Derivative Valuation and Collateralization

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ABSTRACT

This paper presents a model that characterizes a collateral process directly based on Credit Support Annex (CSA). The model is devised that allows for collateralization adhering to bankruptcy laws. We find evidence that there is a strong linkage between market and credit risk. Our study suggests that banks and regulators need to think about an integrated framework to capture interactions of these two types of risk. This requires all profits and losses are gauged in a consistent way across risk types as they tend to be driven by the same economic factors.

Due to the complexity of collateralization, the literature seems to turn away from direct and detailed modeling. For example, Johannes and Sundaresan [2007], and Fuijii and Takahahsi [2012] model collateralization via a cost-of-collateral instantaneous rate. Piterbarg [2010] regards collateral as a regular asset and uses the replication approach to price collateralized derivatives.

This article makes theoretical and empirical contributions to the study of collateralization by addressing several essential questions. First, how does collateralization affect swap rate?

Unlike generic mid-market swap rates, swap premia are determined in a competitive market according to the basic principles of supply and demand. A swap client first contacts a number of swap dealers for a quotation and then chooses the most competitive one. If a premium is too low, the dealer may lose money. If a premium is too high, the dealer may lose the competitive advantage.

This article uses an indirect empirical approach. We define a *swap premium spread* as the premium difference between two swap contracts that have exactly the same terms and conditions but are traded with different CSA counterparties. We reasonably believe that if two contracts are identical except counterparties, the swap premium spread should reflect counterparty credit risk only, as all other risks/costs are identical.

Empirically, we first study the marginal impact of credit risk. Since credit default swap (CDS) premium theoretically reflects the credit risk of a firm, we use the market swap premium spreads as the response variable and the CDS premium differences between two counterparties as the explanatory variable. The estimation result implies that approximately 75% of market spreads can be explained by counterparty credit risk. In other words, counterparty risk alone can provide a good but not overwhelming prediction on spreads.

We then assess the joint effect. Because implied or model-generated spreads take into account both counterparty risk and collateralization, we assign the model-implied spreads as the explanatory variable and the market spreads as the response variable. The new adjusted R^2 is 0.9906, suggesting that counterparty risk and collateralization together have high explanatory power on premium spreads.

Second, we find that the CVA of a collateralized counterparty portfolio is always smaller than the one of the same portfolio without collateralization. We also find that credit risk is negatively correlated with collateralization as an increase in collateralization causes a decrease in credit risk. The empirical tests corroborate our theoretical conclusions that collateralization can reduce CVA charges and mitigate counterparty risk.

Finally, We find that there is a positive correlation between market risk and credit risk as value at risk (VaR) increases after considering counterparty credit risk. We also find that collateralization and market risk have a negative correlation, i.e., collateral posting can actually reduce VaR. This finding contradicts the prevailing belief in the market that collateralization would increase market.

Pricing Collateralized Financial Derivatives

As the only reason for taking collateral is to reduce/eliminate credit risk, collateral analysis should be closely related to credit risk modeling. There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form models. Many practitioners in the market have tended to gravitate toward the reduced-from models given their mathematical tractability and market consistency.

We consider a filtered probability space (Ω, $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq0}$ denotes a filtration. In the reduced-form framework, the stopping or default time τ of a firm is modeled as a Cox arrival process whose first jump occurs at default and is defined by,

$$\tau = \inf\left\{t : \int_0^t h(s,\Gamma_s)ds \geq \Delta\right\} \tag{1}$$

where $h(t)$ or $h(t,\Gamma_t)$ denotes the stochastic hazard rate dependent on an exogenous common state Γ_t, and Δ is a unit exponential random variable independent of Γ_t.

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t) = \exp\left(-\int_t^s h(u)du\right) \tag{2a}$$

The default probability for the period (t, s) is given by

$$q(t,s) := P(\tau \leq s \mid \tau > t) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u)du\right) \tag{2b}$$

Let valuation date be t. Consider a financial contract that promises to pay a $X_T > 0$ at maturity date $T > t$, and nothing before the time.

