Conjectures Emerging Not-So-Unsettled:

Oppermann's, Firoozbakht's, Legendre's, Andrica's, Brocard's, etc.

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ABSTRACT[1]

All of the key Conjectures that may still be of interest beyond, or irrespective of, assuming the validity of RH appear straightforward to tackle with a handful of minimalist instruments based on the RI solely. A host of extra Propositions (zeta based formulae & equivalences) being spawned from the latter pillar as a 'side effect' of importance in its own right.

3/3/22[2]

Background: Conjectures Grand & Less So

Oppermann's Conjecture: At least one prime could be fitted in between every pair of a square and a pronic product for numbers exceeding 1.

$$\pi\{n^2\} - \pi\{n(n-1)\} \geq 1 \leq \pi\{(n+1)^2\} - \pi\{n(n+1)\} \quad (COpp)$$

Legendre's Conjecture:

$$\pi\{(n+1)^2\} - \pi\{n^2\} \geq 1 \qquad (CLeg)$$

Brocard's Conjecture: There exist at least 4 primes in between p_{n+1} and p_n squares

$$\pi\{{p_{n+1}}^2\} - \pi\{{p_n}^2\} \geq 4 \quad (CBroc)$$

Andrica's Conjecture:

$$\forall n\colon\ \sqrt{p_{n+1}} - \sqrt{p_n} < 1 \qquad (CAnd1)$$

[1] *In memoriam* each and every child that may [have] come to be terrified by, let alone murdered in, arbitrarily inflicted war conflicts…

[2] Developed over a time span of 15 through 23 February 2022, postponed on & suspended in light of the ongoing dramatic turmoil.

$$\exists x:\ p_{n+1}^x - p_n^x \equiv \begin{cases} 1 & \leftrightarrow x_{min} \cong .567148 \\ & < 1 \leftrightarrow x < x_{min} \end{cases} \qquad (CAnd2)$$

Cramer's Conjecture:

$$p_{n+1} - p_n = O((logp_n)^2) \qquad (Cram1)$$

$$p_{n+1} - p_n \equiv O(p_n^x)\ \leftrightarrow x \approx .525 \quad (Cram2)$$

Firoozbakht's Conjecture: $p^{1/n}(n)$ is strictly decreasing

$$p_{n+1} < p_n^{1+\frac{1}{n}} \qquad (ConFir)$$

Reducing on Riemann Identity

To begin with, RI can be juxtaposed against an *auxiliary* functor, which metric could effectively be seen as a *partial (interior) zeta*.

$$\exists E_n(\cdot): \frac{p_n^s}{p_n^s - 1} \equiv \frac{E_n}{E_{n-1}} \ \leftrightarrow \prod_1^N \frac{E_n}{E_{n-1}} = \frac{E_N}{E_0} \equiv \zeta_N(s),\ \zeta_T(s) = \zeta(s), T \to +\infty \quad (1)$$

It is a matter of pragmatic convention whether to assume $E_0=1$ or $E_1=1$ depending on how p_0 and p_1 count respectively. In any event, it is always an option to calibrate technically (experimentally) for the 'right' corner, or initial condition. In the meantime, please observe from (1) that the partial *and* complete (conventional, Riemann's) zeta could prove either bounded or otherwise.

Better yet, a reduction of the form (1a) could be considered for a phi being a finite function, preferably a linear operator such that, in Big O notation (the rare incident we invoke asymptotics explicitly in the present exposition):

$$\exists \varphi(\cdot): \forall Y = \varphi X \sim O(X)$$

$$E_n \equiv \varphi p_n^s \ \ iff\ \ E_{n-1} \equiv \varphi(p_n^s - 1) = \varphi p_{n-1}^s \ \ (1a)$$

A functional recursion (2) readily obtains from (1a) while suggesting a somewhat generalized interim form (2a):

$$p_n^s = p_{n-1}^s + 1 \qquad (2)$$

$$\forall k \le n:\ p_n^s = p_k^s + (n-k) = p_0^s + n = p_1^s + (n-1) \qquad (2a)$$

