Inconsistent number systems,nowel development.

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Abstract.In this paper we dealin using paraconsistent first order logic $\overline{LP}^{\#}_{\omega} = \cup_{n<\omega} \overline{LP}^{\#}_{n}$ with restricted modus ponens rule and infinite levels of a contradiction [1]-[4], where $\overline{LP}^{\#}_{n}$ is an paraconsistent first order logic with n levels of a contradiction.

1.Introduction.
2.The Graham Priest argument.
3.Inconsistent logic with restricted modus ponens rule based on RM3-assignment.
4.The da'Costa type paraconsistent logic $\mathbf{C}^{\#}_{\infty}$ with infinite levels of a contradictions and restricted modus ponens rule can to safe Naive Set Theory from a trivality.
5.The paraconsistent set theory $NF^{\#}_{\infty}$ based on logic $C^{\#}_{\infty}$ with infinite levels of a contradiction.
6. The paraconsistent set theory $ZF^{\#}_{\infty}$ based on logic $C^{\#}_{\infty}$ with infinite levels of a contradiction.
7. Nonclassical bivalent propositional language with a strong negation $\neg_{\mathbf{s}}$ and a weak negation $\neg_{w}$.
8.First order paraconsistent propositional logic with zero levels of a contradictions $\mathbf{LP}^{\#}_{0}$.
9.First order paraconsistent quantificational logic with zero levels of a contradictions $\overline{\mathbf{LP}}^{\#}_{0}$.
10. First order paraconsistent propositional logic with one level of a contradictions $\mathbf{LP}^{\#}_{1}$.
11.First order paraconsistent quantificational logic with one level of a contradictions $\overline{\mathbf{LP}}^{\#}_{1}$.
12.First order paraconsistent propositional logic with n levels of a contradictions $\mathbf{LP}^{\#}_{n}$.
13.First order paraconsistent quantificational logic with n levels of a contradictions $\overline{\mathbf{LP}}^{\#}_{n}$.
14.First order paraconsistent propositional logic with countable levels of a contradiction $\mathbf{LP}^{\#}_{\omega}$.
15.First order paraconsistent quantificational logic with countable levels of a contradiction $\mathbf{LP}^{\#}_{\omega}$.
16.Paraconsistent set theory $\mathbf{ZFC}^{\#}_{\boldsymbol{\omega}}$.
17.Inconsistent $w_{\{0\}}$-relations and $w_{\{0\}}$-functions of the order inconsistency zero.
18.Inconsistent $w_{[n]}$-relations and $w_{[n]}$-functions of the order

inconsistency $n \geq 1$.
19.Inconsistent w-Equivalences and w-Orderings of the order inconsistency zero.
20.Inconsistent $w_{[n]}$-Equivalences and $w_{[n]}$-Orderings of the order inconsistency $n \geq 1$.
21.Almost classica w-natural numbers $\mathbb{N}_w^{\mathbf{cl}}$.
22.Inconsistent $w_{\{0\}}$-natural numbers of the order inconsistency zero $\mathbb{N}_{w_{\{0\}}}$.
23.Recursion and the addition operation in $\mathbb{N}_{w_{\{0\}}}$.
24.Inconsistent $w_{[n]}$-natural numbers $\mathbb{N}_{w[n]}$ of the order inconsistency $n \geq 1$.
25.Inconsistent $w_{[n]}$-integers $\mathbb{Z}_{w[n]}$, inconsistent $w_{[n]}$-rationals $\mathbb{Q}_{w[n]}$, and inconsistent $w_{[n]}$-reals $\mathbb{R}_{w[n]}$.
26.Inconsistent $w_{\{0\}}$-integers $\mathbb{Z}_{w_{\{0\}}}$, inconsistent $w_{\{0\}}$-rationals $\mathbb{Q}_{w_{\{0\}}}$ and inconsistent $w_{\{0\}}$-reals $\mathbb{R}_{w_{\{0\}}}$ of the order inconsistency zero.
27.Inconsistent $w_{[n]}$-integers $\mathbb{Z}_{w[n]}$, inconsistent $w_{[n]}$-rationals $\mathbb{Q}_{w[n]}$ and inconsistent $w_{[n]}$-reals $\mathbb{R}_{w[n]}$.

1.Introduction

In this article we dealing using non-classical approach based on paraconsistent set theory $ZFC_\omega^\#$ [1-4].

2.The Graham Priest argument.

In this paper we dealin using paraconsistent first order logic $\overline{LP}_\omega^\# = \bigcup_{n<\omega} \overline{LP}_n^\#$ with restricted modus ponens rule and infinite levels of a contradiction [1]-[4], where $\overline{LP}_n^\#$ is an paraconsistent first order logic with n levels of a contradiction. Let $\overline{\mathcal{L}}_\omega^\# = \mathcal{L}_\omega^\#\left(\overline{LP}_\omega^\#\right)$ be the formal language corresponding to logic $\overline{LP}_\omega^\#$ and let $\overline{\mathcal{F}}_\omega^\#$ be the set of the all wff's of $\overline{\mathcal{L}}_\omega^\#$.

Remark 2.1. In contrast with ordinary classical (unrestricted) modus ponens rule

$$A, A \to B \vdash_{\mathbf{UMP}} B \tag{2.1}$$

the restricted modus ponens rule reads

$$A, A \to B \vdash_{\mathbf{RMP}} B \text{ if and only if } A \notin \Delta_1 \text{and } A \to B \notin \Delta_2, \tag{2.2}$$

where $\Delta_1, \Delta_2 \subsetneqq \overline{\mathcal{F}}_\omega^\#$.Thus it is not in general true by using paraconsistent first order logic $\overline{LP}_\omega^\#$ that if $A \to B$ holds and A holds then B holds.

