1D and 2D GLOBAL STRONG SOLUTIONS OF NAVIER STOKES EXISTENCE AND UNIQUENESS

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Abstract

Consider the Navier-Stokes equation for a one-dimensional and two-dimensional compressible viscous liquid. It is a well-known fact that there is a strong solution locally in time when the initial data is smooth and the initial density is limited down by a positive constant. In this article, under the same hypothesis, I show that the density remains uniformly limited in time from the bottom by a positive constant, and therefore a strong solution exists globally in time. In addition, most existing results are obtained with a positive viscosity factor, but current results are true even if the viscosity factor disappears with density. Finally, I prove that this solution is unique in a class of weak solutions that satisfy the usual entropy inequalities. The point of this work is the new entropy-like inequalities that Bresch and Desjardins introduced into the shallow water system of equations. This discrepancy gives the density additional regularity (assuming such regularity exists first).

1 Prerequisites

Bernoulli's Equation (1.1) Bernoulli's equation is used when you have one or two different points that lie on something known as a streamline. A streamline outlines the path traveled by a given single particle in the vector field provided by the fluid. Bernoulli's equation also states that the fluid must have a constant stable fixed density and that there is no friction present within the system. Without these initial conditions the equation would not work. To conclude Bernoulli's equation serves as a substantial way of viewing the balance of pressure, velocity and elevation within a system involving fluidity. We also use the continuity equation to ensure conservation of mass and conservation of energy. In Bernoulli's equation incompressibility is assumed to be true.

Example: Suppose we are given a massive water tank that is elevated 25 meters in the air, and that the tank is filled with water. Moreover, this tank is also open to the atmosphere. A hole is then poked through the tank allowing the water to flow out. The hole created is 3.5 meters above solid ground. If the whole created is significantly small in comparison to the size of the tank, how quickly will the water flow out of the tank?

$$P_1 + \frac{\rho v_1^2}{2} + \rho g h_1 = P_2 + \frac{\rho v_2^2}{2} + \rho g h_2$$

Since we are given the information that the whole is extremely small, we can deduce that the velocity at our point one is nearly 0, and will therefore cancel out. Also since our system is open to the atmosphere both points one and two will be equal to one, and will again cancel out.

$$v_2 = \sqrt{2(9.81m/s^2)(25m - 3.5m)} = 20.54m/s$$

Laminar and Turbulent Flow (1.2) In the case of laminar flow there is no random fluctuations and hence calculations can be done with relative ease. Whereas turbulent flow is chaotic and fluctuations within the system are present. In laminar flow within a tubed system in which water is flowing constantly there is a concept that arises. This is known as the no slide condition, and this condition works to help us conceptually understand water flow at specific parts of pipe. The condition states that the flow velocity right at either side of the pipe wall will always be equal to zero.

In turbulent flow the no slip condition also satisfies, and works. However, it is important to note that the flow velocity is not represented as a simple parabolic arc, as turbulence has its own effects on the fluid. Turbulence is responsible for the mixture of varying layers of flow within the system. The total momentum transfer caused by this usually leads to the uniformity with the flow velocity within the pipe/system. Keep in mind that this example of flow velocity is time averaged and is not instantaneous.

Pressure drop is significantly important in both laminar and turbulent flow. Pressure is caused by the shear forces acting upon any given flow velocity in any system, usually a tube. The following equation shows the relationship between where the water starts and where it ends.

$$\Delta p = p_{in} - p_{out}$$

Another key aspect of pressure drop would be the fact that laminar flow pressure drops are usually smaller than turbulent flow pressure drops. We can calculate the pressure drop within a laminar tubed system using the Darcy-Weisbach equation.

$$\frac{\Delta p}{L} = f \cdot \frac{\rho}{2} \cdot \frac{u_{avg}^2}{D}$$

This equation is heavily reliant on the average flow velocity "u", the frictional force "f", the density of the given fluid ρ for example water is 1000 Kilograms per cubic meter, the length of the tube or system "L", and the diameter of the tube "D". In order to calculate the frictional force within a laminar system you must find the Reynolds number and use the following equation.

$$f = \frac{64}{Re}$$

Adding this knowledge to the Darcy-Weisbach equation we get the following.

$$\frac{\Delta p}{L} = \frac{64\mu}{\rho u_{avg} D} \cdot \frac{\rho}{2} \cdot \frac{u_{avg}^2}{D}$$

This in turn lets us know that the change in pressure is proportional to the flow velocity. Calculating for the friction within a turbulent flow system is a little more complicated. In order to find the frictional force we must use the Colebrook equation.

$$\frac{1}{\sqrt{f}} = -2log\left(\frac{1}{3.7}\frac{\epsilon}{D} + \frac{2.51}{Re\sqrt{f}}\right)$$

This equation lets us know that the change in pressure is proportional to the flow velocity squared and that in turbulent flow the friction force is dependent on the roughness of the pipe. Moreover, in order to make the process of calculating friction within a turbulent system more convenient and straightforward. Scientists created the Moody diagram which is a graphical representation of the Colebrook equation, and allows for easier interpretation. Flow within a smoothed out tube has a significantly lower friction factor than that of a rough tube. Moreover, there is also a smaller pressure drop in a smoothed out tube as well. Modeling for turbulent flow is a little more complicated as it is necessary to look at turbulent eddies. Large eddies are known for carrying a large sum of kinetic energy, and these larger eddies are vital to the creation of smaller eddies. Once eddies become incredibly small they dissipate in the form of heat this is due to the frictional forces which are in turn caused by the viscosity within the fluid. Another important aspect of eddies to note is their energy cascade. “Big whirls have little whirls that feed on their velocity and little whirls have lesser whirls and so on to viscosity” (Lewis Fry Richardson). In order to solve complex turbulent problems we must include the Navier Stokes equations. Laminar flow example problem: Suppose you are given an artery and you are given the knowledge that the artery has been reduced to half its original value. Find out what the factor of reduction is for the artery, assuming no turbulence occurs.

$$Q = \frac{(P_2 - P_1)\pi r^4}{8\eta l}$$

$$r_2 = 0.75983 r_1$$

Therefore, there is a decrease of 24 percent. Now we will view an example in which there is turbulent flow. Suppose water flows through a 175 mm diameter pipe for which the relative roughness is 0.0007 at a rate of 0.6 m^3/s. Calculate the pressure drop over a 160m pipe. Take mu = 0.0023 kg/ms.

$$D = 0.175m,\ \frac{\epsilon}{D} = 0.0007,\ \dot{v} = 0.6\frac{m^3}{s},\ L = 160m$$

$$\Delta p = \frac{fl}{D}\frac{1}{2}\rho c^2 \quad \dot{v} = Ac \quad \dot{v} = \frac{\pi D^2}{4}c \quad c = \frac{4\dot{v}}{\pi D^2}$$

$$Re = \frac{\rho c D}{\mu} = \frac{\rho 4 \dot{v} D}{\mu \pi D^2} = 1,897,996.84 \quad f = 0.015$$

$$\Delta p = \frac{fl}{D}\frac{1}{2}\rho c^2 = 4,266,912.56$$

Divergence and Curl (1.3) In order to understand the intuition behind divergence and curls we must understand vector fields. A vector field serves as a visualization of a function in which the input and output space both are in the same dimension. We will start by viewing a vector field example listed below.

$$\vec{F}(x,y) = (y-1)\vec{i} + (x+y)\vec{j}$$

Test any specified points that will lie on the vector field to see how it functions as a whole.

$$\vec{F}(2,3) = 2\vec{i} + 5\vec{j}$$

$$\vec{F}(-1,0) = -1\vec{i} - 1\vec{j}$$

$$\vec{F}(-5,2) = \vec{i} - 3\vec{j}$$

$$\vec{F}(1,1) = 2\vec{j}$$

It is important to note that vector fields are not static, they are in fact dynamic and change over time. Divergence serves to tell us about a particular point on a plane and how much the proposed fluid tends to flow out of or into small regions near it. Curl refers to the operator that describes a certain level of circulation within a specified part of a given vector field. Clockwise spin represents positive curl, and vice versa for counterclockwise.

Stokes' theorem (1.4) Generalized formulation

$$\int_{\partial D} \omega = \int_D d\omega$$

Stokes' theorem proposes the relationship between two distinct concepts. The first part of the equation revolves around the surface integral, which is just a definite integral expressed in terms of the surface in which it resides. This differs from the line integral, as instead of representing a curve in the first dimension the surface integral allows second dimensional surfaces. Relating back to Stokes' theorem, the surface integral revolves around the curl of the given function bounded by a closed path on any surface in the second dimension. The second part of the theorem on the other side of the impartiality constitutes the line integral of any given vector function that revolves/resides on the bounded path. We will start by viewing the one dimensional stoke's theorem in which we indicate no differentiation between vector and co vector fields for simplicity. Note that when you are given a point or a set of points and a function you can almost always sum them up, also if you happen to be given a function in the form h dx you can almost always integrate said function over any specified/given interval. The use of the partial derivative symbol serves to create a way of expressing a boundary condition/conditions. This means that if we were given a specific interval for example D = [a, b] in which a and b are the two endpoints/boundaries of the integral. Then, we represent it's boundary ∂D as a union of a and b. The second part of the equation is just taking the differential of the given function. Once we combine these concepts we can get the following via the fundamental theorem of calculus.

$$\sum_{x \in (a,b)} \pm f(x) = \int_D \frac{df}{dx} dx$$

Using [25, (see References)] we can interpret the summation as a form of "zero" dimensional calculation. Using the logic that $\int$ represents one dimensional calculations and $\iint$ represents two dimensional calculations. Applying this logic leads to the following formulation which accurately presents stokes' theorem in one dimension.

$$\sum_{\partial D} f = \int_D d(f)$$

Now we will look at the stokes' theorem in two dimensions. Here ∂ will represent the boundary of a given curve/region in the second dimension R^2. ∇ will denote the way in which we will differentiate the function or functions represented as a gradient. Our function can be any function f represented by the following vector field

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle = f_x dx + f_y dy + f_z dz$$

In order to move forward we must familiarize ourselves with differentiation of vector fields. We can express the following $F = Adx + Bdy$ defined by the following statement.

$$dF = d(Adx + Bdy) = \frac{dB}{dx} dxdy - \frac{dA}{dy} dxdy = \left(\frac{dB}{dx} - \frac{dA}{dy} \right) dxdy$$

Also note that in our formalization $\partial \circ \partial$ will equal zero as having a boundary of another boundary would simply result in both of them cancelling each other out yielding zero. Also take into consideration that $d \circ \nabla$ will also be equal to zero, as if we consider the function $F = \nabla f$ whilst also writing and remembering that $F = Adx + Bdy$ we have that.

$$\frac{dA}{dy} = \frac{dB}{dx}$$

Which follows the fundamental theory that $f_x y = f_y x$, and so conversely this also conceptually works and makes sense, as the following also remains true.

$$dF = \frac{dB}{dx} - \frac{dA}{dy} = 0$$

And that there will always be some f in which $\nabla f = F$. Having this information we can now begin to take the line integral in correspondence to some curve C by finding the equation of the given curve with implicit methods. Which in this case we will represent as some s(t) also with $\partial s(t)$ being a union of a and b, and with the following derivative.

$$s'(t) = x(t)\frac{dt}{dx} + y(t)\frac{dt}{dy}$$

Now combining this with our previous information, we get the following.

$$\int_C A(x,y)dx + B(x,y)dy = \int_a^b A(s(t))x(t) + B(s(t))y(t)dt$$

We also know that our curve also remains in the boundary of a to b, and therefore, we write $\partial(C)$. This is important because it underlines the main theorem for that of line integrals.

$$\int_{\partial C} f = \int_C \nabla(f)$$

Where $\nabla(f) = f_x dx + f_y dy + f_z dz$. Giving us

$$\int_{\partial D} F dr = \iint_D curl(F) dS$$

Applying divergence theorem yields our final formulation for three dimensional Stokes' theorem.

$$\iint_{\partial D} F dS = \iiint_D div(F) dV$$

Now that we have a clear grasp of the Stokes' theorem we can work on using it to further our knowledge of fluids. Example problem: Use Stokes' theorem to evaluate the following.

$$\int_C \vec{F} \cdot d\vec{r}$$

Where

$$\vec{F} = (3yx^2 + z^3)\vec{i} + y^2\vec{j} + 4yx^2 k$$

And C represents a triangle with the vertices (0, 0, 3), (0, 2, 0) and (4, 0, 0). It is also important to note that this triangle C has a counterclockwise rotation if you are above the triangle and looking down towards the xy-plane. First we will test the Collinearity. Collinearity [26, (see References)]

$$a = (a_1, a_2, ..., a_d), b = (b_1, b_2, ..., b_d) \ \ and \ \ c = (c_1, c_2, ..., c_d)$$

Note the following points (-1, 2, 5), (1, 2, -7) and $(4, 3/2, -\frac{1}{2})$ which are 3 points in R^d. a, b, c are collinear if the following is true

$$c = a + \lambda(b - a) \ \ for\ some \ \ \lambda \in R$$

$$(a-c)+\lambda(b-a)=0$$

$$(a_i-c_i)+\lambda(b_i-a_i)=0,\ \ \forall i\in[d]$$

$$\sum_{i=1}^{d}[(a_i-c_i)+\lambda(b_i-a_i)]^2=0$$

$$\sum_{i=1}^{d}[(b_i-a_i)^2\lambda^2+2(a_i-c_i)(b_i-a_i)\lambda+(a_i-c_i)^2]=0$$

$$[\sum_{i=1}^{d}(b_i-a_i)^2]\lambda^2+[2\sum_{i=1}^{d}(a_i-c_i)(b_i-a_i)]\lambda+[\sum_{i=1}^{d}(a_i-c_i)^2]=0$$

$$\lambda=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

It is important to note that in order for λ to exist and also be unique B^2-4AC must be equal to zero, hence a,b,c are collinear if

$$[2\sum_{i=1}^{d}(a_i-c_i)(b_i-a_i)]^2-4[\sum_{i=1}^{d}(a_i-c_i)^2][\sum_{i=1}^{d}(b_i-a_i)^2]=0$$

$$(-1,2,5)\ \ for\ \ a,b,c$$

$$[2\sum_{i=1}^{d}-18]^2-4[\sum_{i=1}^{d}36][\sum_{i=1}^{d}9]=0$$

$$1296d^2-144d[9d]=1296d^2-1296d^2=0$$

Hence collinearity is true. Testing our initial given points results in all of them being collinear. Meaning we can now solve our question.

$$\int_C\vec{F}\cdot d\vec{r}=\iint_S curl\vec{F}\cdot d\vec{S}$$

$$curl\vec{F}=\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3yx^2+z^3 & y^2 & 4yx^2\end{bmatrix}=4x^2\vec{i}+3z^2\vec{j}-3x^2\vec{k}-8yx\vec{j}$$

$$=4x^2\vec{i}+(3z^2-8yx)\vec{j}-3x^2\vec{k}$$

Where our vertices are

$$S=(4,0,0)\ \ R=(0,2,0)\ \ T=(0,0,3)$$

Using this knowledge we can deduce that the following two vectors lie on the plane.

$$\vec{RS}=<4,-2,0>\quad \vec{RT}=<0,-2,3>$$

We must find the cross product of these two vectors in order to obtain the equation.

$$\vec{RS} \cdot \vec{RT} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 0 & -2 & 3 \end{bmatrix} = -6\vec{i} - 12\vec{j} - 8\vec{k}$$

Allow R to be our point for testing, hence the equation of the plane is

$$-6(x-0) - 12(y-0) - 8(z-3) = 0$$

$$-6x - 12y - 8z = -24$$

Simplify

$$3x + 6y + 4z = 12$$

We can then rewrite the equation of the Surface Area in terms of z.

$$z = 3 - \frac{3}{4}x - \frac{3}{2}y$$

Then, define

$$f(x,y,z) = z - 3 + \frac{3}{4}x + \frac{3}{2}y = 0$$

We then must find the unit normal vector of the given surface area.

$$\vec{n} = \frac{\nabla f}{||\nabla f||} = \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1 \right\rangle}{||\nabla f||}$$

Now we will find the dot product of the two.

$$curl\vec{F} \cdot \vec{n} = \left\langle 4x^2, (3(3 - \frac{3}{4}x - \frac{3}{2}y)^2 - 8yx), -3x^2 \right\rangle \cdot \frac{\left\langle \frac{3}{4}, \frac{3}{2}, 1 \right\rangle}{||\nabla f||}$$

$$= \frac{1}{||\nabla f||}\left(3x^2 + \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy - 3x^2\right)$$

$$= \frac{1}{||\nabla f||}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right]$$

Now we can apply Stokes' theorem.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot d\vec{S}$$

$$= \iint_S \frac{1}{||\nabla f||}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right] dS$$

$$= \iint_D \frac{1}{||\nabla f||}\left[\frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy\right] ||\nabla f|| dA$$

$$= \iint\limits_D \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xydA$$

With the limits.

$$0 \le x \le 4 \quad 0 \le y \le 2 - \frac{1}{2}x$$

Then applying the integral again with our new limits,

$$\int_C \vec{F} \cdot d\vec{r} = \iint\limits_S curl\vec{F} \cdot d\vec{S}$$

$$= \int_0^4 \int_0^{2-\frac{1}{2}x} \frac{9}{2}\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xydydx$$

$$= \int_0^4 \left(-\left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^3 - 6xy^2\right)\Bigg|_0^{2-\frac{1}{2}x} dx$$

$$= \int_0^4 \left(3 - \frac{3}{4}x\right)^3 - 6x\left(2 - \frac{1}{2}x\right)^2 dx$$

$$= \int_0^4 \left(3 - \frac{3}{4}x\right)^3 - 24x + 12x^2 - \frac{3}{2}x^3dx$$

$$= \left(-\frac{1}{3}\left(3 - \frac{3}{4}x\right)^4 - 12x^2 + 4x^3 - \frac{3}{8}x^4\right)\Bigg|_0^4 = -5$$

2 1D Global Strong Solutions of Compressible Navier Stokes Existence and Uniqueness

Theorem 2.1 In our case we will obey the following boundary condition. The flow must be isentropic; therefore, entropy within the system will never change. Since the flow is isentropic it must be both reversible and adiabatic. Adiabatic refers to heat not leaving or entering the system.

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho\upsilon = 0,$$

$$\frac{\partial}{\partial t}\rho\upsilon + \frac{\partial}{\partial x}\rho\upsilon^2 + \frac{\partial}{\partial x}\Lambda(\rho) - \frac{\partial}{\partial x}\left(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right) = 0$$

$$(x,t) \in \mathbb{R}\cdot\mathbb{R}_+$$

In our case we treat the pressure of the fluid as $\Lambda(\rho)$ and this function will obey some gamma law.

$$\Lambda(\rho) = \rho^\gamma,\ \gamma > 1$$

In our case we will also include some viscosity coefficient presented as $\Phi(\rho)$ which we will assume is positive and is not dynamic, and therefore, constant. However, it is vital to note how all gasses depend on temperature and density of the given fluid or within the given system. There will be a viscosity present within the system that is of equal energy, and that will be non-existent for when $\rho = 0$. We will also characterize some ρ^κ for when $\kappa < \frac{1}{2}$. Hence, we will now present the main cases that will be explored throughout the proof.

$$\Phi(\rho) = \chi \ \ and \ \ \Phi(\rho) = \chi\sqrt[3]{\rho}$$

It is important to note that χ is just some positive constant. In this proof we will look at a case in which a vacuum never arises. There will be some key fundamental assumptions and theories that we will use to work our way through the proof. The first one is the entropy inequality which was derived by D. Bresch and B. Desjardians. This inequality also works for other density dependent viscosity coefficients. The main aspect of the inequality we will look at, however, will be further regularity of the density within the system. In the case of this proof we will note that the entropy inequality satisfies $\Phi(0) = 0$ for any dimension two or higher, it is important to simultaneously note that for two dimensions the two viscosity coefficients will converge and become one singular coefficient. We will also note that for the one dimensional case the inequality is responsible for giving some control on negative powers provided and applied to the density. Having this knowledge allows for us to determine that no vacuum will be present if it was not present when time was equal to zero.

3 Result

We will now move onto the formalization given specific initial data/conditions. We will assume that the data is positive and that there exists varying limits when

$x = \pm\infty$. We also must ensure that the two velocities of the fluid represented as υ_+ and υ_- are fixed and that our density represented as ρ is both constant and positive. This can be expressed through the following conditions: $\rho_+ > 0$ and $\rho_- > 0$, and we will always refer to $\upsilon(x)$ and $\rho(x)$ as a smooth function that pertains to the idea of monotony within the function. "A monotonic function is a function which is either entirely non-increasing or non-decreasing. A function is monotonic if its first derivative (which need not be continuous) does not change sign." (Wolfram Alpha). These functions will also satisfy the following

$$\frac{\partial}{\partial x}\rho' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

$$\rho(x) = \rho_\pm \;\; when \;\; \pm x \geq 1, \quad \rho(x) > 0 \quad \forall x \in \mathbb{R} \tag{1}$$

$$\upsilon(x) = \upsilon_\pm \;\; when \;\; \pm x \geq 1 \tag{2}$$

From earlier on recall that we stated a gamma function that pertains to our pressure for when $\gamma > 1$ and will assume that there will be a constant which we stated before as χ that will be greater than zero. We can also refer back to the original conditions and note that our pressure coefficient will satisfy the following.

$$\Phi(\rho) \geq \chi\rho^\kappa \quad \forall\rho \leq 1 \quad \exists\kappa \in \left[0, \frac{1}{2}\right), \tag{3}$$

$$\Phi(\rho) \geq \chi \quad \forall\rho \geq 1, \tag{4}$$

$$\frac{1}{N}(\Phi(\rho) - N) \leq \Lambda(\rho) \quad \forall\rho \geq 0 \tag{5}$$

(5) will only restrict the growth for our Φ when a large ρ is present. Example coefficients $\Phi(\rho) = \chi$ and $\Phi(\rho) = \sqrt[3]{\rho}$.

Theorem 3.1 Now we will tackle the main theorem in which we will assume the following conditions for $\rho_i(x)$ and $\upsilon_i(x)$.

$$\begin{aligned} -\lambda_i < -\rho_i(x) \leq -\lambda_i' \leq 0 < \infty, \\ \rho_i - \rho' \in H^1(\mathbb{R}), \\ \upsilon_i - \upsilon' \in H^1(\mathbb{R}), \end{aligned} \tag{6}$$

Where λ_i and λ_i' are constants, and that conditions (3), (4), and (γ) are satisfied, then there will exist a global strong solution represented as (ρ, υ) of our initial data on $\mathbb{R}^+ \cdot \mathbb{R}$ for any $T > 0$ will follow the following. Also note that anything in $*$ is not actually apart of the equation.

$$\rho - \rho' \in L^\infty(0, T; H^1(\mathbb{R})),$$

$$\upsilon - \upsilon' \in L^\infty(0, T; H^1(\mathbb{R}))\{*T : H^1((U) \longrightarrow L^2(L)*\} \cap L^2(0, T; H^2(\mathbb{R})),$$

Since we have already defined $T > 0$ we can deduce that our constants $\lambda(T)$ and $\lambda'(T)$ exists such that

$$-\lambda(T) < -\rho(x,t) \leq -\lambda'(T) \leq 0 < \infty \quad \forall(x,t) \in (0,T) \cdot R$$

In order to progress we must take a look at Lipschitz continuity. Lipschitz continuity states that when you have a continuous Lipschitz function, that function is limited/bounded by the speed at which the function can change. Hence, there will exist a real number so that if you are given any pair of points, the absolute value of the line connecting the points's slope will never be greater than it's real number. This real number may vary; however, a famous one is Lipschitz's constant.

