Trigonometric Functions ⇆ Hyperbolic Functions

Alfredo Dimas Moreira Garcia
E-mail: avaliac@sjc.sp.gov.br

Abstract
Construction of relationships that transform hyperbolic functions into trigonometric functions.

The Pythagorean formula for a right triangle with hypotenuse "h" and side "a" adjacent to angle α and side "b" opposite angle α is:

$$h^2 = a^2 + b^2 \quad 33.20$$

For this triangle we have the trigonometric relations:

$$a = h.cos\alpha \qquad b = h.sen\alpha \quad 33.21$$

Reshaping the Pythagorean formula gives:

$$h^2 = a^2 + b^2 \rightarrow b^2 = h^2 - a^2 = (h+a)(h-a) \rightarrow \left(\frac{h+a}{b}\right)\left(\frac{h-a}{b}\right) = e^{\emptyset}e^{-\emptyset} = 1 \quad 33.22$$

This is divided into the following hyperbolic functions:

$$e^{\emptyset} = \frac{h+a}{b} \quad 33.23$$

$$e^{-\emptyset} = \frac{h-a}{b} \quad 33.24$$

Where applying the trigonometric relations we obtain:

$$e^{\emptyset} = \frac{h+a}{b} = \frac{h+h.cos\alpha}{h.sen\alpha} = \frac{1+cos}{sen\alpha} \quad 33.25$$

$$e^{-\emptyset} = \frac{h-a}{b} = \frac{h-h.cos\alpha}{h.sen\alpha} = \frac{1-cos\alpha}{sen\alpha} \quad 33.26$$

From trigonometry we have:

$$tg\left(\frac{\alpha}{2}\right) = \frac{1-cos\alpha}{sen\alpha} = \frac{sen\alpha}{1+cos\alpha} = \sqrt{\frac{1-cos}{1+cos}} \quad 33.27$$

Applying

$$e^{\emptyset} = \frac{1+cos}{sen\alpha} = \frac{1}{tg\left(\frac{\alpha}{2}\right)} = \frac{1}{\sqrt{\frac{1-cos}{1+cos\alpha}}} = \sqrt{\frac{1+cos}{1-cos}} \quad 33.28$$

$$e^{-\emptyset} = \frac{1-cos}{sen\alpha} = tg\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-cos\alpha}{1+cos}} \quad 33.29$$

From these we get:

$$ln(e^{\emptyset}) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \quad 33.30$$

$$ln(e^{-\emptyset}) = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \quad 33.31$$

From these we obtain the hyperbolic angle ∅:

$$\emptyset = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \quad 33.32$$

$$-\emptyset = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \quad 33.33$$

Denominating the hyperbolic cosine ch∅ as:

$$x = ch\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} + \frac{h-a}{b}\right) = \frac{h}{b} = \frac{h}{h.sen\alpha} = \frac{1}{sen\alpha} = cosec\alpha \quad 33.34$$

And calling the hyperbolic sine sh∅ as:

$$y = sh\emptyset = \frac{e^{\emptyset}-e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} - \frac{h-a}{b}\right) = \frac{a}{b} = \frac{h.cos\alpha}{h.sen\alpha} = \frac{cos\alpha}{sen\alpha} = cotg\alpha \quad 33.35$$

Applying the hyperbolic cosine ch∅ and the hyperbolic sine sh∅ to the unitary hyperbola equation $x^2 - y^2 = 1$ we get:

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = cosec^2\alpha - cotg^2\alpha = 1 \quad 33.36$$

Which is a result of trigonometry.

With the relations of the hyperbolic cosine ch∅ and the hyperbolic sine sh∅ we can define the other relations between the trigonometric functions and the hyperbolic functions:

$$tgh\emptyset = \frac{sh\emptyset}{ch\emptyset} = \frac{\frac{cos\alpha}{sen\alpha}}{\frac{1}{sen\alpha}} = cos\alpha \quad 33.37$$

$$cotgh\emptyset = \frac{ch\emptyset}{sh\emptyset} = \frac{\frac{1}{sen\alpha}}{\frac{cos\alpha}{sen\alpha}} = \frac{1}{cos\alpha} = sec\alpha \quad 33.38$$

$$sech\emptyset = \frac{1}{ch\emptyset} = \frac{1}{\frac{1}{sen\alpha}} = sen\alpha \quad 33.39$$

$$cosech\emptyset = \frac{1}{sh\emptyset} = \frac{1}{\frac{cos\alpha}{sen\alpha}} = \frac{sen\alpha}{cos\alpha} = tg\alpha \quad 33.40$$

$$sech^2\emptyset + tgh^2\emptyset = sen^2\alpha + cos^2\alpha = 1 \quad 33.41$$

$$cotgh^2\emptyset - cosech^2\emptyset = sec^2\alpha - tg^2\alpha = 1 \quad 33.42$$

Construction of the already known relationships that transform the hyperbolic functions into the exponential form of a complex number.

Next, we will use Euler's formulas:

$$e^{i\alpha} = cos\alpha + isen\alpha \qquad e^{-i\alpha} = cos\alpha - isen\alpha \qquad 33.43$$

Reshaping the Pythagorean formula, we get:

$$h^2 = a^2 + b^2 = a^2 - (ib)^2 = (a + ib)(a - ib) \rightarrow \frac{(a+ib)}{h}\frac{(a-ib)}{h} = e^{\emptyset}e^{-\emptyset} = 1 \qquad 33.44$$

This breaks down into the following complex hyperbolic functions:

$$e^{\emptyset} = \frac{a+ib}{h} \qquad 33.45$$

$$e^{-\emptyset} = \frac{a-ib}{h} \qquad 33.46$$

For this triangle we have the trigonometric relations:

$$\frac{a}{h} = cos\alpha \qquad \frac{b}{h} = sen\alpha \qquad 33.47$$

Applying trigonometric relations, we get:

$$e^{\emptyset} = \frac{a+ib}{h} = \frac{a}{h} + i\frac{b}{h} = cos\alpha + isen\alpha \qquad 33.48$$

$$e^{-\emptyset} = \frac{a-ib}{h} = \frac{a}{h} - i\frac{b}{h} = cos\alpha - isen\alpha \qquad 33.49$$

To conform to Euler's formulas we must change the hyperbolic arguments to ∅=iα and thus we obtain the hyperbolic functions written as the exponential form of a complex number:

$$e^{\emptyset} = e^{i\alpha} = cos\alpha + isen\alpha \qquad 33.50$$

$$e^{-\emptyset} = e^{-i\alpha} = cos\alpha - isen\alpha \qquad 33.51$$

Calling the cosseno $chi\alpha$ hyperbolic complex as

$$x = chi\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) + (cos\alpha - isen\alpha)] = cos\alpha \qquad 33.52$$

And naming the sine $shi\alpha$ hyperbolic complex as:

$$y = shi\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) - (cos\alpha - isen\alpha)] = isen\alpha \qquad 33.53$$

Applying the cosine $x = chi\alpha = cos\alpha$ hyperbolic complex and the sine $y = shi\alpha = isen\alpha$ hyperbolic complex in the equation of the unit hyperbola $x^2 - y^2 = 1$ results:

$$x^2 - y^2 = ch^2 i\alpha - sh^2 i\alpha = cos^2\alpha - i^2 sen^2\alpha = cos^2\alpha + sen^2\alpha = 1 \qquad 33.54$$

Which is a result of trigonometry.
With the relationships of the hyperbolic cosine $chi\alpha = cos\alpha$ and the hyperbolic sine $shi\alpha = isen\alpha$ we can define the other relationships between complex trigonometric functions and complex hyperbolic functions.