Remark 2.2. In adition in logic $\overline{LP}_\omega^\#$ we distingvish a strong negation $\neg_s A$ and a weak negation $\neg_w A$. A strong negation that is ordinary classical negation, i.e. $\neg_s A$ holds if and only if $A \vdash B, B \in \overline{\mathcal{F}}_\omega^\#$.A weak negation that is nonclassical negation, i.e. $A \wedge \neg_w A$ might holds.

Remark.2.3. In particular, $((A \wedge \neg_w A) \to B) \wedge (A \wedge \neg_w A)$ might hold while B does not.

Remark.2.4. In particular is that this permit a solution to the following problem raised by Graham Priest [5],[6]. Ordinarily one wants postulates such as the Cancellation Law

$$\forall x \left[\neg(x = 0) \to \forall y \forall z (x \times y = x \times z \to y = z) \right] \tag{2.3}$$

to hold when moving from the classical theory of the rings to nonclassical theory of the inconsistent rings.But canonical inconsistent fields [i.e. inconsistent fields based on canonical inconsistent logic with unique negation ¬] have both

$$\neg(x_{\mathbf{Inc}} = 0) \text{ and } x_{\mathbf{Inc}} = 0 \quad (2.4)$$

for some $x_{\mathbf{Inc}}$ [for examle $(x_{\mathbf{Inc}} = 1) \wedge (x_{\mathbf{Inc}} = 0)$] and therefore

$$\forall y \forall z(x_{\mathbf{Inc}} \times y = x_{\mathbf{Inc}} \times z = 0) \quad (2.5)$$

holding.Yet one does not want to detach every $y = z$ or the theory is trivial. Yet one also does not want to forbid detachment for those x which are classically not identical with zero.

Remark 2.5.Obviously Priest paradox arises from the statement:

$$x \text{ classically not identical with zero.} \quad (2.6)$$

The statement (2.6) completely does not well defined by using canonical inconsistent logic with unique negation ¬. Note that if the statement $\neg(x = 0)$ treated classically i.e. under definition of the strong negation $\neg_s$ (see Remark.2.1), this meant impossibilyty $x = 0$, i.e. the statement $x = 0$ is not holds classically and assuming that both $\neg(x = 0)$ and $x = 0$ holding we conclude that the statement $\neg(x = 0)$ is not holds classically in contrary with Priest assumption (2.6).Cancellation Law (2.3) breaks down.

Remark 2.6.In order to avoid the difficultness mentioned above we apply the logics $\overline{LP}_n^{\#}, n \geq 1$and postulate the Cancellation Law in the following form

$$\forall x\left[\neg_s(x = 0) \to \forall y \forall z(x \times y = x \times z \to y = z)\right], \quad (2.7)$$

and

$$\forall x\left[\neg_w(x = 0) \to \forall y, \forall z(x \times y = x \times z \to y = z)\right]. \quad (2.8)$$

We set now instead (2.4) that

$$\neg_w(x = 0) \text{ and } x = 0. \quad (2.9)$$

From (2.7) one to detach $y = z$ only for such x which are classically not identical with zero as it should be. However if we set $(x_{\mathbf{Inc}} = 1) \wedge (x_{\mathbf{Inc}} = 0) \in \Delta_1$ and $(((x_{\mathbf{Inc}} = 1) \wedge (x_{\mathbf{Inc}} = 0)) \to 0 = 1) \in \Delta_1$ then

$$((x_{\mathbf{Inc}} = 1) \wedge (x_{\mathbf{Inc}} = 0)) \to 0 = 1 \nvdash_{\mathbf{RMP}} 0 = 1 \quad (2.10)$$

as it should be.

3.Inconsistent logic with restricted modus ponens rule based on RM3-assignment.

The classical example of the inconsistent logic with restricted modus ponens rule has been proposed by C.E. Mortensen, see ref.[5-7].

Let $\mathcal{L}$ be an canonical language consisting of simple terms (names), one for each real number; function symbols $+, \times, -, \div;$ atomic predicates $=, <, \in;$ variables $x, y, z, \ldots$ and operators $\neg, \wedge, \forall$.

Remind that any RM3-assignment [5],[6] is a function I assigning to the wff's of $\mathcal{L}$, or the appropriate sublanguage of $\mathcal{L}$ under investigation at the time, values from the set

$\{\mathbf{T},\mathbf{N},\mathbf{F}\}$ in accordance with the following definition:

(i) for any atomic wff with terms $t_1, t_2,$ we have $I(t_1 = t_2), I(t_1 < t_2)$ and $I(t_1 \in t_2)$ all belong to $\{\mathbf{T},\mathbf{N},\mathbf{F}\}$, (read 'true, neuter, false');

(ii) $I(\neg A)$ and $I(A \wedge B)$ are given by the RM3-matrices:

$$\begin{array}{ccccc} \wedge & \mathbf{T} & \mathbf{N} & \mathbf{F} & \neg \\ {}^*\mathbf{T} & \mathbf{T} & \mathbf{N} & \mathbf{F} & \mathbf{F} \\ {}^*\mathbf{N} & \mathbf{N} & \mathbf{N} & \mathbf{F} & \mathbf{N} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{T} \end{array} \tag{3.1}$$

(iii) $I((x)A) = \min\{y$: for some term $t, I(A(t|x)) \ = y\},$ where min is relative to the ordering: false $<$ neuter $<$ true. A sentence A holds in an assignment I iff $I(A) \in \{\mathbf{T},\mathbf{N}\}$.

Let us consider now the classical standard model of the natural numbers, equipped with names for the natural numbers. In view of the Extendability Lemma [8],[9], the set of sentences holding therein can be extended by adding any collection of sentences of the form $\neg(n = n)$ and evaluating in an RM3-assignment. Note that the contradiction does not spread to other sentences of the form $\neg(m = m)$. Similarly, collections of sentences of the form $n = m$ for distinct $n, m,$ may be added with the same result.