Remark 3.2 Suppose that a function f is called Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant L that can be represented by the following.

$$|f(x) - f(y)| \leq L|x - y| \quad \forall (x, y) \in R$$

Suppose that f and g are Lipschitz continuous on R, show that f + g is also Lipschitz continuous on $\mathbb{R}$.

$$|f(x) - f(y)| \leq L|x - y|, \quad \forall (x, y)$$

$$|g(x) - g(y)| \leq L'|x - y|, \quad \forall (x, y)$$

$$\forall (x, y) : |(f + g)(x) - (f + g)(y)| = |(f(x) - f(y)) + (g(x) - g(y))|$$

$$= |(f(x) - f(y)) + (g(x) - g(y))|$$

$$\leq |f(x) - f(y)| + |g(x) - g(y)|$$

$$\leq L|x - y| + L'|x - y| = (L + L')|x - y|$$

$$|x^2 - y^2| = |x + y| \cdot |x - y|$$

So therefore, for any constant λ, there exist $(x, y) \in \mathbb{R}^2$ such that $|x + y| > \lambda$, so $x \longrightarrow x^2$ cannot be Lipschitz on $\mathbb{R}$. With this basic background knowledge of Lipschitz continuity we may now continue.

Now we can finally make a statement of our findings. If our viscosity coefficient which is $\Phi(\rho)$ is greater than or equal to our constant χ which is simultaneously greater than zero this works for all $\rho \geq 0$, if our given value of Φ represents a Lipschitz continuity, and is therefore, only possible if $\gamma \geq 2$. Moreover, the solution we have provided is unique and resides within the class of weak Navier Stokes solutions in one dimension that satisfy the inequality of entropy which we will discuss later on. Also it is vital to note that the assumption we made previously (6) for initial data/conditions implies that the entropy at its initial state and the entropy at a relative state is finite. This only occurs if our $\Phi(\rho)$ satisfies

$$\Phi(\rho) - \chi \geq 0 \quad \rho \geq 1$$

$$\Phi(\rho) - \chi \geq 0 > 0 \quad \forall \rho \geq 0$$

$$\Phi(\rho) \geq \chi > 0 \quad \forall \rho \geq 0, \tag{7}$$

Note that the existence for smooth solutions when time is minuscule has already been solved. Which we will present now.

Remark 3.3 We will claim that our initial ρ_i and our initial υ_i satisfy

(6), and our coefficient Φ will satisfy condition (7), since this is the case we can deduce that there exists some initial $T_i > 0$ that is dependent upon our constants $\lambda_i, \lambda'_i, ||\rho_i - \rho'||H^1$ (Which represents our norm/magnitude of our given vectors) and $\upsilon_i - \upsilon'||H^1$. Such that our initial equations at the very beginning will have a unique solution for (ρ, υ) on the interval $(0, T_i)$ in which there is a satisfaction for

$$\rho - \rho' \in L^\infty(0, T_1; H^1(\mathbb{R})), \quad \partial_t \rho \in L^2((0, T_2) \cdot \mathbb{R})$$
$$\upsilon - \upsilon' \in L^\infty(0, T_1; H^2(\mathbb{R})), \quad \partial_t \upsilon \in L^2((0, T_1) \cdot \mathbb{R})$$

For all solutions satisfying the condition $T_1 < T_i$. Moreover, there will exist a solution for some $\lambda(t) > 0$ and $\lambda'(t) < \infty$ such that there can be the following expression $\lambda(t) \leq \rho(x,t) \leq \lambda'(t) \quad \forall t \in (0, T_i)$. When viewing this previously thought out formulation it becomes abundantly clear that if we were to use a truncated viscosity coefficient such as, $\Phi_\varsigma(\rho)$:

$$\Phi_\varsigma(\rho) = max\left(\Phi(\rho), \frac{1}{\varsigma}\right)$$

Where max represents our function's greatest value, then we can deduce that there will be some approximated solutions for $(\rho_i, \upsilon_\varsigma)$ which we define for some small time $(0, T_i)$ in which T_i could possibly depend on the value given to ς. To prove the previous theory/statement we must show that $(\rho_\varsigma, \upsilon_\varsigma)$ satisfies our previously stated bounds uniformly, this must be with respect to ς and T to prove dependency.

$$\lambda(T) \leq \rho_\varsigma \leq \lambda'(T) \quad \forall t \in [0, T],$$
$$\rho_\varsigma - \rho' \in L^\infty(0, T_1, H^1(\mathbb{R})),$$
$$\upsilon_i - \upsilon' \in L^\infty(0, T, H^1(\mathbb{R}))$$

Now we must prove the entropy inequalities, which will help us determine our bounds for ρ_ς and υ_ς.

4 Entropy Inequalities

We must turn the initial equations to their conservative form:

$$\rho_t = \frac{\partial \rho}{\partial t} = \rho \cdot \frac{\partial}{\partial t} \quad \partial_x(\rho\upsilon) = \rho\upsilon \cdot \frac{\partial}{\partial x}$$

$$Let\ \Upsilon = \begin{bmatrix} \rho \\ \rho\upsilon \end{bmatrix} \quad when\ \iota = \rho\upsilon \longrightarrow \begin{bmatrix} \rho \\ \iota \end{bmatrix}$$

Then we view the flux, which we can represent as $E(\Upsilon)$

$$E(\Upsilon) = \begin{bmatrix} \rho\upsilon \\ \rho\upsilon^2 + \rho^\gamma \end{bmatrix} \ since\ \iota = \rho\upsilon$$

$$\frac{(\rho\upsilon)^2}{\rho} = \frac{\rho^2\upsilon^2}{\rho} = \rho\upsilon^2$$

$$E(\Upsilon) = \begin{bmatrix} \iota \\ \frac{\iota^2}{\rho} + \rho^\gamma \end{bmatrix}$$

The following result has been previously found (2 [see References]). This is a proof of the findings for $H(\Upsilon)$.

$$H(\Upsilon) = \rho\frac{\upsilon^2}{2} + \frac{1}{\gamma-1}\rho^\gamma = \frac{\iota^2}{2\rho} + \frac{1}{\gamma-1}\rho^\gamma$$

$$\rho\frac{\upsilon^2}{2} - \frac{\iota^2}{2\rho} = \frac{1}{\gamma-1}\rho^\gamma - \frac{1}{\gamma-1}\rho^\gamma$$

$$\rho\frac{\upsilon^2}{2} - \frac{\iota^2}{2\rho} = \frac{\upsilon^2\rho}{2} - \frac{\iota^2}{2\rho} = \frac{\upsilon^2\rho^2}{2\rho} - \frac{\iota^2}{2\rho} = 0$$

$$\frac{\upsilon^2\rho^2 - \iota^2}{2\rho} = \upsilon^2\rho^2 - \iota^2 = \rho^2 - \frac{\iota^2}{\upsilon^2} = 0$$

$$\rho^2 = \frac{\iota^2}{\upsilon^2}$$

$$\rho = \frac{\iota}{\upsilon} \quad \upsilon = \frac{\iota}{\rho} \quad \iota = \rho\upsilon$$

$$\rho\frac{\upsilon^2}{2} = \frac{\iota^2}{2\rho}$$

$$\frac{\rho\upsilon^2}{2} = \frac{\rho^2\upsilon^2}{2\rho}$$

$$\frac{1}{2}(\rho\upsilon^2) = \frac{1}{2\rho}(\rho^2\upsilon^2)$$

$$\frac{1}{2}(\rho\upsilon^2) = \frac{1}{2}(\rho\upsilon^2)$$

This result is used for calculating the entropy for the initial system of equations. Since we have stated that (ρ, υ) is smooth, we will then have

$$\frac{\partial\left(\frac{\iota^2}{2\rho} + \frac{1}{\gamma-1}\rho^\gamma\right)}{\partial t} + \frac{\partial\left(\frac{\rho\upsilon^3}{2} + \frac{\gamma\upsilon\rho^\gamma}{\gamma-1} - \Phi(\rho)\upsilon\upsilon_x\right)}{\partial x} + \Phi(\rho)\upsilon_x^2 = 0 \qquad (8)$$

When $I(\Upsilon)$ is represented as

$$I(\Upsilon) = \frac{\rho\upsilon^3(\gamma-1) + 2\gamma\upsilon\rho^\gamma}{2(\gamma-1)}$$

Now we must integrate (8) with respect to x over our limit $\mathbb{R}$. We will begin by solving the integral and then reconstructing it to fit our initial assumption.

$$\int_{\mathbb{R}} \frac{\partial H(\Upsilon)}{\partial t}dx + \int_{\mathbb{R}} \frac{\partial[L(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x]}{\partial x}dx + \int_{\mathbb{R}} \Phi(\rho)\upsilon_x^2 dx$$

We will not focus on $H(\Upsilon)$ for now.

$$\int_{\mathbb{R}} \frac{\partial[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x]}{\partial x} dx + \int_{\mathbb{R}} \Phi(\rho)\upsilon_x^2 dx = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

$$\frac{d}{dx}\int_{\mathbb{R}} [I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x] dx + \frac{d}{dx}\int_{\mathbb{R}} \Phi(\rho)\upsilon^2 dx = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

$$\frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}} [I(\Upsilon) - \Phi(\rho)\upsilon^2] dx + \frac{d}{dx}\int_{\mathbb{R}} \Phi(\rho)\upsilon^2 dx = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

$$\frac{d^2}{dx^2}\int_{\mathbb{R}} [I(\Upsilon) - \Phi(\rho)\upsilon^2] dx + \frac{d}{dx}\int_{\mathbb{R}} \Phi(\rho)\upsilon^2 dx = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

$$\frac{d^2}{dx^2}[I(\Upsilon) - \Phi(\rho)\upsilon^2]x + \frac{d}{dx}[\Phi(\rho)\upsilon^2]x = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

It is important to note that the far left zeros out.

$$\upsilon^2\Phi(\rho) = \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

Having this knowledge, and reapplying the assumption we arrive at the following.

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon) dx + \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx \tag{9}$$

However, it is important to realize that the solutions we wish to achieve are $\rho(x,t)$ and $\upsilon(x,t)$ we also want to note the fact that there is convergence to $\rho_\pm$ and $\upsilon_\pm$ at $\pm\infty$, hence we will not expect there to be a integrable entropy state. Moreover, we must revert and use the relative entropy of the system. Relative entropy is formulated by Kullback–Leibler divergence, which entails that

$$O(\Psi \mid \Psi') = \sum_{x\in X} \Psi(x) log\left(\frac{\Psi(x)}{\Psi'(x)}\right)$$

Hence, we can apply this to our actual functions.

$$\rho(\upsilon - \upsilon')^2 + \Lambda(\rho \mid \rho')$$

$$where \;\; \Lambda(\rho \mid \rho') = \sum_{x\in X} \rho(x) log\left(\frac{\rho(x)}{\rho'(x)}\right)$$

Where $\Lambda(\rho \mid \rho')$ is the relative entropy for $\frac{\rho^\gamma}{\gamma-1}$

$$\Lambda(\rho \mid \rho') = \frac{\rho^\gamma}{\gamma-1} - \frac{\rho^{\gamma'}}{\gamma-1} - \frac{\gamma\rho^{\gamma'-1}}{\gamma-1}(\rho - \rho')$$

We must also determine the concavity or convexity of the function. Recall that a function is strictly convex if

$$\frac{\partial^2 f}{\partial x^2} > 0$$

$$where \;\; \Lambda(\rho) = \rho^{\gamma}$$

So since Λ is strictly convex, we can deduce that $\Lambda(\rho \mid \tilde{\rho})$ is non-negative for every ρ and $\Lambda(\rho \mid \tilde{\rho}) = 0$ if and only if $\rho = \tilde{\rho}$. Recall that $\rho'(x)$ and $\upsilon'(x)$ from (2) and (3) are smooth; therefore, indicating that

1. The functions $\rho'(x)$ and $\upsilon'(x)$ must be continuous, and a limit must exist at all given points

2. The functions $\rho'(x)$ and $\upsilon'(x)$ must be differentiable at every point, meaning that the limit from the left hand side must be equal to that of the right hand side, where

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

3. The functions $\rho'(x)$ and $\upsilon'(x)$ must have usable derivatives, meaning that all derivatives must be equal to zero. Where Λ is the gradient operator

$$\nabla\Lambda(x_0, y_0, ...) = \begin{bmatrix} \frac{\partial \Lambda}{\partial x}(x_0, y_0, ...) \\ \frac{\partial \Lambda}{\partial y}(x_0, y_0, ...) \\ \vdots \end{bmatrix}$$

Recall the Υ matrix, and apply the knowledge of smooth functions which works due to $\rho'(x)$ and $\upsilon'(x)$ being identified as smooth functions. Then rewrite Υ as $\Upsilon' = (\rho', \rho'\upsilon')$ We will also check for a positive constant ξ which will depend on the infimum of the smooth density ρ'. **Remark (4.1)** Example of infimum: Suppose A is a set, A=[1,10,π,55,11.2,$\sqrt{2}$,$\frac{1}{2}$], then the infimum of this set is the greatest lower bound within the set. In this case inf[A] $= \frac{1}{2}$ and in our case we let ρ' be a set. Such that for every ρ and for every $x \in \mathbb{R}$ and for $\gamma > 1$, we have

$$\rho + \Lambda(\rho) \leq \xi \left[1 + \left(\frac{\rho^{\gamma}}{\gamma - 1} - \frac{\rho^{\tilde{\gamma}}}{\gamma - 1} - \frac{\gamma \rho^{\tilde{\gamma}-1}(\rho - \tilde{\rho})}{\gamma - 1} \right) \right] \tag{10}$$

Where ρ^{γ} is our pressure and ρ is the density which is less than or equal to our constant ξ multiplied by the relative entropy add one. Recall that the relative entropy attributed to $\frac{\rho^{\gamma}}{\gamma-1}$, $\Lambda(\rho \mid \rho')$ is the free energy associated with in-equilibrium variability and, of course, non-equilibrium deviation. Notation:

$$\liminf_{n\to\infty} = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$$

$$\liminf_{n\to\infty} x_n \;\; or \;\; \varliminf_{n\to\infty} x_n$$

$$\varliminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{m\geq n} x_m)$$

$$\varliminf_{\rho\to 0} \Lambda(\rho \mid \rho') \geq \xi^{-1}. \tag{11}$$

$y = \varphi x$ where φ is a constant inverse $y = \frac{\varphi}{x}$. The greatest lower bound of every limit point in the function is greater than or equal to the inverse of our constant which depends on the greatest lower bound. When we finally establish the proof of Theorem 3.1, this inequality will be the first one we utilize, and is the usual relative entropy that is used when dealing with the compressible Navier Stokes equations:

Argument 4.1 Allow our defined ρ, υ to be a solution of the initial Navier Stokes equations, and allow satisfaction of the entropy inequality. Not the magnitude, and apply

$$\left|\frac{\partial}{\partial x}\upsilon\right|^2 = \sum_{i=1}^{N}\left|\frac{\partial}{\partial x}\upsilon_i\right|^2$$

$$|\upsilon_x|^2 = \upsilon_x^2$$

$$\partial_t H(\Upsilon) + \partial_x[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x] + \Phi(\rho)|\upsilon_x|^2 \leq 0, \quad (12)$$

We will also note the original $H(\Upsilon \mid \tilde{\Upsilon})$:

$$\int_E \rho_i(x)dx > 0$$

$$H(\Upsilon \mid \tilde{\Upsilon}) = \rho(\upsilon - \tilde{\upsilon})^2 + \Lambda(\rho \mid \tilde{\rho})$$

$$\int_{\mathbb{R}} H(\Upsilon \mid \tilde{\Upsilon}) = \int_{\mathbb{R}} [\rho(\upsilon - \tilde{\upsilon})^2 + \Lambda(\rho \mid \tilde{\rho})]dx$$

$$\textit{Recall that } H(\Upsilon_i \mid \Upsilon') = \frac{\rho_i(\upsilon_i - \upsilon')^2 + 2[\Lambda(\rho_i \mid \rho')]}{2}$$

$$\int_{\mathbb{R}} H(\Upsilon_i \mid \Upsilon')dx = \int_{\mathbb{R}} \left[\frac{\rho_i(\upsilon_i - \upsilon')^2 + 2[\Lambda(\rho_i \mid \rho')]}{2}\right] dx < \infty \quad (13)$$

Then for any time that is greater than zero there will exist a positive constant $\Psi(T)$ that indicates that the following statement is true

$$\int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx$$

On interval $\mathbb{R}$, now we introduce interval $T[0, T]$; therefore, we must apply

$$\int_0^T f dt$$

Giving us

$$\int_0^T \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dx dt$$

Now for $H(\Upsilon \mid \Upsilon')$ we must look for the constant in the supremium, this will give us the largest value in $H(\Upsilon \mid \Upsilon')$ that we can possible get to as our interval varies [0,T]

$$\sup_{[0,T]} \int_{\mathbb{R}} [H(\Upsilon \mid \Upsilon')]dx$$

With this knowledge we can construct the following inequality

$$\sup_{[0,T]} \int_{\mathbb{R}} [H(\Upsilon \mid \Upsilon')]dx + \int_0^T \int_{\mathbb{R}} \Phi(\rho)|\upsilon_x|^2 dxdt \leq \Psi(T). \tag{14}$$

Which means that the highest value on the interval [0,T] applied to the relative entropy add the integral over our space $\mathbb{R}$ applied to the viscosity coefficient multiplied by the non-changing magnitude of υ must be less than or equal to our constant $\Psi(T)$. With this knowledge, note that $\Psi(T)$ depends only on the inequality $T > 0$, Υ', Υ_i, γ, and on our constant N which appears in (γ). Note that when both ρ' and ρ_i are bounded above and below, and are away from zero, it is easy to check that the following must also be true.

$$\Lambda(\rho_i \mid \rho') \leq N(\rho_i - \rho')^2$$

Therefore, (13) will hold under the assumptions provided within Theorem 3.1

Proof of Argument 4.1. First, we must revert back to the knowledge of $H(\Upsilon \mid \Upsilon')$

$$H(\Upsilon \mid \Upsilon') = H(\Upsilon) - H(\Upsilon') - DH(\Upsilon')(\Upsilon - \Upsilon')$$

Then we must apply

$$\partial_t H(\Upsilon \mid \Upsilon')$$
$$H(\Upsilon) \approx \partial_t H(\Upsilon) + \partial_x[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x]$$
$$H(\Upsilon') \approx -[\partial_t H(\Upsilon') + \partial_x[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x]]$$
$$DH(\Upsilon')(\Upsilon - \Upsilon') \approx \partial_x[DI(\Upsilon')(\Upsilon - \Upsilon')]$$

Formulating this, we get

$$\partial_t H(\Upsilon \mid \Upsilon') = \partial_t H(\Upsilon) + \partial_x[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x] - \partial_t H(\Upsilon')$$
$$-\partial_x[I(\Upsilon) - \Phi(\rho)\upsilon\upsilon_x] + \partial_x[DI(\Upsilon')(\Upsilon - \Upsilon')]$$

The second law of thermodynamics along with stability helps us better understand the second part of the proof (9 [see References])

$$-DH(\Upsilon')[D[\Upsilon'_t + E(\Upsilon')_x](\Upsilon - \Upsilon') + [\Upsilon_t + E(\Upsilon)_x]]$$
$$+DH(\Upsilon')[[\Upsilon'_t + E(\Upsilon')_x] + [E(\Upsilon \mid \Upsilon')]_x]$$

Continuation: Note that we can formulate relative flux as follows

$$E(\Upsilon \mid \Upsilon') = \Psi$$

$$where\ \Psi = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)\Lambda(\rho \mid \rho') \end{bmatrix}$$

Where the flux satisfies the following expansion:

Step 1

$$\begin{bmatrix} \rho\upsilon \\ \rho\upsilon^2 + \rho^\gamma \end{bmatrix} - \begin{bmatrix} \rho'\upsilon' \\ \rho'\upsilon^{2'} + \rho^{\gamma'} \end{bmatrix} - D \begin{bmatrix} \rho'\upsilon' \\ \rho'\upsilon^{2'} + \rho^{\gamma'} \end{bmatrix}$$

$$\cdot \left[\begin{bmatrix} \rho\upsilon \\ \rho\upsilon^2 + \rho^\gamma \end{bmatrix} - \begin{bmatrix} \rho'\upsilon' \\ \rho'\upsilon^{2'} + \rho^{\gamma'} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)\Lambda(\rho \mid \rho') \end{bmatrix}$$

Step 2

$$\begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix} - D \begin{bmatrix} \rho'\upsilon' \\ \rho'\upsilon^{2'} + \rho^{\gamma'} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)\Lambda(\rho \mid \rho') \end{bmatrix}$$

Step 3

$$\begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix} - \begin{bmatrix} D(\rho'\upsilon') \\ D(\rho'\upsilon^{2'} + \rho^{\gamma'}) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)\Lambda(\rho \mid \rho') \end{bmatrix}$$

Step 4

$$\begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix}$$

$$- \begin{bmatrix} [D(\rho'\upsilon')][\rho\upsilon - \rho'\upsilon'] \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)p(\rho \mid \rho') \end{bmatrix}$$

$$where\ B = [D(\rho'\upsilon^{2'} + \rho^{\gamma'})][\rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'}]$$

Step 5

$$-D \begin{bmatrix} [\rho'\upsilon'][\rho\upsilon - \rho'\upsilon'] \\ [\rho'\upsilon^{2'} + \rho^{\gamma'}][\rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'}] \end{bmatrix}$$

$$+ \begin{bmatrix} \rho\upsilon - \rho'\upsilon' \\ \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho(\upsilon - \upsilon')^2 + (\gamma - 1)\Lambda(\rho \mid \rho') \end{bmatrix}$$

Step 6

$$-D \cdot \begin{bmatrix} \rho'\upsilon'\rho\upsilon - \rho^{2'}\upsilon^{2'} + \rho\upsilon - \rho'\upsilon' \\ \rho'\upsilon^{2'}\rho\upsilon^2 + \rho'\upsilon^{2'}\rho^\gamma - \rho^{2'}\upsilon^{4'} - \rho'\upsilon^{2'}\rho^\gamma + \rho^{\gamma'}\rho\upsilon^2 + \rho^{\gamma'}\rho^\gamma - \rho^{\gamma'}\rho'\upsilon^{2'} - W \end{bmatrix}$$

$$Where\ W = \rho^{2\gamma'} + \rho\upsilon^2 + \rho^\gamma - \rho'\upsilon^{2'} - \rho^{\gamma'}$$

Where the negative applied on the W only applies to the first term.

