A Method to Prove a Prime Number between 3*N* and 4*N*

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Abstract

In this paper, we will prove that when an integer n >1, there exists a prime number between 3n and 4n. This is another step in the expansion of the Bertrand's postulate - Chebyshev's theorem after the proof of a prime number between 2n and 3n.

Introduction

The Bertrand's postulate - Chebyshev's theorem States that for any positive integer n, there is always a prime number p such that $n < p \leq 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \leq 3n$. In 2011, Andy Loo [3] expanded the theorem to that when $n \geq 2$, there exists a prime number in the interval $(3n, 4n)$. Recently, the author used a different method [4] to prove that a prime number exists between 2n and 3n by analyzing the binomial coefficient $\binom{3n}{n}$. In this paper, we will use the similar way to prove that a prime number exists between 3n and 4n by analyzing the binomial coefficient $\binom{4n}{n}$.

Definition: $\Gamma_{a\geq p>b}\{\binom{4n}{n}\}$ denotes the prime factorization operator of $\binom{4n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{4n}{n}$ in the range of $a \geq p > b$. In this operator, p is a prime number, a and b are real numbers, and $4n \geq a \geq p > b \geq 1$.

It has some properties:

It is always true that $\Gamma_{a\geq p>b}\{\binom{4n}{n}\} \geq 1$ **— (1)**

If there is no prime number in $\Gamma_{a\geq p>b}\{\binom{4n}{n}\}$, then $\Gamma_{a\geq p>b}\{\binom{4n}{n}\} = 1$, or vice versa, if $\Gamma_{a\geq p>b}\{\binom{4n}{n}\} = 1$, then there is no prime number in $\Gamma_{a\geq p>b}\{\binom{4n}{n}\}$. **— (2)**

For example, $\Gamma_{12\geq p>8}\{\binom{16}{4}\} = 11^0 = 1$. No prime number is in $\binom{16}{4}$ in the range of $12 \geq p > 8$.

If there is at least one prime number in $\Gamma_{a\geq p>b}\{\binom{4n}{n}\}$, then $\Gamma_{a\geq p>b}\{\binom{4n}{n}\} > 1$, or vice versa, if $\Gamma_{a\geq p>b}\{\binom{4n}{n}\} > 1$, then there is at least one prime number in $\Gamma_{a\geq p>b}\{\binom{4n}{n}\}$. **— (3)**

For example, $\Gamma_{8\geq p>4}\{\binom{16}{4}\} = 5 > 1$. Prime number 5 is in $\binom{16}{4}$ in the range of $8 \geq p > 4$.

Let $v_p(n)$ be the p-$adic\ valuation$ of n, the exponent of the highest power of p that divides n. We define *R(p)* by the inequalities $p^{R(p)} \leq 4n < p^{R(p)+1}$, and determine the p-$adic\ valuation$ of $\binom{4n}{n}$.

$v_p\left(\binom{4n}{n}\right) = v_p((4n)!) - v_p((3n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{4n}{p^i}\right\rfloor - \left\lfloor\frac{3n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \le R(p)$

because for any real numbers a and b, the expression of $\lfloor a + b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{4n}{n}$, then $v_p\left(\binom{4n}{n}\right) \le R(p) \le \log_p(4n)$, or $p^{v_p\left(\binom{3n}{n}\right)} \le p^{R(p)} \le 4n$ **— (4)**

And if $4n \ge p > \lfloor 2\sqrt{n}\rfloor$, then $0 \le v_p\left(\binom{4n}{n}\right) \le R(p) \le 1$. **— (5)**

From the prime number decomposition, when $n > \lfloor 2\sqrt{n}\rfloor$,

$\binom{4n}{n} = \frac{(4n)!}{n!\cdot(3n)!} = \Gamma_{4n\ge p>n}\{\frac{(4n)!}{n!\cdot(3n)!}\} \cdot \Gamma_{n\ge p>\lfloor 2\sqrt{n}\rfloor}\{\frac{(4n)!}{n!\cdot(3n)!}\} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor\ge p}\{\frac{(4n)!}{n!\cdot(3n)!}\}$.

When $n \le \lfloor 2\sqrt{n}\rfloor$, $\binom{4n}{n} \le \Gamma_{4n\ge p>n}\{\frac{(4n)!}{n!\cdot(3n)!}\} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor\ge p}\{\frac{(4n)!}{n!\cdot(3n)!}\}$.

