_i \|L_{i,j}(f_1) - f_1\|_{C([0,1]^2)} = 0$. Analogously it is possible to check that

$$
\begin{aligned}
|L_{i,j}(f_2;x_1,x_2) - f_2(x_1,x_2)| &\leq \frac{1}{j+2}, \\
|L_{i,j}(f_1^2;x_1,x_2) - f_1^2(x_1,x_2)| &\leq \frac{2}{i+3}, \\
|L_{i,j}(f_2^2;x_1,x_2) - f_2^2(x_1,x_2)| &\leq \frac{2}{j+3}
\end{aligned}
$$

whenever $(i,j) \in \Gamma$ and $x_1,\ x_2 \in [0,1]$. Arguing analogously as in Example 7, we obtain $st_{C_1}^T\text{-}\lim_i \|L_{i,j}(f_r) - f_r\|_{C([0,1]^2)} = 0$, $r = 2,3$. Thus all the hypotheses of Theorem 3 are fulfilled, and hence $st_{C_1}^T\text{-}\lim_i \|L_{i,j}(f) - f\|_{C([0,1]^2)} = 0$ for every $f \in C([0,1]^2)$. However, by construction, it is readily seen that Theorems 1 and 2 are not satisfied.

5. Rates of Triangular A-Statistical Convergence

In this section, using the same techniques as in [18], we study the rates of convergence of a sequence of positive linear operators and for summability matrices, we present four different ways to compute the corresponding rates of triangular A-statistical convergence in Theorem 3.

Definition 2. Let $A = (a_{i,j})$ be a non-negative regular summability matrix and (α_i) be a positive non-increasing sequence. A double sequence $x = (x_{i,j})$ is *triangular A-statistically convergent to a number L with the rate of $o(\alpha_i)$* if, for every $\varepsilon > 0$,

$$
\lim_i \frac{1}{\alpha_i} \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,
$$

where

$$
K_i(\varepsilon) := \{j \in \mathbb{N} : j \leq i,\ |x_{i,j} - L| \geq \varepsilon\}.
$$

In this case, we write

$$
x_{i,j} - L = st_A^T\text{-}o(\alpha_i) \qquad \text{as } i \to \infty.
$$

Definition 3. Let $A = (a_{i,j})$ and (α_i) be as in Definition 2. Then, a double sequence $x = (x_{i,j})$ is *triangular A-statistically bounded with the rate of* $O(\alpha_i)$ if there is $M > 0$ with

$$\lim_i \frac{1}{\alpha_i} \sum_{j \in L_i(M)} a_{i,j} = 0,$$

where

$$L_i(M) := \{ j \in \mathbb{N} : j \leq i, \ |x_{i,j}| \geq M \ \}.$$

In this case, we write

$$x_{i,j} = st_A^T\text{-}O(\alpha_i) \qquad \text{as } i \to \infty.$$

We see from the above definitions that the rate directly effects the entries of the summability method rather than the terms of the double sequence $x = (x_{i,j})$. For example, when one takes the identity matrix I, if $a_{i,i} = o(\alpha_i)$ then $x_{i,j} - L = st_A^T\text{-}o(\alpha_i)$ for any convergent sequence $(x_{i,j} - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergence as follows.

Definition 4. Let $A = (a_{i,j})$ and (α_i) be as in Definition 2. Then, a double sequence $x = (x_{i,j})$ is *triangular A-statistically convergent to a number L with the rate of* $o_k(\alpha_i)$ if, there is $M > 0$ with

$$\lim_i \sum_{j \in M_i(\varepsilon)} a_{i,j} = 0,$$

where

$$M_i(\varepsilon) := \{ j \in \mathbb{N} : j \leq i, \ |x_{i,j} - L| \geq \varepsilon \ \alpha_j \}.$$

In this case, we write

$$x_{i,j} - L = st_A^T\text{-}o_k(\alpha_i) \qquad \text{as } i \to \infty.$$

Definition 5. Let $A = (a_{i,j})$ and (α_i) be as in Definition 2. Then, a double sequence $x = (x_{i,j})$ is *triangular A-statistically bounded with the rate of* $O_k(\alpha_i)$ if there is $M > 0$ with

$$\lim_i \sum_{j \in N_i(M)} a_{i,j} = 0,$$

where

$$N_i(M) := \{ j \in \mathbb{N} : j \leq i, \ |x_{i,j}| \geq M \ \alpha_j \}.$$

In this case, we write

$$x_{i,j} = st_A^T\text{-}O_k(\alpha_i) \qquad \text{as } i \to \infty.$$

Using these definitions we obtain the following auxiliary results.

