What is the Value of the Function x/x at x=0? - What is 0/0?

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Abstract: It will be a very pity that we have still confusions on the very famous problem on 0/0 and the value of the elementary function of x/x at x=0. In this note, we would like to discuss the problems in some elementary and self contained way in order to obtain some good understanding for some general people.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 Introduction

It will be a very pity that we have still confusions on the very famous problem on 0/0 and the value of the elementary function of x/x at x=0. In this note, we would like to discuss the problems in some elementary and self contained way in order to obtain some good understanding for some general people.

Firstly, recall that

$$\frac{0}{0} = 0/0 = 0$$

was stated by the founder Brahmagupta (598 -668 ?) who established four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brāhmasphuṭasiddhānta. We have been, however, considering that his definition $0/0 = 0$ is wrong for over 1300 years, but, we saw that his definition is right and suitable. However, its meaning will be still vague in a sense. When we consider the fractional in the usual way

$$\frac{0}{0} = X,$$

and so,

$$0 = 0 \times X,$$

we can not determine X uniquley, indeed, we can consider such X as any number - **undetermined**.

Meanwhile, from the fact

$$\frac{x}{x} = 1$$

except for $x = 0$, some few people consider that

$$\frac{0}{0} = 1.$$

First of all, we have to recall that in our usual axioms for the number fields $\mathbf{R}$ and $\mathbf{C}$ we do not consider the division by zero $x/0$ and so, when we consider such fractionals, we have to give their meanings (definitions) strictly. However, such an introduction is quite simple and we can introduce the Yamada field contaning the division by zero fractionals $x/0$. See [2] for the details.

This statement is for the general fractions

$$\frac{a}{b}$$

containing the case of $b = 0$.

The very natural and strong motievations of the Yamada field are given by the Moore-Penroze generalized solution for the simple and fundamental equation $bx = a$ and the following the Takahasi uniquess theorem:

Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying

$$F(b,a)F(c,d) = F(bc,ad)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a,b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b,0) = 0.$$

On the long mysterious history of the division by zero, this fact seems to be decisive. Indeed, Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following the fact, we should **define**

$$F(b,0) = \frac{b}{0} = 0.$$

Of course, the division by zero fractionals $\frac{b}{0}$ are not the usual fractions, and so we have to consider the fractionals with the Yamada field laws or with the properties of the Moore-Penrose generalized inverses. Precisely, see the cited references.

Next we shall consider the function case. Note firstly that fraction case as numbers and function case are different essentially. In mathematics, the definition is, of course, very fundamental and important. Many confusions on the division by zero are based on the definitions of the division by zero.

2 Essence of division by zero calculus

We will state very elementary facts and so, in order to state the contents in a self contained way, we state the essence of division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n, \tag{2.1}$$

we will **define**

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n! C_n.$$

However, we can consider the more general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [1, 2, 3, 4].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

3 Hyperbolic case

In the hyperbolic curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{3.1}$$

by the representation by parameter t

$$x = \frac{a}{\cos\theta} = \frac{a}{2}\left(\frac{1}{t} + t\right)$$

and

$$y = \frac{b}{\tan\theta} = \frac{b}{2}\left(\frac{1}{t} - t\right),$$

the origin $(0, 0)$ may be included as the point of the hyperbolic curve, as we see from the cases $\theta = \pm\pi/2$ and $t = 0$.

In addition, from the fact, we will be able to understand that the asymptotic lines are the tangential lines of the hyperbolic curve.

The two tangential lines of the hyperbolic curve with gradient m is given by

$$y = mx \pm \sqrt{a^2m^2 - b^2} \tag{3.2}$$

and the gradients of the asymptotic lines are

$$m = \pm\frac{b}{a}.$$

Then, we have asymptotic lines $y = \pm\frac{b}{a}x$ as tangential lines.

The common points of (3.1) and (3.2) are given by

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2m}{\sqrt{a^2m^2 - b^2}}\right).$$

For the case $a^2m^2 - b^2 = 0$, they are $(0, 0)$.