Step 7

$$-D \cdot \begin{bmatrix} \rho' \upsilon' \rho \upsilon - \rho^{2'} \upsilon^{2'} + \rho \upsilon - \rho' \upsilon' \\ \rho' \upsilon^{2'} \rho \upsilon^2 - \rho^{2'} \upsilon^{4'} + \rho^{\gamma'} \rho \upsilon^2 + \rho^{\gamma'} \rho^{\gamma} - \rho^{\gamma'} \rho' \upsilon^{2'} - \rho^{2\gamma'} + \rho \upsilon^2 + T \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \rho \upsilon^2 - 2\rho \upsilon \upsilon' + \rho \upsilon^2 + \rho^{\gamma} - \rho^{\gamma'} - \gamma \rho^{\gamma'-1}(\rho - \rho') \end{bmatrix}$$

$$where\ T = \rho^{\gamma} - \rho' \upsilon^{2'} - \rho^{\gamma'}$$

Step 8

$$Note\ (\gamma - 1)\Lambda(\rho \mid \rho')$$

$$= (\gamma - 1) \left[\frac{1}{\gamma - 1} \rho^{\gamma} - \frac{1}{\gamma - 1} \rho^{\gamma'} - \frac{\gamma}{\gamma - 1} \rho^{\gamma'-1}(\rho - \rho') \right]$$

Step 9

$$\begin{bmatrix} -D[\rho' \upsilon' \rho \upsilon - \rho^{2'} \upsilon^{2'} + \rho \upsilon - \rho' \upsilon'] = 0 \\ Z \end{bmatrix}$$

$$Z = -D[\rho' \upsilon^{2'} \rho \upsilon^2 - \rho^{2'} \upsilon^{4'} + \rho^{\gamma'} \rho \upsilon^2 + \rho^{\gamma'} \rho^{\gamma} - \rho^{\gamma'} \rho' \upsilon^{2'} - \rho^{2\gamma'} + \rho \upsilon^2 + \rho^{\gamma} - \rho' \upsilon^{2'} - \rho^{\gamma'}]$$

$$= 2\rho \upsilon^2 - 2\rho \upsilon \upsilon' + \rho^{\gamma} - \rho^{\gamma'} - \gamma \rho^{\gamma'-1}(\rho - \rho')$$

As we know Υ is a solution to the original Navier Stokes equations as well as satisfaction of the entropy inequality. We also know that $\Upsilon' = (\rho', \rho' \upsilon')$ which satisfies (2) and (3) (Also note that $\frac{\partial}{\partial t} \Upsilon' = 0$), we deduce

$$\frac{\partial}{\partial t}[H(\Upsilon) - H(\Upsilon')] + \frac{\partial}{\partial x}[DI(\Upsilon')(\Upsilon - \Upsilon')]$$

$$-DH(\Upsilon')[D \left[\frac{\partial}{\partial t} \Upsilon' + \frac{\partial}{\partial x} E(\Upsilon') \right] (\Upsilon - \Upsilon') + \left[\frac{\partial}{\partial t} \Upsilon + \frac{\partial}{\partial x} E(\Upsilon) \right]$$

$$- \left[\frac{\partial}{\partial t} \Upsilon' + \frac{\partial}{\partial x} E(\Upsilon') \right] - \left[\frac{\partial}{\partial x} [E(\Upsilon \mid \Upsilon')] \right]] + \Phi(\rho) \left| \frac{\partial}{\partial x} \upsilon \right|^2$$

$$\leq -D^2 H(\Upsilon') \left[\frac{\partial}{\partial x} E(\Upsilon') \right] (\Upsilon - \Upsilon') - D_2 H(\Upsilon') \left[\frac{\partial}{\partial x} \left(\Phi(\rho) \frac{\partial}{\partial x} \upsilon \right) \right]$$

$$+DH(\Upsilon') \left[\frac{\partial}{\partial x} [E(\Upsilon \mid \Upsilon')] + \frac{\partial}{\partial x} [E(\Upsilon')] \right]$$

$$-\frac{\partial}{\partial x} \left[\left[I(\Upsilon) - \Phi(\rho) \upsilon \frac{\partial}{\partial x} \upsilon \right] - [DI(\Upsilon')(\Upsilon - \Upsilon')] \right]$$

We can find that

$$\left[\frac{\partial}{\partial t} H(\Upsilon) + \frac{\partial}{\partial x} \left(I(\Upsilon) - \Phi(\rho) \upsilon \frac{\partial}{\partial x} \upsilon \right) \right] = 0$$

$$-\frac{\partial}{\partial t} H(\Upsilon') = 0$$

$$-\frac{\partial}{\partial x}\left[I(\Upsilon) - \Phi(\rho)\upsilon\frac{\partial}{\partial x}\upsilon\right] = 0$$

$$\frac{\partial}{\partial x}[DI(\Upsilon')(\Upsilon - \Upsilon')] = 0$$

$$-D^2H(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon' + \frac{\partial}{\partial x}E(\Upsilon')\right](\Upsilon - \Upsilon') = 0$$

$$-DH(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon + \frac{\partial}{\partial x}E(\Upsilon)\right] = 0$$

$$DH(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon' + \frac{\partial}{\partial x}E(\Upsilon')\right] = 0$$

$$DH(\Upsilon')\frac{\partial}{\partial x}[E(\Upsilon \mid \Upsilon')] = 0$$

Giving us the following formulation

$$\Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 \leq 0$$

This can be satisfied via the results found in (12). We can then deduce the following. Recall that

$$\upsilon = \sqrt{\frac{2\upsilon'(\gamma-1)}{D_2\rho(\gamma-1)} - \frac{2\rho^\gamma}{\rho(\gamma-1)}}$$

Therefore, we can solve for υ', finding

$$D_2\left[\frac{\rho\upsilon^{2'}}{2} + \frac{\rho^\gamma}{\gamma-1}\right] = \upsilon'$$

$$D_2H(\Upsilon') = \upsilon'$$

Now we must integrate over the bound of all real numbers [$x \in \mathbb{R}$], we can also have aid in knowing that the supremum of [$\frac{\partial}{\partial x}\Upsilon'$] $\in [-1, 1]$.

pt 1. for integral $x \in \mathbb{R}$ for $\frac{\partial}{\partial t}H(\Upsilon \mid \Upsilon')$

$$\int_{x\in\mathbb{R}}\frac{\partial}{\partial t}H(\Upsilon) + \frac{\partial}{\partial x}\left(I(\Upsilon) - \Phi(\rho)\upsilon\frac{\partial}{\partial x}\upsilon\right) - \int_{x\in\mathbb{R}}\frac{\partial}{\partial t}H(\Upsilon')$$

$$-\int_{x\in\mathbb{R}}\frac{\partial}{\partial x}\left[I(\Upsilon) - \Phi(\rho)\upsilon\frac{\partial}{\partial x}\upsilon\right] + \int_{x\in\mathbb{R}}\frac{\partial}{\partial x}[DI(\Upsilon')(\Upsilon - \Upsilon')]$$

$$-\int_{x\in\mathbb{R}}D^2H(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon' + \frac{\partial}{\partial x}E(\Upsilon')\right](\Upsilon - \Upsilon') - \int_{x\in\mathbb{R}}DH(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon + \frac{\partial}{\partial x}E(\Upsilon)\right]$$

$$+\int_{x\in\mathbb{R}}DH(\Upsilon')\left[\frac{\partial}{\partial t}\Upsilon' + \frac{\partial}{\partial x}E(\Upsilon')\right] + \int_{x\in\mathbb{R}}DH(\Upsilon')\frac{\partial}{\partial x}[E(\Upsilon \mid \Upsilon']$$

pt 2. left of inequality $\Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2$

$$\int_{x\in\mathbb{R}} \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2$$

pt 3. right side of inequality. Since we know $\frac{\partial}{\partial x}\Upsilon' \in [-1,1]$ we can deduce the upper and lower bounds of integration.

$$\int_{-1}^{1} -D^2H(\Upsilon')\left[\frac{\partial}{\partial x}E(\Upsilon')\right](\Upsilon-\Upsilon')dx$$

$$= -\int_{-1}^{1} D^2H(\Upsilon')\left[\frac{\partial}{\partial x}E(\Upsilon')\right](\Upsilon-\Upsilon')dx$$

using the knowledge $-\upsilon'\left[\frac{\partial}{\partial x}\left(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right)\right] = \left(\frac{\partial}{\partial x}\upsilon'\right)\Phi(\rho)\frac{\partial}{\partial x}\upsilon$

$$+\int_{-1}^{1} -\upsilon'\left[\frac{\partial}{\partial x}\left(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right)\right]dx = +\int_{-1}^{1}\left(\frac{\partial}{\partial x}\upsilon'\right)\Phi(\rho)\frac{\partial}{\partial x}\upsilon dx$$

using the identity: $DH(\Upsilon')\frac{\partial}{\partial x}[E(\Upsilon\mid\Upsilon')] = -\frac{\partial}{\partial x}[DH(\Upsilon')]E(\Upsilon\mid\Upsilon')$

$$+\int_{-1}^{1} DH(\Upsilon')\frac{\partial}{\partial x}[E(\Upsilon\mid\Upsilon')]dx = -\int_{-1}^{1}\frac{\partial}{\partial x}[DH(\Upsilon')]E(\Upsilon\mid\Upsilon')dx$$

Recall $DH(\Upsilon')\left[\frac{\partial}{\partial x}E(\Upsilon')\right] = -\frac{\partial}{\partial x}[DH(\Upsilon')]E(\Upsilon')$

$$+\int_{-1}^{1} DH(\Upsilon')\left[\frac{\partial}{\partial x}E(\Upsilon')\right]dx = -\int_{-1}^{1}\frac{\partial}{\partial x}[DH(\Upsilon')]E(\Upsilon')dx$$

Note to the reader that L^∞ represents the real/complex vector space that is formulated and structured based off bounded sequences with some supremum normalization, and L^∞ is equal to $L^\infty(X,\Sigma,\mu)$. These vector spaces are two associated Banach spaces. Also note that within these spaces operations such as scalar multiplication and addition are only applied on a coordinate by coordinate basis. This is taken in respect to the normalization $|x|_\infty = \sup_n |x_n|$. Knowing this, we can now move onto the next part of the proof. Namely,

$$\left|\int_{-1}^{1}\left(\frac{\partial}{\partial x}\upsilon'\right)\Phi(\rho)\frac{\partial}{\partial x}\upsilon dx\right| = \left|\int_{-1}^{1}\left(\frac{\partial}{\partial x}D_2\left[\rho\frac{\upsilon^2}{2}+\frac{1}{\gamma-1}\rho^\gamma\right]\right)\chi\sqrt[3]{\rho}\frac{\partial}{\partial x}\upsilon dx\right|$$

We can find an equation that pertains to the viscosity coefficient. As well as the squared vector norm for υ'.

$$\int_{-1}^{1}\chi\rho^{\frac{1}{3}}dx + \frac{1}{2}\left(\frac{\partial}{\partial x}\upsilon\right)^2\int_{-1}^{1}\chi\sqrt[3]{\rho}dx$$

$$\left|\left|\frac{\partial}{\partial x}\upsilon'\right|\right|_{L^\infty}^2 = \left|\left|\frac{\partial}{\partial x}\upsilon'\right|\right|_\infty^2$$

$$\left\|\frac{\partial}{\partial x}\upsilon'\right\|_{\infty}^{2} = \sqrt[\infty]{\sum_i \frac{\partial}{\partial x}\upsilon_i'^{\infty}} \cdot \sqrt[\infty]{\sum_i \frac{\partial}{\partial x}\upsilon_i'^{\infty}}$$

$$\frac{\partial}{\partial x}\upsilon_j'^{\infty} >> \frac{\partial}{\partial x}\upsilon_i'^{\infty} \quad \forall i \neq j$$

$$\sum_i \frac{\partial}{\partial x}\upsilon_i'^{\infty} = \frac{\partial}{\partial x}\upsilon_j'^{\infty}$$

$$\left\|\frac{\partial}{\partial x}\upsilon'\right\|_{\infty}^{2} = \sqrt[\infty]{\frac{\partial}{\partial x}\upsilon_j'^{\infty}} \cdot \sqrt[\infty]{\frac{\partial}{\partial x}\upsilon_j'^{\infty}} = |\frac{\partial}{\partial x}\upsilon_j'| \cdot |\frac{\partial}{\partial x}\upsilon_j'| = \frac{\partial}{\partial x}\upsilon_j^{2'}$$

$$\int_{-1}^{1} \chi \sqrt[3]{\rho} dx + \frac{1}{2}\left[\left(\frac{\partial}{\partial x}\upsilon\right)^2 \left(\frac{\partial}{\partial x}\upsilon_j^{2'}\right)\right] \int_{-1}^{1} \chi \sqrt[3]{\rho} dx$$

We can now formulate an inequality

$$\left|\int_{-1}^{1} \left(\frac{\partial}{\partial x}D_2\left[\rho\frac{\upsilon^2}{2} + \frac{1}{\gamma - 1}\rho^{\gamma}\right]\right) \chi \sqrt[3]{\rho} \frac{\partial}{\partial x}\upsilon dx\right| \leq \int_{-1}^{1} \chi \sqrt[3]{\rho} dx$$

$$+\frac{1}{2}\left[\left(\frac{\partial}{\partial x}\upsilon\right)^2 \left(\frac{\partial}{\partial x}\upsilon_j^{2'}\right)\right] \int_{-1}^{1} \chi \sqrt[3]{\rho} dx$$

Hence the proof stands. It then becomes quite apparent than when looking at

$$||\Upsilon'||_{W^{1,\infty}}$$

It is known that

$$f \in C_{loc}^{0,1}(\Upsilon) \Leftrightarrow f \in W_{loc}^{1,\infty}(\Upsilon)$$

Meaning

$$C^{0,1} = W^{1,\infty}$$

Hence,

$$||\Upsilon'||_{W^{1,\infty}} = ||\Upsilon'||_{C^{0,1}}$$

we can deduce that there will be some constant that depends on the vector norm. This constant will satisfy the following.

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx + \frac{1}{2}\int_{\mathbb{R}} \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 dx \leq N + N\int_{-1}^{1} |\Upsilon - \Upsilon'|dx$$

$$+N\int_{-1}^{1} |E(\Upsilon \mid \Upsilon')|dx + N\int_{-1}^{1} \Phi(\rho)dx$$

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx + \frac{1}{2}\int_{\mathbb{R}} \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 dx \leq N + N[\int_{-1}^{1} |\Upsilon - \Upsilon'|dx$$

$$+\int_{-1}^{1} |E(\Upsilon \mid \Upsilon')|dx + \int_{-1}^{1} \Phi(\rho)dx]$$

$$\frac{\frac{d}{dt}\int_R H(\Upsilon \mid \Upsilon')dx + \frac{1}{2}\int_R \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 dx}{\int_{-1}^1 |\Upsilon - \Upsilon'|dx + \int_{-1}^1 |E(\Upsilon \mid \Upsilon')|dx + \int_{-1}^1 \Phi(\rho)dx} \leq 2N$$

$$\frac{\frac{d}{dt}\int_R H(\Upsilon \mid \Upsilon')dx + \frac{1}{2}\int_R \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 dx}{2[\int_{-1}^1 |\Upsilon - \Upsilon'|dx + \int_{-1}^1 |E(\Upsilon \mid \Upsilon')|dx + \int_{-1}^1 \Phi(\rho)dx]} \leq N \tag{15}$$

Moreover, we must show that the right side of the inequality can be controlled by $H(\Upsilon \mid \Upsilon')$. Firstly, we can write

$$|E(\Upsilon \mid \Upsilon')| \leq \begin{cases} H(\Upsilon \mid \Upsilon') \\ H(\Upsilon \mid \Upsilon')(\gamma - 1) \end{cases}$$

(12) and (7) then yield the following inequality

$$\left|\int_{-1}^1 \left(\frac{\partial}{\partial x}D_2\left[\rho\frac{\upsilon^2}{2} + \frac{1}{\gamma-1}\rho^\gamma\right]\right)\chi\sqrt[3]{\rho}\frac{\partial}{\partial x}\upsilon dx\right| \leq$$

$$\frac{1}{2}\left[\left(\frac{\partial}{\partial x}\upsilon\right)^2\left(\frac{\partial}{\partial x}\upsilon_j^{2'}\right)\right]\int_{-1}^1 \chi\sqrt[3]{\rho}dx$$

$$\int_{-1}^1 \chi\sqrt[3]{\rho}dx \leq \frac{\left|\int_{-1}^1 \left(\frac{\partial}{\partial x}D_2\left[\rho\frac{\upsilon^2}{2} + \frac{1}{\gamma-1}\rho^\gamma\right]\right)\chi\sqrt[3]{\rho}\frac{\partial}{\partial x}\upsilon dx\right|}{\frac{1}{2}\left[\left(\frac{\partial}{\partial x}\upsilon\right)^2\left(\frac{\partial}{\partial x}\upsilon_j^{2'}\right)\right]}$$

$$\int_{-1}^1 \chi\sqrt[3]{\rho}dx \leq 0$$

$$\textit{Recall } N \geq 0$$

$$\int_{-1}^1 \chi\sqrt[3]{\rho}dx - N \leq 0$$

$$\int_{-1}^1 \chi\sqrt[3]{\rho}dx \leq N + \int_{\mathbb{R}} \Lambda(\rho \mid \rho')dx \tag{16}$$

Using (12) we can formulate a separate inequality

$$\int_{-1}^1 |\Upsilon - \Upsilon'|dx \leq \int_{-1}^1 |\rho - \rho'|dx + \int_{-1}^1 \rho|\upsilon - \upsilon'|dx + \int_{-1}^1 |\upsilon'(\rho - \rho')|dx$$

$$\leq N\int_{-1}^1 (1 + \Lambda(\rho \mid \rho'))dx + \sqrt{\left(\int_{-1}^1 \rho dx\right)\left(\int_{-1}^1 \rho(\upsilon - \upsilon')^2 dx\right)}$$

$$\leq N\int_{-1}^1 (1 + \Lambda(\rho \mid \rho'))dx + \sqrt{\left(\int_{-1}^1 (1 + \Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^1 H(\Upsilon \mid \Upsilon')dx\right)}$$

$$\leq N\int_{-1}^1 H(\Upsilon \mid \Upsilon')dx + N$$

$$\int_{-1}^{1}|\Upsilon-\Upsilon'|dx \leq \int_{-1}^{1}|\rho-\rho'|dx+\int_{-1}^{1}\rho|\upsilon-\upsilon'|dx+\int_{-1}^{1}|\upsilon'(\rho-\rho')|dx$$

$$\leq \sqrt{\left(\int_{-1}^{1}\rho dx\right)\left(\int_{-1}^{1}\rho(\upsilon-\upsilon')^2dx\right)}$$

$$-\sqrt{\left(\int_{-1}^{1}(1+\Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx\right)} \leq -N\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx \leq N$$

$$\int_{-1}^{1}|\Upsilon-\Upsilon'|dx - \int_{-1}^{1}|\rho-\rho'|dx-\int_{-1}^{1}\rho|\upsilon-\upsilon'|dx-\int_{-1}^{1}|\upsilon'(\rho-\rho')|dx \leq 0$$

$$\leq \sqrt{\left(\int_{-1}^{1}\rho dx\right)\left(\int_{-1}^{1}\rho(\upsilon-\upsilon')^2dx\right)}$$

$$-\sqrt{\left(\int_{-1}^{1}(1+\Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx\right)} + N\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx \leq 0 \leq N$$

$$\int_{-1}^{1}|\Upsilon-\Upsilon'|dx - \int_{-1}^{1}|\rho-\rho'|dx-\int_{-1}^{1}\rho|\upsilon-\upsilon'|dx-\int_{-1}^{1}|\upsilon'(\rho-\rho')|dx \leq$$

$$-\sqrt{\left(\int_{-1}^{1}\rho dx\right)\left(\int_{-1}^{1}\rho(\upsilon-\upsilon')^2dx\right)}$$

$$+\sqrt{\left(\int_{-1}^{1}(1+\Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx\right)} \leq N\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx \leq 0 \leq N$$

$$\int_{-1}^{1}|\Upsilon-\Upsilon'|dx - \int_{-1}^{1}|\rho-\rho'|dx-\int_{-1}^{1}\rho|\upsilon-\upsilon'|dx-\int_{-1}^{1}|\upsilon'(\rho-\rho')|dx$$

$$+\sqrt{\left(\int_{-1}^{1}\rho dx\right)\left(\int_{-1}^{1}\rho(\upsilon-\upsilon')^2dx\right)}$$

$$-\sqrt{\left(\int_{-1}^{1}(1+\Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx\right)}$$

$$+N\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx \leq 0 \leq 0 \leq 0 \leq N$$

$$\int_{-1}^{1}|\Upsilon-\Upsilon'|dx - \int_{-1}^{1}|\rho-\rho'|dx-\int_{-1}^{1}\rho|\upsilon-\upsilon'|dx-\int_{-1}^{1}|\upsilon'(\rho-\rho')|dx$$

$$+\sqrt{\left(\int_{-1}^{1}\rho dx\right)\left(\int_{-1}^{1}\rho(\upsilon-\upsilon')^2dx\right)}$$

$$-\sqrt{\left(\int_{-1}^{1}(1+\Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1}H(\Upsilon \mid \Upsilon')dx\right)}$$

$$+N\int_{-1}^{1} H(\Upsilon \mid \Upsilon')dx \leq 0 \leq 0 \leq N$$

$$\int_{-1}^{1} |\Upsilon - \Upsilon'|dx - \int_{-1}^{1} |\rho - \rho'|dx - \int_{-1}^{1} \rho|\upsilon - \upsilon'|dx - \int_{-1}^{1} |\upsilon'(\rho - \rho')|dx$$

$$+\sqrt{\left(\int_{-1}^{1} \rho dx\right)\left(\int_{-1}^{1} \rho(\upsilon - \upsilon')^2 dx\right)}$$

$$-\sqrt{\left(\int_{-1}^{1} (1 + \Lambda(\rho \mid \rho'))\right)\left(\int_{-1}^{1} H(\Upsilon \mid \Upsilon')dx\right)}$$

$$+N\int_{-1}^{1} H(\Upsilon \mid \Upsilon')dx \leq 0$$

$$N \geq 0$$

This leads to (15) becoming

$$N + N\int_{-1}^{1} |\Upsilon - \Upsilon'|dx + N\int_{-1}^{1} |E(\Upsilon \mid \Upsilon')|dx + N\int_{-1}^{1} \Phi(\rho)dx$$

$$-N\int_{-1}^{1} |H(\Upsilon \mid \Upsilon')|dx = N$$

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx + \frac{1}{2}\int_{\mathbb{R}} \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right|^2 dx \leq N + N\int_{-1}^{1} |H(\Upsilon \mid \Upsilon')|dx$$

And using Gronwall's Argument we can deduce Argument 4.1.

$$u'(t) \leq \beta(t)u(t), \quad t \in I^{\infty}$$

$$u(t) \leq u(a)\exp\left(\int_{a}^{t} \beta(s)ds\right) \quad \forall t \in I$$

Note that this also implies that the following is a true statement. Firstly, recall

$$\sup_{[0,T]}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx + \int_{0}^{T}\int_{\mathbb{R}} \Phi(\rho)\left|\frac{\partial}{\partial x}\upsilon\right| dxdt \leq N(T)$$

where we must prove

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx = \sup_{[0,T]}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx$$

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx = \int_{\mathbb{R}} \sup_{[0,T]} H(\Upsilon \mid \Upsilon')dx$$

$$F(x) = \int_{0}^{T} f(t)dt$$

$$Let\ T = x\ and\ 0 = a$$
$$F(x) = \int_a^x f(t)dt$$
$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$
$$F'(x) = \lim_{h\to 0} f(n_h) = f(x)$$

Therefore, the derivative of $\int_a^x f(t)dt$ is f(x). Namely,

$$H(\Upsilon \mid \Upsilon') = \frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx$$
$$H(\Upsilon \mid \Upsilon') = H(\Upsilon \mid \Upsilon')$$

we conclude,

$$\frac{d}{dt}\int_{\mathbb{R}} H(\Upsilon \mid \Upsilon')dx \leq N(T) \tag{17}$$

Even with this result it is important to note that Argument 4.1 does not prove the stability of the initial Navier Stokes equations; however, the following Argument will prove stability for us.

