Folium of Descartes and Division by Zero Calculus - An Open Question

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Abstract: In this note, in the folium of Descartes, with the division by zero calculus we will see some interesting results at the point at infinity with some interesting geometrical property. We will propose an interesting open question.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 Introduction

In this note, in the folium of Descartes, with the division by zero calculus we will see some interesting results at the point at infinity with some interesting geometrical property. We will propose an interesting open question.

2 Essences of division by zero and division by zero calculus

We will state very elementary facts and so, in order to state the contents in a self contained way, we state first the essences of division by zero and division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will **define**

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [2, 3, 4, 5].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

3 Statement of results

For the folium of Descartes $x^3 + y^3 - 3axy = 0, (a > 0)$, we consider the parametric expression

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

Recall its beautiful whole graph, for example, [1, 6]. In particular, the line $x + y = -a$ is the asymptotic line of the curve for $t \to -1$.

At first, surprisingly enough, we have the results by the division by zero calculus, directly at $t = -1$

$$x = 0, \quad y = -a.$$

Next, for the differentials, we have

$$\frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2}, \quad \frac{dy}{dt} = \frac{3a(2t-t^4)}{(1+t^3)^2}, \quad \frac{dy}{dx} = \frac{2t-t^4}{1-2t^3}.$$

Then, by the division by zero calculus, we have, at $t=-1$

$$\frac{dx}{dt} = \frac{1}{9}, \quad \frac{dy}{dt} = \frac{-1}{9}.$$

These results show that

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right) = -1,$$

that is right.

For the whole parameter t over $(-\infty, +\infty)$, the point $(0, -a)$ is only isolated from the folium of Descartes that may be considered as the limit points of the curve for $t \to -1$ from the both sides.

In addition, we note that the curvature and tangential angle are, for the Heaviside step function H

$$\kappa(t) = \frac{2(1+t^3)^4}{3a(1+4t^2-4t^3-4t^5+4t^6+t^8)^{3/2}}$$

and

$$\phi(t) = \arctan\left(\frac{t(t^3-2)}{2t^3-1}\right) + H\left(2t - \frac{1}{2^{1/3}}\right).$$

Therefore,

$$\kappa(-1) = 0, \quad \phi(-1) = -1$$

that may be considered as the reasonable results.

Meanwhile, with the polar expression we have

$$r = \frac{3a \sec\theta \tan\theta}{1+\tan^3\theta}.$$

Then, we have, at $\theta = 3\pi/4$ and $-\pi/4$

$$r = 0.$$

However, this property is popular, since the **real infinity** is represented by zero, in our sense.

Open question

What does the point $(0, -a)$ mean for the beautiful curve of the Descartes folium?

Unbearably fun God's secret:

The division by zero calculus gives a unique finite fixed value at an isolated singular point of a function called a pole and considered as taking infinity there. Actually at an isolated singular point the function takes a unique finite fixed value. It has been discovered that the value has various natural meanings in many cases. However, some of them will have mysterious meanings. We want to find the meanings of those values. From elementary school students to geniuses and prophets they will be able to enjoy to look for their meanings.

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