Now, of course, a more general function *phi(n)*, possibly a *non*linear operator, could comply just as well with the exact same RI structural requirement (with an eye on an interim-canceling property which meets the *orduale*, "simplicity-in-completeness" criterion). In this case, (1a) would be reworked as something like (1b), with commonality of the functional stretchings implying that of the arguments net of a residuale *R(n)* which, depending on the nature of the transforms, could be either a period or an integration constant, or the like. In other words:

$$\varphi_n\{p_n^s\} = \varphi_n\{(p_{n+1}^s - 1)\} \leftrightarrow \{p_n^s\} = \{(p_{n+1}^s - 1)\} + R_n$$
$$= \{p_k^s\} + (n-k) - \sum_{i=1}^{n} R_i \qquad (1b)$$

On the other hand, a stretching would have to act consistently on both sides. E.g. if taking on a (1c) form, then again, our *identity-based* approach would suggest (1d):

$$\varphi_n\{(a+b)\} \equiv a^s + b^s \rightarrow \varphi_n\{(p_n - p_{n-k})\} \equiv \varphi_n\{(p_n - p_{n-k})\} \leftrightarrow \{p_n^s\}$$
$$= \{p_k^s\} + \{p_n^s - p_k^s\} \sim \{p_k^s\} + (n^s - k^s) = \{p_1^s\} + (n_p^s - 1^s),$$
$$n = n_p \equiv p_n \in \boldsymbol{N} \cap \boldsymbol{P} \quad (1d)$$

Though possibly looking trivial under some oversimplifying assumption as well as controversial in regards of resummation-based calibration, still (1d) accommodates the highly productive generating pattern, in particular the unity sign reversal $p_n=2m_n\pm1$—more so under the RH, or Re(s)=1/2.

Somehow, (2a) suggests (3):

$$p_n^s - 1 = p_{n-1}^s \leftrightarrow \prod_{n}^{T} \frac{p_n^s}{p_n^s - 1} = \prod_{n=1}^{T} \frac{p_n^s}{p_{n-1}^s} = \frac{p_T^s}{p_1^s} = (\frac{p_T}{p_1})^s \equiv \zeta(s) = 1 + (T-1) * p_1^{-s} \qquad (3)$$

Less awkward, if we invoke (1a) again, phi could be calibrated for based on:

$$E_T \sim \zeta(s) \equiv \begin{cases} \varphi p_T^s \\ \varphi_T p_T^s \end{cases} \rightarrow \varphi \sim \zeta(s) * p_T^{-s} \qquad (4a)$$

$$E_n = \varphi p_n^s \sim \zeta(s) * (\frac{p_n}{p_T})^s \qquad (4b)$$

$$\prod \frac{E_n}{E_{n-1}} = \frac{E_T}{E_1} = (\frac{p_T}{p_1})^s = p_T^s = E_T = \zeta(s) \qquad (4c)$$

Seems like, a linear phi (unless a Mikusinski-type operator) can be assumed away as canceling out. But that might imply the trivial controversy of zeta collapsing to an infinite power of unity, *s* irrespective. This entire setup could be rethought as follows (in many modes, linear and otherwise, with a phi-hjatt being non-unitary/non-degenerate), e.g.:

$$\exists\widehat{\varphi_n}: \frac{p_n^s}{p_n^s-1} \equiv \widehat{\varphi_n} = \frac{E_n}{E_{n-1}} \;\rightarrow\; \prod \frac{E_n}{E_{n-1}} = \zeta(s) = \prod \widehat{\varphi_n} \qquad (5a)$$

$$p_n^s = \frac{\widehat{\varphi_n}(s)}{\widehat{\varphi_n}(s)-1}\,,\;\; p_n^s - 1 = \frac{1}{\widehat{\varphi_n}(s)-1} \qquad (5L)$$

$$p_n^s = \widehat{\varphi_n}\{p_{n-1}^s\} = \widehat{\varphi_n}^{[n-k]}\{p_k^s\} = \widehat{\varphi_n}^{[n-1]}\{p_1^s\} \quad (5NL)$$

Rethinking Along Orduale/Residuale Lines

The entire setup could be reapproached in a more straightforward, less *azimuthale* fashion. For starters, consider an *orduale* representation pertaining to identity-based (i.e. ensured) fitting (6).

$$\forall n, k \le n \in \boldsymbol{N}\ \exists\varphi: \frac{P_n}{P_k} \equiv \begin{cases}\varphi\zeta \\ \zeta^{\varphi}\end{cases} \quad , \quad \forall P_x \in \boldsymbol{P} \qquad (6)$$