Lemma 3. *Let $(x_{i,j})$ and $(y_{i,j})$ be double sequences. Assume that $A = (a_{i,j})$ is a non-negative regular summability matrix, and let (α_i) and (β_i) be positive non-increasing sequences. If $x_{i,j} - L_1 = st_A^T\text{-}o(\alpha_i)$ and $y_{i,j} - L_2 = st_A^T\text{-}o(\beta_i)$, then we have*

(i) $(x_{i,j} - L_1) \mp (y_{i,j} - L_2) = st_A^T\text{-}o(\gamma_i)$ *as* $i \to \infty$ *, where* $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$,

(ii) $\lambda(x_{i,j} - L_1) = st_A^T\text{-}o(\alpha_i)$ *as* $i \to \infty$ *for any real number* λ.

Furthermore, similar results hold with the symbol o replaced by O.

Proof. (i) Assume that $x_{i,j} - L_1 = st_A^T\text{-}o(\alpha_i)$ and $y_{i,j} - L_2 = st_A^T\text{-}o(\beta_i)$. Also, for $\varepsilon > 0$, define

$$\begin{aligned}
K_i &:= \{\, j \in \mathbb{N} : j \le i, \ \ |(x_{i,j} - L_1) \mp (y_{i,j} - L_2)| \ge \varepsilon\},\\
K_i^1 &:= \left\{ j \in \mathbb{N} : j \le i, \ \ |x_{i,j} - L_1| \ge \frac{\varepsilon}{2}\right\},\\
K_i^2 &:= \left\{ j \in \mathbb{N} : j \le i, \ \ |y_{i,j} - L_2| \ge \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Then observe that

$$K_i \subset K_i^1 \cup K_i^2,$$

which gives, for all $i \in \mathbb{N}$,

$$\sum_{j \in K_i} a_{i,j} \le \sum_{j \in K_i^1} a_{i,j} + \sum_{j \in K_i^2} a_{i,j}. \tag{18}$$

Since $\gamma_i = \max\{\alpha_i, \beta_i\}$, by (18), we get

$$\frac{1}{\gamma_i} \sum_{j \in K_i} a_{i,j} \le \frac{1}{\alpha_i} \sum_{j \in K_i^1} a_{i,j} + \frac{1}{\beta_i} \sum_{j \in K_i^2} a_{i,j}. \tag{19}$$

Taking in (19) the limit as $i \to \infty$, we deduce

$$\lim_i \frac{1}{\gamma_i} \sum_{j \in K_i} a_{i,j} = 0,$$

which completes the proof of (i). The proof of (ii) is similar. □

With an analogous technique, it is possible to prove the following

Lemma 4. *Let $(x_{i,j})$ and $(y_{i,j})$ be double sequences, $A = (a_{i,j})$ be a non-negative regular summability matrix and (α_i) and (β_i) be positive non-increasing sequences. If $x_{i,j} - L_1 = st_A^T\text{-}o_k(\alpha_i)$ and $y_{i,j} - L_2 = st_A^T\text{-}o_k(\beta_i)$, then we have*

(i) $(x_{i,j} - L_1) \mp (y_{i,j} - L_2) = st_A^T\text{-}o_k(\gamma_i)$ *as* $i \to \infty$, *where* $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$,

(ii) $\lambda(x_{i,j} - L_1) = st_A^T\text{-}o_k(\alpha_i)$ *as* $i \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol o_k replaced by O_k.

Now we have the following result.