The binomial default rule considers only two possible states: default or survival. For a discrete one-payment period (t, T) economy, at time T the contract either defaults with the default probability $q(t,T)$ or survives with the survival probability $p(t,T)$. The survival payoff is equal to X_T and the default payoff is a fraction of X_T: φX_T, where φ is the recovery rate. The value of this defaultable contract at time t is the discounted expectation of all the possible payoffs and is given by

$$V^N(t) = E\{D(t,T)\left[p(t,T) + \varphi(T)q(t,T)\right]X_T \mid \mathcal{F}_t\} = E\left[D(t,T)I(t,T)X_T \mid \mathcal{F}_t\right] \tag{3}$$

where $E\{\bullet \mid \mathcal{F}_t\}$ is the expectation conditional on $\mathcal{F}_t$, $D(t,T)$ denotes the risk-free discount factor at time t for maturity T and $I(t,T) = \left[p(t,T) + \varphi(T)q(t,T)\right]$ can be regarded as a risk-adjusted discount ratio.

Suppose that there is a CSA agreement between a bank and a counterparty in which the counterparty is required to deliver collateral when the mark-to-market (MTM) value arises over the effective threshold H.

The default payment under a CSA can be mathematically expressed as

$$P^D(T) = C(T) + \varphi(T)\left(X_T - C(T)\right) = \varphi(T)X_T + C(T)(1 - \varphi(T)) \tag{4}$$

where $C(T)$ is the collateral amount at T.

It is worth noting that the default payment in equation (4) is always greater than the original recovery, i.e., $P^D(T) > \varphi(T)X_T$ because $\varphi(T)$ is always less than 1. Said differently, the default payoff of a collateralized contract is always greater than the default payoff of the same contract without a CSA. That

is why the major benefit of collateralization should be viewed as an improved recovery in the event of a default.

According to CSA, if the contract value $V^C(t)$ is less than the effective threshold $H(t)$, no collateral is posted; otherwise, the required collateral is equal to the difference between the contract value and the effective threshold. The collateral amount posted at time t can be expressed mathematically as

$$C(t) = \max(V^C(t) - H(t), 0) \tag{5}$$

where $H(t) = 0$ corresponds to full-collateralization; $H(t) > 0$ represents partial-collateralization; and $H(t) < 0$ reflects over-collateralization.

For a discrete one-payment period (t, T) economy, at time T, if the contract survives, the survival value is the promised payoff X_T and the collateral taker returns the collateral to the provider. If the contract defaults, the default payment is defined in (4) where the future value of the collateral is $C(T) = C(t)/D(t,T)$. Since the most predominant form of collateral is cash according to ISDA [2013], it is reasonable to consider the time value of money only for collateral assets. The large use of cash means that collateral is both liquid and not subject to large fluctuations in value. The value of the CSA contract is the discounted expectation of all the payoffs and is given by

$$V^C(t) = E\{D(t,T)[q(t,T)(C(T) + \varphi(T)(X_T - C(T))) + p(t,T)X_T] | \mathcal{F}_t\} \tag{6}$$

After some simple mathematics, we have the following proposition

The value of a collateralized single-payment contract is given by

$$V^C(t) = E[F(t,T)X_T | \mathcal{F}_t] - G(t,T) \tag{7a}$$

where

$$F(t,T) = \left(1_{V^N(t) \leq H(t)} + 1_{V^N(t) > H(t)} / \bar{I}(t,T)\right) I(t,T) D(t,T) \tag{7b}$$

$$G(t,T) = 1_{V^N(t) > H(t)} H(t) \bar{q}(t,T)(1 - \varphi(T)) / \bar{I}(t,T) \tag{7c}$$

where $\bar{I}(t,T) = E(I(t,T) | \mathcal{F}_t)$. $I(t,T)$ and $V^N(t)$ are defined in (3).

The pricing in (7) is relatively straightforward. We first compute $V^N(t)$ and then test whether its value is greater than $H(t)$. After that, the calculations of $F(t,T)$, $G(t,T)$ and $V^C(t)$ are easily obtained.