Argument 4.2 We will presume the viscosity coefficient to have both a continuous first derivative and a continuous second derivative. Better known as a C^2 function, and we will allow (ρ, υ) to be a solution to the initial Navier Stokes equation. Knowing this, we can write the following

$$\rho - \rho' \in L^\infty((0,T); H^1(\mathbb{R})), \quad \upsilon - \upsilon' \in L^2((0,T); H^2(\mathbb{R})), \quad 0 < \sigma \leq \rho \leq \Sigma \tag{18}$$

Then we can formulate the following inequality involving N(T)

$$\tau'(\rho) = \frac{\Phi(\rho)}{\rho^2}$$
$$\int \tau'(\rho) = \int \frac{\Phi(\rho)}{\rho^2}dx$$
$$\tau(\rho) = \int \frac{\Phi(\rho)}{\rho^2}dx \tag{19}$$

Then we can deduce that there is a constant N(T) that satisfies the following inequality

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\frac{(\rho(\upsilon - \upsilon')^2)}{2} + \Lambda(\rho \mid \rho')\right] dx + \int_0^T \int_{\mathbb{R}} \Phi(\rho) \left|\frac{\partial}{\partial x}\upsilon\right|^2 dxdt \leq N(T)$$

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\frac{\rho \left|(\upsilon - \upsilon')^2 + \frac{\partial}{\partial x}(\tau(\rho))\right|^2}{2} + \Lambda(\rho \mid \rho')\right] dx$$

$$+\int_0^T \int_{\mathbb{R}} \frac{\partial}{\partial x}(\tau(\rho))\frac{\partial}{\partial x}(\rho^\gamma)dxdt \leq N(T)$$

Then,

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\frac{\rho \left| (\upsilon - \upsilon')^2 + \frac{\partial}{\partial x}\left[\int \frac{\Phi(\rho)}{\rho^2}dx\right]\right|^2}{2} + \Lambda(\rho \mid \rho') \right] dx$$
$$+\int_0^T \int_{\mathbb{R}} \frac{\partial}{\partial x}\left[\int \frac{\Phi(\rho)}{\rho^2}dx\right]\frac{\partial}{\partial x}(\rho^\gamma)dxdt \leq N(T) \tag{20}$$

Recall that N(T) depends exclusively on $T > 0$, (ρ', υ'), Υ_0, γ, and the constant N found in (7). Also note that $\Phi(\rho)$ is non-negative, and therefore, (20) intuitively suggests that $\tau(\rho)$ is increasing over the bounds. The argument must cater, and thusly imply

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\frac{\rho \left| (\upsilon - \upsilon')^2 + \frac{\partial}{\partial x}\left[\int \frac{\Phi(\rho)}{\rho^2}dx\right]\right|^2}{2} + \Lambda(\rho \mid \rho') \right] dx - N(T)$$

$$\leq -\int_0^T \int_{\mathbb{R}} \int \left[\frac{d}{dx}\frac{\Phi(\rho)}{\rho^2}\right] dx \frac{\partial}{\partial x}\rho^\gamma dxdt$$

$$\sup_{[0,T]} \int_{\mathbb{R}} \left[\frac{\rho \left| (\upsilon - \upsilon')^2 + \frac{\partial}{\partial x}\left[\int \frac{\Phi(\rho)}{\rho^2}dx\right]\right|^2}{2} + \Lambda(\rho \mid \rho') \right] dx \leq N(T)$$

Then, in combination with results yielded from Argument 4.1 it is valid to write

$$\left|\left|\sqrt{\rho}\frac{\partial}{\partial x}(\tau(\rho))\right|\right|_{L^\infty(0,T;L^2(\Omega))} = 2\left|\left|\Phi(\rho)\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}\right|\right|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)$$

$$Let \left|\left|\frac{\Phi(\rho)}{\rho\sqrt{\rho}}\right|\right| = ||u||$$

$$Let\ 2\left|\left|\Phi(\rho)\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}\right|\right| = 2||k||$$

$$||u||_{L^\infty(0,T;L^2(\Omega)^N)} = \left(\int_0^T \left|\left|\frac{\Phi(\rho)}{\rho\sqrt{\rho}}(t)\right|\right|^P_{L^2(\Omega)^N} dt\right)^{\frac{1}{P}}$$

$$2||k||_{L^\infty(0,T;L^2(\Omega)^N)} = 2\left(\int_0^T \left|\left|\Phi(\rho)\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}(t)\right|\right|^P_{L^2(\Omega)^N} dt\right)^{\frac{1}{P}}$$

$$\left(\int_0^T \left|\left|\frac{\Phi(\rho)}{\rho\sqrt{\rho}}(t)\right|\right|^P_{L^2(\Omega)^N} dt\right)^{\frac{1}{P}} - 2\left(\int_0^T \left|\left|\Phi(\rho)\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}(t)\right|\right|^P_{L^2(\Omega)^N} dt\right)^{\frac{1}{P}}$$

$$= 0 \leq N(T) \; Hence, \; 0 \leq N(T)$$

This inequality will be used throughout the proof of Theorem 3.1. It is vital to acknowledge that there is a necessary feature of the proof, this being a new entropy inequality. The inequality we will use was first derived by Bresch and Desjardins in (10 [see References]) represented in two dimensions and higher. Moreover, in 1 dimension these calculations are much easier. In order to find the stable inequality one must know the needed regularity applied to ρ and υ. Unfortunately unlike (14) there is no conspicuous way to regularize the initial Navier Stokes equations, whilst also pertaining to preservation that must be upholded in order to derive (19). However, recall that (18) is the natural regularity for strong solutions to the initial Navier Stokes equations, and therefore, it becomes noticeable that this is enough to justify our computational methods and results. This will be explored throughout the proof.

Proof. We have to show that the following is bounded.

$$\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{\rho|\upsilon-\upsilon'|^2}{2}+\frac{\partial}{\partial x}\int\frac{(\upsilon-\upsilon')\Phi(\rho)}{\rho}dx+\int\frac{\Phi(\rho)}{2\rho}dx\right]dx+\frac{d}{dt}\int_{\mathbb{R}}\Lambda(\rho\mid\rho')dx$$

Step one. From our results found by (17), we have already established that the following is true.

$$\rho(\upsilon-\upsilon')\frac{d}{dx}\tau(\rho)+\frac{1}{2}\rho\left[\frac{d}{dx}\tau(\rho)\right]^2\cdot\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}\rho|\upsilon-\upsilon'|^2\right]dx$$

$$+\frac{d}{dt}\int_{\mathbb{R}}\Lambda(\rho\mid\rho')dx-N(T)\leq 0$$

$$\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}\rho|\upsilon-\upsilon'|^2\right]dx+\frac{d}{dt}\int_{\mathbb{R}}\Lambda(\rho\mid\rho')dx\leq N(T) \tag{21}$$

Step two. Next we must show that:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\rho\frac{\left[\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx\right]^2}{2}dx&=-\int\rho^2\frac{\Phi(\rho)}{\rho^2}\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx\frac{\partial^2}{\partial x^2}\upsilon dx\\ &-\int\left(2\rho\frac{\Phi(\rho)}{\rho^2}+\rho^2\frac{2\rho\Phi(\rho)}{\rho^4}\frac{\partial}{\partial x}\rho\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx\frac{\partial}{\partial x}\upsilon dx\right)\end{aligned} \tag{22}$$

Proof.

$$\frac{d}{dt}\int_{\mathbb{R}}\rho\frac{(\tau(\rho))_x^2}{2}dx+\int(2\rho\tau'(\rho)+\rho^2\tau''(\rho))\rho_x(\tau(\rho))_x\upsilon_x dx$$

$$=-\int\rho^2\tau'(\rho)(\tau(\rho))_x\upsilon_{xx}dx$$

$$0=-\int\rho^2\tau'(\rho)(\tau(\rho))_x\upsilon_{xx}dx$$

$$0=-\frac{d}{dx}\int\rho^2\tau'(\rho)(\tau(\rho))\upsilon_{xx}dx$$

$$0 = -\frac{d}{dx} \cdot \frac{d^2}{dx^2} \int \rho^2 \tau'(\rho)(\tau(\rho))\upsilon dx$$

$$0 = -\rho^2(\tau(\rho))\upsilon \cdot \frac{d}{dx} \cdot \frac{d^2}{dx^2} \int \tau'(\rho) dx$$

$$0 = \int \tau'(\rho) dx$$

$$0 = \tau(\rho)$$

Recall

$$\tau(\rho) = \int \frac{\Phi(\rho)}{\rho^2} dx = 0$$

Hence, the proof stands.

Recall that for the first initial Navier Stokes equation the density is bounded within $L^2((0,T) \cdot \mathbb{R})$. In that case we can write the right hand side as

$$\int \rho^2 \tau'(\rho)(\tau(\rho))_x \upsilon_{xx} dx = \int \rho^2 \tau'(\rho)(\tau(\rho))_{xx} \upsilon_x dx$$

$$\frac{d}{dx} \cdot \frac{d^2}{dx^2} \int \rho^2 \tau'(\rho)(\tau(\rho))\upsilon dx = \frac{d^2}{dx^2} \cdot \frac{d}{dx} \int \rho^2 \tau'(\rho)(\tau(\rho))\upsilon dx$$

This means that the derivatives are interchangeable, hence we may right

$$\int \rho^2 \tau'(\rho)(\tau(\rho))_{xx} \upsilon_x dx$$

However, recall that there are not any bounds that pertain to $\frac{\partial^2}{\partial x^2}\rho$. Moreover, it is vital to use this when justifying the derivation and calculation of (20):

Firstly, we garner attention to the fact that (22) only makes sense if (ρ, υ) satisfies the inequality of (18). Also realize that both ρ' and υ' are constant if they are not within the bounds of (-1,1). Results from (20) imply that $\rho_x \in L^\infty((0,T); L^2(\mathbb{R}))$ and $\upsilon_x \in L^2((0,T); H^1(\mathbb{R}))$. This knowledge yields that we only need to use the continuity equation within our computations. Hence forth, the result of (22) with the underlying assumption of (18) can be achieved via regularizing our continuity equation.

Step 3. The derivative of the cross product is then evaluated:

$$\begin{aligned} \frac{d}{dt} \int \rho(\upsilon - \upsilon') \frac{\partial}{\partial x}(\tau(\rho)) dx &= \int \frac{\partial}{\partial x}(\tau(\rho)) \frac{\partial}{\partial t}(\rho(\upsilon - \upsilon')) dx \\ + \int \rho(\upsilon - \upsilon') \frac{\partial}{\partial t} \frac{\partial}{\partial x}(\tau(\rho)) dx &= \int \frac{\partial}{\partial x}(\tau(\rho)) \frac{\partial}{\partial t}(\rho(\upsilon - \upsilon')) dx \\ &- \int (\rho(\upsilon - \upsilon'))_x \tau'(\rho) \frac{\partial}{\partial t} \rho dx \end{aligned} \tag{23}$$

Proof:

$$\frac{d}{dt} \cdot \frac{d}{dx} \int \rho(\upsilon - \upsilon') \int \frac{\Phi(\rho)}{\rho^2} dx dx = \frac{d}{dt} \cdot \frac{d}{dx} \int \int \frac{\Phi(\rho)}{\rho^2} dx (\rho(\upsilon - \upsilon')) dx$$

$$+\frac{d}{dt}\cdot\frac{d}{dx}\int\rho(\upsilon-\upsilon')\int\frac{\Phi(\rho)}{\rho^2}dxdx=\frac{d}{dt}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho(\upsilon-\upsilon'))dx$$

$$-\frac{d}{dt}\cdot\frac{d}{dx}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho}dx$$

$$\frac{d}{dt}\cdot\frac{d}{dx}\int\rho(\upsilon-\upsilon')\int\frac{\Phi(\rho)}{\rho^2}dxdx-\frac{d}{dt}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho(\upsilon-\upsilon'))dx$$

$$-\frac{d}{dt}\cdot\frac{d}{dx}\int\rho(\upsilon-\upsilon')\int\frac{\Phi(\rho)}{\rho^2}dxdx+\frac{d}{dt}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho(\upsilon-\upsilon'))dx$$

$$=\frac{d}{dt}\cdot\frac{d}{dx}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho}dx=0$$

$$0=\frac{d}{dt}\cdot\frac{d}{dx}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho}dx=0$$

$$-\frac{d}{dt}\cdot\frac{d}{dx}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho}dx=0$$

$$-\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho}dx=0$$

$$\frac{1}{\rho}\cdot\int\Phi(\rho)dx=0$$

$$\int\Phi(\rho)dx=0$$

Now we can formulate the following with equation (2).

$$\textit{Recall }(2)\ \frac{\partial}{\partial t}\rho(\upsilon-\upsilon')=(\Phi(\rho)\upsilon_x)_x-\frac{\partial}{\partial x}\rho^\gamma-\frac{\partial}{\partial x}(\rho\upsilon^2)+\frac{\partial}{\partial x}(\rho\upsilon)\frac{\partial}{\partial x}(\rho)\cdot\upsilon'$$

Multiplying by $\frac{\partial}{\partial x}\tau(\rho)$ we get

$$\frac{d}{dx}\cdot\frac{d}{dt}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho(\upsilon-\upsilon'))dx=\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\Phi(\rho)\upsilon)dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho^\gamma)dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho\upsilon^2)dx$$

$$+\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx$$

$$\frac{d}{dx}\cdot\frac{d}{dt}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho(\upsilon-\upsilon'))dx=\frac{d}{dx}\cdot\frac{d}{dt}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho\upsilon)dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dt}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho\upsilon')dx=\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\Phi(\rho)\upsilon)dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho^\gamma)dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho\upsilon^2)dx$$

$$+\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon' dx$$

Then,

$$\frac{d}{dx}\cdot\frac{d}{dt}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho^2}dx=-\frac{d}{dx}\int(\rho\upsilon)^2\frac{\Phi(\rho)}{\rho^2}dx+\frac{d}{dx}\cdot\frac{d}{dt}\int\Phi(\rho)\upsilon\upsilon' dx$$

$$Recall\ \frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon' dx=0$$

$$\frac{d}{dx}\cdot\frac{d}{dt}\int(\rho(\upsilon-\upsilon'))\frac{\Phi(\rho)}{\rho^2}dx+\frac{d}{dx}\int\upsilon^2\Phi(\rho)dx=\frac{d}{dx}\cdot\frac{d}{dt}\int\Phi(\rho)\upsilon\upsilon' dx=0$$

Hence, the continuity equation will yield the following.

$$\int\frac{\partial}{\partial x}(\rho(\upsilon-\upsilon'))\tau'(\rho)\frac{\partial}{\partial t}\rho dx$$

$$-\int\left(\frac{\partial}{\partial x}(\rho\upsilon)\right)^2\tau'(\rho)dx+\int\frac{\partial}{\partial x}(\rho\upsilon')\frac{\partial}{\partial x}(\rho\upsilon)\tau'(\rho)dx$$

The previous questions will always hold after it has been identified that ρ and υ contain satisfaction towards (18).

Step 4. We can deduce that if τ and Φ show satisfaction towards (18) then the following is true.

$$\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx\Phi(\rho)\upsilon dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\left(2\frac{\Phi(\rho)}{\rho}+\rho^2\tau''(\rho)\right)\rho\int\frac{\Phi(\rho)}{\rho^2}dx\upsilon dx$$

$$=\frac{d}{dx}\cdot\frac{d^2}{dx^2}\int\Phi(\rho)\int\frac{\Phi(\rho)}{\rho^2}dx\upsilon dx$$

$$\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial^2}{\partial x^2}(\Phi(\rho)\upsilon)dx=\int\rho^2\tau'(\rho)\frac{\partial}{\partial x}\tau(\rho)\frac{\partial^2}{\partial x^2}\upsilon dx$$

$$+\int(2\rho\tau'(\rho)+\rho^2\tau''(\rho))\frac{\partial}{\partial x}\rho\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\upsilon dx$$

(22) and (23) then yield

$$\frac{d}{dt}\int\rho\frac{(\tau(\rho))_x^2}{2}dx+\frac{d}{dt}\int\rho(\upsilon-\upsilon')\frac{\partial}{\partial x}\tau(\rho)dx$$

$$=\frac{d}{dt}\left\{\rho(\upsilon-\upsilon')\frac{\partial}{\partial x}\tau(\rho)+\rho\frac{\left|\frac{\partial}{\partial x}\tau(\rho)^2\right|}{2}dx\right\}$$

$$\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial t}(\rho(\upsilon-\upsilon'))dx=-\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\rho^\gamma dx$$

$$\frac{d}{dt}\left\{\rho(\upsilon-\upsilon')\frac{\partial}{\partial x}\tau(\rho)+\rho\frac{\left|\frac{\partial}{\partial x}\tau(\rho)^2\right|}{2}dx\right\}+\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\rho^\gamma dx$$

$$\frac{d}{dt}\left\{\rho(\upsilon-\upsilon')\frac{\partial}{\partial x}\tau(\rho)+\rho\frac{\left|\frac{\partial}{\partial x}\tau(\rho)^2\right|}{2}dx\right\}+\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\rho^\gamma dx$$

$$=-\frac{d}{dx}\cdot\frac{d}{dx}\int\int\frac{\Phi(\rho)}{\rho^2}dx(\rho\upsilon^2)dx+\frac{d}{dx}\int\upsilon^2\Phi(\rho)dx$$

$$+\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\upsilon\frac{\Phi(\rho)}{\rho}[\rho\upsilon'-\rho\upsilon']dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\frac{\Phi(\rho)}{\rho^2}[-\rho(\rho\upsilon^2)+(\rho\upsilon^2)]dx+\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx+\frac{d}{dx}\int\Phi(\rho)\upsilon^2dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx=0$$

Now using (20) we may arrive at the following.

$$\begin{aligned}&\frac{d}{dt}\left\{\int\rho(\upsilon-\upsilon')\frac{\partial}{\partial x}\tau(\rho)+\rho\frac{\left|\frac{\partial}{\partial x}\tau(\rho)\right|^2}{2}dx\right\}+\int\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\rho^\gamma dx\\&=\int\Phi(\rho)(\upsilon_x)^2dx-\int\Phi(\rho)\upsilon_x\upsilon_x'dx-\int\rho\frac{\partial}{\partial x}(\tau(\rho))\upsilon\upsilon_x'dx\end{aligned}\tag{24}$$

Proof:

$$0\geq\frac{d}{dx}\int\Phi(\rho)\upsilon^2dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\rho\int\frac{\Phi(\rho)}{\rho^2}dx\upsilon\upsilon'dx$$

$$-\frac{d}{dx}\int\Phi(\rho)\upsilon^2dx\geq-\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\rho\int\frac{\Phi(\rho)}{\rho^2}dx\upsilon\upsilon'dx$$

$$\int\Phi(\rho)dx\geq\frac{-\frac{d}{dx}\cdot\frac{d}{dx}\int\Phi(\rho)\upsilon\upsilon'dx-\frac{d}{dx}\cdot\frac{d}{dx}\int\rho\int\frac{\Phi(\rho)}{\rho^2}dx\upsilon\upsilon'dx}{-\frac{d}{dx}\upsilon^2}$$

Hence forth, we know that there are particular bounds for υ_x' in (-1,1) and we also know that there are specified bounds attributed to Argument 4.1 and inequality (16), and therefore, it becomes apparent that the right hand-side of the following equality is bounded by the following.

$$C\int_{\mathbb{R}}\Phi(\rho)|\upsilon_x|^2dx+N\int_{-1}^{1}\Phi(\rho)dx+N\int\rho\left|\frac{\partial}{\partial x}\tau(\rho)\right|^2dx+N\int_{-1}^{1}\rho\upsilon^2dx$$

$$-N\int\Phi(\rho)|\upsilon_x|^2dx-N\int_{\mathbb{R}}\Lambda(\rho,\rho')dx$$

$$-N\int_{\mathbb{R}}\rho\left|(\upsilon-\upsilon')+\frac{\partial}{\partial x}\tau(\rho)\right|^2dx-N(T)\leq0$$

$$N\int_{\mathbb{R}}\Phi(\rho)|\upsilon_x|^2dx+N\int_{-1}^{1}\Phi(\rho)dx+N\int\rho\left|\frac{\partial}{\partial x}\tau(\rho)\right|^2dx+N\int_{-1}^{1}\rho\upsilon^2dx$$

$$\leq N\int\Phi(\rho)|\upsilon_x|^2dx+N\int_{\mathbb{R}}\Lambda(\rho,\rho')dx+N\int_{\mathbb{R}}\rho\left|(\upsilon-\upsilon')+\frac{\partial}{\partial x}\tau(\rho)\right|^2dx+N(T)$$

Lastly, with (24) and (21)

$$\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')+\frac{\partial}{\partial x}\tau(\rho)\right|^2+\Lambda(\rho\mid\rho')\right]dx+\int_{\mathbb{R}}\frac{\partial}{\partial x}\tau(\rho)\frac{\partial}{\partial x}\rho^{\gamma}dx$$

$$\leq\int_{\mathbb{R}}\Phi(\rho)|\upsilon_x|^2dx+N\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')+\frac{\partial}{\partial x}\tau(\rho)\right|^2+\Lambda(\rho,\rho')\right]dx+N(T)$$

Proof:

$$\frac{d}{dt}\cdot\frac{d}{dx}\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')+\int\frac{\Phi(\rho)}{\rho^2}dx\right|^2+\Lambda(\rho\mid\rho')\right]dx$$

$$+\frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}\int\frac{\Phi(\rho)}{\rho^2}dx\rho^{\gamma}dx$$

$$\leq\frac{d}{dx}\int_{\mathbb{R}}\Phi(\rho)|\upsilon|^2dx$$

$$+\frac{d}{dx}N\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')+\int\frac{\Phi(\rho)}{\rho^2}dx\right|^2+\Lambda(\rho,\rho')\right]dx+N(T)$$

$$\frac{d}{dt}\cdot\frac{d}{dx}\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')+\int\frac{\Phi(\rho)}{\rho^2}dx\right|^2+\Lambda(\rho\mid\rho')\right]dx$$

$$+\frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}\int\frac{\Phi(\rho)}{\rho^2}dx\rho^{\gamma}dx$$

$$-\frac{d}{dx}\int_{\mathbb{R}}\Phi(\rho)|\upsilon|^2dx-\frac{d}{dx}N\int_{\mathbb{R}}\left[\frac{1}{2}\rho\left|(\upsilon-\upsilon')-\int\frac{\Phi(\rho)}{\rho^2}dx\right|^2+\Lambda(\rho,\rho')\right]dx$$

$$\leq N(T)$$

Now when using the bounds found on the viscosity derived in Argument 4.1 and through the use of Gronwall's inequality we find (19).

