Pati-Salam GUT from Grassmann number factorization in $SU(2)$ supergauge theories

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ABSTRACT: This paper will propose a new construction of the $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$ Pati-Salam gauge symmetry. It is based on a particular construction of supersymmetric theory where the vector multiplet is in the adjoint representation of the $SU(2)$ group. A factorization of the Grassmann numbers from the commutator of vector multiplets will give new non-trivial terms which will correspond to a $SU(4)_{PS}$ gauge theory.

KEYWORDS: Pati-Salam, $SU(2)$ supergauge, GUT

Contents

1 Introduction

For simplicity this paper focuses on a $\mathcal{N}=1$ Yang-Mills supersymmetric theory whose vector multiplet is in the adjoint representation of the $SU(2)$ group. The following developments can be extended to higher supersymmetry models as long as the vector multiplet is in the adjoint representation of $SU(2)$ [1]. Long known results [1–3] give us a Lagrangian of this form :

$$\mathcal{L}^{YM}_{\mathcal{N}=1} = \underbrace{\frac{1}{32\pi}\mathrm{Im}(\tau\int d^2\theta \mathrm{Tr} W^\alpha W_\alpha)}_{\textbf{Gauge lagrangian}} + \overbrace{\int d^2\theta d^2\overline{\theta}\mathrm{Tr}\Phi^\dagger e^{\mathcal{V}}\Phi}^{\textbf{Matter lagrangian}} \tag{1.1}$$

where Φ is the chiral multiplet (in an unknown representation), $\mathcal{V}$ is the vector multiplet (in the adjoint representation of the $SU(2)$ group) and $W_\alpha = -\frac{1}{4}\overline{DD}\left(e^{-\mathcal{V}}D_\alpha e^{\mathcal{V}}\right)$ is the superpotential[2]. A computation developed in [1] allows to explicitly write equation 1.1 this way:

$$\mathcal{L}_{\textbf{gauge}} = \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} + \frac{1}{2}D^2\right) + \frac{\Theta}{32\pi^2}g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{1.4}$$

$$\mathcal{L}_{\textbf{matter}} = (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + f^\dagger f + i\sqrt{2}gz^\dagger\lambda\psi - i\sqrt{2}g\overline{\psi}\overline{\lambda}z + gz^\dagger Dz \tag{1.5}$$

where $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$, $D_\mu z = \partial_\mu z - ig[v_\mu, z]$ and $\Phi = z + \sqrt{2}\theta\psi - \theta^2 f$.
The vecteur multiplet $\mathcal{V} = \theta\sigma^\mu\overline{\theta}v_\mu + i\theta^2\overline{\theta}\overline{\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2 D$ is in the Wess-Zumino gauge.

The terms we see here are the terms that have survived after integration of Grassmann numbers. The problem is now to determine if other terms could be derived from these two equations. To survive the integration one needs exactly the same coefficient as the integrator (for example $\theta^2\overline{\theta}^2$ for the matter Lagrangian). A term lower in number of Grassmann or conjugate Grassmann numbers (like $\theta^2\overline{\theta}$) does not survive the integration. A higher term is necessarily zero because $\theta^3 = \overline{\theta}^3 = 0$ by definition. However, if there was a factorization it would allow to lower the power of Grassmann or conjugate Grassmann numbers, preventing the term from being zero, and with the perfect combination make it integrable.
This is what we mean by factorization of Grassmann numbers, a non-trivial identity allowing to decrease the powers of Grassmann numbers to make terms integrable in equation 1.1.
In this paper we will in the first section use the assumed identity to show the appearance of new terms. The purpose of this first demonstration is to show how new $SU(4)$ gauge terms appear as if by magic from a simple identity, and to convince the reader of the power of this identity. The second part will then demonstrate this identity and why there is this mysterious appearance of the $SU(4)$ gauge group.

[1] The notations are the same as in [1]
[2] The Grassmann numbers follow these classic identities

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2 \tag{1.2}$$

and

$$\overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\theta}^2 \tag{1.3}$$

