New trigonometric integrals with Barnes function

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Abstract

This time, I talk about some integrals in the continuity of my precedent paper Corrections about V. S. Adamchik's papers (1).Now we can deduce from the three integrals, six new integrals and I give the general formulas in terms of Barnes function.

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G\,(1+z) = (2\,\pi)^{\frac{z}{2}}\,\mathrm{e}^{-\frac{z(1+z)}{2}-\frac{\gamma\,z^2}{2}} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^{k} \mathrm{e}^{-z+\frac{z^2}{2\,k}} \qquad (2)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \qquad (3)$$

$$G\,(1+z) = G\,(z)\,\Gamma\,(z) \qquad (4)$$

$$\log\left(G\,(1+z)\right) = \frac{z\log\left(2\,\pi\right)}{2} - \frac{z\,(1+z)}{2} + z\log\left(\Gamma\,(1+z)\right) - \int_0^z \log\left(\Gamma\,(t+1)\right)\,\mathrm{d}t \qquad (5)$$

$$\int_0^z \log\left(\Gamma\,(t+1)\right)\,\mathrm{d}t = \frac{z\log\left(2\,\pi\right)}{2} - \frac{z\,(1+z)}{2} + z\log\left(\Gamma\,(1+z)\right) - \log\left(G\,(z)\right) - \log\left(\Gamma\,(z)\right) \qquad (6)$$

3 Identity and rules

We know that $\frac{1}{\mathrm{e}^{2\,\pi\,x}-1}=\frac{1}{(\mathrm{e}^{\pi\,x}-1)(\mathrm{e}^{\pi\,x}+1)}=\frac{1/2}{\mathrm{e}^{\pi\,x}-1}$-$\frac{1/2}{\mathrm{e}^{\pi\,x}+1}$

And in general in terms of a: $\frac{1}{\mathrm{e}^{a\pi\,x}+1}=\frac{1}{\mathrm{e}^{a\pi\,x}-1}$-$\frac{2}{\mathrm{e}^{2\,a\pi\,x}-1}$

So we need this identity for the general formula.

And now several rules for calculation:

$\zeta\left(1,-2,\frac{1}{2}\right)=-\frac{3\,\zeta(1,-2)}{4}$

$\sum_{r=1}^{k-1}\zeta\left(1,s,\frac{r}{k}\right)=\left(k^{s}-1\right)\zeta\left(1,s\right)+k^{s}\zeta\left(s\right)\log\left(k\right)$ with s=-2

$\zeta\left(1,-2,1+t\right)=t^{2}\log\left(t\right)+\zeta\left(1,-2,t\right)$

4 The first integral

$$\int_{0}^{\infty}\frac{x^{2}}{\mathrm{e}^{a\,\pi\,x}+1}\arctan\left(\frac{x}{z}\right)\mathrm{d}x$$

Let A be the Glaisher–Kinkelin's constant (7).

a and z are both positiv number.

I use the identity and the general formula of $\int_{0}^{\infty}\frac{x^{2}}{\mathrm{e}^{a\pi\,x}-1}\arctan\left(\frac{x}{z}\right)\mathrm{d}x$

So I deduce the general formula of $\int_{0}^{\infty}\frac{x^{2}}{\mathrm{e}^{a\pi\,x}+1}\arctan\left(\frac{x}{z}\right)\mathrm{d}x$

$-\frac{z^{3}\log(2\,az)}{6}+\frac{11\,z^{3}}{36}+\frac{z}{6\,a^{2}}+\frac{z^{2}}{a}\log\left(\Gamma\left(\frac{az}{2}\right)\right)-\frac{\log(\Gamma(az))z^{2}}{a}-4\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right)z}{a^{2}}-4\,\frac{z\log\left(G\left(\frac{az}{2}\right)\right)}{a^{2}}-2\,\frac{z\log(A)}{a^{2}}+2\,\frac{z\log(\Gamma(az))}{a^{2}}+2\,\frac{z\log(G(az))}{a^{2}}-4\,\frac{\zeta\left(1,-2,\frac{az}{2}\right)}{a^{3}}+\frac{\zeta(1,-2,az)}{a^{3}}$

5 The second integral

$$\int_{0}^{\infty}\frac{x^{2}+z^{2}}{\mathrm{e}^{a\,\pi\,x}+1}\arctan\left(\frac{x}{z}\right)\cos\left(2\,\arctan\left(\frac{x}{z}\right)\right)\mathrm{d}x$$

a and z are both positiv number.