5 Theorem 3.1 Proof

We demonstrate the existence component of Theorem 3.1 in this section, which will be summarized in the following assertion:

Proposition 5.1 Assume that viscosity Φ fulfills (4)-(5), and that initial data (ρ_i, υ_i) satisfies (6). Then, for any and all $T > 0$, there emerges some constants $N(T)$, $\lambda(T)$, and $\lambda'(T)$ such that for every strong solution (ρ, υ) of the initial Navier Stokes equations with preliminary data (ρ_i, υ_i), outlined on $(0, T)$, and satisfying

$$\rho - \rho' \in L^\infty(0, T, dom\left(\frac{-f'}{f}\right) = \overline{C_c^1(\mathbb{R})}), \quad \frac{\partial}{\partial t}\rho \in L^2((0, T) \cdot \mathbb{R})$$

$$\rho - \rho' \in L^\infty(0, T, H_0^1(\mathbb{R})), \quad \frac{\partial}{\partial t}\rho \in L^2((0, T) \cdot \mathbb{R})$$

$$\upsilon - \upsilon' \in L^2(0, T, H^2(\mathbb{R})), \quad \frac{\partial}{\partial t}\upsilon \in L^2((0, T), \cdot\mathbb{R}),$$

Since ρ and ρ^{-1} are bounded, we can easily deduce that the following bounds must simultaneously hold

$$-\lambda(T) < -\rho(t) \leq \lambda'(T) \leq 0 \quad \forall t \in [0, T],$$

$$\lim_{p\to\infty}\left(\int_{H^1(\mathbb{R})} |\rho - \rho'|^p dx\right)^{\frac{1}{p}} = \sup_{x\in H^1(\mathbb{R})} |\rho - \rho'| \leq N(T)$$

$$\lim_{p\to\infty}\left(\int_{H^1(\mathbb{R})} |\upsilon - \upsilon'|^p dx\right)^{\frac{1}{p}} = \sup_{x\in H^1(\mathbb{R})} |\upsilon - \upsilon'| \leq N(T)$$

Furthermore, we can deduce that the given constants $N(T)$, $\lambda(T)$ and $\lambda'(T)$ depend on Φ; however, this is only through the results yielded from (4) and (5), in particular the constant N.

Proof of Theorem 4.1. We will start by introducing and defining $\Phi_\ell(\rho)$ to be an approximation that induces a positive value catered towards the viscosity coefficient.

$$\Phi_\ell(s) = max\left(\Phi(s), \frac{1}{n}\right)$$

$$max\left(\Phi(s), \frac{1}{n}\right)$$

$$= \begin{cases} \Phi(s) \\ \frac{1}{n} \end{cases}$$

$$\Phi_\ell(s)$$

$$= \begin{cases} \Phi(s) \\ \frac{1}{n} \end{cases}$$

We will bring notice to the fact that Φ_ℓ verifies

$$2\Phi + 1 \leq \Phi_\ell \leq 0 \; or \; \Phi \leq \Phi_\ell \leq \Phi + 1$$

To be more precise Φ_ℓ has satisfaction for (4) and (5) in which some constants arise that are specially independent on ℓ. Furthermore, $\forall \ell > 0$, we will let

$(\rho_\ell, \upsilon_\ell)$ be strong solutions to the initial Navier Stokes equations with the fact that $\Phi = \Phi_\ell$:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho\upsilon = 0$$

$$\frac{\partial}{\partial t}\rho\upsilon + \frac{\partial}{\partial x}\rho\upsilon^2 + \frac{\partial}{\partial x}\Lambda(\rho) = \frac{\partial}{\partial x}\left(\Phi_\ell(\rho)\frac{\partial}{\partial x}\upsilon\right)$$

We note that this solution will arise and prove to exist for at least a small time of data $(0, T_i)$ this formulation was proposed in Remark 3.3 (T_i could potentially depend on ℓ). Moreover, proposition 5.1 must then imply that $\forall T > 0$ there will exist some constants $N(T)$, $\lambda'(T)$ and $\lambda(T) > 0$, these being independent on ℓ, such that

$$\lambda(T) \leq \rho_\ell(t) \leq \lambda'(T) \quad \forall t \in [0, T],$$

$$\lim_{p\to\infty}\left(\int_0^T |\rho_\ell - \rho'|^p_{H^1(\mathbb{R})}dx\right)^{\frac{1}{p}} \leq N(T)$$

$$\lim_{p\to\infty}\left(\int_0^T |\upsilon_\ell - \upsilon'|^p_{H^1(\mathbb{R})}dx\right)^{\frac{1}{p}} \leq N(T)$$

In this specified case we will take $T_i = \infty$ in Remark 3.3 $\forall\ell$. Moreover, note that the bound for the density resides $\in \ell$ for when $T > 0$. When we take ℓ to be large. More specifically $\ell \geq \frac{1}{\lambda(T)}$, it can be easily seen that $(\rho_\ell, \upsilon_\ell)$ is a solution to the initial Navier Stokes equations on the interval $[0, T]$ including our non-truncated viscosity coefficient $\Phi(\rho)$. This is gathered from the uniqueness to the solution of Remark 3.3, hence, we see the desired global solution to the initial Navier Stokes equations. When passing the limit $\in \ell$.

As a result, the remainder of this section is devoted to proving Proposition 5.1. To begin, we show that ρ is evenly constrained from above and below by some positive constants. Then, using some common parabolic equations reasoning, we'll look into the regularity of the velocity.

5.1 A Priori Estimate

Sine we have already identified the initial data (ρ_i, υ_i) shows satisfaction towards (6), we yield

$$\int \rho_i(\upsilon_i - \upsilon')^2 dx < \infty \; and \; \int_\Omega \rho_i \left|\frac{\partial}{\partial x}(\tau(\rho_i))\right|^2 dx < +\infty$$

Recall first integral

$$\int_{\mathbb{R}}\left[\frac{1}{2}\rho_i(\upsilon_i - \upsilon')^2 + \Lambda(\rho_i \mid \rho')\right]dx < \infty$$

$$\frac{1}{2}\int_{\mathbb{R}} \rho_i(\upsilon_i - \upsilon')^2 + \Lambda(\rho_i \mid \rho')dx < \infty$$

$$\int_{\mathbb{R}} \rho_i(\upsilon_i - \upsilon')^2 + \Lambda(\rho_i \mid \rho')dx < \infty$$

$$Hence, \int_{\mathbb{R}} \rho_i(\upsilon_i - \upsilon')^2 dx < \infty$$

Second integral

$$N \int \rho \left| \frac{\partial}{\partial x} \tau(\rho) \right|^2 dx \leq N(T)$$

$$N \cdot \frac{d}{dx} \int \rho \left| \tau(\rho) \right|^2 dx \leq N(T)$$

$$\int \rho \left| \tau(\rho) \right|^2 dx \leq 0$$

Giving us

$$\int_{\Omega} \rho_i \left| \frac{\partial}{\partial x} (\tau(\rho_i)) \right|^2 dx < +\infty$$

Furthermore, because (ρ, υ) satisfies (18), we can apply the inequalities in Arguments 4.1 and 4.2. We arrive at the following conclusions, which we will employ throughout the demonstration of proposition 5.1. Defining L^1_{loc} *Note. Let us first consider a function $f \in L^1_{loc}(\Omega)$, where Ω is an open subset of R^N. One cannot, for an arbitrary $x \in \Omega$, give a meaning to f(x).

$$\begin{aligned}
\lim_{p \to \infty} \left(\int_0^T |\sqrt{\rho}\upsilon - \sqrt{\rho}\upsilon'|^p_{L^2(\Omega)} dx \right)^{\frac{1}{p}} &\leq N(T), \\
\lim_{p \to \infty} \left(\int_0^T |\rho|^p_{L^2_{loc} \cap L^{\gamma}_{loc}(\Omega)} dx \right)^{\frac{1}{p}} &\leq N(T), \\
\lim_{p \to \infty} \left(\int_0^T |\rho - \rho'|^p_{L^1(\Omega)} dx \right)^{\frac{1}{p}} &\leq N(T), \\
\sqrt{\sum_{k=1}^{n} \int_0^T \left| \sqrt{\Phi(\rho)} \frac{\partial}{\partial x} \upsilon_k \right|^2_{L^2(\Omega)}} dx &\leq N(T)
\end{aligned} \tag{25}$$

And

$$\begin{aligned}
\lim_{p \to \infty} \left(\int_0^T \left| \Phi(\rho) \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\rho}} \right) \right|^p_{L^2(\Omega)} dx \right)^{\frac{1}{p}} &\leq N(T), \\
\sqrt{\sum_{k=1}^{n} \int_0^T \left| \sqrt{\Phi(\rho)} \frac{\partial}{\partial x} \left(\rho^{\frac{\gamma-1}{2}} \right)_k \right|^2_{L^2(\Omega)}} dx &\leq N(T)
\end{aligned} \tag{26}$$

5.2 Uniform Bounds for the Density

The first argument establishes that no vacuum states can exist. There exists a constant rho that is bigger than zero at all times, such that

$$Recall\ -\Lambda(T) < -\rho(x,t) \leq -\Lambda(T) \leq 0 < \infty \quad \forall(x,t) \in [0,T] \cdot \mathbb{R}$$

$$Hence,\ \rho(x,t) \geq \Lambda(T) \quad \forall(x,t) \in [0,T] \cdot \mathbb{R}$$

The two arguments will prove to be vital for the proof of this proposition. Firstly,

Argument 5.1 $\forall T > 0$, there is bound to exist some $\beta > 0$ and D(T) such that there is satisfaction for $\forall x_i \in \mathbb{R}$ and $t_i > 0$, then there exists $x_1 \in [x_i - D(T), x_i + D(T)]$ with

$$\rho(x_1, t_i) > \beta$$

(11 [see References]) yields this result. In this case we will be focusing on completeness, and hence, we will provide a proof. In the proof we will allow $\beta > 0$ such that the following is true.

$$\frac{1}{\gamma-1}\rho^{\gamma} - \frac{1}{\gamma-1}\rho^{\gamma'} - \frac{\gamma}{\gamma-1}\rho^{\gamma'-1}(\rho-\rho') \geq \frac{1}{2N}$$

$$\frac{2\rho^{\gamma}}{\gamma-1} - \frac{2\rho^{\gamma'}}{\gamma-1} - \frac{2\gamma\rho^{\gamma'-1}(\rho-\rho')}{\gamma-1} \geq \frac{1}{N}$$

$$N\left[\frac{2\rho^{\gamma}}{\gamma-1} - \frac{2\rho^{\gamma'}}{\gamma-1} - \frac{2\gamma\rho^{\gamma'-1}(\rho-\rho')}{\gamma-1}\right] \geq 1$$

$$N \geq \frac{1}{2\Lambda(\rho \mid \rho')} \quad \forall \rho < \beta$$

(β exists thanks to (11)). Then, if

$$\sup_{x\in[x_i-D, x_i+D]} \rho(x, t_i) < \beta$$

We can then easily deduce

$$\int \Lambda(\rho \mid \rho')dx \geq N^{-1}D$$

$$\frac{1}{D}\int \Lambda(\rho \mid \rho')dx \geq \frac{1}{N}$$

$$\frac{N}{D}\int \Lambda(\rho \mid \rho')dx \geq 1$$

$$\frac{N}{D}\int \Lambda(\rho \mid \rho')dx - 1 \geq 0$$

$$\int \Lambda(\rho \mid \rho')dx - 1 \geq 0$$

An appropriate choice of D generates a disparity because the integral on the left hand side is constrained by a constant (see Argument 4.1).

Argument 5.2 Now we will allow the following to ensue, in which we denote that the infimum of the function resides within the positive part of the function.

$$let\ z(x,t) = \inf(\rho(x,t),1)$$

$$z(x,t) = \rho(x,t)_+$$

Moreover, we can deduce that there exists $\psi > 0$ with a simultaneous constant N(T), such that the following stands.

$$\lim_{p\to\infty}\left(\int_0^T \left|\frac{\partial}{\partial x}\frac{1}{z^\psi}\right|^p_{L^2(\mathbb{R})} dx\right)^{\frac{1}{p}} \leq N(T)$$

Using (4) we arrive at

$$\lim_{p\to\infty}\left(\int_0^T \left|z^{\frac{2\alpha-3}{2}}\frac{\partial}{\partial x}z\right|^p_{L^2(\Omega)} dx\right)^{\frac{1}{p}} - \lim_{p\to\infty}\left(\int_0^T \left|\frac{\partial}{\partial x}z^{\frac{2\alpha-1}{2}}\right|^p_{L^2(\Omega)} dx\right)^{\frac{1}{p}} = 0 \leq N,$$

And this result fundamentally follows with the knowledge that $\psi + \alpha = \frac{1}{2} > 0$

Proposition 5.2 proof. We will use Argument 5.1 and 5.2 and the Poincaré inequality, we can yield $\frac{1}{z^\psi}$, and deduce that it is bounded in $L^\infty((0,T)\cdot\mathbb{R})$. Firstly, we will recall that if there is some p, such that $1 \leq p < \infty$, as well as a subset Ω bounded in a particular direction. We get that there exists some constant N, that will depend exclusively on Ω and p. Moreover, for every function u of the space $W_0^{1,p}(\Omega)$ with zero-trace. The Poincaré inequality states:

$$||u||_{L^p(\Omega)} \leq N||\nabla u||^p_L(\Omega)$$

Hence, in this case we get:

$$\frac{1}{z^\psi}(x,t) \leq N(T) \quad \forall(x,t) \in \mathbb{R}\cdot(0,T)$$

Therefore, this will yield Proposition 5.2 with the constant $\lambda(T) = \frac{1}{N(T)^{\frac{1}{\psi}}}$, and then we know we must find a bound for the density residing within L^∞ space.

Proposition 5.3 $\forall T > 0$, $\lambda'(T)$ must exists, and therefore, the following stands.

$$\rho(x,t) \leq \lambda'(T) \quad \forall(x,t) \in \mathbb{R}\cdot(0,T)$$

Now we can allow $w = \frac{1}{2}(\gamma - 1)$, then (19) with the knowledge of (4) and with the use of (20) we can yield $\frac{\partial}{\partial x}\rho^w$ bounded in the space $L^2((0,T)\cdot\mathbb{R})$. Moreover, for all the compact subsets K residing in $\mathbb{R}$, we have

$$\int_K \sqrt{\left(\sum_{i=1}^N \left|\frac{\partial}{\partial x}\rho_i^w\right|^2\right)} dx = \int_K \sqrt{\left(\sum_{i=1}^N \left|\rho^{w-1}\frac{\partial}{\partial x}\rho_i\right|^2\right)} dx$$

$$\leq \sqrt{\left(\int_K \rho^{1+2w} dx\right)\left(\int_K \frac{1}{\rho^3}\left(\frac{\partial}{\partial x}\rho\right)^2 dx\right)}$$

$$\leq \sqrt{\left(\int_K \rho^{\gamma} dx\right)\left(\int_K \rho\left[\frac{\Phi(\rho)}{\rho^2}\right]^2\left(\frac{\partial}{\partial x}\rho\right)^2 dx\right)}$$

Then using (10) we arrive at

$$\int_K \sqrt{\sum_{i=1}^{N}\left|\frac{\partial}{\partial x}\rho_i^w\right|^2} dx \leq$$

$$N\sqrt{\left[\sqrt{\sum_{i=1}^{N}|K_i|^2} + \int_K \Lambda(\rho \mid \rho')dx\right]\left[\int_k \left[\frac{\Phi(\rho)}{\rho}\right]^2\left(\frac{\partial}{\partial x}\rho\right)^2 dx\right]}$$

Since we already know that

$$-\rho^{\gamma} + \rho^{w} \leq 1$$

We can then gather that ρ^w is bounded within $L^\infty(0,T;W^{1,1}_{loc}(R))$, and $|K|$ is the only factor that affects the $W^{1,1}(K)$ norm of $\rho^w(t,\cdot)$. As a result of the Sobolev embedding, Proposition 5.3 is obtained. Note two specified embedding theorems, the Sobolev embedding theorem and the interpolation theorem. The proof of the Sobolev embedding theorem (27 [see References]) is based on the theorem conducted for $U \subset \mathbb{R}^n$ as well as the creation on X of an atlas whose domains of charts are geodesic balls of definite radius $\rho, 0 < \rho < \delta$ of which is uniformly locally limited, that is, such that there is an integer k in which each position $x \in X$ has a neighborhood with at maximum k of the examined balls, as well as a nonempty intersection. The Sobolev embedding hypothesis holds for topologies with boundaries if the conical constraint is reformulated appropriately, such as for compacted topologies with C^1 boundary. For compacted riemannian geometry, or compact riemannian manifolds with a border that fulfills some form of cone condition, such as C^1, Kondrakov's theorem holds.

Kondrakov's theorem

For Riemannian manifolds where the Sobolev embedding hypothesis applies, the multiplication and composition theorems are applicable. For a manifold with a volume of finite nature (for example compact), the multiplication hypothesis is valid in the format provided in problem V13 (27 [see References]), and furthermore in the format provided in problem V13,2 (27 [see References]).

multiplication theorem and composition theorem

In its most general form, the interpolation theorem holds for C^∞ compacted

Riemannian manifolds both with and without boundaries, with the extra hypothesis $f_X ud\mu = 0$ during the latter case. The proof for $(j = 1, m = 2, a = \frac{1}{2})$ in problem V12 is applicable for any C^∞ riemmanian manifold, using a proof that is very similar. As may be observed by following the specifics of Aubin's proof, many other examples are valid for non-compact manifolds (27 (pp. 94-95)[see References]).

Proof showing if the interpolation theorem holds for riemmanian manifolds without boundaries underneath the constraint (27 (Aubin, p. 94)[see References]) $f_X ud\mu = 0$, it also holds for riemmanian manifolds containing boundaries if $j > 0$. To begin, we'll assume that X is compact without the need for a boundary, and that $u \in D(X)$ and

$$\int_X ud\mu = k \neq 0,$$

Where $d\mu$ is a volume element. Set

$$v = u - \frac{k}{VolX},$$

$$VolX = \int_X d\mu;$$

Then $v \in D(X)$ and

$$\int_X vd\mu = 0.$$

The interpolation theorem then gives us the following for such functions.

$$||\nabla^j v||_{Lp} \leq c||\nabla^m v||^a_{L^r}||v||^{1-a}_{L^q}.$$

In our case we already our embedded case to be an interpolation.

Proposition 5.4 There is indeed a constant N(T) that ensures

$$\lim_{p\to\infty}\left(\int_0^T |\rho(x,t) - \rho'(x)|^p_{H^1(\mathbb{R})}dx\right)^{\frac{1}{p}} \leq N(T).$$

The proof of Proposition 5.3 yields

$$J = \int \left(\frac{\partial}{\partial x}\rho\right)^2 dx \leq H = \lambda^{3'} \int \frac{1}{\rho^3}\left(\frac{\partial}{\partial x}\rho\right)^2 dx \leq L$$

Recall

$$\frac{\rho}{\Phi(\rho)\Phi(\rho)} \cdot \frac{\Phi(\rho)\Phi(\rho)}{\rho^4} = \frac{1}{\rho^3}$$

$$L = \lambda^{3'} \int \frac{\rho(\tau'(\rho))^2}{(\Phi(\rho))^2}\left(\frac{\partial}{\partial x}\rho\right)^2 dx$$

$$L \leq \chi\lambda^{3'}\left(\frac{1}{\inf(1, \lambda^{2\alpha})}\right)\int \rho\left(\frac{\partial}{\partial x}\int \frac{\Phi(\rho)}{\rho^2}dx\right)^2 dx \leq N(T)$$

$$J - H + L - \chi\lambda^{3'} \left(\frac{1}{\inf(1, \lambda^{2\alpha})}\right) \int \rho \left(\frac{\partial}{\partial x} \int \frac{\Phi(\rho)}{\rho^2} dx\right)^2 dx \leq 0 \leq 0 \leq N(T)$$

And then the result will follow simultaneously.

5.3 Uniform Bounds for the Velocity

Proposition 5.5 There exists a constant that will satisfy the following.

$$\sqrt{\sum_{k=1}^{n} \int_0^T |[\upsilon - \upsilon']_k|^2_{H^2(\mathbb{R})} dx} \leq N(T),$$

$$\sqrt{\sum_{k=1}^{n} \int_0^T \left|\frac{\partial}{\partial t}\upsilon_k\right|^2_{L^2(\mathbb{R})} dx} \leq N(T)$$

To be more specific we note that $\upsilon - \upsilon' \in C^0(0, T; H^1(\mathbb{R})$. Note that C^0 space simply refers to the space of continuous functions. To prove this, we must show that $\upsilon - \upsilon'$ is bounded, and therefore, we must show that it resides within $L^2(0, T; H^1(\mathbb{R}))$. Recall that since the inequality $\rho \geq \lambda > 0$ is true, we can deduce that we must use the information gathered in (4). Moreover, it becomes quite apparent that there will exists some arbitrary constant that satisfies an inequality. In particular we have $\chi' > 0$ such that the following must be true.

$$\frac{1}{\chi'}\Phi(\rho(x,t)) \geq 0 \quad \forall (x,t) \in \mathbb{R} \cdot [0,T],$$

$$\Phi(\rho(x,t)) \geq \chi' \quad \forall (x,t) \in \mathbb{R} \cdot [0,T],$$

Hence, (14) will yield the knowledge that $\frac{\partial}{\partial x}\upsilon$ is bounded within the space $L^2((0,T) \cdot \mathbb{R})$ and simultaneously note that $\upsilon - \upsilon'$ is also bounded within the space $L^\infty(0, T; L^2(\mathbb{R}))$. Henceforth, we may also deduce that $\upsilon - \upsilon'$ is also bounded within $L^2(0, T; H^1(\mathbb{R}))$. Note that with this knowledge there is also the fact that $\frac{\partial}{\partial t}\rho$ is bounded within the space $L^2((0,T) \cdot \mathbb{R})$. Since we have already made it clear that $\rho - \rho'$ is bounded within $L^\infty(0, T; H^1)$, it will then follow that the following will also stand. Recall space $C_p^{s_j}$. Hence we can deduce that our ρ resides within $C^{s_0}((0,T) \cdot \mathbb{R})$ for when some $s_0 \in (0,1)$. Moreover, we will now define the second part of the initial Navier Stokes equations as follows:

$$\gamma\rho^{\gamma-2}\frac{\partial}{\partial x}\rho - \frac{\partial}{\partial x}\left[\frac{\Phi(\rho)}{\rho}\frac{\partial}{\partial x}\upsilon\right] + \frac{\partial}{\partial t}\upsilon - \frac{\partial}{\partial x}\upsilon'\upsilon + (\upsilon - \upsilon')\frac{\partial}{\partial x}\upsilon = \frac{\partial}{\partial x}\int \frac{\Phi(\rho)}{\rho^2} dx \quad (27)$$

It then becomes apparent to recall the fact that τ arises from the new entropy inequality that was defined in (Argument 4.2).