2 Development of the factorized terms

By development of the terms inside the integrals of equation 1.1, we obtain :

$$e^{-\mathcal{V}} D_\alpha e^{\mathcal{V}} = D_\alpha \mathcal{V} - \frac{[\mathcal{V}, D_\alpha \mathcal{V}]}{2} + \frac{[\mathcal{V}, [\mathcal{V}, D_\alpha \mathcal{V}]]}{3!} - \frac{[\mathcal{V}, [\mathcal{V}, [\mathcal{V}, D_\alpha \mathcal{V}]]]}{4!}.. \tag{2.1}$$

as the term in the superpotential[3] for the gauge term and

$$e^{\mathcal{V}} = 1 + \mathcal{V} + \frac{\mathcal{V}^2}{2} + \frac{\mathrm{Tr}\mathcal{V}[\mathcal{V}, \mathcal{V}]}{3!} + \frac{\mathcal{V}\mathrm{Tr}\mathcal{V}[\mathcal{V}, \mathcal{V}]}{4!} \tag{2.2}$$

as the term wrapped between chiral multiplets in the matter term[4].
The commutators deduced here (which are the only derivable ones in the available equations) allow a factorization of Grassmann numbers in the specific case in which they are applied here. We give for now these identities:

$$\boxed{\textbf{The gauge bosons identity: } [v_\mu, v_\nu] \propto \theta\sigma^\mu\overline{\theta}\frac{\mathscr{A}_\mu}{\theta^2\overline{\theta}^2}}$$

with $\mathscr{A}_\mu = \lambda_m \mathscr{A}^m_\mu$ the $SU(4)$ gauge bosons with λ_m the 15 Gell-Mann matrices for $SU(4)$.

$$\boxed{\textbf{The scalar bosons identity: } [v_\mu, v_\nu] \propto \frac{S_6}{\theta^2\overline{\theta}^2} \simeq \frac{\mathscr{S} + \Delta_R + \Delta_L + \Phi_6}{\theta^2\overline{\theta}^2}}$$

where their representations in $SU(4) \times SU(2) \times SU(2)$ is $\mathscr{S} \sim (6,1,1)$, $\Delta_L \sim (6,3,1)$, $\Delta_R \sim (6,1,3)$ and $\Phi_6 \sim (6,3,3)$[5].

$\boxed{⚠: v_\mu \text{ must be here a 4-vector in the adjoint representation of } SU(2)}$[6]

In this part we will give the set of factorized terms deduced by the two identities given without explanation for the moment. The goal is to show quickly how these identities give surprisingly new terms with a precise physical meaning, i.e. terms of gauges and Higgs potentials for spontaneous symmetry breakings. We will start with the superpotential and then the matter term.

2.1 Superpotential factorized terms

We start with the superpotential $-\frac{1}{4}\overline{DD}e^{-\mathcal{V}} D_\alpha e^{\mathcal{V}}$ and just with the gauge bosons identity. To get a factorization in this one we need a commutator of vector bosons. The first ones to appear are

[3] using

$$e^{-\mathcal{V}} D_\alpha e^{\mathcal{V}} = (-1)^{m+1} \sum_{m=0}^{\infty} \frac{1}{m!} [\mathcal{V}, D_\alpha]_m$$

formula with $[\mathcal{V}, D_\alpha]_m = [\mathcal{V}, [\mathcal{V}, D_\alpha]_{m-1}]$ and $[\mathcal{V}, D_\alpha]_0 = D_\alpha$ [4]

[4] using $\mathcal{V}^3 = \mathcal{V}^b \mathcal{V}^a (T^a_{\mathbf{Adj}})_{bc} \mathcal{V}^c = \mathrm{Tr}\mathcal{V}[\mathcal{V}, \mathcal{V}]$ [4]

[5] notation is inspired by [5]

[6] we focused only on the vector boson because of a property that will be proved in one of my next articles

in[7] :

$$[\mathcal{V}, D_\alpha \mathcal{V}] = \overline{\theta}^2 (\sigma^{\nu\mu}\theta)_\alpha [v_\mu, v_\nu] + 2i\theta\sigma^\mu\overline{\theta}(\sigma^{\mu\nu}\theta)_\alpha \overline{\theta}^2 [\partial_\mu v_\nu, v_\rho]^{8}$$

$$[\mathcal{V}, D_\alpha \mathcal{V}] = (\sigma^{\nu\mu}\theta)_\alpha \theta\sigma^\mu\overline{\theta}\frac{\mathscr{A}_\mu}{\theta^2} + 2i(\sigma^{\mu\nu}\theta)_\alpha \overline{\theta}^2 \partial_\nu \mathscr{A}_\mu$$

Finally

$$\frac{1}{4}\frac{\overline{DD}[\mathcal{V}, D_\alpha \mathcal{V}]}{2} = -i(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu) \tag{2.3}$$

is the only remaining term at this order.
At the next order we have :

$$-\frac{1}{4}\frac{\overline{DD}[\mathcal{V}, [\mathcal{V}, D_\alpha \mathcal{V}]]}{3!} = \frac{1}{3!}(\sigma^{\mu\nu}\theta)_\alpha [v_\mu, \mathscr{A}_\nu] + \frac{1}{3!}(\sigma^{\mu\nu}\theta)_\alpha \theta\sigma^\rho[\overline{\lambda}, \mathscr{A}_\rho] \tag{2.4}$$