Irrespective of whether or not the above fudge-calibrations act as [non-linear] operators or mere multipliers/powers, one other (in fact, closely collated) approach of a residuale nature could be embarked on (7) suggesting that, for any number of objects, it is conceivable to find a *rho-aggregate* (a Lame function or constant elasticity of substitution) rendering them one, or making their ad-hoc *complete* set *simply* intra-related. (In a sense, this could be an extension of *groups* with rho acting as a particular *generalized operation*, as was proposed in Shevenyonov (2019).)

$$\forall\Delta\exists\rho: P_n^{\rho} \equiv P_k^{\rho} + \Delta^{\rho} \qquad (7)$$

One can also make sure the above generalizes Andrica's conjecture (*CAnd2*) in a number of ways albeit interrelated.

Now, if we were to refer again to the very definition of the zeta as in RI, extra convenience can be salvaged from (8).

$$\exists B(\cdot): \zeta(s) \equiv \frac{B_2}{B_1} * \frac{B_3}{B_2} \dots \frac{B_T}{B_{T-1}} = \frac{B_T}{B_1} = \frac{\varphi\widetilde{P_T}}{\varphi\widetilde{P_1}} \;\leftrightarrow\; \widetilde{P_n} = \widetilde{P_{n-\Delta}} * \zeta, \qquad \Delta = T-1 \rightarrow \infty$$

$$\widetilde{P_n} \equiv P_n^s = P_1^s * \zeta^{\frac{n-1}{T-1}} = P_k^s * \zeta^{\frac{n-k}{T-1}}(s) \qquad (8)$$

$$P_n = \zeta^{\frac{n-1}{T-1}} \sim 1^{\frac{n-1}{s}} \qquad if\ P_1 = 1 \qquad (8a)$$

Please check this holds identically for *n=T* and *k=1*, even as the actual calibrating parameter could (or may have to) be different, if one were to arrive at a working closed form as well as recurrent and efficient formula for generating primes based on the indices alone. Still, the structure and order of magnitude should suffice for inference in *orduale* (weak, monotonous, ol-multiplicative) terms while staying more cautious when it comes to *residuale* (stronger-form, ol-additive) implications. (The archaic-Latin "*ol*-" being the core/weaker root-stem of "ultra/ultimate") For instance, depending on how successful the convention of P_1=1works out, the generating structure is captured in (8a). Or, invariant thereto (and perhaps the particular *s*-value beyond power-of-unity), a relation/ratio for any two primes could be inferred as one ultimate alternative to conventional differentials being tested by the grand conjectures. In passing, the P_1 could *capture*/hide/subsume whatever calibration factor best fits.

At this rate, Andrica's conjecture (which does resemble phi-tweaking as in RI) could be addressed as follows.

$$\exists x: P_{n+1}^x - P_n^x \equiv 1 = 1^x = P_n^x * \left(\zeta^{\frac{1}{[T-1]*s}} - 1\right) \leftrightarrow (\frac{P_{n+1}}{P_n})^x = \zeta^{\frac{1}{[T-1]*s}} \sim 1^{1/s} \quad (8b)$$

On second thoughts, (8) can be reworked with respect to the interior as opposed to the corners, in *partial*-zeta terms:

$$P_n^s = P_k^s * \zeta_n{}^{\frac{n-k}{n-1}} \quad (8')$$

In which light, (8b) takes on a modified representation (8c):

$$(\frac{P_{n+1}}{P_n})^x = \zeta_n{}^{\frac{1}{[n-1]*s}} \quad (8c)$$

Again, anywhere near the extreme, *x* could take on nearly any value under-unity (if only because of the unity-reversion or conversion in the RHS as per any reasonable *s*-value above unity). *Within* the *interior n<T*, however, the lower bound is higher than that at around .50457 as a tossup in between the P2/P2=2/1=2 versus P3/P2=3/2=1.5 corners with the same weights attached to [complete or extreme-case] zeta taken to the respective powers (as adjusted to an allowance for a partial-to-complete zeta).

$$x_{min} > \frac{3/2}{\frac{3}{2} + 2/1} * \frac{log\zeta^{\frac{1}{2}}(2)}{\log(3/2)} + \frac{2/1}{\frac{3}{2} + 2/1} * \frac{log\zeta^{\frac{1}{2}}(2)}{\log(2/1)} = .50457 \quad (8d)$$

This proves (*CAnd2*), whilst (*CAnd1*) could build on (8b) by substituting *x=1/2*, such that the setup becomes a matter of infinity-times-zero comparable to unity in the extreme case, and less per any interior cases:

Now, how about Firoozbakht's conjecture? Just replace/specify *x=1/n* to arrive at (8e):

$$P_n^{1/n} = P_k^{1/n} * \zeta^{\frac{1}{[T-1]*sn}} \sim const^{1/n} \quad (8e)$$

As long as an arbitrary initial/interior reference value P_k is viewed as an exogenous function, (*ConFir*) does hold with an eye on the resultant function being strictly decreasing.