Our goal is to define some bounds on υ, hence we need to control the right hand side or parts on the left that contribute to the right hand side of (27). The first term we will look at is $-\gamma\rho^{\gamma-2}\frac{\partial}{\partial x}\rho$ is bounded within the space

$L^\infty(0,T;L^2(\mathbb{R})$ this can be found when reviewing proposition 5.4. Next, if we look at $\frac{\partial}{\partial x}\upsilon'\upsilon$ it becomes clear that there are bounds within $L^2((0,T)\cdot\mathbb{R})$ due to the fact that υ resides within L^∞ space. Moreover, in order to deduce the rest we could use the Hölder's inequality and the interpolation inequality. Hence,

$$\sqrt{\sum_{k=1}^{n}\int_0^T \left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\frac{\partial}{\partial x}\upsilon\right|^2_{\sqrt[3]{L^4(\mathbb{R})}}dx}$$

$$\leq \lim_{p\to\infty}\left(\int_0^T\left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\right|^p_{L^2(\mathbb{R})}dx\right)^{\frac{1}{p}}\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{L^4(\mathbb{R})}dx}$$

$$\leq \sup_{x\in L^2}\left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\right|\left[\sqrt{\sum_{k=1}^{n}\left|\frac{\partial}{\partial x}\upsilon\right|^2_{L^2}}\right]^{\frac{2}{3}}$$

$$\left[\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})}dx}\right]^{\frac{1}{3}}$$

$$\leq N\left[\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})}dx}\right]^{\frac{1}{3}}$$

Proof.

$$\sqrt{\sum_{k=1}^{n}\int_0^T \left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\frac{\partial}{\partial x}\upsilon\right|^2_{\sqrt[3]{L^4(\mathbb{R})}}dx}$$

$$\leq \lim_{p\to\infty}\left(\int_0^T\left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\right|^p_{L^2(\mathbb{R})}dx\right)^{\frac{1}{p}}\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{L^4(\mathbb{R})}dx}$$

$$-\sup_{x\in L^2}\left|\frac{\partial}{\partial x}\int\frac{\Phi(\rho)}{\rho^2}dx-(\upsilon-\upsilon')\right|\left[\sqrt{\sum_{k=1}^{n}\left|\frac{\partial}{\partial x}\upsilon\right|^2_{L^2}}\right]^{\frac{2}{3}}$$

$$+N\left[\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})}dx}\right]^{\frac{1}{3}}\leq 0\leq 0$$

In this specified case we make use of results from (19) as well as proposition 5.2. Henceforth, regularity results in the form of (27) which is that of a parabolic equation. Also it becomes apparent that our diffusion coefficient resides within the space $C^{s_0}((0,T)\cdot\mathbb{R})$. This then yields

$$\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})}dx}-N\left[\sqrt{\sum_{k=1}^{n}\int_0^T\left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})}dx}\right]^{\frac{1}{3}}\leq N,$$

Therefore,

$$\sqrt{\sum_{k=1}^{n}\int_0^T \left|\frac{\partial}{\partial x}\upsilon\right|^2_{W^{1,\frac{4}{3}}(\mathbb{R})} dx} \leq N$$

It emerges from Sobolev inequalities that $\frac{\partial}{\partial x}\upsilon$ is restricted in $L^2(0,T;L^\infty(\mathbb{R}))$.

Finally, we can now see that the right hand side and the parts of the left hand side that revolves around the right hand side in (27) is bounded in the space $L^2(0,T;L^2(\mathbb{R}))$, and through the results of classical regularity for parabolic equations we get that $\upsilon - \upsilon'$ is bounded within $L^2(0,T;H^2(\mathbb{R}))$ and we get that $\frac{\partial}{\partial t}\upsilon$ is bounded within $L^2(0,T;L^2(\mathbb{R}))$, which then concludes our proof.

Now we can easily deduce that Propositions 5.2, 5.3, 5.4, and 5.5 all follow from 5.1.

6 Uniqueness

In the final section, we prove that the global strong solution is unique among a broad class of weak solutions that satisfy the standard entropy inequality. The following is a rewrite of the result:

Proposition 6.1 Firstly, we will assume

$$\Phi(\rho) \geq \chi > 0 \quad \forall \rho \geq 0$$

Then, the following stands

$$\sum_{i=1}^{n} |[\Phi(\rho) - \Phi(\tilde{\rho})]_i| \leq N \sum_{i=1}^{n} |[\rho - \tilde{\rho}]_i| \quad \forall \rho, \tilde{\rho} \geq 0$$

Consider that $\gamma \geq 2$ is true, and that (ρ, υ) is the result to Theorem 3.1's initial Navier Stokes equations.

Assuming $(\tilde{\rho}, \tilde{\upsilon})$ is a weak resolution of the initial Navier Stokes equations with beginning data (ρ_i, υ_i) that satisfies the entropy inequality (12) and relative entropy constraint (14), and if the following is also true.

$$\lim_{x\to\pm\infty}(\tilde{\rho}\tilde{\upsilon} - \tilde{\rho}\upsilon_\pm - \rho_\pm\tilde{\upsilon} + \rho_\pm\upsilon_\pm) = 0$$

Then, we can deduce

$$(\tilde{\rho}, \tilde{\upsilon}) = (\rho, \upsilon)$$

It's worth noting that we don't have to assume that $\tilde{\rho}$ will not vanish. As a result of the following reasoning, this argument will be made:

Argument 6.1 We will allow the fact that $\tilde{\Upsilon}$ be equal to $(\tilde{\rho}, \tilde{\rho}\tilde{\upsilon})$, and the fact that this will represent a weak solution to the initial Navier Stokes equations. With a present satisfaction towards (12), and we will also let Υ be equal to $(\rho, \rho\upsilon)$ with the knowledge that this represents the global strong solution to the initial Navier Stokes equations, in which there is a present satisfaction towards

(8). In this case we have the underlying assumption that $\tilde{\Upsilon}$ and Υ exist such that the following stand.

$$\lim_{x\to\pm\infty}(\tilde{\rho}-\rho)=0, \quad \lim_{x\to\pm\infty}(\tilde{\upsilon}-\upsilon)=0 \tag{28}$$

Therefore, we have

$$\frac{d}{dt}\int_{\mathbb{R}} H(\tilde{\Upsilon}\mid \Upsilon dx+\int_{\mathbb{R}}\Phi(\tilde{\rho})\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]^2 dx+\int_{\mathbb{R}}\frac{\partial}{\partial x}\upsilon[\Phi(\tilde{\rho})]\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]dx \\ -\int_{\mathbb{R}}\frac{1}{\rho}\left[\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon)\right](\tilde{\rho}-\rho)(\upsilon-\tilde{\upsilon})dx\leq N\int\left|\frac{\partial}{\partial x}\upsilon\right| H(\tilde{\Upsilon}\mid\Upsilon)dx \tag{29}$$

The structure of the equation, rather than the qualities of the solutions, is used to prove this argument. We'll put it off until the end of this segment.

Now we will prove Proposition 5.1. Moreover, it is vital that we show that a specified two terms that can be found within (29) can be controlled via the relative entropy represented as $H(\tilde{\Upsilon}\mid\Upsilon)$ and as well as the viscosity. Alos since we already know that $\gamma\geq 2$ and that $\rho\geq\lambda'>0$, we will allow the fact that there will exist some N that satisfies

$$\frac{\Lambda(\tilde{\rho}\mid\rho)}{\left[\sum_{i=1}^{n}[\tilde{\rho}-\rho]_i\right]^2}\geq N \quad \forall\tilde{\rho}\geq 0$$

Hence,

$$\sum_{i=1}^{n}\left[\int_{\mathbb{R}}\frac{\partial}{\partial x}\upsilon[\Phi(\tilde{\rho})-\Phi(\rho)]\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]dx\right]_i\leq N\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}\upsilon\right|$$

$$\sqrt{\int_{\mathbb{R}}\left[\sum_{i=1}^{n}[\tilde{\rho}-\rho]_i\right]^2dx}\sqrt{\int_{\mathbb{R}}\left[\sum_{i=1}^{n}\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]_i\right]^2dx}$$

$$\leq N\left[\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}\upsilon\right|\right]^2\sqrt{\int_{\mathbb{R}}\left[\sum_{i=1}^{n}[\tilde{\rho}-\rho]_i\right]^2dx}+\frac{1}{4}\int_{\mathbb{R}}\Phi(\tilde{\rho})\left[\sum_{i=1}^{n}\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]_i\right]^2dx$$

$$\leq N\left[\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}\upsilon\right|\right]^2\int_{\mathbb{R}}H(\tilde{\Upsilon}\mid\Upsilon)dx+\frac{1}{4}\int_{\mathbb{R}}\Phi(\tilde{\rho})\left[\sum_{i=1}^{n}\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]_i\right]^2dx$$

Which works in the case of two specific cases in (29). We can see that if we had $\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon)$ bounded in $L^\infty((0,T)\cdot\mathbb{R})$, a comparable calculation would occur for the last given term. Writing, on the other hand,

$$\frac{\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon)-\Phi'(\rho)\left(\frac{\partial}{\partial x}\rho\right)\left(\frac{\partial}{\partial x}\upsilon\right)}{\frac{\partial^2}{\partial x^2}\upsilon}=\Phi(\rho)$$

It then becomes quite apparent that $\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon)$ can only be bounded within the space $L^2((0,T)\cdot\mathbb{R})$. Hence, with this knowledge it becomes vital to control $|\tilde{\upsilon}-\upsilon|$ in the environment of L^∞, this can then only be made possible by the ensuing argument.

Argument 6.2 We will allow $\tilde{\rho}\geq 0$ to satisfy $\int \Lambda(\tilde{\rho}\mid\rho')dx < +\infty$. Then a specified constant will be dependent on it, and therefore, for a regular defined function s, we get the following.

$$\sup_{x\in\mathbb{R}}|s| \leq N\sqrt{\int_{\mathbb{R}}\tilde{\rho}\left[\sum_{i=1}^{n}|s_i|\right]^2 dx} + N\sqrt{\int_{\mathbb{R}}\left[\sum_{i=1}^{n}\left|\frac{\partial}{\partial x}s_i\right|\right]^2 dx}$$

Henceforth, through the use of Argument 6.2 with the knowledge that our regular function $s+\upsilon=\tilde{\upsilon}$, we can yield:

$$\int_{\mathbb{R}}\frac{1}{\rho}\left[\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right](\tilde{\rho}-\rho)(\upsilon-\tilde{\upsilon})dx$$

$$\leq \sqrt{\sum_{i=1}^{n}\left|\frac{1}{\rho}\left[\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right](\tilde{\rho}-\rho)(\upsilon-\tilde{\upsilon})\right|^2}\sqrt{\sum_{i=1}^{n}|\tilde{\rho}-\rho|^2\sup_{x\in\mathbb{R}}|\upsilon-\tilde{\upsilon}|}$$

$$\leq N\sqrt{\sum_{i=1}^{n}\left|\frac{1}{\rho}\left[\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right]\right|^2}\sqrt{H(\tilde{\Upsilon}\mid\Upsilon)}$$

$$\left(\sqrt{H(\tilde{\Upsilon}\mid\Upsilon)}+\sqrt{\int_{\mathbb{R}}\left[\sum_{i=1}^{n}\left|\frac{\partial}{\partial x}(\upsilon-\tilde{\upsilon})_i\right|\right]^2 dx}\right)$$

$$\leq N\left[\sum_{i=1}^{n}\left|\frac{1}{\rho}\left[\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon\right]\right|^2\right]H(\tilde{\Upsilon}\mid\upsilon)$$

$$+\frac{1}{4}\int_{\mathbb{R}}\Phi(\tilde{\rho})\left[\sum_{i=1}^{n}\left|\frac{\partial}{\partial x}(\upsilon-\tilde{\upsilon})_i\right|\right]^2 dx$$

Moreover, (29) becomes

$$\frac{\frac{1}{2}\int_{\mathbb{R}}\Phi(\tilde{\rho})\left[\frac{\partial}{\partial x}(\tilde{\upsilon}-\upsilon)\right]^2 dx + \frac{d}{dt}\int_{\mathbb{R}}H(\tilde{\Upsilon}\mid\Upsilon)dx}{\int H(\tilde{\Upsilon}\mid\Upsilon)dx}\leq N(t)$$

Where $N(t)\in L^1(0,T)$. Through the use of Gronwall's Argument, and the knowledge of the fact that $H(\tilde{\Upsilon}\mid\Upsilon)$ when $t=0$, and hence the expression also equals zero. This then yields Proposition 6.1.

Proof of Argument 6.2. Using the information gatered by (11), we can deduce that there will exist some $\beta>0$ and a constant N that satisfies

$$\frac{\sum_{i=1}^{n}|\{x\in\mathbb{R};\tilde{\rho}\leq\beta\}_i|}{\int_{\mathbb{R}}\Lambda(\tilde{\rho}\mid\rho')dx}\leq N$$

Now we will take $\frac{D}{N} - 1 = \int \Lambda(\tilde{\rho} \mid \rho')dx$. Moreover, $\forall x_i \in \mathbb{R}$ it is a proven fact that $\tilde{\rho}$ in the interval $(x_i - \frac{D}{2}, x_i + \frac{D}{2})$ is large than our previously defined β. Where β is a set of measurements of at least one, hence, we may denote this set by μ, giving us that $\mu = (x_i - \frac{D}{2}, x_i + \frac{D}{2}) \cap \{\tilde{\rho} \geq \beta\}$. Therefore, $\forall x \in \mu$ the following must be true.

$$\sum_{i=1}^{n} |s(x_i)_i| \leq \sum_{i=1}^{n} |s(x)| + \int_{x_i}^{x} \sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s(y)_i \right| dy \leq \sum_{i=1}^{n} |s(x)_i|$$

$$+\sqrt{D} \sqrt{\int_{\mathbb{R}} \left[\sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s(y)_i \right| \right]^2 dy}$$

$$\sum_{i=1}^{n} |s(x_i)_i| \leq \int_{x_i}^{x} \sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s(y)_i \right| dy$$

$$-\sqrt{D} \sqrt{\int_{\mathbb{R}} \left[\sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s(y)_i \right| \right]^2 dy} \leq 0$$

Now we can integrate everything with respect to $x \in \mu$ giving us

$$\sum_{i=1}^{n} |s(x_i)_i| \leq \frac{1}{\sum_{i=1}^{n} |\mu_i|} \int_{\mu} \sum_{i=1}^{n} |s_i| dx + \sqrt{D} \sqrt{\int_{\mathbb{R}} \left[\sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s_i \right| \right]^2 dx}$$

$$\leq \frac{1}{\sqrt{\sum_{i=1}^{n} |\mu_i|}} \sqrt{\int_{\mu} \left[\sum_{i=1}^{n} |s_i| \right]^2 dx} + \sqrt{D} \sqrt{\int_{\mathbb{R}} \left[\sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s_i \right| \right]^2 dx}$$

$$\sum_{i=1}^{n} |s(x_i)_i| \leq \frac{1}{\sum_{i=1}^{n} |\mu_i|} \int_{\mu} \sum_{i=1}^{n} |s_i| dx - \frac{1}{\sqrt{\sum_{i=1}^{n} |\mu_i|}} \sqrt{\int_{\mu} \left[\sum_{i=1}^{n} |s_i| \right]^2 dx} \leq 0$$

Now we can finally note that since $\tilde{\rho} \geq \beta \in \mu$ the following must simultaneously stand.

$$\sum_{i=1}^{n} |s(x_i)_i| \leq \frac{1}{\sqrt{\beta} \sqrt{\sum_{i=1}^{n} |\mu_i|}} \sqrt{\int_{\mu} \tilde{\rho} \left[\sum_{i=1}^{n} |s_i| \right]^2 dx} + \sqrt{D} \sqrt{\int_{\mathbb{R}} \left[\sum_{i=1}^{n} \left| \frac{\partial}{\partial x} s_i \right| \right]^2 dx}$$

Also we can note that since $\sum_{i=1}^{n} |\mu_i| \geq 1$, the result will follow.

Proof of the Argument 6.1. It is useful to remember that the initial Navier Stokes equations can be rearranged in the manner

$$\frac{\partial}{\partial t} \Upsilon_i + \frac{\partial}{\partial x} E_i(\Upsilon) - \frac{\partial}{\partial x} \left[G_{ij}(\Upsilon) \frac{\partial}{\partial x} \left(D_j H(\Upsilon) \right) \right] = 0$$

Where we can define $G(\Upsilon)$ as a matrix that is symmetric and positive, whilst it is also important to note that DH is the derivative with respect to Υ of the previously defined entropy $H(\Upsilon)$. Also recall that there is the flux that is associated and reliant on $H(\Upsilon)$ this being $E(\Upsilon)$. Therefore, the entropy exists and is thus equivalent to the existence of a previously defined entropy flux function known as $I(\Upsilon)$. Such that the following most be true.

$$D_j I(\Upsilon) - \sum_i D_i H(\Upsilon) D_j E_i(\Upsilon) = 0 \quad \forall \Upsilon \tag{30}$$

Then we can denote that the strong solution to the initial Navier Stokes equations satisfies

$$\frac{\partial}{\partial t} H(\Upsilon) + \frac{\partial}{\partial x} I(\Upsilon) - DH(\Upsilon) \frac{\partial}{\partial x}(G(\Upsilon) \frac{\partial}{\partial x} DH(\Upsilon)) = 0$$

Note that $E(\Upsilon) = (\rho \upsilon, \rho \upsilon^2 + \rho^\gamma)$ and $\frac{G(\Upsilon)}{\Phi(\rho)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ Moreover, using (30) gives us

$$\frac{\partial}{\partial t} H(\tilde{\Upsilon} \mid \Upsilon) = \left[\frac{\partial}{\partial t} H(\tilde{\Upsilon}) + \frac{\partial}{\partial x} I(\tilde{\Upsilon}) - \frac{\partial}{\partial x}(G(\tilde{\Upsilon}) \frac{\partial}{\partial x} DH(\tilde{\Upsilon})) DH(\tilde{\Upsilon}) \right]$$

$$- \left[\frac{\partial}{\partial t} H(\Upsilon) + \frac{\partial}{\partial x} I(\Upsilon) - \frac{\partial}{\partial x}(G(\Upsilon) \frac{\partial}{\partial x} DH(\Upsilon)) DH(\Upsilon) \right] - \frac{\partial}{\partial x}[I(\tilde{\Upsilon}) - I(\Upsilon)]$$

$$-D[K] + \frac{\partial}{\partial x}[G(\tilde{\Upsilon}) \frac{\partial}{\partial x} DH(\tilde{\Upsilon}) - G(\Upsilon) \frac{\partial}{\partial x} DH(\Upsilon)]$$

$$[DH(\tilde{\Upsilon}) - DH(\Upsilon)] + \frac{\partial}{\partial x}(G(\Upsilon) \frac{\partial}{\partial x} DH(\Upsilon)) DH(\tilde{\Upsilon} \mid \Upsilon)$$

Where

$$K = DH(\Upsilon) \left[\frac{\partial}{\partial t} \Upsilon + \frac{\partial}{\partial x} E(\Upsilon) - \frac{\partial}{\partial x}(G(\Upsilon) \frac{\partial}{\partial x}(DH(\Upsilon))) \right] (\tilde{\Upsilon} - \Upsilon)$$

$$+H(\Upsilon) \left[\frac{\partial}{\partial t} \tilde{\Upsilon} + \frac{\partial}{\partial x} E(\tilde{\Upsilon}) - \frac{\partial}{\partial x}(G(\tilde{\Upsilon}) \frac{\partial}{\partial x}(DH(\tilde{\Upsilon}))) \right]$$

$$-H(\Upsilon) \left[\frac{\partial}{\partial t} \Upsilon + \frac{\partial}{\partial x} E(\Upsilon) - \frac{\partial}{\partial x}(G(\Upsilon) \frac{\partial}{\partial x}(DH(\Upsilon))) \right]$$

$$-\frac{\partial}{\partial x}[I(\Upsilon)(\tilde{\Upsilon} - \Upsilon)] - H(\Upsilon) \frac{\partial}{\partial x}[E(\tilde{\Upsilon} \mid \Upsilon)]$$

Where the relative flux has been defined previously. Now we will use the fact that $\tilde{\Upsilon}$ and Υ are solutions to the initial Navier Stokes equations that satisfy our previously defined natural entropy inequality. Hence, we can now formulate the following.

$$\frac{\partial}{\partial t} H(\tilde{\Upsilon} \mid \Upsilon) + \frac{\partial}{\partial x}[I(\tilde{\Upsilon}) - I(\Upsilon)] - \frac{\partial}{\partial x}[DI(\Upsilon)(\tilde{\Upsilon} - \Upsilon)] - DH(\Upsilon) \frac{\partial}{\partial x}[E(\tilde{\Upsilon} \mid \Upsilon)]$$

$$-\frac{\partial}{\partial x}[G(\tilde{\Upsilon}) \frac{\partial}{\partial x} DH(\tilde{\Upsilon}) - G(\Upsilon) \frac{\partial}{\partial x} DH(\Upsilon)][DH(\tilde{\Upsilon})$$

$$-DH(\Upsilon)] - \frac{\partial}{\partial x}(G(\Upsilon)\frac{\partial}{\partial x}DH(\Upsilon))DH(\tilde{\Upsilon} \mid \Upsilon) \leq 0$$

Now if we integrate the inequality with respect to x and using the results gathered from (28), we yield

$$\frac{d}{dt}\int_{\mathbb{R}} H(\tilde{\Upsilon} \mid \Upsilon)dx + \frac{d}{dx}\int [DH(\Upsilon)]E(\tilde{\Upsilon} \mid \Upsilon)dx + \frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}[G(\tilde{\Upsilon})DH(\tilde{\Upsilon})$$

$$-G(\Upsilon)DH(\Upsilon)][DH(\tilde{\Upsilon}) - DH(\Upsilon)]dx$$

$$-\frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}[G(\Upsilon)DH(\Upsilon)]DH(\tilde{\Upsilon} \mid \Upsilon)dx \leq 0$$

Finally, we will check that the following equality's stand.

$$\frac{\frac{\partial}{\partial x}[DH(\Upsilon)]E(\tilde{\Upsilon} \mid \Upsilon)}{(\frac{\partial}{\partial x}\upsilon)(\gamma - 1)} - \frac{\rho(\upsilon - \tilde{\upsilon})^2}{(\gamma - 1)} = \Lambda(\rho \mid \tilde{\rho}),$$

$$\frac{\rho\left[\frac{\partial}{\partial x}[G(\Upsilon)\frac{\partial}{\partial x}DH(\Upsilon)]DH(\tilde{\Upsilon} \mid \Upsilon)\right]}{(\frac{\partial}{\partial x}(\Phi(\rho)\frac{\partial}{\partial x}\upsilon))(\tilde{\rho} - \rho)(\upsilon - \tilde{\upsilon})} = 1,$$

$$[G(\tilde{\Upsilon})\frac{\partial}{\partial x}DH(\tilde{\Upsilon}) - G(\Upsilon)\frac{\partial}{\partial x}DH(\Upsilon)]\frac{\partial}{\partial x}[DH(\tilde{\Upsilon}) - DH(\Upsilon)]$$

$$-[\Phi(\tilde{\rho})\frac{\partial}{\partial x}\tilde{\upsilon} - \Phi(\rho)\frac{\partial}{\partial x}\upsilon]\frac{\partial}{\partial x}[\tilde{\upsilon} - \upsilon]$$

$$+\Phi(\tilde{\rho})\left[\frac{\partial}{\partial x}\tilde{\upsilon} - \frac{\partial}{\partial x}\upsilon\right]^2 + \frac{\partial}{\partial x}\upsilon[\Phi(\tilde{\rho}) - \Phi(\rho)]\frac{\partial}{\partial x}[\tilde{\upsilon} - \upsilon] = 0 = 0$$

It then follows that the following stands

$$[\frac{d}{dt}\int_{\mathbb{R}} H(\tilde{\Upsilon} \mid \Upsilon)dx + \frac{d}{dx}\int_{\mathbb{R}}\Phi(\tilde{\rho})[(\tilde{\upsilon} - \upsilon)]^2dx + \frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}\upsilon[\Phi(\tilde{\rho})$$

$$-\Phi(\rho)][(\tilde{\upsilon} - \upsilon)]dx - \frac{d}{dx}\cdot\frac{d}{dx}\int_{\mathbb{R}}\frac{1}{\rho}[(\Phi(\rho)\upsilon)](\tilde{\rho} - \rho)(\upsilon - \tilde{\upsilon})dx]$$

$$\frac{1}{\int\sum_{i=1}^{n}\left|\frac{\partial}{\partial x}\upsilon_i\right| H(\tilde{\Upsilon} \mid \Upsilon)dx} \leq N$$

This then exemplifies argument.

A Proof of Equality 24

Argument A.1 We will allow (ρ, υ) to show satisfaction towards (18) and then yield the following piecewise function.

$$O = \left[\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho\upsilon = 0, \rho(x, 0) = \rho_i(x)\right] \tag{31}$$

Hence, it can be seen that (ρ, υ) shows satisfaction towards (22).