We set $X = [\mathcal{V}, D_\alpha \mathcal{V}]$ and at the fourth order we have:

$$[\mathcal{V}, [\mathcal{V}, X]] = -[\mathcal{V}, [X, \mathcal{V}]] - [X, [\mathcal{V}, \mathcal{V}]] \to [\mathcal{V}, [\mathcal{V}, X]]_{\theta^2\overline{\theta}^2} = [[\mathcal{V}, \mathcal{V}], X]_{\theta^2\overline{\theta}^2}$$

through Jacobi identity.
We use the value of this commutator:

$$[\mathcal{V}, \mathcal{V}] = [\theta\sigma^\mu\overline{\theta} v_\mu, \theta\sigma^\mu\overline{\theta} v_\mu] = \theta^2\overline{\theta}^2 [v_\mu, v_\nu] = \theta\sigma^\mu\overline{\theta}\mathscr{A}_\mu$$

Finaly at this order:

$$[\mathcal{V}, [\mathcal{V}, X]] = (\sigma^{\nu\mu}\theta)_\alpha \overline{\theta}^2 [\mathscr{A}_\mu, \mathscr{A}_\nu] + 2i\theta\sigma^\mu\overline{\theta}(\sigma^{\mu\nu}\theta)_\alpha \overline{\theta}^2 \partial_\nu [\mathscr{A}_\rho, \mathscr{A}_\mu]$$

$$\frac{1}{4}\frac{\overline{DD}[\mathcal{V}, [\mathcal{V}, X]]}{4!} = -\frac{1}{4!}((\sigma^{\mu\nu}\theta)_\alpha [\mathscr{A}_\mu, \mathscr{A}_\nu] - \overbrace{\cancel{\frac{1}{4!}(2i\theta\sigma^\mu\overline{\theta}(\sigma^{\mu\nu}\theta)_\alpha \partial_\nu [\mathscr{A}_\rho, \mathscr{A}_\mu])}}^{\textbf{one too much non-factorizable } \overline{\theta}} \tag{2.5}$$

The sum below gives the needed gauge term:

$$\frac{1}{4}\overline{DD}\left(\frac{X}{2} + \frac{[\mathcal{V}, [\mathcal{V}, X]]}{4!}\right) = -i(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu) - \frac{(\sigma^{\mu\nu}\theta)_\alpha [\mathscr{A}_\mu, \mathscr{A}_\nu]}{4!} = -i(\sigma^{\mu\nu}\theta)_\alpha \mathscr{A}_{\mu\nu}$$

with $\mathscr{A}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - \frac{i}{4!}[\mathscr{A}_\mu, \mathscr{A}_\nu]$.
We scale the superfield with a gauge coupling constant $\mathcal{V} \to 2g\mathcal{V}$ and set $\mathfrak{g} = \frac{g^2}{6}$, we then have :

$$\mathscr{A}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - i\mathfrak{g}[\mathscr{A}_\mu, \mathscr{A}_\nu]$$

[7] using :

$$\mathcal{V} = \theta\sigma^\mu\overline{\theta} v_\mu + i\theta^2\overline{\theta\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2(D - i\partial^\mu v_\mu)$$

$$D_\alpha \mathcal{V} = (\sigma^\mu\overline{\theta})_\alpha v_\mu + 2i\theta_\alpha\overline{\theta\lambda} - i\overline{\theta}^2\lambda_\alpha + \theta_\alpha\overline{\theta}^2 D + 2i(\sigma^{\mu\nu}\theta)_\alpha\overline{\theta}^2\partial_\mu v_\nu + \theta^2\overline{\theta}^2(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha$$

[8] we will use :

$$[\partial A, B] = \partial[A, B]$$

And because $\sigma^{\mu\nu}\sigma^{\rho\sigma} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}$, the lagragian with this superpotential gives:

$$\frac{1}{32\pi}\mathrm{Im}\left(\tau\int d\theta^2 W^\alpha W_\alpha\right)_{\mathbf{gauge}} = \underbrace{-6\mathfrak{g}\mathrm{Tr}\left(\mathscr{A}_{\mu\nu}\mathscr{A}^{\mu\nu}\right) + \frac{9\Theta}{2\pi^2}\mathfrak{g}^2\mathrm{Tr}\left(\mathscr{A}_{\mu\nu}\tilde{\mathscr{A}}^{\mu\nu}\right)}_{\textbf{SU(4) Gauge densities}} \tag{2.6}$$