As far as Cramer's is concerned, by invoking (8) again, observe:

$$P_{n+1} - P_n = P_n * \left(\zeta^{\frac{1}{[T-1]*s}} - 1\right) = P_1 * \zeta^{\frac{n-1}{[T-1]*s}} * \left(\zeta^{\frac{1}{[T-1]*s}} - 1\right) \quad (8f)$$

Not only does (8f) resemble *Euler's beta*-like density (as if to allow for oscillations), it also hints at surefire applicability of the aforementioned *rho-residuale*! For one thing, (*Cram2*) follows from the early interior case around n-1=n/2, i.e. n=2 or 3 (at the extreme, the gap collapses to *o(zeta^(1/s))* in small-o). In partial-zeta terms, (8f) becomes (8f*):

$$\Delta P_n \sim \begin{cases} P_1 * \zeta_2^{\frac{1}{s}} * \left(\zeta_2^{\frac{1}{s}} - 1\right), & \zeta_2 = \dfrac{P_1^s P_2^s}{(P_1^s - 1)(P_2^s - 1)} \sim \dfrac{2^s}{2^s + 1 - (1^s + 2^s)} = \dfrac{2^s}{1 - 1^s} \\ P_1 * \zeta_3^{\frac{1}{2s}} * \left(\zeta_3^{\frac{1}{2s}} - 1\right), & \zeta_3 = \zeta_2 * \dfrac{P_3^s}{P_3^s - 1} = \dfrac{6^s}{(3^s - 1)(1 - 1^s)} = \dfrac{6^s}{3^s + 1^s - 1} \end{cases}$$

$$\Delta P_n \sim P_1 * \begin{cases} \left(\frac{2^s}{1-1^s}\right)^{1/s} * \left[\left(\frac{2^s}{1-1^s}\right)^{\frac{1}{s}} - 1\right] \sim 4T^{\frac{2}{s}} - 2T^{1/s} \to 2, \; s \in \boldsymbol{R} \gg 0 \\ \left(\frac{6^s}{3^s+1^s-1}\right)^{1/s} * \left[\left(\frac{6^s}{3^s+1^s-1}\right)^{\frac{1}{s}} - 1\right] \sim 2, \; s \in \boldsymbol{R} \end{cases} \quad n = 2, 3 \quad \text{(8f*)}$$

Both these values about stand the scrutiny of the square root of either 2 or 3, and both P_2 and P_3 for that matter. A still more-rigorous procedure could apply identically (8f**):

$$\exists x: \; \zeta_n^{\frac{1}{[n-1]s}} * \left(\zeta_n^{\frac{1}{[n-1]s}} - 1\right) \cong \zeta_n^{\frac{x}{s}} \; \leftrightarrow x \cong \frac{\frac{1}{s} * log\zeta_n}{\frac{2}{[n-1]s} * log\zeta_n + \varepsilon_n} \sim \frac{n-1}{2} \quad (8f **)$$

Again, for the case n=2, 3, *x* seems to be a fairly accurate [lower-bound] predictor of the experimental (empirical, phenomenological) value of .525 under epsilon tending to *o*(1) asymptotically. By contrast, the more accurate estimate as in (8f***) may have to be considered in the more general domains of (*n, s*), leaving *x* largely intact structurally.

$$x' \approx \left(\frac{2}{n-1} + \varepsilon_n * \varphi_n\right)^{-1} \sim x, \qquad \varphi_n \sim \begin{cases} (log2T)^{-1} \sim o(1) \\ (log2)^{-1} = 3.3219 \end{cases} \quad (8f ***)$$

In the meantime, one may have come to wonder just whence all these mysterious conflations around ½ arise. It happens, a peculiar recursion could be proposed for Andrica's and RH alike:

$$x = t \sim \sqrt{t - 1/2^2} \approx .56 \ldots \leftrightarrow \frac{s - \frac{1}{2}}{i} = t \sim s\bar{s} = \left(\frac{1}{2} + it\right)\left(\frac{1}{2} - it\right) \leftrightarrow s \approx \frac{1}{2} + is\bar{s} \ (8f')$$

The 'naive' conjecture posits a product of the RH conjugates in the RHS.