Thus, $\binom{4n}{n} \le \Gamma_{4n\ge p>n}\{\frac{(4n)!}{n!\cdot(3n)!}\} \cdot \Gamma_{n\ge p>\lfloor 2\sqrt{n}\rfloor}\{\frac{(4n)!}{n!\cdot(3n)!}\} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor\ge p}\{\frac{(4n)!}{n!\cdot(3n)!}\}$.

Since all prime numbers in $n!$ are not in the range of $4n \ge p > n$,

$\Gamma_{4n\ge p>n}\{\frac{(4n)!}{n!\cdot(3n)!}\} = \Gamma_{4n\ge p>n}\{\frac{(4n)!}{(3n)!}\}$.

Referring to **(5)**, $\Gamma_{n\ge p>\lfloor 2\sqrt{n}\rfloor}\{\frac{(4n)!}{n!\cdot(3n)!}\} \le \prod_{n\ge p} p$.

It has been proved [5] that $\prod_{n\ge p} p < 2^{2n-3}$ when $n \ge 3$.

Thus for $n \ge 3$, $\binom{4n}{n} < \Gamma_{4n\ge p>n}\{\frac{(4n)!}{(3n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor\ge p}\{\frac{(4n)!}{n!\cdot(3n)!}\}$ **— (6)**

Proposition

For every integer $n > 1$, there exists at least a prime number p such that $3n < p \le 4n$.

Proof:

By induction on n, for n=2, $\binom{4n}{n} = \binom{8}{2} = 28 > \frac{4^{4n-3}}{n\cdot 3^{3n-3}} = \frac{512}{27} \approx 18.96$

If $\binom{4n}{n} > \frac{4^{4n-3}}{n\cdot 3^{3n-3}}$ for n stands, then for n+1,

$$\binom{4(n+1)}{(n+1)} = \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \binom{4n}{n}$$

$$> \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \frac{4^{4n-3}}{n\cdot 3^{3n-3}} = \frac{4}{3}\cdot\frac{4n+3}{3n+2}\cdot\frac{4n+2}{3n+1}\cdot\frac{4n+1}{n}\cdot \frac{4^{4n-3}}{(n+1)\cdot 3^{3n-3}}$$

$$> \frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{1}\cdot\frac{4^{4n-3}}{(n+1)\cdot 3^{3n-3}} = \frac{4^{4(n+1)-3}}{(n+1)\cdot 3^{3(n+1)-3}}$$

Thus for $n \ge 2$, $\binom{4n}{n} > \frac{4^{4n-3}}{n\cdot 3^{3n-3}}$ **— (7)**

Applying **(7)** into **(6)**:

For $n \ge 3$, $\frac{4^{4n-3}}{n\cdot 3^{3n-3}} < \binom{4n}{n} < \Gamma_{4n\ge p>n}\{\frac{(4n)!}{(3n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor 2\sqrt{n}\rfloor\ge p}\{\frac{(4n)!}{n!\cdot(3n)!}\}$ **— (8)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(n) \le \left\lfloor \frac{n}{3} \right\rfloor + 2 \le \frac{n}{3} + 2$. **— (9)**

Referring to **(4)** and **(9)**,

$$\Gamma_{\lfloor 2\sqrt{n} \rfloor \ge p}\left\{\frac{(4n)!}{n!\cdot(3n)!}\right\} = \Gamma_{\lfloor 2\sqrt{n} \rfloor \ge p}\left\{\binom{4n}{n}\right\} \le (4n)^{\pi(2\sqrt{n})} \le (4n)^{\frac{2\sqrt{n}}{3}+2} \quad \text{— (10)}$$

Applying **(10)** into **(8)**: $\frac{4^{4n-3}}{n(3^{3n-3})} < \Gamma_{4n \ge p > n}\left\{\frac{(4n)!}{(3n)!}\right\} \cdot 2^{2n-3} \cdot (4n)^{\frac{2\sqrt{n}}{3}+2}$