I use the identity and the general formula of $\int_0^{\infty} \frac{x^2+z^2}{e^{a\,\pi\,x}-1} \arctan\left(\frac{x}{z}\right) \cos\left(2 \arctan\left(\frac{x}{z}\right)\right) dx$

So I deduce the general formula of $\int_0^{\infty} \frac{x^2+z^2}{e^{a\,\pi\,x}+1} \arctan\left(\frac{x}{z}\right) \cos\left(2 \arctan\left(\frac{x}{z}\right)\right) dx$

$$\frac{2\,z^3 \log(2\,za)}{3} - \frac{29\,z^3}{36} - \frac{z}{6\,a^2} - \frac{z^2 \log(2)}{2\,a} + 4\,\frac{z \log\left(\Gamma\left(\frac{az}{2}\right)\right)}{a^2} + 4\,\frac{z \log\left(G\left(\frac{az}{2}\right)\right)}{a^2} + 2\,\frac{z \log(A)}{a^2} - 2\,\frac{z \log(\Gamma(za))}{a^2} - 2\,\frac{z \log(G(za))}{a^2} + 4\,\frac{\zeta\left(1,-2,\frac{az}{2}\right)}{a^3} - \frac{\zeta(1,-2,za)}{a^3}$$

6 The third integral

$$\int_0^{\infty} \frac{1}{e^{a\,\pi\,x}+1} \log\left(x^2+z^2\right) \sin\left(2 \arctan\left(x/z\right)\right) \left(x^2+z^2\right) dx$$

a and z are both positiv number.

I use the identity and the general formula of $\int_0^{\infty} \frac{\log\left(x^2+z^2\right)}{e^{a\,\pi\,x}-1} \sin\left(2 \arctan\left(\frac{x}{z}\right)\right)\left(x^2+z^2\right) dx$

So I deduce the general formula of $\int_0^{\infty} \frac{\log\left(x^2+z^2\right)}{e^{a\,\pi\,x}+1} \sin\left(2 \arctan\left(\frac{x}{z}\right)\right)\left(x^2+z^2\right) dx$

$$-z^3 \log(2\,za) + \frac{3\,z^3}{2} + \frac{z}{3\,a^2} + \frac{2\,\log(2) z}{3\,a^2} - \frac{\log(a) z}{3\,a^2} - 8\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right) z}{a^2} - 8\,\frac{z \log\left(G\left(\frac{az}{2}\right)\right)}{a^2} - 4\,\frac{z \log(A)}{a^2} + 4\,\frac{\log(\Gamma(za)) z}{a^2} + 4\,\frac{z \log(G(za))}{a^2}$$

7 Integrals in terms of hyperbolic sine

We know the identity: $\frac{1}{\sinh(\pi\,x)}=\frac{2}{e^{\pi\,x}-1}-\frac{2}{e^{2\,\pi\,x}-1}$

Now in terms of a: $\frac{1}{\sinh(a\pi\,x)}=\frac{2}{e^{a\pi\,x}-1}-\frac{2}{e^{2\,a\pi\,x}-1}$

I use the same principle and we have three general formulas.

8 The fourth integral

$$\int_0^{\infty} \frac{x^2}{\sinh(a\pi\, x)} \arctan\left(\frac{x}{z}\right) \mathrm{d}x$$

a and z are both positiv number.