This raises the following question [5]. If we add, for example, $0 = 2$ to the standard model of the natural numbers, then, in virtue of the substitutivity of identity and the fact that $\neg(0 = 2)$ also holds, have we not imported the further sentence $\neg(0 = 0)$? The answer is no, and it illustrates the generality of the Extendability Lemma.

The rule of substitutivity of identity (SI) in the form if $t_1 = t_2$ holds, then Ft_1 holds iff Ft_2 holds (all terms $t_1, t_2,$ with t_2 replacing t_1 in F_1 at least one place) does not always hold in RM3-assignments. What is the case, if the sentences holding in an RM3-assignment include those holding in the standard model of the natural numbers, is that

$(t_1 = t_2 \wedge Ft_1) \to Ft_2$ holds, since it holds in the standard model.

Remark 3.1.But it is not in general true that if $A \to B$ holds and A holds then B holds.

In particular, $((A \wedge \neg A) \to B) \wedge (A \wedge \neg A)$ might hold while B does not, i.e.

$$(A \wedge \neg A) \to B \nvdash_{RM3} B, \tag{3.2}$$

where by $\vdash_{RM3}$ we denote the rule of conclusion corresponding to RM3-assignments.

However, this leads to no loss of information from classical arithmetic, since we do have

that if $(A \to B) \wedge A$ holds, and if moreover $(A \to B) \wedge A$ holds back in the standard model

for arithmetic, then B holds (trivial).

Remark 3.2.(i) A special case of interest is this: if $t_1 = t_2 \wedge Ft_1$ holds and if moreover $\neg(t_1 = t_2)$ and $\neg Ft_1$ both do not hold, then Ft_2 holds. (Reason: for then $t_1 = t_2 \wedge Ft_1$ holds back in the classical complete subtheory, wherein Ft_2 could be detached.)

(ii) Thus the rule SI does not hold in all RM3-assignments.

4. The da Costa type paraconsistent logic $\mathbf{C}_{\infty}^{\#}$ with infinite levels of a contradictions and restricted modus ponens rule can to save Naive Set Theory from a trivality.

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It well known that canonical da Costa's paraconsistent logics is invalid in order to obtain non trivial paraconsistent set theory,see [8]. In order to resolve this tension we consider the da Costa type paraconsistent logics $\mathbf{C}_{\infty}^{\#}$ with infinite levels of a contradictions and restricted modus ponens rule mentioned above.

We remind that da Costa paraconsistent set theory is a paraconsistent set theory whose underlying logic is one of da Costa's paraconsistent logics $C_{n}^{=}, 1 \leq n \leq \omega$.

Definition 4.1.The postulates of $C_{\omega}^{=}$ are those of the positive intuitionistic first-order logic with equality, plus:

(1) $\neg\neg A \Rightarrow A,$

(2) $A \vee \neg A.$

(3) unrestricted modus ponens rule : $A, A \Rightarrow B \vdash_{\mathbf{UMP}} B.$

Definition 4.2.The postulates of $C_{n}^{=}, 1 \leq n \leq \omega$, are those of $C_{\omega}^{=},$ plus:

(1) $B^{(n)} \wedge (A \Rightarrow B) \wedge (A \Rightarrow \neg B) \Rightarrow \neg A,$

(2) $A^{(n)} \wedge B^{(n)} \Rightarrow (A \Rightarrow B)^{(n)} \wedge (A \wedge B)^{(n)} \wedge (A \vee B)^{(n)},$

(3) $(\forall x)(A(x))^{(n)} \Rightarrow ((\forall x)A(x))^{(n)} \wedge ((\exists x)A(x))^{(n)},$

where $A^{(n)}$ defined as follows: $A^{1} = A^{0} = \neg(A \wedge \neg A), A^{n+1} = (A^{n})^{0}, A^{(n)} = A^{1} \wedge \ldots \wedge A^{n},$

(4) unrestricted modus ponens rule :

$A, A \Rightarrow B \vdash_{\mathbf{UMP}} B.$

In each , $C_{n}^{=}, 1 \leq n \leq \omega, \neg^{*}A$ is defined as $\neg A \wedge A^{(n)},$and it is proved that satisfies all the properties of the classical negation. Then classical logic can be obtained inside these systems; consequently, they are finitely trivializable. For, from any formula of the form $A \wedge \neg A \wedge A^{(n)}$ one can deduce any formula whatsoever. Nonetheless, $C_{\omega}^{=}$ is not finitely trivializable. Moreover, each system terns in the hierarchy $C_{1}^{=}, C_{2}^{=}, \ldots, C_{n}^{=}, \ldots, C_{\omega}^{=}$ is strictly stronger than the following ones. Thus, we may construct a hierarchy of da Costa's paraconsistent set theories in which, at least intuitively, it seems that each system may admit more nonclassical sets than the preceding ones.

Definition 4.3.Let $\mathcal{L}_{\omega} = \mathcal{L}_{\omega}(C_{\omega}^{=})$ be the formal language corresponding to logic $C_{\omega}^{=}$ and let $W_{\omega} = W_{\omega}(\mathcal{L}_{\omega})$ be the set of the all wff's of $\mathcal{L}_{\omega}$The postulates of $C_{\omega}^{\#}$ are those of the postulates of $C_{\omega}^{=}$ but with restricted modus ponens rule : $A, A \Rightarrow B \vdash_{\mathbf{RMP}_{\omega}} B$ if and only if $(A,B) \in \Delta_{\omega} \times \Delta_{\omega}$ [instead unrestricted modus ponens rule (3)], where $\Delta_{\omega} \subsetneqq W_{\omega}$.