We discuss a special case where $H(t) = 0$ corresponding to full-collateralization. Suppose that default probabilities are uncorrelated with interest rates and payoffs[1]. From Proposition 1, we can easily obtain $V^C(t) = V^F(t)$ where $V^F(t) = E\left[D(t,T)X_T \mid \mathcal{F}_t\right]$ is the risk-free value. That is to say: the value of a fully-collateralized contract is equal to the risk-free value. This conclusion is in line with the results of Johannes and Sundaresan [2007], Fuijii and Takahahsi [2012], and Piterbarg [2010].

The value of a collateralized multiple-payment contract is given by

$$V^C(t) = \sum_{i=1}^{m} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)X_i \mid \mathcal{F}_t\right] - \sum_{i=0}^{m-1} E\left[\prod_{j=0}^{i-1}\left(F(T_j,T_{j+1})\right)G(T_i,T_{i+1}) \mid \mathcal{F}_t\right] \qquad (8a)$$

where

$$F(T_j,T_{j+1}) = \left(1_{J(T_j,T_{j+1}) \leq H(T_j)} + 1_{J(T_j,T_{j+1}) > H(T_j)} / \bar{I}(T_j,T_{j+1})\right) I(T_j,T_{j+1})D(T_j,T_{j+1}) \qquad (8b)$$

$$G(T_j,T_{j+1}) = 1_{J(T_j,T_{j+1}) > H(T_j)} H(T_j)\bar{q}(T_j,T_{j+1})\left(1 - \varphi(T_{j+1})\right) / \bar{I}(T_j,T_{j+1}) \qquad (8c)$$

$$J(T_j,T_{j+1}) = E\left[D(T_j,T_{j+1})I(T_j,T_{j+1})\left(V^C(T_{j+1}) + X_{j+1}\right) \mid \mathcal{F}_{T_j}\right] \qquad (8d)$$

where $\prod_{j=0}^{i-1} F(T_j,T_{j+1}) = 1$ is an empty product when $i = 0$. Empty product allows for a much shorter mathematical presentation of many subjects.

Empirical Results

Impact of collateralization on swap rate

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[1] Moody's Investor's Service [2000] presents statistics that suggest that the correlations between interest rates, default probabilities and recovery rates are very small and provides a reasonable comfort level for the uncorrelated assumption.

Unlike the generic benchmark swap rates, swap premia are determined according to the basic principles of supply and demand. The swap market is highly competitive. In a competitive market, prices are determined by the impersonal forces of demand and supply, but not by the manipulations of powerful buyers or sellers.

A swap premium is supposed to cover the expected profit and all the expenses, including the cost of bearing unsecured credit risk. Unfortunately, however, we do not know what percentage of the market swap premium is allocated to the unsecured credit risk, which makes a direct verification impossible.

To circumvent this difficulty, we design an indirect verification process in which we select some CSA swap pairs such that the two contracts in each pair have exactly the same terms and conditions but are traded with different counterparties under different collateral agreements. It is reasonable to believe that the difference between the two contracts in each pair is solely attributed to collateralized counterparty credit risk, as all the other risks/costs are identical. Therefore, by accounting for credit risk and collateralization, we can efficiently compare the implied spreads with the market spreads for these pairs.

Given the above information, we are able to compute the collateralized swap rates. We find that the model-implied spreads fluctuate randomly around the market spreads. We refer to the differences between the model-implied premium spreads and the market quoted premium spreads as the *model-market premium spread differentials*.

Next, we examine the effects of credit risk and collateralization, alone or combined, on swap premium spreads. First, we study the marginal effect of credit risk. For each swap pair, we obtain the counterparty CDS premia. Presumably, the differences in CDS premia should mainly represent the differences in counterparty risk. To determine the strength of the statistical relationship between market premium spreads and counterparty risk, we present the estimate of the following regression model.

$$Y = a + bX + \varepsilon \tag{9}$$

where Y is the market swap premium spread, X is the difference between the counterparty CDS premia, a is the intercept, b is the slope, and ε is the regression residual.