Since for $n \ge 3$, both $2^{2n-3} > 0$ and $(4n)^{\frac{2\sqrt{n}}{3}+2} > 0$

$$\Gamma_{4n \ge p > n}\left\{\frac{(4n)!}{(3n)!}\right\} > \frac{4^{4n-3}}{n(3^{3n-3})(2^{2n-3})(4n)^{\frac{2\sqrt{n}}{3}+2}} = \frac{27\cdot\left(\frac{4}{3}\right)^{3n}}{2\cdot(4n)^{\frac{2\sqrt{n}}{3}+3}} = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} \quad \text{— (11)}$$

Let $f(x) = \frac{u}{w}$ where x, u, w are real numbers and $x \ge 42$, $u = \frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}$, $w = (4x)^{\frac{2\sqrt{x}+9}{3}}$

$$\frac{du}{dx} = \left(\frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}\right)' = \frac{27}{2}\cdot\left(\frac{4}{3}\right)^{3x}\cdot 3\cdot ln\left(\frac{4}{3}\right) = u\cdot 3\cdot ln\left(\frac{4}{3}\right)$$

$$\frac{dw}{dx} = \left((4x)^{\frac{2\sqrt{x}+9}{3}}\right)' = \left((4x)^{\frac{2\sqrt{x}+9}{3}}\right)\left(\frac{ln(4x)}{3\sqrt{x}} + \frac{2\sqrt{x}+9}{3x}\right) = w\left(\frac{ln(x)+ln(4)+2}{3\sqrt{x}} + \frac{3}{x}\right)$$

$$f'(x) = \left(\frac{u}{w}\right)' = \frac{w(u)' - u(w)'}{w^2} = \frac{u}{w}\left(3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x}\right)$$

Let $f_1(x) = 3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x}$

Since $f_1'(x) = \frac{ln(x)+ln(4)}{6x\sqrt{x}} + \frac{3}{x^2} > 0$, when $x > 1$, $f_1(x)$ is a strictly increasing function.

When $x = 42$, $f_1(x) = 3\cdot ln\left(\frac{4}{3}\right) - \frac{ln(x)+ln(4)+2}{3\sqrt{x}} - \frac{3}{x} \approx 0.863 - 0.367 - 0.071 = 0.425 > 0$.

Thus, when $x \ge 42$, $f_1(x) > 0$.

Since when $x \ge 42$, u, w, and $f_1(x)$ are greater than zero, $f'(x) = \frac{u}{w}\cdot f_1(x) > 0$.

Thus $f(x)$ is a strictly increasing function for $x \ge 42$. Then when $x \ge 42$, $f(x+1) > f(x)$.

Let $x = n \ge 42$, then $f(n+1) > f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}}$

Since for $n = 42$, $f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{126}}{(168)^{\frac{2\sqrt{42}+9}{3}}} \approx \frac{7.457E+16}{1.952E+16} > 1$, and since

$f(n+1) > f(n)$, by induction on n, when $n \ge 42$, $f(n) = \frac{27}{2}\cdot\frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} > 1$. **— (12)**

Applying **(12)** to **(11)**: When $n \geq 42$, $\Gamma_{4n\geq p>n}\{\frac{(4n)!}{(3n)!}\} > \frac{27}{2} \cdot \frac{\left(\frac{4}{3}\right)^{3n}}{(4n)^{\frac{2\sqrt{n}+9}{3}}} > 1$.

Thus when $n \geq 42$,

$\Gamma_{4n\geq p>n}\{\frac{(4n)!}{(3n)!}\}$

$= \Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{3n\geq p>2n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{2n\geq p>\frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{3n}{2}\geq p>\frac{4n}{3}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{4n}{3}\geq p>n}\{\frac{(4n)!}{(3n)!}\} > 1.$

If there is any prime number p such that $3n \geq p > 2n$, then $(4n)!$ has a factor of p in this range, and $(3n)!$ also has the same factor of p. Thus, they cancel to each other in $\frac{(4n)!}{(3n)!}$ with no prime number in this range. Referring to **(2)**, $\Gamma_{3n\geq p>2n}\{\frac{(4n)!}{(3n)!}\} = 1$.

If there is any prime number p such that $\frac{3n}{2} \geq p > \frac{4n}{3}$, then $(4n)!$ has the product of $p \cdot 2p$, and $(3n)!$ also has the same product of $p \cdot 2p$. Thus, they cancel to each other in $\frac{(4n)!}{(3n)!}$ with no prime number in this range. Referring to **(2)**, $\Gamma_{\frac{3n}{2}\geq p>\frac{4n}{3}}\{\frac{(4n)!}{(3n)!}\} = 1$.