Definition 4.4.Let $\mathcal{L}_{n} = \mathcal{L}_{n}(C_{n}^{=})$ be the formal language corresponding to logic $C_{n}^{=}$ and let $W_{n} = W_{n}(\mathcal{L}_{n})$ be the set of the all wff's of $\mathcal{L}_{\omega}$The postulates of $C_{n}^{\#}, 1 \leq n \leq \omega$, are those of $C_{n}^{=}, 1 \leq n \leq \omega,$but with restricted modus ponens rule : $A, A \Rightarrow B \vdash_{\mathbf{RMP}_{n}} B$ if and only if $(A,B) \in \Delta_{n} \times \Delta_{n}$instead unrestricted modus ponens rule (4), where $\Delta_{n} \subsetneqq W_{n}(C_{n}^{=})$

Definition 4.5.The postulates of $C_{\infty}^{\#},$are $\bigcup_{n=1}^{\omega} C_{n}^{\#}$ with restricted modus ponens rule :

$A, A \Rightarrow B \vdash_{\mathbf{RMP}_{\infty}} B$ if and only if $(A,B) \in \bigcup_{n=1}^{\omega} \Delta_{n} \times \Delta_{n}.$

We remind that da Costa paraconsistent set theories NF_{n}^{C} are constructed very similarly to $NF.$The main postulates of NF_{ω}^{C} are the following [8]:

I.Extensionality

$$\forall\alpha\forall\beta\forall x[x \in \alpha \Leftrightarrow x \in \beta \Rightarrow \alpha = \beta]. \tag{4.1}$$

II.**Abstraction**

$$\exists\alpha\forall x[x \in \alpha \Leftrightarrow F(x)], \tag{4.2}$$

where α does not occur free in $F(x)$ and $F(x)$ is stratified or it does not contain any formula of the form $A \Rightarrow B$.

5. The paraconsistent set theory $NF_{\infty}^{\#}$ based on logic $C_{\infty}^{\#}$ with infinite levels of a contradiction.

Definition 5.1.The main postulates of $NF_{\omega}^{\#}$ are those of the postulates of NF_{ω}^{C} but with logic of type $C_{\omega}^{\#}$ instead logic $C_{\omega}^{=}$.

Definition 5.2.The main postulates of $NF_{n}^{\#}, 1 \leq n \leq \omega,$ are those of the postulates of NF_{n}^{C} but with logic of type $C_{n}^{\#}, 1 \leq n \leq \omega,$ instead logic $C_{n}^{=}$.

Definition 5.3.The main postulates of $NF_{\infty}^{\#}$, are those of the postulates of NF_{ω}^{C} but with logic of type $C_{\infty}^{\#}$.

Da Costa's paraconsistent set theories of type NF_{ω}^{C} and $NF_{n}^{C},\ 1 \leq n \leq \omega$. has been studying A.I. Arruda [8].A.I. Arruda has been proved that da Costa's formulation of the axiom schema of abstraction (1.2.2) for the systems $NF_{n},\ 1 \leq n < \omega,$ leads to the trivialization of the systems (see [8]).

Remark 5.1.Note that in NF_{ω}^{C}, the restrictions regarding the use of non-stratified formulas obstruct a direct proof of the paradox of Curry. Russell's set $\Re$, defined as $\hat{x}\neg(x \in x)$, exists as well as many other non-classical sets. The paradox of Russell in the form $\Re \in \Re \wedge \neg(\Re \in \Re)$ is derivable but apparently, it causes no ham to the system.

Due to its weakness, the primitive negation of $NF_{\omega}^{C}, \neg,$ is almost useless for set-theoretical purposes. Thus, let us define

$$\sim A \text{ for } A \Rightarrow \forall x\forall y[x \in y \wedge x = y]. \tag{5.1}$$

The universal set $\mathbf{V}$ is defined as $\hat{x}(x = x)$, the empty set $\emptyset$ as $\hat{x} \sim(x = x)$, and the complement of a set α, $\bar{\alpha}$, as $\hat{x}\sim(x \in a)$.

Theorem 5.1.[8]. In $NF_{\omega}^{C}, \sim$ is a minimal intuitionistic negation.

Corollary 1. $\vdash A \Rightarrow (\sim A \Rightarrow \sim B)$, $\vdash (A \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim A)$.

Corollary 2. All the theorems of NF whose proofs depend only on the laws of the minimal intuitionistic first-order logic with equality and on the postulates of extensionality and abstraction of NF are valid in NF_{ω}^{C}.

Theorem 5.2.[8].(Cantor's Theorem) $NF_{\omega}^{C} \vdash \sim(\alpha \leq P(\alpha))$.

Corollary.[8].(Cantor's Paradox) $NF_{\omega}^{C} \vdash (\mathbf{V} \leq P(\mathbf{V}))\ \wedge \sim(\mathbf{V} \leq P(\mathbf{V}))$.

Remark 5.2.Note that Cantor's paradox does not trivialize NF_{ω}^{C}, since from A and $\neg A$ we cannot obtain any formula B whatsoever. For instance, apparently, we cannot obtain any formula of the form $\neg B$, where B is a nonatomic formula.

Theorem 5.3.[8].(i) $NF_{\omega}^{C} \vdash \forall\alpha\forall\beta[(\alpha = \beta) \wedge \sim(\alpha = \beta)]$,(ii) $NF_{\omega}^{C} \vdash [(\alpha \in \beta) \wedge \sim(\alpha \in \beta)]$, (iii) $NF_{\omega}^{C} \vdash [(\alpha \in \alpha) \wedge \sim(\alpha \in \alpha)]$.

Proof. By the corollaries of theorems 5.1 and 5.2, we obtain

$$x = x \Rightarrow \delta,$$
$$\delta \Leftrightarrow \forall\alpha\forall\beta[(\alpha \in \beta) \wedge (\alpha = \beta)] \quad (5.24)$$

Thus, as $x = x,$ then $\forall\alpha\forall\beta(\alpha = \beta)$. By the same corollaries we also obtain $\forall\alpha\forall\beta[\sim(\alpha = \beta)]$. The proof of part (ii) is similar to that of part (i). Part (iii) is an immediate consequence of part (ii).