On the other hand, based on the limit convention (*X*), (8g) obtains:

$$\frac{x^\rho - 1}{\rho} \xrightarrow[\rho \to 0]{} logx \qquad (X)$$

$$\zeta^{\frac{1}{[T-1]*s}} - 1 \sim \frac{log\zeta^{\frac{1}{s}}}{[T-1]} = \frac{log\zeta}{[T-1]*s}$$

$$\zeta^{\frac{n-1}{[T-1]*s}} \sim \left(\frac{log\zeta}{[T-1]*s} + 1\right)^{n-1} \approx \frac{[n-1]*log\zeta}{[T-1]*s} + 1$$

$$logP_n = logP_1 + \frac{[n-1]*log\zeta}{[T-1]*s} = \frac{[n-1]*log\zeta}{[T-1]*s}$$

$$\Delta P_n \approx [n-1]\left(\frac{log\zeta}{[T-1]*s}\right)^2 = \begin{cases} O(log^2 P_n) \\ o(log^2 P_n) \end{cases} \qquad (8g)$$

Please note that the Big O result comprises the interior (smaller *n*) cases, while small-o anywhere near the extreme (larger *n*). Incidentally, as will be shown in the subsequent section, this very result (*Cram1*) could be discerned more directly from RI equivalences.

Furthermore, Brocard's conjecture can be tackled in all too straightforward a way (even though we have alternate, more elegant means on hand).

$$\pi\{{p_{n+1}}^2\} - \pi\{{p_n}^2\} = \frac{{p_{n+1}}^2}{2logp_{n+1}} - \frac{{p_n}^2}{2logp_n} > \frac{{p_{n+1}}^2 - {p_n}^2}{2logp_n}$$

$$= \frac{P_1}{2logp_n}\left[\zeta^{\frac{2n}{[T-1]*s}} - \zeta^{\frac{2(n-1)}{[T-1]*s}}\right] \sim \frac{{\zeta_n}^{\frac{2}{[1-1/n]*s}} - {\zeta_n}^{2/s}}{\frac{2}{s} * log\zeta_n}$$

$$Z^{\frac{1}{1-1/n}} - Z \geq 4logZ \quad IF \quad Z^{\frac{1}{n-1}} \geq \frac{4}{\pi(Z)} + 1 < \frac{4}{Z} + 1, \qquad IF\ Z \equiv {\zeta_n}^{2/s} \geq 4$$

$$Z^{\frac{n}{n-1}} > Z + 4 \quad IF \quad \frac{n}{n-1} > \frac{\log(Z+4)}{logZ} > \frac{log2Z}{logZ} = 1 + \frac{log2}{Z} > \frac{3}{2} \quad (9)$$

Apparently, for *n* large enough and tending to T (effective infinity), the criterion in (9) reduces to (9a):

$$o(\pi(Z))\sim o\left(\pi\left(\zeta^{\frac{2}{s}}\right)\right)\geq 4 \quad (9a)$$

But, this extreme case might be of lesser relevance compared to an interior, even as it calls for the PND frequency-equivalence (the *pi*) to take on extremely large values. Anyway, as imperfect as (9) might be, it seems to be pointing to n>>3 being the required possibility, according as (*Broc*) postulates, with Z>4 requirement met for most cases we have observed, yet probably only per select cases of zeta-complete as opposed to partial.