Thus, when $n \geq 42$,

$\Gamma_{4n\geq p>n}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{2n\geq p>\frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \cdot \Gamma_{\frac{4n}{3}\geq p>n}\{\frac{(4n)!}{(3n)!}\} > 1.$ **— (13)**

Referring to **(1)**, $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} \geq 1$, $\Gamma_{2n\geq p>\frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} \geq 1$, and $\Gamma_{\frac{4n}{3}\geq p>n}\{\frac{(4n)!}{(3n)!}\} \geq 1$.

If $\Gamma_{2n\geq p>\frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} = 1$ or $\Gamma_{\frac{4n}{3}\geq p>n}\{\frac{(4n)!}{(3n)!}\} = 1$, it will drop out from **(13).**

If $n \geq 42$ and $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} > 1$, then referring to **(3)**, there exists at least a prime number p such that $3n < p \leq 4n$. **— (14)**

$\Gamma_{2n\geq p>\frac{3n}{2}}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4\cdot(\frac{n}{2})\geq p>3\cdot(\frac{n}{2})}\{\frac{(4n)!}{(3n)!}\}$.

If $\frac{n}{2} \geq 21$ and, $\Gamma_{4\cdot(\frac{n}{2})\geq p>3\cdot(\frac{n}{2})}\{\frac{(4n)!}{(3n)!}\} > 1$, let $m_1 = \frac{n}{2}$, then when $m_1 \geq 21$, there exists at least a prime number p such that $3m_1 < p \leq 4m_1$. Since $n \geq 42 > m_1 \geq 21$, the statement is also valid for n. Thus, when $n \geq 42$, if $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} > 1$, then $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} > 1$, there exists at least a prime number p such that $3n < p \leq 4n$. **— (15)**

$\Gamma_{\frac{4n}{3}\geq p>n}\{\frac{(4n)!}{(3n)!}\} = \Gamma_{4\cdot(\frac{n}{3})\geq p>3\cdot(\frac{n}{3})}\{\frac{(4n)!}{(3n)!}\}$.

If $\frac{n}{3} \geq 14$ and, $\Gamma_{4\cdot(\frac{n}{3})\geq p>3\cdot(\frac{n}{3})}\{\frac{(4n)!}{(3n)!}\} > 1$, let $m_2 = \frac{n}{3}$, then when $m_2 \geq 14$, there exists at least a prime number p such that $3m_2 < p \leq 4m_2$. Since $n \geq 42 > m_2 \geq 14$, the statement is also valid for n. Thus, when $n \geq 42$, if $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} > 1$, then $\Gamma_{4n\geq p>3n}\{\frac{(4n)!}{(3n)!}\} > 1$, there exists at least a prime number p such that $3n < p \leq 4n$. **— (16)**

From the right side of **(13)**, at least one of these 3 factors is greater than one when $n \geq 42$. From **(14)**, **(15)**, and **(16)**, when $n \geq 42$ and any one of these 3 factors is greater than one, there exists at least a prime number p such that $3n < p \leq 4n$. **— (17)**

Table 1 shows that when $2 \leq n \leq 42$, there is a prime number p such that $3n < p \leq 4n$. **— (18)**

Thus, the proposition is proven by combining **(17)** and **(18)**: For every integer n>1, there exists at least a prime number p such that $3n < p \leq 4n$. **— (19)**

Table 1: For $2 \leq n \leq 42$, there is a prime number p such that $3n < p \leq 4n$.

$3n$	6	9	12	15	18	21	24	27	30	33	36	39	42	45
p	7	11	13	17	19	23	29	31	37	41	43	47	53	59
$4n$	8	12	16	20	24	28	32	36	40	44	48	52	56	60
$3n$	48	51	54	57	60	63	66	69	72	75	78	81	84	87
p	61	67	71	73	79	83	83	89	89	97	97	101	101	103
$4n$	64	68	72	76	80	84	88	92	96	100	104	108	112	116
$3n$	90	93	96	99	102	105	108	111	114	117	120	123	126	
p	103	107	107	109	109	113	113	127	127	131	131	137	139	
$4n$	120	124	128	132	136	140	144	148	152	156	160	164	168	

References

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