Theorem 4. *Let $(L_{i,j})$ be a sequence of positive linear operators from $C(D)$ into itself, and $A=(a_{i,j})$ be a nonnegative regular summability matrix method. Assume that the following conditions hold:*

(i) $\|L_{i,j}(f_0) - f_0\| = st_A^T\text{-}o(\alpha_i)$ *as* $i \to \infty$,

(ii) $\omega(f;\delta_{i,j}) = st_A^T\text{-}o(\beta_i)$ *as* $i \to \infty$, *where* $\delta_{i,j} := \sqrt{\|L_{i,j}(\varphi)\|}$, *with* $\varphi(u,v) = (u-x)^2 + (v-y)^2$. *Then for any* $f \in C(D)$ *we get*

$$\|L_{i,j}(f) - f\| = st_A^T\text{-}o(\gamma_i) \quad \text{as } i \to \infty,$$

where $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$. *Furthermore, similar results holds when the symbol* o *is replaced by* O.

Proof. Let $f \in C(D)$ and $(x,y) \in D$ be fixed. Using linearity and positivity of $L_{i,j}$ we have, for any $(i,j) \in \mathbb{N}^2$ and $\delta > 0$,

$$\begin{aligned}
&|L_{i,j}(f;x,y) - f(x,y)| \\
&\quad = |L_{i,j}(f(u,v) - f(x,y);x,y) - f(x,y)(L_{i,j}(f_0;x,y) - f_0(x,y))| \\
&\quad \leq L_{i,j}(|f(u,v) - f(x,y)|;x,y) + N|L_{i,j}(f_0;x,y) - f_0(x,y)| \\
&\quad \leq L_{i,j}\left(\left[1 + \frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta}\right]\omega(f;\delta);x,y\right) \\
&\qquad + N|L_{i,j}(f_0;x,y) - f_0(x,y)| \\
&\quad \leq \omega(f;\delta)\, L_{i,j}\left(\left(1 + \frac{(u-x)^2 + (v-y)^2}{\delta^2}\right);x,y\right) \\
&\quad + N|L_{i,j}(f_0;x,y) - f_0(x,y)| \\
&\quad \leq \omega(f;\delta)\,|L_{i,j}(f_0;x,y) - f_0(x,y)| + \frac{\omega(f;\delta)}{\delta^2} L_{i,j}(\varphi;x,y) + \omega(f;\delta) \\
&\qquad + N|L_{i,j}(f_0;x,y) - f_0(x,y)|,
\end{aligned}$$

where $N := \|f\|_{C(D)}$. Taking the supremum over $(x,y) \in D$ in both sides of the above inequality, we obtain, for any $\delta > 0$,

$$\begin{aligned}
\|L_{i,j}f - f\|_{C(D)} &\leq \omega(f;\delta)\,\|L_{i,j}f_0 - f_0\|_{C(D)} + \frac{\omega(f;\delta)}{\delta^2}\,\|L_{i,j}\varphi\|_{C(D)} \\
&\quad + \omega(f;\delta) + N\,\|L_{i,j}f_0 - f_0\|_{C(D)}.
\end{aligned}$$

Now, if we take $\delta := \delta_{i,j} := \sqrt{\|L_{i,j}(\varphi)\|}$, then we may write

$$\|L_{i,j}f - f\|_{C(D)} \leq \omega(f;\delta)\,\|L_{i,j}f_0 - f_0\|_{C(D)} + 2\omega(f;\delta) + N\,\|L_{i,j}f_0 - f_0\|_{C(D)}$$

and hence

$$\|L_{i,j}f - f\|_{C(D)} \leq S\{\omega(f;\delta)\,\|L_{i,j}f_0 - f_0\|_{C(D)} + \omega(f;\delta) + \|L_{i,j}f_0 - f_0\|_{C(D)}\}, \tag{20}$$

where $S = \max\{2, N\}$. For a given $r > 0$, define the following sets:

$$U_i := \{j \in \mathbb{N} : j \leq i,\ \|L_{i,j}(f) - f\|_{C(D)} \geq r\},$$
$$U_i^1 := \left\{j \in \mathbb{N} : j \leq i,\ \omega(f;\delta)\,\|L_{i,j}f_0 - f_0\|_{C(D)} \geq \frac{r}{3S}\right\},$$
$$U_i^2 := \left\{j \in \mathbb{N} : j \leq i,\ \omega(f;\delta) \geq \frac{r}{3S}\right\},$$
$$U_i^3 := \left\{j \in \mathbb{N} : j \leq i,\ \|L_{i,j}f_0 - f_0\|_{C(D)} \geq \frac{r}{3S}\right\}.$$