The results of this regression are shown in Exhibit 7. It can be seen that the adjusted R^2 is 0.7472, implying that approximately 74% of market premium spreads can be explained by credit risk alone. We provide empirical evidence that counterparty credit risk alone plays a significant but not overwhelming role in determining credit-related spreads when contracts are under collateral agreements. Moreover, the slope is 0.0127, suggesting that a CDS spread of about 100 bps translates into a swap spread of about 1.27 bps. Finally the small p-value indicates that the changes in CDS premia are closely related to the changes in market premium spreads.

Exhibit 1: Marginal credit risk regression results

This exhibit presents the regression results for the following regression model:

MarketSwapPremiumSpreads = a + b * DifferencesBetweenCounterpartyCDSs + ε

where the market swap premium spreads are used as the dependent variable and the differences between the counterparty CDS premia as the explanatory variable.

Slope	Intercept	Adjusted R^2	Significance F	T value	P value
0.0127	-2.7E-04	0.7472	1.1E-05	18.6	1.51E-66

According to ISDA Margin Survey (ISDA [2013], 73.7% of OTC derivatives are subject to collateral agreements. For large firms, the figure is 80.7%. Accounting for collateralization has become increasingly important in pricing OTC derivatives. Since the implied spreads generated by our model take into account both credit risk and collateralization, the statistical relationship between the market spreads and the model-implied spreads should refer to the joint effect of counterparty credit risk and collateralization on market spreads. Thus, we present another regression model where the market spreads are regressed on the implied spreads. The regression results are shown in Exhibit 8.

Exhibit 2: Credit risk and collateralization combined regression results

This exhibit presents the regression results for the following regression model:

$$MarketSwapPremiumSpreads = a + b * ImpliedSwapPremiumSpreads + \varepsilon$$

where the market swap premium spreads are used as the dependent variable and the model-implied swap premium spreads as the explanatory variable.

Slope	Intercept	Adjusted R^2	Significance F	T value	P value
0.9857	4.48E-05	0.9906	3.21E-08	25.4	3.16E-110

Exhibit 2 shows that the adjusted R^2 value is 0.9906, implying that approximately 99% of the market spreads can be explained by the implied spreads. Also, the small p-value suggests that the changes in implied spreads are strongly related to the changes in market spreads.

The empirical results shed light on the economic and statistical significance of collateralization. The increase in the explanatory power of swap premium spreads bears an interesting finding: It seems that credit risk alone has a modest explanatory power on premium spreads. Only the combination of credit risk and collateralization can sufficiently explain them.

Impact of collateralization on credit risk

In this subsection, we study how collateralization affects credit risk by measuring CVA changes due to collateral posting. CVA is the market price of counterparty credit risk that has become a central part of counterparty credit risk management.

From the same dataset above, we find that there are a total of 3052 counterparties having live trades as of May 11, 2012. 516 of them have CSA agreements. We randomly select one counterparty portfolio that contains 476 interest rate swaps, 36 interest rate swaptions and 223 interest rate caps/floors. First, we compute the risk-free value $V^F = 2,737,702$ that is relatively straightforward as the risk-free portfolio value is what trading systems or pricing models normally report.

Second, we assume that there is counterparty credit risk but no collateral agreement. Based on the pricing model proposed by Xiao [2015], we compute the risky value of the portfolio as $V^N = 2,688,014$ after considering counterparty credit risk. By definition, the CVA without collateralization is equal to $CVA_N = V^F - V^N = 49,688.$

Next, we further assume that there is a CSA agreement in which the threshold is 2 million and the MTA is 100,000. The risky value of the portfolio is calculated as $V^C = 2,725,094$ according to Proposition 2. The CVA with collateralization is given by $CVA_C = V^F - V^C = 12,608$. Similarly, we can compute the CVAs under different collateral arrangements and present the results in Exhibit 9.

Exhibit 3. Impact of Collateralization on CVA

This exhibit shows that CVA increases with collateral threshold. The infinite collateral threshold is equivalent to no collateral agreement and the zero-value collateral threshold corresponds to full collateralization. An increase in collateral threshold leads to a decrease in collateralization.