Remark 5.3. We introduce now the logic of type $C^{\#}_{\omega}$ with $\Delta_{\omega} = \Delta^{\delta}_{\omega}$ such that $(x = x, \delta) \notin \Delta^{\delta}_{\omega}$. Thus in $NF^{\#}_{\omega}$ with $\Delta_{\omega} = \Delta^{\delta}_{\omega}$ Theorem 1.2.3 no longer holds.

Remark 5.4. Note that: (i) By Theorem 1.2.3, it could seem that NF^{C}_{ω} is trivial. Nonetheless, apparently this is not the case.

(ii) However, though it is nontrivial, NF^{C}_{ω} is without interest, for not only are every two sets identical, but also every set belongs and does not belong to itself.

Remark 5.5.In order to avoid the results mentioned in Theorem 5.3, one could think of introducing more restrictions in da Costa's formulation of the axiom schema of abstraction when $F(x)$ is non-stratified. Nonetheless, we be live that this is a worthless effort. For:

(i)The only non-stratified formula used in the proof of Cantor's Theorem (which is fundamental in the proof of Theorem 5.3) is a non-stratified formula of the form $\alpha \in \beta$. Then, the new restrictions must avoid those nonstratified atomic formulas of the form $\alpha \in \beta$ which determine a set.

(ii) A new proof of Theorem 5.3 may be obtained in the following way: in NF the formula $y = \{x\}$ cannot determine a relaticm because $\langle x,y\rangle = \langle x,y\rangle \wedge (y = \{x\})$ is non-stratified. But, such a formula does not contain any subformula of the form $A \Rightarrow B$; then, in NF^{C}_{ω} it determines a relation S such that $S \in 1 - 1$, see [7],pp.12. With such a relation we prove that $\alpha \leq P(\alpha)$. In NF^{C}_{ω} we also prove that $\sim(\alpha \leq P(\alpha))$. Then, these new restrictions must also avoid that those non-stratified formulas whose atomic sub formulas are of the form $\alpha = \beta$ determine a set.

(iii) From the above remarks (i) and (ii) we conclude that, in order to avoid the counterintuive results mentioned in Theorem 5.3, the axiom schema of abstraction in NF^{C}_{ω} should be formulated as in NF.

Remark 5.6.Due to the paradoxes obtained in NF^{C}_{ω}, we conclude that in these system the axiom schema of abstraction should be formulated as in NF. Thus, if we want these theories to be paraconsistent set theories, we need to postulate directly the existence of contradictory sets. .Apparently, we may postulate the existence of Russell's set without any problem. Nonetheless, due to the two above considerations about the non-stratified formulas that lead to the proof of the paradox of identity, we believe that, besides Russell's set, very few other non-classical sets may exist in $NF_n,\ 1 \leq n \leq \omega$.

Definition 5.4.[8].Let us denote by $\mathbf{DC}_n$ any da Costa set theory based on the respective $C^{=}_{n}$, where Russell's class is a set. Thus, in $\mathbf{DC}_n, 1 < n < \omega,$ the defined negation $\neg^{*}A \Leftrightarrow \neg A \wedge A^{(n)}$ is a classical negation; and in $\mathbf{DC}_{\omega}$ the defined negation $\sim A \Leftrightarrow A \Rightarrow \forall x\forall y[x \in y \wedge x = y]$ is a minimal intuitionistic negation.

Theorem 5.4.[8].Let $\Re$ be Russell's set. In $\mathbf{DC}_n, 1 < n < \omega, \cup\cup\Re$ is the universal set.

Definition 5.5.[8].Let $\mathbf{DC}_\omega^\mathbf{V}$ be a $\mathbf{DC}_\omega$ with universal set $\mathbf{V}$ defined as $\hat{x}(x = x)$.
Let us define $x \neq \mathbf{V}$ for $\exists y[\sim(y \in x)]$. We introduce now the postulate:
P1.$\forall x[(x = \mathbf{V}) \vee (x \neq \mathbf{V})]$.
Theorem 1.2.5.[8]. In $\mathbf{DC}_\omega^\mathbf{V} + \mathbf{P}1$ is derivable $\cup\cup\Re = \mathbf{V}$.

6. The paraconsistent set theory $ZF_\infty^\#$ based on logic $C_\infty^\#$ with infinite levels of a contradiction.

In this section da Costa's set theories of type ZF we denoted by $ZF_n, 1 \leq n \leq \omega$.
A.I. Arruda has been proved that da Costa's set theories of type ZF incompatible with the existence of Russell's set $\Re$ [8].
Le us consider the set theories $ZF_n, 1 \leq n \leq \omega$, in which the axioms of pairing and inion are postulated in general, and in which we also postulate the existence of the espty set amI of RusselltsS set. Moreover, let us suppose that there is no universal set, i.e.,
S_n. $\forall x \exists y[\neg^*(y \in x)]$, in $ZF_n, 1 \leq n < \omega$;
S_ω. $\forall x \exists y\ [\sim(y \in x)]$, in ZF_ω.
Theorem 6.1.[8].The set theories $ZF_n, 1 \leq n < \omega$ plus S_n are trivial.
Proof. By S$_n$ there exists y such that $\neg^*(y \in \cup\cup\Re)$. By part II of Lemma 4.3 [7], and part I of Lemma 4.4 [8], we obtain $\forall x(x \in \cup\cup\Re)$. Consequently,