We see that from the moment the gauge bosons identity is considered new gauge terms appear almost magically from the supersymmetric lagrangian only and give precisely an $SU(4)$ field strength tensor term, a $SU(2)/SU(4)$ gauge bosons and a $SU(4)/\lambda$ coupling (equation 2.4).
We now use the scalar bosons identity to derive the other accessible terms.
At first accessible order, two terms survive:

$$[\mathscr{V}, D_\alpha\mathscr{V}] = \bar{\theta}^2(\sigma^{\nu\mu}\theta)_\alpha[v_\mu, v_\nu] + 2i\theta\sigma^\mu\bar{\theta}(\sigma^{\mu\nu}\theta)_\alpha\bar{\theta}^2[\partial_\mu v_\nu, v_\rho] + i(\sigma^{\mu\nu})_\alpha\bar{\theta}^2\theta^2\bar{\theta}^2[\partial_\mu v_\nu, \partial^\rho v_\rho]$$

$$-\frac{1}{4}\overline{DD}[\mathscr{V}, D_\alpha\mathscr{V}] = \xcancel{-\frac{1}{4}\overline{DD}(\sigma^{\nu\mu}\theta)_\alpha\frac{S_6}{\theta^2} + -\frac{1}{4}\overline{DD}\left(2i(\sigma^{\mu\nu}\theta)_\alpha\theta\sigma^\mu\bar{\theta}\frac{\partial_\mu S_6}{\theta^2}\right)} + i(\sigma^{\mu\nu})\Box S_6$$

$$\frac{1}{4}\frac{\overline{DD}[\mathscr{V}, D_\alpha\mathscr{V}]}{2} = i(\sigma^{\mu\nu})\frac{m^2 S_6}{2}\text{[9]} \tag{2.7}$$

To the next order we have such terms :

$$-\frac{1}{4}\frac{\overline{DD}[\mathscr{V}, [\mathscr{V}, D_\alpha\mathscr{V}]]}{3!} = (\sigma^{\mu\nu}\theta)_\alpha\frac{[D, S_6]}{3!} - i(\sigma^{\mu\nu}\theta)_\alpha\frac{[\lambda, S_6]}{3!\theta} - 2i(\sigma^{\mu\nu}\theta)_\alpha\frac{[\partial_\mu S_6, v_\nu]}{3!} \tag{2.8}$$

Using the same trick as for gauge bosons identity we determine first the value of the basic vector multiplet commutator :

$$[\mathscr{V}, \mathscr{V}] = S_6$$

With a full scalar boson identity we have one possible term :

$$-\frac{1}{4}\overline{DD}[\mathscr{V}, [\mathscr{V}, X]] = -\frac{1}{4}\overline{DD}\left(\xcancel{(\sigma^{\nu\mu}\theta)_\alpha\frac{[S_6, S_6]}{\theta^2} + 2i\theta\sigma^\mu\bar{\theta}(\sigma^{\mu\nu}\theta)_\alpha\frac{\partial_\mu[S_6, S_6]}{\theta^2}}\right) + i(\sigma^{\mu\nu})[\partial_\mu S_6, \partial^\mu S_6]$$

Again with Klein Gordon equation:

$$[\partial_\mu S_6, \partial^\mu S_6] = [\Box S_6, S_6] = -m^2[S_6, S_6]$$

So

$$\frac{1}{4}\frac{\overline{DD}[\mathscr{V}, [\mathscr{V}, X]]}{4!} = i(\sigma^{\mu\nu})m^2\frac{[S_6, S_6]}{4!} \tag{2.9}$$

We then try the mixed states, that is one commutator is with scalar bosons identity factorized and the other with gauge bosons identity. We get :

$$\frac{1}{4}\frac{\overline{DD}[\mathscr{V}, [\mathscr{V}, X]]_{\mathbf{mixed}}}{4!} = -2i(\sigma^{\mu\nu}\theta)_\alpha\frac{[\partial_\mu S_6, \mathscr{A}_\nu]}{4!} \tag{2.10}$$

[9] using Klein-Gordon equation:

$$(\Box + m^2)\phi$$

for a scalar field.
We summarized here by $\Box S_6 = -m^2 S_6 = -m^2_{\mathscr{S}}\mathscr{S} - m^2_L\Delta_L - m^2_R\Delta_R - m^2_\Phi\Phi_6$ for simplicity.