Again, the representation can be rethought more elegantly, by making use of (8g):

$$\Delta P_n \approx [n-1]\left(\frac{log\zeta}{[T-1]*s}\right)^2 = (logZ^{1/2})^2 = \frac{1}{(n-1)} * log^2 P_n \quad (8g*)$$

$$\pi\{p_{n+1}{}^2\} - \pi\{p_n{}^2\} > \frac{2P_n\Delta P_n}{2logP_n} = \frac{P_n log^2 P_n}{(n-1)logP_n} = \frac{P_n logP_n}{(n-1)} \geq 4 \ \ IFF\ P_n logP_n \geq 4(n-1) \quad (9b)$$

$$IFF \quad \Delta P_n \pi(P_n) \equiv 2m * \pi(P_n) > 4 \ \leftrightarrow IFF \ \ \pi(P_n) > 2 \qquad (9c)$$

Obviously, (9c) always holds for large enough primes, i.e. in excess of 3, as shown before. One other way of positing this would be to re-qualify (8g*) to account for the selfsame prime delta of *2m* (the larger the *m*, the lower the interval PND density):

$$P_n > e^{\sqrt{2m(n-1)}}, \qquad m \in \boldsymbol{N} \geq 1 \quad (8h)$$

Following prior exposition above, the remaining two conjectures by Legendre-Oppermann (*LO*/*OL*) set of conjectures should be a "piece of pie."

$$\pi\{(n+1)^2\} - \pi\{n(n+1)\} = \frac{(n+1)^2}{2\log(n+1)} - \frac{n(n+1)}{logn + \log(n+1)} > \frac{(n+1)}{2\log(n+1)} = \frac{\pi(n+1)}{2}$$

$$\pi\{n^2\} - \pi\{n(n-1)\} > \frac{\pi(n)}{2} \quad (9d)$$

Needless to say, both sides of (9d) feature values in excess of 1 for large enough n>2. QED for both (*LO/OL*).

Zeta Based Formulae & Equivalences

To start with, we herein propose a set of *rho-transitions* with rho tending to zero, so as to shed some prior light on just how counterintuitive some of the relationships sought after might emerge. Aside from (X) above, consider the following.

$$\forall x:\ e^x = (1+\rho x)^{1/\rho} \leftrightarrow loge^x = \frac{e^{x\rho}-1}{\rho} = x \quad (X1)$$

$$\forall x, x_0: \Delta logx = \frac{x^\rho - x_0^\rho}{\rho} = \frac{(x/x_0)^\rho - 1}{\rho} \qquad (X2)$$

$$log\prod y = \frac{(\prod y^\rho) - 1}{\rho} = \sum \frac{(y^\rho - 1)}{\rho} \leftrightarrow \sum (y^\rho - 1) = (\prod y)^\rho - 1 \quad (X3)$$

Evidently, (X3) comes very close to unearthing the interlinkage between additivity (generalized as residuality) versus multiplicity (orduality)! We will, however, be interested in one-to-one or at least onto-correspondences, or per-element comparisons, if we are to infer meaningful, closed-form expressions generating primes over and above whatever has been proposed in literature. While at it, consider how (X3) can be narrowed down to (X3*):

$$\forall k: logy^k = \frac{y^{k\rho} - 1}{\rho} = k(y^\rho - 1) \leftrightarrow \frac{y^{k\rho} - 1}{k\rho} = y^\rho - 1 \quad (X4)$$

$$\rho x = \log(1+\rho x) \qquad (X5a)$$

$$x^\rho = e^{x^\rho} - 1 \qquad (X5b)$$

Interestingly enough, (X4) depicts *k-invariance*.

Now, why don't we zoom in on the innermost interchange between the elements of objects as diverse as sums and products? Suppose (P0), which hints at RI *inter alia*:

$$\exists x, y: \sum x \equiv \prod y \qquad (P0)$$

It can be shown (less trivial) and readily verified (very easily by [re]summation or multiplication) that:

$$x_n = \varphi logy_n, \qquad \varphi = \frac{\Pi}{log\Pi} = \frac{\Sigma}{log\Sigma} \quad (P1)$$

When applied to RI, the phi-fudge amounts to nothing other than an equivalent of PND frequency over *n=zeta*:

$$\varphi_{RI} = \frac{\zeta(s)}{log\zeta(s)} = \pi(\zeta) \equiv \pi_s \qquad (P1')$$

One is now in a position to appreciate that:

$$n^{-s} = \pi_s log(1 - P_n^{-s})^{-1} \qquad (P2)$$

Based on (X), one arrives at (P2a), which it is natural to linearize toward (P2b).