$$y \in \cup\cup\Re \wedge \neg^*(y \in \cup\cup\Re), \qquad (6.1)$$

and this formula trivializes the system.
Theorem 6.2.[8]. The paradox of identity is derivable in ZF_ω plus S$_\omega$.
Proof. By S$_\omega$ there exists y such that $\sim(y \in \cup\cup\Re)$. Using part II of Lemma 4.3 [8], we obtain $y \in \cup\cup\Re$. Consequently, by the definition of ~ it follows $\forall x \forall y(x \in y \wedge x = y)$ and therefore, the paradox of identity, $\forall x \forall y(x = y)$,follows.
Theorem 6.3.[8].. The systems ZF_n ,$1 \leq n < \omega$ with Russell's set and the axiom schema of separation postulate for all sets are trivial.
Proof.If the axiom schema of separation is postulated for all sets then there exists a subset α of $\Re$ such that (1) $\forall x\left[x \in \alpha \Leftrightarrow (x \in \Re) \wedge (x \in x)^{(n)}\right]$. From (1) we obtain (2) $\alpha \in \alpha \Leftrightarrow \neg(\alpha \in \alpha) \wedge (\alpha \in \alpha)^{(n)}$. Consequently, we have $(\alpha \in \alpha) \wedge \neg^*(\alpha \in \alpha)$, and this formula trivializes the system.
Definition 6.1.We introduce now paraconsistent logic of type $C_\infty^\#, = \bigcup_{n=1}^{\omega} C_n^\#$ with restricted modus ponens rule : $A, A \Rightarrow B \vdash_{\mathbf{RMP}_\infty} B$ if and only if $(A,B) \in \Delta_\infty = \bigcup_{n=1}^{\omega} \Delta_n$, and we choose a set Δ_∞ such that

$$(\alpha \in \alpha) \wedge \neg^*(\alpha \in \alpha) \vdash_{\mathbf{RMP}_\infty} \varnothing, \qquad (6.2)$$

i.e.,from $(\alpha \in \alpha) \wedge \neg^*(\alpha \in \alpha)$ we cannot obtain any formula whatsoever.
Definition 6.2.(i)The main postulates of $ZF_n^\#$ are those of the postulates of $ZF_n^\#$ but with logic of type $C_n^\#$ instead logic $C_n^=$.(ii) The main postulates of $ZF_\infty^\#$ are $\bigcup_{n=1}^{\omega} ZF_n^\#$ with logic of type $C_\infty^\#$ mentioned above in Definition 6.1.

7. Nonclassical bivalent propositional language with a strong negation $\neg_s$ and a weak negation $\neg_w$.

Remind that any syntactic system comprises a vocabulary and a grammar.The vocabulary of a syntactic system is a nonempty set of elements called words. An expression is any finite sequence of words. If A is the expression $\langle e_1,\ldots,e_n\rangle$ we shall simply write it as $e_1\ldots e_n$. In addition, we define the operation of concatenation: the concatenation AB of two expressions A and B is defined by $\langle e_1,\ldots,e_m\rangle\,\langle e_{m+1},\ldots,e_n\rangle = \langle e_1,\ldots,e_m,e_{m+1},\ldots,e_n\rangle$.With any syntactic system there is associated a wellordering of the expressions, called the alphabetical order. The grammar of the system consists in the division of the set of expressions into the class of nouns, the class of sentences, classes of functors of various kinds (and possibly a remainder of expressions that have no significant role at all). When a noun or sentence belongs to the vocabulary itself, it is generally called atomic; expressions that are not words are called molecular. When the system is defined, the grammatical division of the vocabulary may be given at once, and used to define the molecular nouns and sentences. As an example, we take the language of the propositional calculus.

Definition 7.1. A propositional classical syntactic system (**PCLS**) is a triple $\langle \mathbf{A}, L, \mathbf{S}\rangle$, where:

(a) $\mathbf{A}$ is a set, at most denumerable (the atomic sentences);

(b) L is a set of four distinct elements $\{\wedge, \neg_{\mathbf{s}},), (\}$ (logical signs), disjoint from $\mathbf{A}$;

(c) $\mathbf{S}$ (the set of sentences) is the smallest set including $\mathbf{A} \subsetneq \mathbf{S}$ and such that if A,B are in $\mathbf{S}$, so are (i) $\neg_{\mathbf{s}}A$ and (ii) $A \wedge B$.

Definition 7.2. A propositional nonclassical syntactic system (**PNCLS**) is a triple $\langle \mathbf{A}^{\#}, L^{\#}, \mathbf{S}^{\#}\rangle$, where:

(a) $\mathbf{A}^{\#}$ is a set, at most denumerable (the atomic sentences);

(b) $L^{\#}$ is a set of four distinct elements $\{\wedge, \neg_{\mathbf{s}}, \neg_{w},), (\}$ (logical signs), disjoint from $\mathbf{A}^{\#}$;

(c) $\mathbf{S}^{\#}$ (the set of sentences) is the smallest set including $\mathbf{A}^{\#} \subsetneq \mathbf{S}^{\#}$ and such that if A,B are in $\mathbf{S}$, so are (i) $\neg_{\mathbf{s}}A$, (ii) $\neg_{w}A$ and (iii) $A \wedge B$.

Remind that a valuation of a syntactic system is a function that assigns $\mathbf{T}$ (true) to some of its sentences, and/or $\mathbf{F}$ (false) to some of its sentences. We do not rule out that not all sentences are assigned $\mathbf{T}$ or $\mathbf{F}$, nor that no sentence is assigned $\mathbf{T}$ (respectively, $\mathbf{F}$), nor that some sentences are assigned something else. Precisely, a valuation maps a nonempty subset of the set of sentences into the set $\{\mathbf{T},\mathbf{F}\}$.

Definition 7.3. We call a valuation bivalent iff it maps all the sentences into $\{\mathbf{T},\mathbf{F}\}$.

In general, some of the symbols have an intended meaning, and this leads to a distinction between admissible and inadmissible valuations. A language $\mathcal{L}$ comprises exactly a syntactic system (its syntax) and nonempty class of valuations of that syntactic system (its admissible valuations). The expressions of the syntax of $\mathcal{L}$ are also called expressions of $\mathcal{L}$. As an example we consider again the propositional calculus.

In that subject, one is generally concerned with a kind of language that we shall call a bivalent propositional language.