Now we sum up all the term together :

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i(\sigma^{\mu\nu}\theta)_\alpha \mathscr{G} + \theta^2(\sigma^\mu \partial_\mu \overline{\lambda})_\alpha$$

$$\mathscr{G} = \left(\mathscr{A}_{\mu\nu} - V(S_6) + C(S_6) + \frac{[\lambda, S_6]}{3!\theta} + i\frac{[v_\mu, A_\nu]}{3!} + i\theta\sigma^\mu \frac{[\overline{\lambda}, \mathscr{A}_\mu]}{3!}\right)$$

with $V(S_6) = \frac{m^2}{2}S_6 + \frac{m^2}{4!}[S_6, S_6] - i\frac{1}{3!}[D, S_6]$ the superhiggs potential terms, $C(S_6) = \frac{2}{4!}[\partial_\mu S_6, \mathscr{A}_\nu] + \frac{2}{3!}[\partial_\mu S_6, v_\nu]$ the superhiggs coupling terms and $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]$ the $SU(2)$ field strength tensor.
Then we have

$$\int d^2\theta d^2\overline{\theta} W_\alpha W^\alpha = \left(\sigma^{\mu\nu}\right)^2 (\mathscr{G}F_{\mu\nu} - \mathscr{G}^2) + i\sigma^{\mu\nu}(F_{\mu\nu} - i\mathscr{G})D - i\frac{[\lambda, S_6]}{3!}\sigma^{\mu\nu}\sigma^\rho\partial_\rho\overline{\lambda} - i\sigma^{\mu\nu}\sigma^\rho\lambda\frac{[\overline{\lambda}, \mathscr{A}_\rho]}{3!} + \dots$$

... summarizes the already known terms.
Taking the last term and combining it with already known terms for λ gives us a new covariant derivative: $\mathscr{D}_\mu\overline{\lambda} = \partial_\mu\overline{\lambda} - ig[v_\mu, \lambda] + i4\mathfrak{g}\sigma^{\alpha\beta}[\mathscr{A}_\mu, \overline{\lambda}]$.
We take some terms of $(\sigma^{\mu\nu}\theta)^2$ in the superpotential:

$$-\overbrace{\frac{S_6}{3!}}^{V(S_6)}\overbrace{\left(\frac{2}{4!}[\partial_\mu S_6, \mathscr{A}_\nu] + \frac{2}{3!}[\partial_\mu S_6, v_\nu]\right)}^{C(S_6)} - i\overbrace{\frac{[S_6, S_6]}{4!}}^{V(S_6)}\frac{[v_\mu, \mathscr{A}_\mu]}{3!} - i\overbrace{\frac{[S_6, S_6]}{4!}}^{V(S_6)}\overbrace{\frac{[\mathscr{A}_\mu, \mathscr{A}_\nu]}{4!}}^{\mathscr{A}_{\mu\nu}} - i\overbrace{\frac{[S_6, S_6]}{4!}}^{V(S_6)}\overbrace{\frac{[v_\mu, v_\nu]}{2}}^{F_{\mu\nu}} \tag{2.11}$$

factoring out m^2.
We see that these terms correspond to terms to be found out in a squared 'covariant derivative' term

$$(\partial_\mu S_6 - ix[v_\mu, S_6] - iy[\mathscr{A}_\mu, S_6])^2$$

As far as I know, I have not managed to put this perfectly in the desired form but in my opinion the terms shown above should have a similar meaning although the result is highly non-trivial and this term is missing: $\partial_\mu S_6 \partial^\mu S_6$!
We still write in the overall term : $\left(\mathscr{D}_\mu S_6 \mathscr{D}^\mu S_6\right)_{\mathbf{pseudo}}$ to advise of some 'covariant derivative'-like terms.
Other terms have no known meaning and so are not shown out.

2.2 Matter factorized terms

We now determine the new matter terms of our given identites.
At the first order with a commutator we have[10] :

$$\frac{\Phi^\dagger \mathrm{Tr}\mathscr{V}[\mathscr{V}, \mathscr{V}]\Phi|_{\theta^2\overline{\theta}^2}}{3!} = \frac{1}{6}z^\dagger v_\mu \mathscr{A}_\mu z + i\frac{\sqrt{2}}{6}z^\dagger \lambda S_6 \psi - i\frac{\sqrt{2}}{6}\overline{\psi}\overline{\lambda}S_6 z + \frac{1}{6}iz^\dagger v_\mu S_6 \partial_\mu z \tag{2.12}$$

[10] we will use:

$$\Phi = z + \sqrt{2}\theta\psi + \theta^2 f + i\theta\sigma^\mu\overline{\theta}\partial_\mu z - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\overline{\theta} - \frac{1}{4}\theta^2\overline{\theta}^2\partial_\mu\partial^\mu z$$