4 Parabolic case

For the envelop of the lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{4.1}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{4.2}$$

The origin of this parabolic function is excluded from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis.

– This function may be considered as a function with zero gradient and passing the point at infinity; however, the point at infinity is represented by 0, the origin; that is, the line may be considered as the x axis. Furthermore, then we can consider the x axis as a tangential line of the parabolic function, because they are gradient zero at the point at infinity. –

Furthermore, we can say that the x axis $y = 0$ and the parabolic function have the zero gradient at **the origin**, since $\tan(\pi/2) = 0$; that is, in the reasonable sense the x axis is a tangential line of the parabolic function.

Indeed, we will see the surprising property that the gradient of the parabolic function at the origin is zero. We have many examples, see [1].

Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is a native tangential line of the parabolic function at the origin. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (4.2) for the line (4.1) is given by

$$\left(\frac{p}{m}, \frac{2p}{m}\right)$$

and it is $(0, 0)$ for $m = 0$.

We can see that the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is the reflected one of the point at infinity.

5 On the function x/x at x=0

Apparently, by our division by zero calculus, for the function

$$f(x) = \frac{x}{x},$$

we have

$$f(0) = \left(\frac{x}{x}\right)_{x=0} = 1.$$

However, when we write the result as

$$f(0) = \left(\frac{x}{x}\right)_{x=0} = \frac{0}{0},$$

we have the contradiction

$$f(0) = 0.$$

Therefore, we can not write so. We have to consider the difference

$$\left(\frac{x}{x}\right)_{x=0}$$

and

$$\frac{0}{0}.$$

In this note, we shall refer to some interesting property of the function

$$f(x) = \frac{x}{x}.$$

We will consider the function with the parameter representation by t

$$x = t - \frac{1}{t}, \quad y = t^2 + \frac{1}{t^2}.$$

For $t \neq 0$, it represents the function

$$y = x^2 + 2.$$

Note that for $t = 1$, it represents the point

$$(0, 2).$$

However, by the division by zero calculus it represents the point for $t = 0$

$$(0, 0).$$

What does the point $(0, 0)$ mean? However, we can see its reason completely that the origin represents the point at infinity and the function passes the point at infinity. We can see its total figure on the horn torus on which the point at infinity and zero point are attaching. See the cited references.

Here, we see that for $t \neq 0$,

$$y = x^2 + 2 = \left(t - \frac{1}{t}\right)^2 + 2 = t^2 - 2t \times \frac{1}{t} + \frac{1}{t^2} + 2$$

$$= t^2 - 2\frac{t}{t} + \frac{1}{t^2} + 2.$$

Then, if we use the result

$$\left(\frac{t}{t}\right)_{t=0} = 1,$$

by the division by zero calculus, we obtain the result

$$(0, 0).$$

Meanwhile, when we use the result of the division by zero

$$\left(\frac{t}{t}\right)_{t=0} = 0,$$

we have the result

$$(0, 2).$$

Therefore, the important case appears **by mixing both results of the division by zero and division by zero calculus.**

6 Conclusion

We introduced the new tangential lines by the concept of the division by zero. We referred to the relation of the point at infinity and the origin.

For the function x/x we should consider both cases at $x = 0$ as the values 1 and 0. Then, we should check the results for the both cases. Of course, we can not consider always the cancellation as in

$$\left(\frac{x}{x}\right)_{x=0} = 1.$$

If we do not consider the division by zero calculus, we can not consider the case $t = 0$ and we can not catch the point at infinity that may be considered as the point of the function on the horn torus.

References

[1] H. Okumura, *Geometry and division by zero calculus,* International Journal of Division by Zero Calculus, **1**(2021), 1-36.

[2] S. Saitoh, *Introduction to the Division by Zero Calculus*, Scientific Research Publishing, Inc. (2021), 202 pages.

[3] S. Saitoh, *History of Division by Zero and Division by Zero Calculus*, International Journal of Division by Zero Calculus, **1** (2021), 1-38.

[4] S. Saitoh, *Division by Zero Calculus - History and Development*, Scientific Research Publishing, Inc. (2021.11), 332 pages.