Definition 7.4. $\mathcal{L}_0$ is a classical bivalent propositional language with a strong negation $\neg_{\mathbf{s}}A$ iff its syntax is a **PCLS** and its admissible valuations are the functions

$v : \mathbf{S} \to \{\mathbf{T},\mathbf{F}\}$

such that for all sentences A,B of $\mathcal{L}$:

(i) $v(A) \in \{\mathbf{T},\mathbf{F}\}$;

(ii) $v(\neg_s A) = \mathbf{T}$ iff $v(A) = \mathbf{F}$;
(iii) $v(A \wedge B) = \mathbf{T}$ iff $v(A) = v(B) = \mathbf{T}$.

Example 7.1. Classical bivalent propositional language with a strong negation $\neg_s$ and with just two atomic sentences, p and q. This language has just four admissible valuations,
which are partially depicted by the following truth table:

$$\begin{array}{ccccccc} * & p & q & \neg_s p & \neg_s q & p \wedge \neg_s q & p \wedge \neg_s p \\ v_1 & \mathbf{T} & \mathbf{T} & & & & \\ v_2 & \mathbf{T} & \mathbf{F} & & & & \\ v_3 & \mathbf{F} & \mathbf{T} & & & & \\ v_4 & \mathbf{F} & \mathbf{F} & & & & \end{array} \quad (7.1)$$

Definition 7.5. $\mathcal{L}_0^{\#}$ is a nonclassical bivalent propositional language with a strong negation $\neg_s A$ and a weak negation $\neg_w$, corresponding to praconsistent logic with zero levels of a contradictions $\mathbf{LP}_0^{\#}$ (see sect.8) iff its syntax is a **PNCLS** and its admissible valuations are the functions $v^{\#} : \mathbf{S}^{\#} \to \{\mathbf{T},\mathbf{F}\}$, where $\mathbf{S}^{\#} = \mathbf{S}_1^{\#} \cup \mathbf{S}_2^{\#}, \mathbf{S}_1^{\#} \cap \mathbf{S}_2^{\#} = \varnothing$ such that for all sentences A, B of $\mathcal{L}_0^{\#}$:
(i) $v^{\#}(A) \in \{\mathbf{T},\mathbf{F}\}$;
(i) $v^{\#}(A \wedge B) = \mathbf{T}$ iff $v^{\#}(A) = v^{\#}(B) = \mathbf{T}$.
(ii) for any $A \in \mathbf{S}_1^{\#} : v^{\#}(\neg_s A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(iii) for any $A \in \mathbf{S}_1^{\#} : v^{\#}(\neg_w A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(iv) for any $A \in \mathbf{S}_2^{\#} : v^{\#}(\neg_w A) = \mathbf{T}$ iff $v^{\#}(\neg_s A) = \mathbf{F}$ and
$v^{\#}(\neg_w A) = \mathbf{F}$ iff $v^{\#}(\neg_s A) = \mathbf{T}$;

Remark 7.1. Note that for $A \in \mathbf{S}_1^{\#}$ any admissible valuation has the same truth tables for both strong negation $\neg_s A$ and a weak negation $\neg_w A$. However for $A \in \mathbf{S}_1^{\#}$ any admissible valuation has the truth tables

Example 7.2. Nonclassical bivalent propositional language with a strong negation $\neg_s$ and
with just two atomic sentences, p and q. This language has just four admissible valuations,
which are partially depicted by the following truth table:

$$\begin{array}{ccccccc} * & p \in \mathbf{S}_1^{\#} & q \in \mathbf{S}_2^{\#} & \neg_s p & \neg_w p & \neg_s q & \neg_w q \\ v_1 & \mathbf{T} & \mathbf{T} & \mathbf{F} & \mathbf{F} & \mathbf{T} & \mathbf{F} \\ v_2 & \mathbf{T} & \mathbf{T} & & & & \\ v_3 & \mathbf{F} & \mathbf{F} & & & & \\ v_4 & \mathbf{F} & \mathbf{F} & & & & \end{array} \quad (7.2)$$

Definition 7.6. $\mathcal{L}_1^{\#}$ is a nonclassical bivalent propositional language with a strong negation $\neg_s A$ and a weak negation $\neg_w$, corresponding to praconsistent logic with one level of a contradictions iff its syntax is a **PNCLS** and its admissible valuations are the functions $v^{\#} : \mathbf{S}^{\#} \to \{\mathbf{T},\mathbf{F}\}$, where $\mathbf{S}^{\#} = \mathbf{S}_1^{\#} \cup \mathbf{S}_2^{\#\#} \cup \mathbf{S}_3^{\#}, \mathbf{S}_1^{\#} \cap \mathbf{S}_2^{\#} = \varnothing$ such that for all sentences
A, B of $\mathcal{L}_1^{\#}$:
(i) $v^{\#}(A) \in \{\mathbf{T},\mathbf{F}\}$;

(ii) for any $A \in \mathbf{S}^{\#}$: $v^{\#}(\neg_{\mathbf{s}}A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(iii) $v^{\#}(A \wedge B) = \mathbf{T}$ iff $v^{\#}(A) = v^{\#}(B) = \mathbf{T}$.
(iv) for any $A \in \mathbf{S}_1^{\#}$: $v^{\#}(\neg_w A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(v) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}(\neg_w A) = \mathbf{T}$ iff $\neg_w A \in \mathbf{S}_2^{\#}$ and $v^{\#}(A) = \mathbf{T}$;
(vi) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}(\neg_w A \wedge A) = \mathbf{T}$ iff $\neg_w A \in \mathbf{S}_2^{\#}$ and $v^{\#}(A) = \mathbf{T}$;
(vii) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}([(\neg_w A \wedge A) \wedge \neg_w(\neg_w A \wedge A)]) = \mathbf{F}$ and $v^{\#}(A) \in \{\mathbf{T},\mathbf{F}\}$;
Note that the property (vii) means that $\neg_w(\neg_w A \wedge A) \notin \mathbf{S}_2^{\#}$.