At the next order, just one term survives:

$$\frac{\Phi^\dagger \mathcal{V}\mathrm{Tr}\mathcal{V}[\mathcal{V},\mathcal{V}]\Phi|_{\theta^2\bar{\theta}^2}}{4!} = \frac{1}{4!} z^\dagger v_\mu^2 S_6 z \tag{2.13}$$

Those are, as far as I know the only terms I could found using the two identities I gave. We can now summarize the overall lagrangian :

$$\boxed{\begin{aligned}\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + \frac{1}{2}D^2\right) + \frac{\Theta}{32\pi^2}g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} - 6\mathfrak{g}\mathrm{Tr}\left(\mathscr{A}_{\mu\nu}\mathscr{A}^{\mu\nu}\right) + \frac{9\Theta}{2\pi^2}\mathfrak{g}^2\mathrm{Tr}\left(\mathscr{A}_{\mu\nu}\tilde{\mathscr{A}}^{\mu\nu}\right)\\ (D_\mu z)^\dagger D^\mu z - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + f^\dagger f + i\sqrt{2}gz^\dagger\lambda\psi - i\sqrt{2}g\overline{\psi}\overline{\lambda}z + gz^\dagger Dz + \mathcal{V}(S_6) + \left(\mathcal{D}_\mu S_6\mathcal{D}^\mu S_6\right)_{\mathbf{pseudo}}\\ \frac{4}{3}g^3z^\dagger v_\mu\mathscr{A}_\mu z + i\frac{4\sqrt{2}}{3}g^3z^\dagger\lambda S_6\psi - i\frac{4\sqrt{2}}{3}g^3\overline{\psi}\overline{\lambda}S_6z + \frac{4}{3}g^3iz^\dagger v_\mu S_6\partial_\mu z + \frac{2}{3}g^4z^\dagger v_\mu^2S_6z + \ldots\end{aligned}} \tag{2.14}$$

with $\mathcal{D}_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - ig[v_\mu,\lambda] + i4\mathfrak{g}\sigma^{\alpha\beta}[\mathscr{A}_\mu,\bar{\lambda}]$, $D_\mu\psi = \partial_\mu\psi - ig[v_\mu,\psi]$ and $\mathcal{V}(S_6) = V(S_6)^2$

We have some interesting conclusions :

- The gauginos λ and superhiggs bosons S_6 couple to the $SU(4)_{PS}$ and $SU(2)$ gauge bosons
- The higgsinos ψ and higgs bosons z couple to the $SU(2)$ gauge bosons only.

We summarized other terms in the superpotential by ...

3 The Levi-Civita identity

This section aims at showing the two identities given and used in the last section. The equations 1.2 and 1.3 give us a precise relationship between Grassmann numbers and 2d Levi-Civita symbols. By definition the indices of the Levi-Civita symbols are equal to 1 or 2 with the following relation :

$$\varepsilon_{ij} = \mathrm{sign}(j-i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For the need of our work, we must have indices that runs from 1 to 3, $\{i,j\} \in [1,3]$. Without loss of generality instead of 1 and 2 only, the indices can run with 2 and 3 only, or 1 and 3 only, with the same constraints $\varepsilon_{ij} = \mathrm{sign}(j-i)$, that is : $\varepsilon_{23} = -\varepsilon_{32} = \varepsilon_{13} = -\varepsilon_{31} = 1$.

This construction extends the symbols without loosing the basic property of them, including the equation 3.1 which will be of use later

$$\varepsilon_{ij}\varepsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{3.1}$$

Recalling equations 1.2 and 1.3 we can construct the extended 2d Levi-Civita symbols as :

$$\varepsilon_{\alpha\beta} = \frac{\theta_\alpha\theta_\beta}{\theta^2} \qquad \varepsilon_{\dot{\alpha}\dot{\beta}} = -\frac{\overline{\theta}_{\dot{\alpha}}\overline{\theta}_{\dot{\beta}}}{\overline{\theta}^2} \tag{3.2}$$

As we can see this identity has in numerator powers of Grassmann or conjugate Grassmann numbers which could divide terms in powers of Grassmann numbers that are too high. It remains to be determined how this happens. Let's take the commutator of the vector representation, it gives :

$$[v_\mu^i, v_\nu^i] = [T^i_{\mathrm{Adj}}, T^j_{\mathrm{Adj}}]v_\mu^i v_\nu^j \tag{3.3}$$

then in $SU(2)$

$$[T^i_{\text{Adj}}, T^j_{\text{Adj}}] = i\varepsilon_{ijk}T^k_{\text{Adj}} = \varepsilon_{ijk}\varepsilon_{klm} \tag{3.4}$$

By definition of the symbols we have :

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \varepsilon_{ij}\varepsilon_{l\dot{m}} \tag{3.5}$$

Where the 2d Levi-Civita symbols are those extended to three different indices. It is to be observed, that it is perfectly possible theoretically to make appear a factorization of the Grassmann and conjugate Grassmann numbers from the operator commutator in the adjoint representation of the $SU(2)$ group. The strong point of this factorization is its strong specificity, it works only for the $SU(2)$ group and only for operators in the adjoint representation, which limits the field of acceptable supersymmetric theories.