The general formula is:

$$-\frac{z^3 \log(2)}{3} + 2\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right) z^2}{a} - \frac{\log(\Gamma(az)) z^2}{a} + \frac{z^2 \log(2\,\pi)}{2\,a} - 8\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right) z}{a^2} - 8\,\frac{z \log\left(G\left(\frac{az}{2}\right)\right)}{a^2} + \frac{z}{2\,a^2} - 6\,\frac{z \log(A)}{a^2} + 2\,\frac{\log(\Gamma(az)) z}{a^2} + 2\,\frac{z \log(G(az))}{a^2} - 8\,\frac{\zeta\left(1, -2, \frac{az}{2}\right)}{a^3} + \frac{\zeta(1,-2,az)}{a^3}$$

9 The fifth integral

$$\int_0^{\infty} \frac{x^2 + z^2}{\sinh(a\pi\, x)} \arctan\left(\frac{x}{z}\right) \cos\left(2\,\arctan\left(\frac{x}{z}\right)\right) \mathrm{d}x$$

a and z are both positiv number.

The general formula is:

$$\frac{4\,z^3 \log(2)}{3} - \frac{z^2 \log(\pi)}{a} - 2\,\frac{z^2 \log(2)}{a} + \frac{z^2 \log(az)}{2\,a} + 8\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right) z}{a^2} + 8\,\frac{z \log\left(G\left(\frac{az}{2}\right)\right)}{a^2} - \frac{z}{2\,a^2} + 6\,\frac{z \log(A)}{a^2} - 2\,\frac{\log(\Gamma(az)) z}{a^2} - 2\,\frac{z \log(G(az))}{a^2} + 8\,\frac{\zeta\left(1, -2, \frac{az}{2}\right)}{a^3} - \frac{\zeta(1,-2,az)}{a^3}$$

10 The sixth integral

$$\int_0^{\infty} \frac{\log\left(x^2 + z^2\right)\left(x^2 + z^2\right)}{\sinh(a\pi\, x)} \sin\left(2\,\arctan\left(\frac{x}{z}\right)\right) \mathrm{d}x$$

a and z are both positiv number.

The general formula is:

$$-16\,\frac{\log\left(\Gamma\left(\frac{az}{2}\right)\right) z}{a^2} - 16\,\frac{z \log\left(G\left(\frac{az}{2}\right)\right)}{a^2} + 4\,\frac{\log(\Gamma(az)) z}{a^2} + 4\,\frac{z \log(G(az))}{a^2} + \frac{z}{a^2} - 12\,\frac{z \log(A)}{a^2} + \log(2)\left(-2\,z^3 + 2\,\frac{z^2}{a} + \frac{4\,z}{3\,a^2}\right) + 2\,\frac{z^2 \log(\pi)}{a} - \frac{\log(a) z}{a^2}$$

11 Applications

First example

Consider and calculate the closed form

$$\int_0^\infty \frac{x^2 \arctan(2\,x)}{e^{3\,\pi\,x}+1}\,dx$$

So we see a=3 and z=1/2

We obtain

$$\frac{35}{864}+\frac{\log\left(\Gamma\left(\frac{1}{4}\right)\right)}{12}-\frac{\log(\pi)}{24}-\frac{2\,\log(2)}{27}-\frac{\log(3)}{48}-\frac{K}{18\,\pi}-\frac{\log(A)}{36}+\frac{4\,\zeta(1,-2,1/4)}{27}$$

Where K is the Catalan's constant (8)

Second example

Consider and calculate the closed form

$$\int_0^\infty \frac{\arctan(3\,x)\cos(2\,\arctan(3\,x))}{e^{4\,\pi\,x}+1}\left(x^2+\frac{1}{9}\right)dx$$

So we see z=1/3 and a=4

We obtain

$$-\frac{223}{7776}-\frac{\log\left(\Gamma\left(\frac{1}{3}\right)\right)}{9}+\frac{\log(\pi)}{18}+\frac{25\,\log(2)}{216}-\frac{11\,\log(3)}{1296}-\frac{\pi\sqrt{3}}{432}+\frac{\Psi\left(1,\frac{1}{3}\right)\sqrt{3}}{288\,\pi}-\frac{\log(A)}{72}-\frac{5\,\zeta(1,-2,1/3)}{64}-\frac{\zeta(1,-2)}{18}$$