$$\frac{\rho n^{-s}}{\pi_s} = 1 - (1 - P_n^{-s})^{\rho} \qquad (P2a)$$

$$P_n^s \sim \pi_s n^s \qquad (P2b)$$

In a sense, the pi-frequency acts as an activator of a “*primalize*” type, whilst its inverse as a “[re]naturalize” operator. A convention could come in handy whereby any natural either proves a respective prime simultaneously or it does not, in which event the pi(s) takes on corner (superficially or phenomenologically Boolean type) values of 1 versus 0. A summation of these over a particular interval spanning 1 to N, though, returns PND *pi equivalence*:

$$\sum_{k=1}^{N} \pi_s^{1/s} 1_k \sim \pi(N) \qquad (P3)$$

Somewhat more rigorously, (P4) holds and can be checked in a number of ways, not least inferred trivially from (P2), (X):

$$P_n^{-s} = 1 - [1 - \rho \frac{n^{-s}}{\pi_s}]^{1/\rho} = 1 - e^{-\frac{n^{-s}}{\pi_s}} \qquad (P4)$$

At the same time, by substituting things back into RI, it occurs that:

$$\zeta(s) \equiv \prod \frac{1}{(1 - P_n^{-s})} = \prod e^{\frac{n^{-s}}{\pi_s}} = e^{\sum n^{-s}/\pi_s} = e^{log\zeta} = \zeta, \qquad QED$$

Note in passing that the respective terms on both sides throughout are co-distributed monotonously and can thus be compared (juxtaposed), even though they might call for per-element (interior, interim) adjustment/fitting factors or functors so as to carefully account for and arrive at productive prime-generating forms. Arguably, any phi-adjustments would do as long as the corners garner a unity ratio, i.e. the product of fitting terms cancels out.

$$\exists \varphi_n \equiv \frac{A_n}{A_{n-1}}: \prod \frac{1}{(1 - P_n^{-s})} \equiv \prod e^{\varphi_n \frac{n^{-s}}{\pi_s}}, \qquad \prod \varphi_n = \frac{A_T}{A_1} \equiv 1$$

$$\exists \varphi_n(\cdot): \quad P_n^{-s} \equiv 1 - e^{-\varphi_n \frac{n^{-s}}{\pi_s}} \qquad (P4')$$

For instance, the *A*-terms could be distributed as the combinatorial coefficients of a binomial decomposition of 2^T, or their normalized representation ((P5) being but a wild guess, I accede).

$$\varphi_k = \frac{A_k}{A_{k-1}} = \frac{\binom{T}{k}}{\binom{T}{k-1}}, \forall k = \overline{0,T}\ 1 = \frac{\binom{T}{k}}{\binom{T}{T-k}} = \frac{A_k}{A_{T-k}} \qquad (P5)$$

From combining (P4) and (*X*), (P2b) results as an actual relationship rather than a linearized one, if only subject to Big O notation due to rho/rho inconclusivity as opposed to explicit asymptotics we make little if any use of above and beyond rho-transforms.

Recall now (P2b) and how it plugs in on RI:

$$\zeta \equiv \sum n^{-s} = \sum \pi_s P_n^{-s} \ \leftrightarrow \ \sum P_n^{-s} \sim \frac{\zeta}{\pi_s} = log\zeta(s) \qquad (P6)$$

If the pi-functor cannot be rearranged on both sides as common, it acts identically as a "renaturalize" ("deprimalize") operator. This borne in mind, one immediately induces recurrent open-forms and close-form cross-correspondences (P7a-b):

$$\zeta(s) \equiv \sum n^{-s} \sim e^{\sum P_n^{-s}} = e^{\sum n^{-s}/\pi_s} \qquad (P7a)$$

$$e^{\sum n^{-s}} \sim (\sum n^{-s})^{\pi_s} \ \leftrightarrow \ e^{\zeta} \sim \zeta^{\pi_s} \qquad (P7b)$$

In fact, the latter is an identity.

At this point, it could be rewarding to introduce and compare primality versus naturality *weights/shares*, defined as below:

$$\alpha_n \equiv \frac{n^{-s}}{\zeta}, \zeta \equiv \prod \zeta^{\alpha_p} \ \leftrightarrow \ \alpha_p = -\frac{\log(1-P_n^{-s})}{log\zeta} = -\frac{\log(1-P_n^{-s})}{\sum P_n^{-s}}, \widehat{\alpha_p} \equiv \frac{P_n^{-s}}{log\zeta} \qquad (A)$$