3.1 Spinor algebra and alternative representation of Grassmann numbers

We use the gamma matrices in six dimensions $\mathscr{G}(\mathbb{R}^6)$ with the property

$$\{\Gamma_m, \Gamma_n\} = \delta_{mn}$$

That is, they act under $SO(6) \simeq SU(4)/\mathbb{Z}_2$ [6]. We can then construct raising and lowering operators like appendix B of [7]:

$$\Gamma^{\pm}_i = \frac{1}{2}(\Gamma_{2i} \pm i\Gamma_{2+i})$$

By definition the raising and lowering operators respect the following identities :

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$$

$$\{\Gamma^{a+}, \Gamma^{b+}\} = \{\Gamma^{a-}, \Gamma^{b-}\} = 0$$

and specifically $(\Gamma^{a+})^2 = 0$ and $(\Gamma^{a-})^2 = 0$. Which is exactly the same algebra as the Grassmann numbers [8] so that we can set the equivalence: $\Gamma^+_i = \theta_i$ and $\Gamma^-_i = \overline{\theta}_i$

3.2 Representation of the factorized terms

Using equations 3.3, 3.4, 3.5 and the gamma matrices operators equivalence we get for the commutator term the equations:

$$[v^\alpha_\mu, v^\beta_\nu] = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\delta}\dot{\tau}} v^\alpha_\mu v^\beta_\nu = \frac{\theta_\alpha\theta_\beta\overline{\theta}_{\dot{\delta}}\overline{\theta}_{\dot{\tau}}}{\theta^2\overline{\theta}^2} = \frac{\Gamma^+_\alpha\Gamma^+_\beta\Gamma^-_{\dot{\delta}}\Gamma^-_{\dot{\tau}}}{\theta^2\overline{\theta}^2} v^\alpha_\mu v^\beta_\nu \tag{3.6}$$

or

$$[v^\alpha_\mu, v^\beta_\nu] = -(\varepsilon_{\beta\delta}\varepsilon_{\dot{\alpha}\dot{\tau}} + \varepsilon_{\delta\alpha}\varepsilon_{\dot{\beta}\dot{\tau}}) v^\alpha_\mu v^\beta_\nu = \frac{i}{2}(\theta\sigma_\mu\overline{\theta})\sigma^\mu_{\delta\dot{\tau}} \left(\frac{\Gamma^+_\alpha\Gamma^-_{\dot{\beta}}}{\theta^2\overline{\theta}^2} + \frac{\Gamma^+_\beta\Gamma^-_{\dot{\alpha}}}{\theta^2\overline{\theta}^2}\right) v^\alpha_\mu v^\beta_\nu \text{ [11]} \tag{3.7}$$

[11] using [1]:

$$\theta_\alpha\overline{\theta}_{\dot{\alpha}} = -\frac{i}{2}(\theta\sigma_\mu\overline{\theta})\sigma^\mu_{\alpha\dot{\alpha}}$$

using the Jacobi identity.

We start with equation 3.6. By definition a 4-vector has the representation $(2,2)$ under the group $SU(2)_L \times SU(2)_R$. Then the product of two 4-vectors becomes

$$(2,2) \otimes (2,2) = (1,1) \oplus (1,3) \oplus (3,1) \oplus (3,3)$$

The raising and lowering operators acting on the $SU(4)_{PS}$ group, we extend the group to $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$ where the four vector are invariant under $SU(4)_{PS}$ and have the overall representation $(1,2,2)$.

The product of the two vectors have become a scalar whose representation under $SU(4)_{PS}$ is modified by the raising and lowering operators. We see that there is six different spinors that come out of those spinor operators. So the overall representation of 3.6 is :

$$S_6 \sim \overbrace{(6,1,1)}^{\mathscr{S}} + \overbrace{(6,3,1)}^{\Delta_L} + \overbrace{(6,1,3)}^{\Delta_R} + \overbrace{(6,3,3)}^{\Phi_6}$$

under $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$.