Proof. We will allow s_ψ to be denoted as the convolution of any function defined by s. This is done through a mollifier. Henceforth, convoluting (31) by the mollifer, we arrive at

$$\rho_\psi\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\upsilon\right)-\frac{\partial}{\partial x}\rho_\psi\upsilon+\frac{\partial}{\partial x}(\rho\upsilon)_\psi=0$$

Now that we have the fact that ρ_ψ is a smooth function, we can perform a set of calculations that will inevitably result in

$$\frac{d}{dt}\int\rho_\psi\frac{\left[\frac{\partial}{\partial x}\int\frac{\Phi(\rho_\psi)}{\rho_\psi^2}dx\right]^2}{2}dx-\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int\rho_\psi^2\frac{\Phi(\rho_\psi)}{\rho_\psi^2}\int\frac{\Phi(\rho_\psi)}{\rho_\psi^2}dx\upsilon dx$$
$$-\frac{d}{dx}\cdot\frac{d}{dx}\cdot\frac{d}{dx}\int(2\rho_\psi\frac{\Phi(\rho_\psi)}{\rho_\psi^2}+\rho_\psi^2\tau'^{\prime}(\rho_\psi))(\rho_\psi)\int\frac{\Phi(\rho_\psi)}{\rho_\psi^2}dx\upsilon dx$$
$$=\frac{d}{dx}\cdot\frac{d}{dx}\int\rho_\psi\int\frac{\Phi(\rho_\psi)}{\rho_\psi^2}dx\frac{\Phi(\rho_\psi)}{\rho_\psi^2}b_\psi dx \qquad (32)$$

Now it becomes apparent that in order to pass to the limit as $\psi\to 0$, we note that $\rho_\psi-\rho'\to\rho-\rho'\in L^\infty(0,T;H^1(\mathbb{R}))$ strong, which is sufficient to take the limit in (32)'s left hand side (Note that it implies the strong convergence in $L^\infty(0,T;L^\infty(\mathbb{R}))$). We simply need to demonstrate that b_ψ gets to zero in $L^2(0,T;H^1(\mathbb{R}))$ strong to demonstrate that the right hand side goes to zero (And thus in $L^2(0,T;L^\infty(\mathbb{R}))$). We compose

$$\frac{1}{2}\left[\frac{\partial}{\partial x}b_\psi-\rho_\psi\frac{\partial^2}{\partial x^2}\upsilon-(\rho\frac{\partial^2}{\partial x^2}\upsilon)_\psi+\frac{\partial^2}{\partial x^2}\rho_\psi\upsilon-\left(\frac{\partial^2}{\partial x^2}\rho\upsilon\right)_\psi\right]$$
$$=\frac{\partial}{\partial x}\rho_\psi\frac{\partial}{\partial x}\upsilon-\left(\frac{\partial}{\partial x}\rho\frac{\partial}{\partial x}\upsilon\right)_\psi$$

Because of the significant convergence of $\rho_\psi\in L^\infty(0,T;H^1(\mathbb{R}))$, the first two terms converge to zero. Hence, for the last term, we can say that $\frac{\partial}{\partial x}\rho\in L^\infty(0,T;L^2(\mathbb{R}))$ and $\upsilon\in L^2(0,T;W^{1,\infty}(\mathbb{R}))$ therefore, the convergence to zero within the space $L^2((0,T)\cdot\mathbb{R})$ is proven.

2D Global Strong Solutions of g-Navier Stokes Existence and Uniqueness

7 Inauguration

We will allow a specified Ω to be bounded within $dom(\mathbb{R}^2)$, in which Π represents the smooth boundary characteristic. Throughout this proof of $2D$ Global Strong Solutions of g-Navier Stokes Existence and Uniqueness, we will allow the solutions to cater towards a non-autonomous case, meaning a smooth manifold is at play. Also note that velocity is still represented as υ, pressure as Λ, and Φ serving as the viscosity coefficient in this case we take $\Phi > 0$. Also recall that since this is in the two dimensional case υ becomes a velocity vector $\upsilon = \upsilon(x,t)$, Λ becomes the pressure vector $\Lambda = \Lambda(x,t)$, and υ_i represents our initial velocity. Henceforth, we can now present the equation.

$$\begin{cases} \upsilon = 0 & \in ((0,T)\cdot\Gamma), \\ div(g\upsilon) = 0 & \in ((0,T)\cdot\Omega), \\ \upsilon_i(x) = \upsilon(x,0) & \in \Omega, \\ \frac{\partial}{\partial t}\upsilon - \chi(\Delta\upsilon) + div(\upsilon^2) + div(\Lambda) = f(t) & \in ((0,T)\cdot\Omega) \end{cases} \tag{1}$$

Notice that in the $2D$ case we are using a different form of the Navier Stokes equations. More precisely, we use g-Navier Stokes equations. This is because when gravity is equivalent to a constant we arrive at the normal Navier Stokes equations, and using $2D$ g-Navier Stokes equations helps us understand $3D$ Navier Stokes on thin domains. Make note that $2D$ g-Navier Stokes equations arrive when $3D$ Navier Stokes equations are reliant on specified thin domains.

Hence, the main goal of this proof is to find strong solutions to the $2D$ non-autonomous g-Navier Stokes equations. Moreover, we first must state the following.

$$g \in W^{1,\infty}(\Omega)$$

$$-d_i < -g(x) \leq -D_i \leq 0 \quad \forall (x_x, x_y) \in \Omega \wedge -\frac{1}{\sqrt{\varphi_i}}\left[\lim_{p\to\infty}\sum_{i=1}^{n}|div(g)_i|^p\right]^{\frac{1}{p}} < d_i \tag{2}$$

In this case we allow $\varphi_1 > 0$ to be the first eigenvalue attributed to the operator belonging to the g-Navier Stokes equations, and whilst residing in Ω.

8 Groundwork

Now we will view some supplementary information related to the previously stated terms of the g-Navier Stokes equations. Allow $\overline{C_c^1(\mathbb{R})} = H_0^1(\Omega, g) = \sqrt{H_0^1(\Omega, g)} = H_0^1(\Omega)$ and $\sqrt{L^2(\Omega, g)} = L^2(\Omega)$ be related assets. In which we may represent the dot products as

$$(\upsilon, \Phi)_g = \int_\Omega g\left[\sum_{i=1}^{2}\upsilon_i\Phi_i\right]dx \quad \forall(\upsilon,\Phi) \in (L^2(\Omega))^2$$

and the fact that

$$((\upsilon,\Phi))_g=\sum_{j=1}^{2}\int_\Omega g[\nabla\upsilon_j\nabla\Phi_j]dx$$

$$((\upsilon,\Phi))_g=\int_\Omega g[\nabla\upsilon_1\nabla\Phi_1+\nabla\upsilon_2\nabla\Phi_2]dx$$

$$\int_\Omega g\left[\sum_{i=1}^{2}\upsilon_i\Phi_i\right]dx=\int_\Omega g[\nabla\upsilon_1\nabla\Phi_1+\nabla\upsilon_2\nabla\Phi_2]dx$$

$$(\Phi_1,\Phi_2),(\upsilon_1,\upsilon_2)\in(H_0^1(\Omega))^2$$

And the following norms

$$\left[\sum_{i=1}^{n}|\upsilon_i|\right]^2=|\upsilon|^2=(\upsilon,\upsilon)_g$$

$$\left[\sum_{i=1}^{n}|\upsilon_i|\right]^2=||\upsilon||^2=((\upsilon,\upsilon))_g$$

Fully note that these norms are thanks to a particular assumption, we will also bring attention to the fact that usual norms residing $\in(L^2(\Omega))^2\wedge(H_0^1(\Omega))^2$ can be expressed by $||\cdot||$ and $|\cdot|$. In this case the dot in the norm represents the place where the variable is "plugged in". Now we allow

$$\Gamma=\left\{\sum_{i=1}^{n}\nabla_i(g\upsilon)_i=0|\upsilon\in(C_0^\infty(\Omega))^2\right\}$$

The said closure within the space $\Gamma\in L^2(\Omega,g)$ may be denoted by Π_g. Moreover, we may also denote $\Gamma\in H_0^1(\Omega,g)$ as Θ_g. Hence, it becomes clear that $\Theta_g\subset\Pi_g\equiv\Pi_g'\subset\Theta_g'$.

Remark 8.1 We will now take a look at an example that will help us better understand the above equivalency. We will start by considering the map

$$\xi:\ell_r\to\ell_s:g\mapsto g.$$

Note that the following map is well defined, that is that the following must stand.

$$g\in\ell_r\Rightarrow g\in\ell_s;$$

In order to see this at play we must observe the following inequalities catering towards $g\in\ell_r$:

$$||g||_s<||g||_r<\infty.$$

This in turn helps us show that $g\in\ell_s$. Hence, it becomes noticeable that the map is injective. Suppose that $\xi(g)=\xi(f)$, then we will note that the proof of injectivity satisfies $g=\xi(g)=\xi(f)=f$. Now for the case of continuity we may just use the fundamental definition presented by epsilon of continuity residing in metric spaces, and in this case this is applied to the map ξ. Lastly, for density we can chose any ϵ whilst simultaneously choosing any $g \in \ell_2$. Then, it is fair to construct the following function. $f \in \ell_1$ such that it becomes clear that $||g - f||_2$ must be less than epsilon. Moreover, recall how every value was arbitrary, and therefore, it is proven that ℓ_1 is dense in ℓ_2. End of Remark 8.1.

With the knowledge provided by Remark 8.1 we can deduce that our functions injections are dense and continuous. We will allow $||\cdot||_* \in \Theta_g$ and the fact that $< \cdot, \cdot\cdot >$ can be served as a representation of the duality pairing between Θ_g and Θ_g'.

Remark 8.2 Note that we must understand duality pairing to continue. We may start by defining some vector space defined by X, hence one may consider that there must exist an algebraic dual space defined by X' that encapsulates all linear functionals. Note that there is no semblance of continuity thus far. Knowing this, we may form a representation of the pairing via $< x, f >$ which will follow

$$< x, f >:= f(x)(x \in X, f \in X')$$

However, it is vital to note that then above paintings are not the only ones that are possible. Rather if we wanted to we could replace X' with some other vector space defined by Y for which it becomes clear that we can form some bilinear mapping that will represent X and Y. Defined by

$$< \cdot, \cdot >: X \times Y \to \mathcal{F}$$

Where the given $\mathcal{F}$ is a representation of the underlying field within the system. We may also define Y to contain sufficiently many vectors which will allow separation of points of X. This pairing is well defined; however, it becomes aparent that this pairing is less natural than the pairing formed between X and X'. Now if we note that the vector space X has a topological structure, for example if we define X as a Banach space, then we can rather consider the pairing between X and its topological dual X^*, which can again be defined as

$$< x, f >:= f(x)(x \in X, f \in X^*)$$

. This is the rather obvious choice when it comes to pairing within Banach spaces. We note that what is happening here is the consideration of pairing between X and a subsequent subspace $X^* \subseteq X'$. If, however, we note that X is just a topological vector space, and the fact that X^* might possibly be a trivial vector space. Even with this, we may still form a pairing. If we define X to be a Hilbert space, however, then there arises another pairing that could be possibly considered. This being the pairing that comes up when viewing the inner product on X. In fact, something note worthy occurs, as we may note that this turns out to be the exact same pairing as the one formed between X and its topological dual space. End remark.

Henceforth, we may now go back to our example, and note that we can now formulate the following trilinear form ϱ. Note that general trilinear forms come in the form of the following. Where for three Schwartz functions a,b, and c defined on $\mathbb{R}^2$ we write

$$\ell(a,b,c) := \int_{(\mathbb{R}^2)^3} a(x)b(y)c(z)\delta(a+b+c)\det\begin{bmatrix}1 & 1 & 1\\ x & y & z\end{bmatrix}^{-1} dxdydz$$

And hence, in our case we get

$$\varrho(\upsilon,\Phi,\Psi) = \sum_{i,j=1}^{2} \frac{d\Phi_j}{dx_i}\int_\Omega g[\upsilon_i\Psi_j]dx$$

Now, note that if the integrals do indeed make sense. It becomes trivial to find whether $(\upsilon,\Phi,\Psi)\in\Theta_g$, then the following stands.

$$\sum_{i,j=1}^{2} \frac{d\Phi_j}{dx_i}\int_\Omega g[\upsilon_i\Psi_j]dx = -\sum_{i,j=1}^{2} \frac{d\Psi_j}{dx_i}\int_\Omega g[\upsilon_i\Phi_j]dx$$

$$\sum_{i,j=1}^{2} \frac{d\Phi_j}{dx_i}\int_\Omega g[\upsilon_i\Psi_j]dx + \sum_{i,j=1}^{2} \frac{d\Psi_j}{dx_i}\int_\Omega g[\upsilon_i\Phi_j]dx = 0$$

$$\varrho(\upsilon,\Phi,\Psi) = -\varrho(\upsilon,\Psi,\Phi)$$

Hence, the following must be true.

$$\varrho(\upsilon,\Phi,\Phi) = \sum_{i,j=1}^{2} \frac{d\Phi_j}{dx_i}\int_\Omega g[\upsilon_i\Phi_j]dx = 0, \quad \forall(\upsilon,\Phi)\in\Theta_g$$

We will set $(\Upsilon|\Theta_g \to \Theta_g')$ by taking the formulation formed by the inner product $||\Upsilon\upsilon||||\Phi|| =< \Upsilon\upsilon,\Phi >= ((\upsilon,\Phi))_g$, now we allow $(\Lambda| < \Theta_g,\Theta_g >)$ still mapped to Θ_g' by the formulation $||\Lambda(\upsilon,\Phi)||||\Psi|| = \sum_{i,j=1}^{2}\frac{d\Phi_j}{dx_i}\int_\Omega g[\upsilon_i\Phi_j]dx = \varrho(\upsilon,\Phi,\Psi)$. We will now denote $\varepsilon(\Upsilon) = \{\upsilon\in\Theta_g|\Upsilon\upsilon\in\Pi_g\}$, then we may bring notice to the equality

$$\varepsilon(\Upsilon) = \{\upsilon|\upsilon\in H^2(\Omega,g)\wedge\upsilon\in\Theta_g\}$$

And the fact that

$$-\frac{\Upsilon\upsilon}{\Delta\upsilon} = \ell_g \quad \forall\upsilon\in\varepsilon(\Upsilon)$$

Where it becomes important to note that ℓ_g can be represented as

$$L^2(\Omega,g) = L^2(\Omega,g)_{\Pi_g} + L^2(\Omega,g)_{\Pi_g^\perp}$$

$$\ell = \ell_g + \ell_{g^\perp}$$

Hence, ℓ_g serves as the orthogonal projector from $L^2(\Omega,g)$ onto Π_g. Moreover, note that the Ladyzhenskaya inequality for when $n=2$ states

$$||u||_{L^4} \le N\sqrt{||u||_{L^2}||\nabla u||_{L^2}}$$

And note that Hölder's Inequality states

$$\sum_{k=1}^{n}|a_kb_k| \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}|b_k|^p\right)^{\frac{1}{p}}$$

Therefore, using these two inequalities yields

$$\left(\sum_{i=1}^{n}|\upsilon_i|^4\right)^{\frac{1}{4}} \leq N\sqrt{\sum_{i=1}^{n}|\upsilon_i|\sum_{i=1}^{n}|\nabla\upsilon_i|} \quad \forall\upsilon \in H_0^1(\Omega)$$

Also through the interpolation inequalities it becomes quite easy to prove that the following is true.

Argument 8.1 If we note that $n = 2$, then it becomes apparent that we can formulate the following.

$$\sum_{i=1}^{n}|\varrho(\upsilon,\Phi,\Psi)_i| \leq \begin{cases} N_1\sqrt{\sum_{i=1}^{n}|\upsilon_i|\sum_{i=1}^{n}||\upsilon_i||}\sum_{i=1}^{n}||\Phi_i|| \\ \sqrt{\sum_{i=1}^{n}|\Psi_i|\sum_{i=1}^{n}||\Psi_i||} \quad \forall(\upsilon,\Phi,\Psi)\in\Theta_g, \\ N_2\sqrt{\sum_{i=1}^{n}|\upsilon_i|\sum_{i=1}^{n}||\upsilon_i||}\sum_{i=1}^{n}||\Phi_i|| \\ \sqrt{\sum_{i=1}^{n}|\Upsilon\Psi_i|\sum_{i=1}^{n}|\Psi_i|} \quad \forall\upsilon\in\Theta_g,\Phi\in\varepsilon(\Upsilon),\Psi\in\Pi_g, \\ N_3\sqrt{\sum_{i=1}^{n}|\upsilon_i|\sum_{i=1}^{n}|\Upsilon\upsilon_i|} \\ \sum_{i=1}^{n}||\Phi_i||\sum_{i=1}^{n}|\Psi_i| \quad \forall\upsilon\in\varepsilon(\Upsilon),\Phi\in\Theta_g,\Psi\in\Pi_g, \\ N_4\sum_{i=1}^{n}|\upsilon_i|\sum_{i=1}^{n}||\Phi_i|| \\ \sqrt{\sum_{i=1}^{n}|\Psi_i|\sum_{i=1}^{n}|\Upsilon\Psi_i|} \quad \forall\upsilon\in\Pi_g,\Phi\in\Theta_g,\Psi\in\varepsilon(\Upsilon) \end{cases} \tag{3}$$

N just represents the appropriate constants in each case

Argument 8.2 Now we will let $\upsilon \in L^2(0,T;\varepsilon(\Upsilon)) \cap L^\infty(0,T;\Theta_g)$, then we can say that the function Λ can be defined via the following.

$$(\Lambda\upsilon(t),\Phi)_g = \sum_{i,j=1}^{2}\frac{d\upsilon(t)_j}{dx_i}\int_\Omega g[\upsilon(t)_i\Phi_j]dx \quad \forall\Phi\in\Pi_g, a.e.\ t\in[0,T]$$

We also note that the above belongs to the space $L^4(0,T;\Pi_g)$. Moreover, it also becomes apparent that it belongs to the space $L^2(0,T;\Pi_g)$ as well.

Proof. Through the use of Argument 8.1, we can deduce that for $a.e.\ t \in [\varpi, T]$. We may write the following.

$$\sum_{i=1}^{n}|\Lambda\upsilon(t)_i| \leq N_3\sqrt{\sum_{i=1}^{n}|\upsilon(t)_i|\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|}\sum_{i=1}^{n}||\upsilon(t)_i||$$

$$-N_3'\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^{\frac{3}{2}}\sqrt{\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|} \leq 0$$

Hence,

$$\int_0^T\left[\sum_{i=1}^{n}|\Lambda\upsilon(t)_i|\right]^4 dt \leq N_3'\int_0^T\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^6\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2 dt$$

$$-N\left[\lim_{p\to\infty}\left(\int_0^T\sum_{i=1}^{n}||\upsilon_i||^p dt\right)^{\frac{1}{p}}\right]^6\int_0^T\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2 dt\le 0\le+\infty$$

Moreover, this concludes the proof.

Argument 8.3 We will allow $\upsilon\in L^2(0,T;\Theta_g)$, which then leads to the following definition of the function $N\upsilon$.

$$(N\upsilon(t),\Phi)_g=\left(\left(\sum_{i=1}^{n}\frac{\nabla g}{g}_i\nabla_i\right)\upsilon,\Phi\right)_g=\sum_{i,j=1}^{2}\frac{d\upsilon_j}{dx_i}\int_\Omega g\left[\frac{\nabla g}{g}_i\Phi_j\right]dx\quad\forall\Phi\in\Theta_g$$

Note that this belongs to the space $L^2(0,T;\Pi_g)$, and this then brings us to note that this must also reside within the space $L^2(0,T;\Theta_g')$. Moreover, we may write

$$\sum_{i=1}^{n}|N\upsilon(t)_i|\le\sum_{i=1}^{n}\frac{\sum_{i=1}^{n}|\nabla g_i|}{d_i}\sum_{i=1}^{n}||\upsilon(t)_i||\quad a.e.\ t\in(0,T)$$

And the fact that

$$\sum_{i=1}^{n}||N\upsilon(t)_i||_*\le\sum_{i=1}^{n}\frac{\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}}\sum_{i=1}^{n}||\upsilon(t)_i||\quad a.e.\ t\in(0,T)$$

Since we note that

$$\upsilon=\frac{-g\Delta\upsilon-g\left(\sum_{i=1}^{n}\frac{\nabla g}{g}_i\nabla_i\right)\upsilon}{-\left(\sum_{i=1}^{n}\nabla_i g\nabla_i\right)}$$

We can then express

$$(-\Delta\upsilon,\Phi)_g-((\upsilon,\Phi))_g+(\Upsilon\upsilon,\Phi)_g=\left(\left(\sum_{i=1}^{n}\frac{\nabla g}{g}_i\nabla_i\right)\upsilon,\Phi\right)_g$$

$$-\left(\left(\sum_{i=1}^{n}\frac{\nabla g}{g}_i\nabla_i\right)\upsilon,\Phi\right)_g=0\quad\forall(\upsilon,\Phi)\in\Theta_g$$

9 Strong Solutions

Interpretation 9.1 Given the knowledge that $f\in L^2(0,T;\Pi_g)$ and that $\upsilon_i\in\Theta_g$. Now we note that there exists a strong solution on the interval $(0,T)$ of the initial equation (1), which we may recall is a function in which $\upsilon\in L^2(0,T;\varepsilon(\Upsilon))\cap L^\infty(0,T;\Theta_g)$ with the additional knowledge of $\upsilon(0)=\upsilon_i$, and such that

$$(\upsilon(t),\Phi)_g=-\int\chi(\upsilon(t),\Phi)_g dt-\int\chi(C\upsilon(t),\Phi)_g dt$$

$$+\int\sum_{i,j=1}^{2}\frac{d\upsilon(t)_j}{dx_i}\int_\Omega g[\upsilon(t)_i\Phi_j]dxdt+\int(f(t),\Phi)_g dt\quad\forall\Phi\in\Theta_g\wedge a.e.\ t\in(0,T)\tag{4}$$

Remark 9.1 Now that we have a stable understanding via the use of the above definition, we can deduce the strong solution $\upsilon \in L^2(0, T; \varepsilon(\Upsilon))$ and the fact that

$$\upsilon = \int f dt - \int \upsilon(\chi\Upsilon + \Lambda + N)dt \in L^2(0, T; \Pi_g)$$

Now we note that because of the results yielded within Arguments 8.2 and 8.3, and by a previously well defined Argument ($Philip 1.2 in [15]$). We may deduce that $\upsilon \in N([0, T]; \Theta_g)$. Henceforth this adds meaning to the equality of $\upsilon(0) = \upsilon_i$. Moreover, it becomes quite aparent that if υ is considered a strong solution to the second dimensional g-Navier Stokes equation (1), then υ must satisfy the following. Which is useful for describing the energy equality experienced.

$$\sqrt{L} = \sum_{i=1}^{n} |\upsilon(s)_i|$$

$$L = \left[\sum_{i=1}^{n} |\upsilon(t)_i|\right]^2 + 2\chi \int_{\nu}^{t} \left[\sum_{i=1}^{n} ||\upsilon(r)_i||\right]^2 dr$$

$$+2\chi \int_{\nu}^{t} \sum_{i,j=1}^{2} \frac{d\upsilon(r)_j}{dx_i} \int_{\Omega} g \left[\frac{\nabla g}{g}_i \upsilon(r)_j\right] dxdr - 2\int_{\nu}^{t} (f(r), \upsilon(r))_g dr$$

$$\forall 0 \leq \nu < t \leq T$$

We will no show a series of proofs for the estimate of the size of the given solution for its derivatives of our partial differential equations. In this case of course, we provide estimates for strong solutions to (1).