$$\sum_n \alpha_n = 1 = \sum_p \alpha_p = \sum_p \widehat{\alpha_p} \qquad (B)$$

$$(\frac{P_n}{n})^{-s} = \pi_s^{-1} = \frac{\widehat{\alpha_p} log\zeta}{\alpha_n \zeta} = \frac{\widehat{\alpha_p}}{\alpha_n} * \pi_s^{-1} \ \leftrightarrow \ \widehat{\alpha_p} = \alpha_n \ OR \ \pi_s^{-1} = T^{\pm 1} \qquad (P8a)$$

$$\alpha_n log\zeta = P_n^{-s} \ \leftrightarrow log\zeta = \frac{P_k^{-s}}{\alpha_k} = \sum P_n^{-s} \ \ \forall k \qquad (P8b)$$

$$\zeta = \frac{k^{-s}}{\alpha_k} = \sum n^{-s} \quad \forall k \qquad (P8c)$$

Please observe that (P8b-c) denote *k-invariance*, while together pointing to [non-linear] recursion (P8d):

$$log\frac{n^{-s}}{\alpha_n} = \frac{P_n^{-s}}{\alpha_n} \qquad (P8d)$$

While at it, (A) suggests cross-correspondence (P8e) while pointing to cross-index equivalence due to (P4):

$$log\zeta = \frac{P_n^{-s}}{\widehat{\alpha_p}} = -\frac{log(1-P_n^{-s})}{\alpha_p} \;\leftrightarrow\; \alpha_p = \widehat{\alpha_p} \qquad (P8e)$$

Transitivity with reference to (P8a) suggests complete and simple equivalence of the indices attempted:

$$\alpha_p = \widehat{\alpha_p} = \alpha_n \qquad (P8f)$$

Arguably, with the aid of the above, proofs for the conjectures could be reconsidered along new, possibly more facile lines.

After-Math

Incidentally, little had I known that a minor handful of the results obtained are known in the literature in a somewhat weak, asymptotic representation (please see Appendix for some standard prior/exogenous results). For instance, the prime summation is referred to as the "*prime zeta* function" *P(s)*, with a result similar to (P6) being obtained in a less-than-efficient manner, by building on cumbersome derivations. I remain hopeful the present paper makes an early step toward a clearer vision which is yet to develop.

On the other hand, some of the results presented herewith reveal a striking similarity to my earlier as well as heretofore-unpublished apparatuses, notably *P-calculus* (Shevenyonov 2016), *L-gebra* (Shevenyonov 2022). To illustrate:

$$P^{[x]}(n) = P_2^{[x]}(k) + P_3^{[x]}(n-k) \qquad (P9a)$$

$$L_n^s = L_{sn}, \quad \Delta^k X = X - k \qquad (P9b)$$

Compare with some of the findings as implied by the prior exposition herein:

$$\exists\varphi: P_n \to n, \varphi^{-1}: n \to P_n$$

$$\varphi(P_n^s) = n \pm const, \qquad P_n^s = \varphi^{-1}n = \pi_s n^s \;\leftrightarrow \varphi^{-1} \sim \pi_s n^{s-1}, \qquad const \sim \varphi P_k^s - k \quad (P9c)$$

$$P_n^s = P_k^s + \varphi^{-1}(n-k) \qquad (P9d)$$

APPENDIX: Select Standard/Reference Results

Based on the "prime zeta," it has been known that:

$$P(s) \equiv \sum P_n^{-s} \sim log\zeta(s) \sim log\frac{1}{s-1}, \qquad s \to 1 \quad (S1)$$

I shall focus on the latter asymptotic correspondence, which I presume may have been discerned from the rather daunting, awkward representation for zeta building on the so-called "Stieltjes constants" (gamma-n), with gamma-null referring to "Euler-Mascheroni constant" that's fairly unwieldy in its own right:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}(1-s)^n \quad (S2a)$$

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m} \frac{log^m k}{k}\right) - \frac{log^{n+1} m}{n+1} \quad (S2b)$$

$$\gamma_0 = \sum_{k=1}^{\infty} \frac{1}{\pi(k)} - log\infty \quad (S2c)$$

While the core result (S1) is trivially implied from substituting s=1 in (S2a), the coefficients are intractable, the derivation counterintuitive and hardly productive when it comes to understanding the nature of primes. In contrast, my approach bypasses azimuthality while generating a plethora of formal results in addition to a schema for as convenient a proving of the grand conjectures that have long dangled inconclusive.

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