Effective Threshold	0	2.1 Million	4.1 Million	6.1 Million	8.1 Mil	Infinite (∞)
CVA	0	12,608	23,685	33,504	42,254	49,688

Exhibit 9 tells us that collateral posting can reduce CVA. Full collateralization makes a portfolio appear to be risk-free. An increase in collateral threshold leads to a rise in unsecured credit exposure, and thereby an increase in CVA. In particular, CVA reaches the maximum when the threshold is infinite representing no collateral arrangement.

We extend our analysis to other CSA portfolios. The results hold across different CSA counterparties and collateral agreements. Our findings show that collateral posting can reduce credit risk and CVA. The results also suggest a negative correlation between collateralization and CVA as an increase in collateralization causes a decrease in CVA charge and vice-versa. These findings improve our understanding of the relationship between collateralization and CVA.

Impact of collateralization on market risk

We study how collateralization impacts market risk by gauging VaR changes due to collateral arrangements. VaR is the regulatory measurement for assessing market risk. It is defined as the maximum loss likely to be suffered on a portfolio for a given probability defined as a confidence level over a given period time. In its most general form, VaR measures 10-day 99th percentile of potential loss that can be incurred.

There are three commonly used methodologies to calculate VaR – parametric, historical simulation and Monte Carlo simulation. Parametric model estimates VaR directly from the standard deviation of portfolio returns typically assuming returns are normally distributed. Historical simulation calculates VaR from the distribution of actual historical returns. Whilst Monte Carlo simulation computes VaR from a distribution constructed from random outcomes. In this paper, we calculate historical VaR.

For monitoring market risk, many organizations segment portfolios in some manner. They may do so by traders and trading desks. Typically, a market risk portfolio contains derivatives across multiple counterparties, while a counterparty portfolio comprises transactions among different traders and trading desks. We fortuitously select a trading portfolio and then partition it into single-counterparty sub-portfolios. One sub-portfolio contains 172 interest rate swaps, 68 caps and floors, 25 European swaptions and 17 cancelable swaps.

First, we compute the VaR of the sub-portfolio without considering credit risk. The calculated result as of May 11, 2012 is -386,570. Here the VaR is a negative value by definition. However, the industry convention is to report VaR as a positive number that is the amount of money one can lose. Thus, people in the market usually say that the VaR is 386,570 in this case. This convention can be confusing in places. If one says that VaR increases, the numbers actually become smaller or move into the left tail.

For many reasons, both historical and practical, market and credit risk have often been treated as if they are unrelated source of risk: the risk types have been measured separately, managed separately, and economic capital against each risk type has been assessed separately.

However, market risk and credit risk actually reinforce each other. Default-induced credit losses can be driven by market price changes. At the same time, the changes in prices depend on the rating migration or increase in the default perception of the firm. The Basel Committee on Banking Supervision (see BCBS [2009]) also finds that those banks that were more severely affected in the global financial crisis measured their market and credit risk separately, whereas banks that used an integrated approach to market and credit risk measurement were much less impacted. Measuring market risk without considering credit risk may mask the significant impact of credit risk and often leads to underestimation of risk.

Therefore, we further calculate VaR by taking credit risk into account. Let us first assume that there is no CSA agreement. Each trade is valued by the risky model developed by Xiao [2015]. The VaR with credit risk is computed as -418,948. It can be seen that VaR has increased from 386,570 to 418,948 after accounting for credit risk. The above empirical results show that market fluctuation has a larger impact on VaR when credit risk is taken into account. Our research highlights the linkage between market and credit risk.

Next, we measure VaR again by considering both counterparty credit risk and collateralization. Assume that there is a CSA arrangement. The VaR values under different effective collateral thresholds are displayed in Exhibit 10.

Exhibit 4. Impact of Collateralization on VaR

This exhibit shows that VaR increases with collateral threshold. The infinite collateral threshold is equivalent to no collateral agreement and the zero-value collateral threshold corresponds to full collateralization.

Effective Threshold	0	2.1 Million	4.1 Million	6.1 Mil	Infinite (∞)
VaR	-386,570	-398,235	-404,401	-410,756	-418,950

Exhibit 10 tells us that collateral posting actually can reduce VaR. The results show that market risk exposure rises as collateral threshold increases. In particular, VaR reaches the maximum when the

threshold is infinite representing no collateral arrangement. We come to the same conclusion after repeating this test for other portfolios.