Or if you prefer,

$$-\frac{223}{7776}-\frac{\log\left(\Gamma\left(\frac{1}{3}\right)\right)}{9}+\frac{\log(\pi)}{18}+\frac{25\,\log(2)}{216}-\frac{11\,\log(3)}{1296}-\frac{\pi\sqrt{3}}{432}+\frac{\Psi\left(1,\frac{1}{3}\right)\sqrt{3}}{288\,\pi}-\frac{\log(A)}{72}-\frac{5\,\zeta(1,-2,1/3)}{64}+\frac{\zeta(3)}{72\,\pi^2}$$

Where $\zeta(3)$ is the Apery's constant (9) and I use the relation $\zeta(1,-2)=-\frac{\zeta(3)}{4\,\pi^2}$

And $\Psi\left(1, \frac{1}{3}\right)$ is the trigamma function at 1/3. (10)

Third example

Consider and calculate the closed form

$$\int_0^\infty \frac{1}{e^{\frac{1}{2}\pi x}+1} \log\left(x^2+25\right) \sin\left(2 \arctan\left(x/5\right)\right) \left(x^2+25\right) \mathrm{d}x$$

So we see z=5 and a=1/2

We obtain

$\frac{1135}{6} - 200 \log\left(\Gamma\left(1/4\right)\right) + 100 \log\left(\pi\right) + \frac{310 \log(2)}{3} + 80 \log\left(3\right) - 125 \log\left(5\right) + 40 \frac{K}{\pi} - 20 \log\left(A\right)$

Fourth example

Consider and calculate the closed form

$$\int_0^\infty \frac{x^2}{\sinh\left(2\pi x\right)} \arctan\left(\frac{x}{3}\right) \mathrm{d}x$$

So we see z=3 and a=2

We obtain

$\frac{3}{8} + \frac{9 \log(\pi)}{4} - \frac{19 \log(2)}{4} + \frac{9 \log(3)}{8} + \frac{\log(5)}{8} - \frac{9 \log(A)}{2} - \frac{7 \zeta(1,-2)}{8}$

Or if you prefer

$\frac{3}{8} + \frac{9 \log(\pi)}{4} - \frac{19 \log(2)}{4} + \frac{9 \log(3)}{8} + \frac{\log(5)}{8} - \frac{9 \log(A)}{2} + \frac{7 \zeta(3)}{32 \pi^2}$

Fifth example

Consider and calculate the closed form

$$\int_0^\infty \frac{x^2+4}{\sinh\left(3\pi x\right)} \arctan\left(\frac{x}{2}\right) \cos\left(2 \arctan\left(x/2\right)\right) \mathrm{d}x$$

So we see z=2 and a=3

We obtain

$-\frac{1}{9} - \frac{4\ \log(\pi)}{3} + \frac{182\ \log(2)}{27} - \log(3) - \frac{37\ \log(5)}{27} + \frac{4\ \log(A)}{3} + \frac{7\,\zeta(1,-2)}{27}$

Or if you prefer

$-\frac{1}{9} - \frac{4\ \log(\pi)}{3} + \frac{182\ \log(2)}{27} - \log(3) - \frac{37\ \log(5)}{27} + \frac{4\ \log(A)}{3} - \frac{7\,\zeta(3)}{108\,\pi^2}$

Sixth example

Consider and calculate the closed form

$$\int_0^\infty \frac{1}{\sinh\left(\frac{\pi x}{2}\right)} \log\left(x^2+9\right) \sin\left(2\ \arctan\left(x/3\right)\right) \left(x^2+9\right) \mathrm{d}x$$

So we see z=3 and a=1/2

We obtain

$144\ \log\left(\Gamma\left(1/4\right)\right) - 72\ \log\left(\pi\right) - 108\ \log\left(2\right) - 48\ \frac{K}{\pi}$

12 References

(1): Denis Gallet,Corrections about V. S. Adamchik's papers (2022)

(2): E. W. Barnes. The Theory of the G-function. Quart. J. Pure Appl. Math. 31, pages 264–314, 1899

(3),(4),(5) and (6): https://dlmf.nist.gov/5.17

(7): https://mathworld.wolfram.com/Glaisher-KinkelinConstant.html

(8): https://mathworld.wolfram.com/CatalansConstant.html

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