Argument 9.1 We know that υ is a strong solution to (1), and therefore, we may deduce

$$\int_0^T \left[\sum_{i=1}^{n} |\upsilon(t)_i|\right]^2 dt - B_1(\upsilon_i, f, \chi, T, \varphi_1) \leq 0, \tag{5}$$

$$\sup_{\nu\in[0,T]} \left[\sum_{i=1}^{n} |\upsilon(\nu)_i|\right]^2 - B_2(\upsilon_i, f, \chi, T, \varphi_1) \tag{6}$$

Now we provide a proof through the use of equation (4). In which case we will replace the term Φ by $\upsilon(t)$. Giving us,

$$\begin{aligned}(\upsilon(t), \upsilon(t))_g = -\int \chi((\upsilon(t), \upsilon(t)))_g dt - \int \chi(N\upsilon(t), \upsilon(t))_g dt \\ -\int \sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_{\Omega} g[\upsilon(t)_i \upsilon(t)_j] dxdt + \int (f(t), \upsilon(t))_g dt\end{aligned} \tag{7}$$

Recall that

$$\sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_{\Omega} g[\upsilon(t)_i \upsilon(t)_j] dx = 0$$

And

$$(N\upsilon(t), \upsilon(t))_g = \sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_\Omega g \left[\frac{\nabla g}{g}_i \upsilon(t)_j\right] dx$$

Then from equation (7) we get

$$(\upsilon(t), \upsilon(t))_g = -\int \chi((\upsilon(t), \upsilon(t)))_g dt - \int \chi \sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_\Omega g \left[\frac{\nabla g}{g}_i \upsilon(t)_j\right] dxdt$$

$$+ \int (f(t), \upsilon(t))_g dt$$

Henceforth, giving us

$$\sum_{i=1}^{n} |\upsilon(t)_i| = \sqrt{L} \tag{8}$$

Where

$$L = -2\int \chi \left[\sum_{i=1}^{n} ||\upsilon(t)_i||\right]^2 dt + 2\int (f(t), \upsilon(t))_g dt$$

$$-2\int \chi \sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_\Omega g \left[\frac{\nabla g}{g}_i \upsilon(t)_j\right] dxdt$$

Now using Argument 8.3 and through the use of the Cauchy inequality. Which states

$$| < u, v > | \leq ||u||||v||$$

Moreover, we have

$$\sum_{i=1}^{n} |\upsilon(t)_i| \leq \sqrt{K}$$

Where

$$K = -2\int \chi \left[\sum_{i=1}^{n} ||\upsilon(t)_i||\right]^2 dt + 2\int \iota\chi \left[\sum_{i=1}^{n} ||\upsilon(t)_i||\right]^2 dt + \int \frac{\left[\sum_{i=1}^{n} |f(t)_i|\right]^2}{2\iota\chi\varphi_1} dt$$

$$+2\int \chi \frac{\lim_{p\to\infty} \left(\sum_{i=1}^{n} |\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}} \left[\sum_{i=1}^{n} ||\upsilon(t)_i||\right]^2 dt$$

Moreover,

$$\sum_{i=1}^{n} |\upsilon(t)_i| \leq \sqrt{\int \frac{\left[\sum_{i=1}^{n} |f(t)_i|\right]^2}{2\iota\chi\varphi_1} dt - 2\int \chi(\delta_i - \iota) \left[\sum_{i=1}^{n} ||\upsilon(t)_i||\right]^2 dt} \tag{9}$$

Where it becomes vital to note $\delta_i = 1 - \frac{\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}} > 0 \wedge \iota > 0$ It then becomes noticeable that this is chosen for the case in which $\delta > \iota$. Hence, when integrating with respect to t from 0 to T, we can deduce (5). Moreover, we also integrate with respect to t of the equation (13) from 0 to ν, in which $-\nu < -T < 0$. Now we can find

$$-\frac{1}{2\iota\chi\varphi_1}\int_0^{\nu}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2 dt \leq \sum_{i=1}^{n}|\upsilon_{i_i} - \left[\sum_{i=1}^{n}|\upsilon(\nu)_i|\right]^2$$

$$\int_0^{\nu}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2 dt \leq -2\iota\chi\varphi_1\sum_{i=1}^{n}|\upsilon_{i_i} + 2\iota\chi\varphi_1\left[\sum_{i=1}^{n}|\upsilon(\nu)_i|\right]^2$$

$$\sum_{i=1}^{n}|f(\nu)_i| = \sqrt{\frac{d}{dx} - 2\iota\chi\varphi_1\sum_{i=1}^{n}|\upsilon_{i_i} + \frac{d}{dx}2\iota\chi\varphi_1\left[\sum_{i=1}^{n}|\upsilon(\nu)_i|\right]^2}$$

Then, we arrive at (6).

Argument 9.2 We recall the fact that υ is a solution of (1), and hence we may formulate the following.

$$\sup_{t\in[0,T]}\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^2 \leq B_3(B_1, B_2), \tag{10}$$

$$\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i| \leq \sqrt{\frac{d}{dx}B_4(B_1, B_2)} \tag{11}$$

The proof then follows. In which we take equation (4) and replace all Φ by $\Upsilon\upsilon(t)$. Moreover, we get

$$\begin{aligned}(\upsilon(t), \Upsilon\upsilon(t))_g = -\int \chi((\upsilon(t), \Upsilon\upsilon(t)))_g dt - \\ \int \chi((N\upsilon(t), \Upsilon\upsilon(t)))_g dt - \int \sum_{i,j=1}^{2}\frac{d\upsilon(t)_j}{dx_i}\int_\Omega g[\upsilon(t)_i\Upsilon\upsilon(t)_j]dxdt \\ + \int (f(t), \Upsilon\upsilon(t))_g dt\end{aligned} \tag{12}$$

Since we note the following

$$((\alpha, \gamma))_g = \sum_{i=1}^{n}||\Upsilon\alpha_i||\sum_{i=1}^{n}||\gamma_i|| \quad \forall(\alpha, \gamma) \in \Theta_g$$

Then this must mean that the following stands.

$$\sum_{i=1}^{n}||\upsilon(t)_i|| = \sqrt{U} \leq +\frac{1}{2}\int \chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2 dt + 2\int\frac{1}{\chi}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2 dt \tag{13}$$

Where

$$U=-2\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt-2\int\chi(\upsilon(t),\Upsilon\upsilon(t))_g dt$$

$$-2\int\sum_{i,j=1}^{2}\frac{d\upsilon(t)_j}{dx_i}\int_{\Omega}g[\upsilon(t)_i\Upsilon\upsilon(t)_j]dxdt+2\int(f(t),\Upsilon\upsilon(t))_g dt$$

Now through the use of Argument 8.1 and 8.2, (13) can be construed and defined as

$$\sum_{i=1}^{n}||\upsilon(t)_i||\leq\sqrt{Q} \tag{14}$$

Where

$$Q=-2\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt$$

$$+\frac{1}{2}\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt+2\int\frac{1}{\chi}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2dt$$

$$+2\int N_3\sqrt{\sum_{i=1}^{n}|\upsilon(t)_i|}\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^{\frac{3}{2}}\sum_{i=1}^{n}||\upsilon(t)_i||dt$$

$$+2\int\frac{\chi\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i}\sum_{i=1}^{n}|\upsilon(t)_i|\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|dt$$

Now we will make use of Young's inequality which states

$$a^{\alpha}b^{\beta}\leq\alpha a+\beta b\quad 0\leq\alpha,\beta\leq 1,\alpha+\beta=1$$

And the already defined Cauchy inequality. Hence, this gives us

$$\sum_{i=1}^{n}||\upsilon(t)_i||\leq\sqrt{V} \tag{15}$$

Where

$$V=-2\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt$$

$$+\frac{1}{2}\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt+2\int\frac{1}{\chi}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2dt$$

$$+\frac{1}{2}\int\chi\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2dt+2\int N_3'\left[\upsilon(t)_i|\right]^2\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^4dt$$

$$+2\int \frac{\chi \lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}}\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2 dt$$

$$+\frac{1}{2}\int \frac{\chi \lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}\sqrt{\varphi_1}}{d_i}\left[\sum_{i=1}^{n}|\upsilon(t)_i|\right]^2 dt$$

Now we can deduce

$$\sum_{i=1}^{n}||\upsilon(t)_i|| \leq \sqrt{O} \tag{16}$$

Where

$$O = -\int \chi\left(1-\frac{\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}}\right)\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2 dt$$

$$+\int \frac{2}{\chi}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2 dt + 2\int N_3'\left[\sum_{i=1}^{n}|\upsilon(t)_i|\right]^2\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^4 dt$$

$$+\int \frac{\chi \lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{2d_i\sqrt{\varphi_1}}\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^2 dt$$

Now we will take out the term $\chi\left(1-\frac{\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}}\right)\left[\sum_{i=1}^{n}|\Upsilon\upsilon(t)_i|\right]^2$ for the moment. This in turn will give us a differential inequality defined as

$$y'(x) > f(x, y(x))$$

Which we can write as

$$y \leq \int \alpha dt + \int \theta y dt$$

Where we may define

$$y(t) = \left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^2$$

$$\alpha(t) = \frac{2}{\chi}\left[\sum_{i=1}^{n}|f(t)_i|\right]^2$$

$$\theta(t) = \left(2N_3'\left[\sum_{i=1}^{n}|\upsilon(t)_i|\right]^2\left[\sum_{i=1}^{n}||\upsilon(t)_i||\right]^2 + \frac{\chi \lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{2d_i\sqrt{\varphi_1}}\right)$$

Now we may apply the knowledge from Gronwall's inequality to arrive at

$$y(t) \leq \frac{\int \alpha(t)\left[\int_0^t \theta(\xi)d\xi\right]dt}{\left[\int_0^t \theta(\xi)d\xi\right]},$$

$$\ln(y(t)) \leq \int_0^t \theta(\xi)d\xi \left[y(0) + \int_0^t \alpha(\nu)d\nu\right]$$

Or we may note

$$\ln[\sum_{i=1}^{n} ||\upsilon(t)_i||] \leq \frac{1}{2} [||\upsilon_{i_i}||]^2 [J] + \frac{1}{\chi}$$
$$\int_0^t \left[\sum_{i=1}^{n} |f(\nu)_i|\right]^2 \left[\int_0^t \left(2N_3' \left[\sum_{i=1}^{n} |\upsilon(\tau)_i|\right]^2 \left[\sum_{i=1}^{n} ||\upsilon(\tau)_i||\right]^2 + K\right) d\tau\right] d\nu \quad (17)$$

Where

$$J = \int_0^t \left(2N_3' \left[\sum_{i=1}^{n} |\upsilon(\tau)_i|\right]^2 \left[\sum_{i=1}^{n} ||\upsilon(\tau)_i||\right]^2 + \frac{\chi \lim_{p\to\infty} \left(\sum_{i=1}^{n} |\nabla g_i|^p\right)^{\frac{1}{p}}}{2d_i\sqrt{\varphi_1}}\right) d\tau$$

And

$$K = \frac{\chi \lim_{p\to\infty} \left(\sum_{i=1}^{n} |\nabla g_i|^p\right)^{\frac{1}{p}}}{2d_i\sqrt{\varphi_1}}$$

Now through the use of Argument 9.1 we may deduce (10). Hence, we will come back to (16), which we note that when this is integrated on the bounds 0 and T, we get (11).

Theorem 9.1 We will allow f to reside in the space $L^2(0, T; \Pi_g)$ and the fact that $\upsilon_i \in \Theta_g$ are present. Then it becomes apparent that υ is a unique strong solution of (1) on the interval (0,T). Proof.

9.1 Uniqueness

We will allow the two defined terms υ, Φ to be strong solutions of (1) with non-changing initial datum, and we will ensure that the equality $\Psi = \upsilon - \Phi$ is present. Moreover, we can use the existing energy equality to obtain

$$2\chi \int_0^t \left[\sum_{i=1}^{n} ||\Psi(\nu)_i||\right]^2 d\nu + 2\chi \int_0^t \sum_{i,j=1}^{2} \frac{d\Psi(\nu)_j}{dx_i} \int_\Omega g \left[\frac{\nabla g}{g}_i \Psi(\nu)_j\right] dxd\nu$$

$$+2\int_0^t \sum_{i,j=1}^{2} \frac{d\Phi(\nu)_j}{dx_i} \int_\Omega g[\Psi(\nu)_i\Psi(\nu)_j]dxd\nu = \left[\sum_{i=1}^{n} |\Psi(t)_i|\right]^2$$

Using Argument 8.1, we get

$$\sum_{i=1}^{n} \left| 2\int_0^t \sum_{i,j=1}^{2} \frac{d\Phi(\nu)_j}{dx_i} \int_\Omega g[\Psi(\nu)_i\Psi(\nu)_j]dxd\nu_i \right|$$

$$-2N_1\int_0^t\sum_{i=1}^{n}|\Psi(\nu)_i|\sum_{i=1}^{n}||\Psi(\nu)_i||\sum_{i=1}^{n}||\Phi(\nu)_i||d\nu\leq-\chi\int_0^t\left[\sum_{i=1}^{n}||\Psi(\nu)_i||\right]^2d\nu$$

$$-\frac{N_1^2}{\chi}\int_0^t\left[\sum_{i=1}^{n}||\Phi(\nu)_i||\right]^2\left[\sum_{i=1}^{n}|\Psi(\nu)_i|\right]^2d\nu\leq 0$$

And through the use of Argument 8.3, we arrive at

$$\sum_{i=1}^{n}\left|2\chi\int_0^t\sum_{i,j=1}^{2}\frac{d\Psi(\nu)_j}{dx_i}\int_\Omega g\left[\frac{\nabla g}{g}_i\Psi(\nu)_j\right]dxd\nu\right|$$

$$-2\chi\frac{\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}}{d_i\sqrt{\varphi_1}}\int_0^t\sum_{i=1}^{n}||\Psi(\nu)_i||\sum_{i=1}^{n}|\Psi(\nu)_i|d\nu\leq$$

$$-\chi\int_0^t\left[\sum_{i=1}^{n}||\Psi(\nu)_i||\right]^2d\nu-\frac{\chi\left[\lim_{p\to\infty}\left(\sum_{i=1}^{n}|\nabla g_i|^p\right)^{\frac{1}{p}}\right]^2}{d_i^2\sqrt{\varphi_1}}\int_0^t\left[\sum_{i=1}^{n}|\Psi(\nu)_i|\right]^2d\nu$$

$$\leq 0$$

Moreover, through the use of Gronwall's inequality we complete the proof.

9.2 Existence

We will start by approaching the proof in terms of a Galerkin scheme. Hence, we will allow $\Phi_1, \Phi_2, ...,$ be a linearly independent spanning set of Θ_g in which specified eigenfunctions reside that are of the defined operator Υ. This can then be defined as an orthonormal component inside of Π_g. Moreover, we may denote

$$\Theta_d = span\{\Phi_1, ..., \Phi_d\}$$

We will also note the projector

$$\ell_d\upsilon=\sum_{j=1}^{d}(\upsilon,\Phi_j)\Phi_j$$

We can also define

$$\upsilon^d(t)=\sum_{j=1}^{d}\sigma_{d,j}(t)\Phi_j$$

Where we note that the term $\sigma_{d,j}$ is used in order to justify the following expression.

$$\begin{gathered}(\upsilon^d(t),\Phi_j)_g=-\int\chi(\Upsilon\upsilon^d(t),\Phi_j)_gdt-\int\chi(N\upsilon^d(t),\Phi_j)_gdt\\+\int\sum_{i,j=1}^{2}\frac{d\upsilon^d(t)_j}{dx_i}\int_\Omega g[\upsilon^d(t)_i\Phi_{j_j}]dxdt+\int(f(t),\Phi_j)_gdt\quad\forall(j=1,...,m)\end{gathered}\tag{18}$$

And now we will recall our initial condition $\upsilon^d(0) = \ell_d \upsilon_i$. This system of Ordinary Differential equations resides within the unknown $(\sigma_{d,1}(t), ..., \sigma_{d,d}(t))$ and fulfills a fundamental theorem called the Peano Theorem which states the following. Let $D \subseteq_{open} \mathbb{R} \times \mathbb{R}$ with the following being true.

$$f : D \to \mathbb{R}$$

With a continuous function and satisfaction towards

$$y'(x) = f(x, y(x))$$

Also note that there exists a continuous, explicit first-order differential equations on D, then we note that every initial value problem will follow $y(x_0) = y_0$ For when f and the knowledge that $(x_0, y_0) \in D$ has some local solution

$$z : I \to \mathbb{R}$$

Where it becomes apparent that I is a neighbourhood of $x_0 \in \mathbb{R}$, such that the following equality stands.

$$z'(x) = f(x, z(x)) \quad \forall x \in I$$

Recall that in the case of this formulation the solution does not need to be unique. With the same initial values (x_0, y_0). This then gives rise to a plethora of differing solutions labelled z.

Henceforth, back to our case the approximate solutions υ_d must exist. We will now perform a Priori Estimates.

$$\begin{aligned}(\upsilon^d(t), \Upsilon\upsilon^d(t))_g = &-\int \chi((\upsilon^d(t), \Upsilon\upsilon^d(t)))_g dt - \int \chi(\upsilon^d(t), \Upsilon\upsilon^d(t))_g dt \\ &-\int \sum_{i,j=1}^{2} \frac{d\upsilon^d(t)_j}{dx_i} \int_\Omega g[\upsilon^d(t)_i \Upsilon\upsilon^d(t)_j] dx dt + \int (f(t), \Upsilon\upsilon^d(t))_g dt\end{aligned} \tag{19}$$

We note that the above formulation is very similar to (12). We note that we formulate exactly what was stated in Argument 9.2. For when we note the term υ^d, in which case we replace υ_i by υ_i^d. Moreover, we note that there exists an equality, $\upsilon_i^d = \ell_d \upsilon_i$. Recall that ℓ_d is the orthogonal projector within Θ_g. Hence,

$$\sum_{i=1}^{n} ||\upsilon_{i_i}^d|| = \sum_{i=1}^{n} ||[\ell_d \upsilon_i]_i|| - \sum_{i=1}^{n} ||\upsilon_{i_i}|| \leq 0$$

We will now determine the bounds for υ^d, and in this case these will be the exact same bounds in which υ in (10) and (11). Or we may also allow υ^d to be in a bounded set of the space $L^2(0, T; \varepsilon(\Upsilon)) \cap L^\infty(0, T; \Theta_g)$ (20). Now we will observe the fact that (18) can be rewritten in an equivalent form

$$\upsilon^d = -\int \chi\Upsilon\upsilon^d dt - \int \chi C\upsilon^d dt - \int \ell_d \Lambda(\upsilon^d, \upsilon^d) dt + \int \ell_d f(t) dt$$

Therefore, we will note that because of Argument 8.2 we get the fact that the following is bounded.

$$\{(\upsilon^d)'\} \in L^2(0,T;\Pi_g)$$

We'll now use the rules of limit computing to create a relation that holds for finite numbers and extrapolate to zero (or infinity). We may now conclude that there must exist $\upsilon \in L^2(0,T;\varepsilon(\Upsilon)) \cap L^\infty(0,T;\Theta_g)$ with the fact that $\upsilon' \in L^2(0,T;\Pi_g)$, and that the term is also a sub sequence of $\{\upsilon^d\}$. Henceforth, we arrive at the following formulations labeled (21).$\{\upsilon^d\}$converges weakly to $\upsilon \in L^2(0,T;\varepsilon(\Upsilon))$, and *weakly** to $\upsilon' \in L^\infty(0,T;\Theta_g)$,$\{(\upsilon^d)'\}$ converges weakly to $\upsilon' \in L^2(0,T;\Pi_g)$.

Since it is clear that Ω is bounded, we can take notice that it is beneficial to use the compactness Argument (11,Chapter III. Theorem 2.1 (28 [see References])) which will allow for the existence of some sub sequence υ^d. This sequence has strong convergence towards $\upsilon \in L^2(0,T;\Theta_g)$ and also $L^2(0,T;\Pi_g)$. Moreover, we can also pass through the limit that resides within the knowledge of the non linearity of ϱ. This proof has been found to be the exact same as the one found in [9,chapter III (28 [see References])].

Argument 9.3 If we note that υ^d satisfies strong convergence towards $\upsilon^2 \in L^2(0,T;\Pi_g)$. Mroever, we may define some vector function η with certain sub parts that belong to the space $C^1([0,T]\cdot(\Omega'))$, then we can deduce

$$\int_0^T \sum_{i,j=1}^{2} \frac{d\upsilon^d(t)_j}{dx_i} \int_\Omega g[\upsilon^d(t)_i \eta(t)_j]dxdt \to \int_0^T \sum_{i,j=1}^{2} \frac{d\upsilon(t)_j}{dx_i} \int_\Omega g[\upsilon(t)_i \eta(t)_j]dxdt$$

Now we can let ζ be a continuously differentiable function that resides within the bounds $[0,T]$ with the knowledge that $\zeta(T) = 0$. We can then multiply (18) by $\zeta(t)$, and also integrate. Giving us

$$(\upsilon^d(T), \zeta'(T)\eta_j)_g = \chi(\Upsilon\upsilon^d(T), \eta_j\zeta(T))_g + \chi(C\upsilon^d(T), \eta_j\zeta(T))_g$$
$$+ \sum_{i,j=1}^{2} \frac{d\upsilon^d(T)_j}{dx_i} \int_\Omega g[\upsilon^d(T)_i[\eta_j\zeta(T)]_j]dx + (f(T), \eta_j\zeta(T))_g$$

Now we may pass to the limit, giving us

$$(\upsilon(T), \Phi\zeta'(T))_g = \chi(\Upsilon\upsilon(T), \Phi\zeta(T))_g + \chi(\upsilon(T), \Phi\zeta(T))_g$$
$$+ \sum_{i,j=1}^{2} \frac{d\upsilon(T)_j}{dx_i} \int_\Omega g[\upsilon(T)_i \Phi\zeta(T)_j]dx - \frac{d}{dx}(\upsilon_i, \Phi)_g \zeta(0) - (f(T), \Phi\zeta(T))_g \quad (22)$$

Note that this formulation will hold $\forall \Phi \in \Theta_g'$. We can also bring notice to the fact that υ shows satisfaction towards (9.1) in the distributive sense. Finally, we must show that υ shows satisfaction towards $\upsilon(0) = 0$. To prove this we must multiply (18) by $\zeta(t)$ and of course integrate. After integration of the first term we arrive at the following.

$$(\upsilon(T), \Phi\zeta'(T))_g = \chi(\Upsilon\upsilon(T), \Phi\zeta(T))_g + \chi(\upsilon(T), \Phi\zeta(T))_g$$
$$+ \sum_{i,j=1}^{2} \frac{d\upsilon(T)_j}{dx_i} \int_\Omega g[\upsilon(T)_i \Phi\zeta(T)_j] dx - \frac{d}{dx}(\upsilon(0), \Phi)_g \zeta(0) - (f(T), \Phi\zeta(T))_g \tag{23}$$

In contrast to (3.19), we now deduce

$$(\upsilon(0) - \upsilon_i, \Phi)\zeta(0) = 0$$

We now note that it is possible to choose some ζ with $\zeta(0) \neq 0$. This then proves that $\upsilon(0) = \upsilon_i$. In turn, this completes the proof.

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