Our findings suggest that there are important interactions between market and credit risk. We find a positive correlation between market and credit risk as they increase or decrease together. We also find that collateralization is actually negatively correlated with market risk, i.e., an increase in collateralization causes a drop in market risk. Said differently, collateral posting can reduce market risk. Our research contributes to the understanding of the interaction between market and credit risk.

Conclusion

This article presents a new model for pricing collateralized financial contracts based on the fundamental principal and legal structure of CSA. The model can back out market prices. This is very useful for pricing outstanding collateralized derivatives.

Empirically, we use a unique proprietary dataset to measure the effect of collateralization on pricing and compare it with model-implied prices. The empirical results show that the model-implied prices are quite close to the market-quoted prices, suggesting that the model is fairly accurate on pricing collateralized derivatives.

We find strong evidence that counterparty credit risk alone plays a significant but not overwhelming role in determining credit-related spreads. Only the joint effect of collateralization and credit risk has high explanatory power on unsecured credit costs. This finding suggests that failure to properly account for collateralization may result in significant mispricing of derivatives.

We also find evidence that there is a strong linkage between market and credit risk. Our research results suggest that banks and regulators need to think about an integrated framework to capture material interactions of these two types of risk. This requires all profits and losses are gauged in a consistent way across risk types as they tend to be driven by the same economic factors. Our finding leads to an improved

understanding of the interaction between market and credit risk and how this interaction is related to risk measurement and management.

References

Collin-Dufresne, P. and Solnik, B. "On the term structure of default premia in the swap and LIBOR markets." Journal of Finance 56 (2001), pp.1095-1115.

Duffie, D. and Huang, M. "Swap rates and credit quality." Journal of Finance, 51(1996), pp. 921-949.

Duffie, D., and K.J. Singleton, K.J. "Modeling term structure of defaultable bonds." Review of Financial Studies, 12(1999), pp. 687-720.

Edwards, F. and Morrison, E. "Derivatives and the bankruptcy code: why the special treatment?" Yale Journal on Regulation, 22 (2005), pp. 101-133.

Feldhutter, P. and Lando, D. "Decomposing swap spreads. Journal of Financial Economics," 88 (2008), pp. 375-405.

FinPricing, 2017, Market Data, https://finpricing.com/lib/EqBarrier.html

Fuijii, M. and Takahahsi, A. "Collateralized credit default swaps and default dependence: Implications for the central counterparties." Journal of Credit Risk, 8(2012), pp. 1-10.

Garlson, D. "Secured creditors and the eely character of bankruptcy valuations." 41 American University Law Review, 63 (1991-1992), pp. 63-106.

Grinblatt, M., "An analytic solution for interest rate swap spreads." Review of International Finance, 2 (2001), pp. 113-149.

He, H. "Modeling Term Structures of Swap Spreads." Working paper, Yale School of Management, Yale University, (2001).

Johannes, M. and Sundaresan, S. "The impact of collateralization on swap rates." Journal of Finance, 62 (2007), pp. 383-410.

Liu, J., Longstaff, F. and Mandell, R. "The Market Price of Risk in Interest Rate Swaps: The Roles of Default and Liquidity Risks," The Journal of Business, 79 (2006), pp. 2337-2360.

Minton, B. "An Empirical Examination of Basic Valuation Models for Plain Vanilla U.S. Interest Rate Swaps." Journal of Financial Economics, 44 (1997), pp. 251-277.

Piterbarg, V. "Funding beyond discounting: collateral agreements and derivatives pricing." Risk Magazine, 2010 (2010), pp. 97-102.

Routh, R. and Douglas, M. "Default interest payable to oversecured creditor subject to reasonableness limitation." ABF Journal, Vol. 4, No. 3 (2005), pp. 1-10.

Sorensen, E. and Bollier, T. "Pricing swap default risk." Financial Analysts Journal, 50 (1994), pp. 23-33.