It follows from (20) that

$$U_i \subset U_i^1 \cup U_i^2 \cup U_i^3.$$

Also define the sets:

$$U_i^4 := \left\{j \in \mathbb{N} : j \leq i,\ \omega(f;\delta) \geq \sqrt{\frac{r}{3S}}\right\},$$
$$U_i^5 := \left\{j \in \mathbb{N} : j \leq i,\ \|L_{i,j}f_0 - f_0\|_{C(D)} \geq \sqrt{\frac{r}{3S}}\right\}.$$

Then, observe that $U_i^1 \subset U_i^4 \cup U_i^5$. So, we have $U_i \subset U_i^2 \cup U_i^3 \cup U_i^4 \cup U_i^5$. Now, since $\gamma_i := \max\{\alpha_i, \beta_i\}$ for each $i \in \mathbb{N}$, we get

$$\begin{aligned}\frac{1}{\gamma_i}\sum_{j\in U_i} a_{i,j} &\leq \frac{1}{\beta_i}\sum_{j\in U_i^2} a_{i,j} + \frac{1}{\alpha_i}\sum_{j\in U_i^3} a_{i,j} \\ &\quad + \frac{1}{\beta_i}\sum_{j\in U_i^4} a_{i,j} + \frac{1}{\alpha_i}\sum_{j\in U_i^5} a_{i,j}.\end{aligned}$$

Letting $i \to \infty$ and using (i) and (ii), we obtain

$$\lim_i \frac{1}{\gamma_i}\sum_{j\in U_i} a_{i,j} = 0.$$

This completes the proof. $\square$

The following analogue also holds.

Theorem 5. *Let $(L_{i,j})$ be a sequence of positive linear operators from $C(D)$ into itself, and $A = (a_{i,j})$ be a nonnegative regular summability matrix method. Assume that the following conditions hold:*

(i) $\|L_{i,j}(f_0) - f_0\| = st_A^T\text{-}o_k(\alpha_i)$ *as* $i \to \infty$,

(ii) $\omega(f;\delta_{i,j}) = st_A^T\text{-}o_k(\beta_i)$ *as* $i \to \infty$, *where* $\delta_{i,j} := \sqrt{\|L_{i,j}(\varphi)\|}$ *with* $\varphi(u,v) = (u-x)^2 + (v-y)^2$. *Then, for any* $f \in C(D)$,

$$\|L_{i,j}(f) - f\| = st_A^T - o_k(\gamma_i) \quad \text{as } i \to \infty,$$

where $\gamma_i := \max\{\alpha_i, \beta_i\}$ *for each* $i \in \mathbb{N}$. *Similar results hold when* o_k *is replaced by* O_k.

6. Possible Further Developments

Here we introduce a further extension of the notion of A-statistical convergence, which includes the triangular A-statistical convergence as a particular case. The idea is to consider a general infinite subset $H \subset \mathbb{N}^2$, defined by means a relation which links the indexes i and j. Put

$$H := \{(i,j) \in \mathbb{N}^2 : \Phi(i,j) \leq 0\},$$

where $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is a function satisfying suitable assumptions. We then define the *Φ-A-density* of H as

$$\delta_A^{\Phi}(H) := \lim_i \sum_{j \in H_i} a_{i,j},$$

and the corresponding notion of Φ-A-statistical convergence as follows. Let $A = (a_{i,j})$ be a non negative regular summability matrix. A sequence $x = (x_{i,j})$ is said to be *Φ-A-statistical convergent* to a real number L provided that for every $\varepsilon > 0$

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} : \Phi(i,j) \leq 0, \, |x_{i,j} - L| \geq \varepsilon\}$.

Note that, when we choose $\Phi(i,j) = j - i$, we reduce to the case of triangular A-statistical convergence. Other interesting choices may be given by $\Phi(i,j) = j - \alpha(i)$, where $\alpha : \mathbb{N} \to \mathbb{N}$ is a suitable increasing function. All the theory developed in the previous sections can be carried on also in the setting of $\Phi - A$-statistical convergence.

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