We conclude then that :

$$[v_\mu^\alpha, v_\nu^\beta] = \frac{\mathscr{S} + \Delta_L + \Delta_R + \Phi_6}{\theta^2 \overline{\theta}^2}$$

We now work out the equation 3.7. Our massless 4-vectors can be described through the spinor-helicity formalism shown in [9], this gives

$$v^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} v^\mu = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

We set the notations : $\lambda^\alpha = v\rangle$, $\lambda_\alpha = \langle v$, $\tilde{\lambda}_{\dot{\alpha}} = v]$ and $\tilde{\lambda}^{\dot{\alpha}} = [v$. So we get : $v_{\alpha\dot{\alpha}} = v]\langle v$.

We see that only one sigma matrix that appears, so we can describe just one of the two 4-vector as a spinor product. That is:

$$\sigma^\mu_{\delta\dot{\tau}} v_\mu^\alpha v_\mu^\beta = \alpha\rangle[\alpha v_\mu^\beta = v_\mu^\alpha \beta\rangle[\beta \simeq (v\rangle[v)v_\mu$$

That is we have a product of two Weyl spinors and a 4-vector. By definition $\Gamma^+_\alpha \Gamma^-_{\dot{\beta}}$ and $\Gamma^+_\beta \Gamma^-_{\dot{\alpha}}$ cannot give an non-zero result at the same time, and we set without loss of generality that $(v\rangle[v)v_\mu$ is annihilated by $\Gamma^+_\beta \Gamma^-_\alpha$. It can be interpreted in two ways : the spinors are in the lowest spinor state and then are annihilated by Γ^-_α or the spinors are in such state $\Gamma^+_\beta \zeta = v\rangle$ and then are annihilated because Γ^+_β cannot be acted twice. We actually use both. We embeed our spinors in the $SU(4)$ representation, to do that we describe the chiral and anti-chiral part as such

$$\lambda^{\{s_1,s_2,s_3\}} = (\Gamma_1^+)^{s_1+1/2}(\Gamma_2^+)^{s_2+1/2}(\Gamma_3^+)^{s_3+1/2}\zeta$$

with ζ the lowest spinor state $\Gamma_i^- \zeta = 0$ and with $s_i = \pm\frac{1}{2}$. By definition a chiral Weyl spinor $\Gamma\lambda = \lambda$ has an even number of $s_i = -\frac{1}{2}$ and an anti-chiral $\Gamma\tilde{\lambda} = -\tilde{\lambda}$ has an odd number[7]. Considering we have three raising operator we can put one on the chiral spinor and none or two on the anti-chiral spinor to satisfy the constraint $\Gamma^+_\beta \Gamma^-_\alpha \sigma^\mu_{\delta\dot{\tau}} v_\mu^\alpha v_\mu^\beta = 0$. We then have with a renormalization constant:

$$v\rangle[v = \frac{1}{\sqrt{2}}\left(1 + \Gamma_i^+ \Gamma_j^+\right)\Gamma_\beta^+ \zeta^2$$

and finally:

$$[v_\mu^\alpha, v_\nu^\beta] = \frac{i}{\sqrt{8}} \frac{(\theta\sigma^\mu\overline{\theta})}{\theta^2\overline{\theta}^2} \left(\Gamma_\alpha^+\Gamma_{\dot{\beta}}^- \left(1+\Gamma_i^+\Gamma_j^+\right)\Gamma_\beta^+\zeta^2\right) v_\mu$$

We see that there is a multiplication of a $SU(4)$ fundamental and anti-fundamental representation minus the $SU(4)$ singlet:

$$4 \otimes \overline{4} \ominus 1 = 15$$

We then get 4-vectors in the 15 representation of $SU(4)$, namely the adjoint representation of $SU(4)$. That is we got $SU(4)$ gauge bosons out of this factorization.

$$[v_\mu^\alpha, v_\nu^\beta] = \frac{\theta\sigma^\mu\overline{\theta}}{\theta^2\overline{\theta}^2}\mathscr{A}_\mu$$

We arrive at the final conclusion of this factorization. A factorization of the commutator by $\frac{\theta\sigma^\mu\overline{\theta}}{\theta^2\overline{\theta}^2}$ gives us the gauge bosons of the $SU(4)$ group and a factorization by $\frac{1}{\theta^2\overline{\theta}^2}$ gives us various scalar bosons which could allow successive spontaneous symmetry breaking.

4 Conclusion

Through this short article it has been proposed a certain type of factorization that allows us to deduce new terms from some supersymmetric theories. Several points of interest are to be noted:

1. Field strength tensor terms of the $SU(4)$ group appear in a perfect way.
2. Several scalar potentials can be deduced
3. Each of the new scalars are coupled with the $SU(2)$ and $SU(4)$ gauge bosons.

Lots of new terms have been overlooked for the conciseness of the article but new physical effects are expected from this terms. More precise developments are planned to find meaning to those new terms.

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