Definition 7.7.Abbreviation $\alpha^{\{k+1\}}$ stands for $(\alpha^{\{k\}})^{\{1\}} = \alpha^{\{k\}} \wedge \neg_w \alpha^{\{k\}}, 1 \leq k \leq n$, where $\alpha^{\{0\}} \triangleq \alpha, \alpha^{\{1\}} = (\alpha \wedge \neg_w \alpha)$.

Definition 7.8. $\mathcal{L}_n^{\#}$ is a nonclassical bivalent propositional language with a strong negation $\neg_{\mathbf{s}}A$ and a weak negation $\neg_w$, corresponding to praconsistent logic with n levels of a contradictions iff its syntax is a **PNCLS** and its admissible valuations are the
functions $v^{\#} : \mathbf{S}^{\#} \to \{\mathbf{T},\mathbf{F}\}$, where $\mathbf{S}^{\#} = \mathbf{S}_1^{\#} \cup \mathbf{S}_2^{\#}, \mathbf{S}_1^{\#} \cap \mathbf{S}_2^{\#} = \varnothing$ such that for all sentences A, B of $\mathcal{L}_n^{\#}$:
(i) $v^{\#}(A) \in \{\mathbf{T},\mathbf{F}\}$;
(ii) for any $A \in \mathbf{S}^{\#}$: $v^{\#}(\neg_{\mathbf{s}}A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(iii) $v^{\#}(A \wedge B) = \mathbf{T}$ iff $v^{\#}(A) = v^{\#}(B) = \mathbf{T}$.
(iv) for any $A \in \mathbf{S}_1^{\#}$: $v^{\#}(\neg_w A) = \mathbf{T}$ iff $v^{\#}(A) = \mathbf{F}$;
(v) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}(\neg_w A) = \mathbf{T}$ iff $\neg_w A \in \mathbf{S}_2^{\#}$ and $v^{\#}(A) = \mathbf{T}$;
(vi) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}(\neg_w A \wedge A) = \mathbf{T}$ iff $\neg_w A \in \mathbf{S}_2^{\#}$ and $v^{\#}(A) = \mathbf{T}$;
(vii) $A \in \mathbf{S}_2^{\#}$: $v^{\#}(A^{\{k\}}) = \mathbf{T}$ iff $\neg_w(A^{\{k\}}) \in \mathbf{S}_2^{\#}$ for any $k \leq n$ and $v^{\#}(A) = \mathbf{T}$;
(vii) for any $A \in \mathbf{S}_2^{\#}$: $v^{\#}(A^{\{n+1\}}) = \mathbf{F}$ and $v^{\#}(A) \in \{\mathbf{T},\mathbf{F}\}$;
Note that the property (vii) means that $\neg_w(A^{\{n\}}) \notin \mathbf{S}_2^{\#}$.

Remark 7.1.

The most important concept as in classical case is that of satisfaction.

Definition 7.9. A set X of sentences of $\mathcal{L}_n^{\#}$ is satisfied by an admissible valuation $v^{\#}$ of $\mathcal{L}_n^{\#}$ iff $v^{\#}(A) = \mathbf{T}$ for every $A \in X$. We shall also say "$v^{\#}$ satisfies A" when $v^{\#}$ satisfies $\{A\}$, and "X (respectively, A) is satisfiable (in $\mathcal{L}_n^{\#}$)" when some admissible valuation of $\mathcal{L}_n^{\#}$
satisfies X (respectively, A).

Definition 7.9. A is a valid sentence (in symbols, $\Vdash A$) in $\mathcal{L}_n^{\#}$ iff every admissible valuation of $\mathcal{L}_n^{\#}$ satisfies A.

Definition 7.10. X is an unassailable set of sentences of $\mathcal{L}_n^{\#}$ iff X is (a set of sentences of $\mathcal{L}_n^{\#}$) such that every admissible valuation of $\mathcal{L}^{\#}$ satisfies some member of X. Thus A is valid iff $\{A\}$ is unassailable; unassailability is a generalization of validity. Note that "X is unassailable" is not the same as "no admissible valuation assigns $\mathbf{F}$ to every member of X" unless all the admissible valuations are bivalent. (This is why we could not use "not falsifiable" instead of the contrived term "unassailable.")

Definition 7.11. X semantically entails A ($X \Vdash_n A$) in $\mathcal{L}_n^{\#}$ iff every admissible valuation of $\mathcal{L}^{\#}$ that satisfies X also satisfies A.

We write "$A \Vdash_n B$" for "$\{A\} \Vdash_n B$"; is called the (double) turnstile. It is fairly easy to see that A in $\mathcal{L}_n^{\#}$ if and only if $\varnothing \Vdash_n A$ in $\mathcal{L}_n^{\#}$, because all admissible valuations of $\mathcal{L}_n^{\#}$ satisfy all sentences in the empty set, vacuously.

Syntactic transformations may preserve certain semantic properties. We call a

mapping
f of sets of sentences to sentences truth-preserving in language $\mathcal{L}_n^\#$ when if v satisfies $X,$
then v satisfies $f(X)$ holds for all arguments X of f and all admissible valuations v of $\mathcal{L}_n^\#$.
Similarly, we say that f preserves validity in $\mathcal{L}_n^\#$ when the following property:

if A for all sentences A in $X,$ then $f(X),$ and if $B,$ then $f(B)$

holds for all arguments X, B of f and all admissible valuations v of $\mathcal{L}_n^\#$. The first part of the following theorem says that a truth-preserving transformation also preserves validity.

Theorem 7.1. (a) If $X \Vdash_n f(X)$ for every argument X of $f,$ then f preserves validity.
(i) If $A \in X,$ then $X \Vdash_n A.$
(ii) If $X \subseteq Y,$ and $X \Vdash_n A,$ then $Y \Vdash_n A.$
(iii) If $X \Vdash_n A$ for every $A \in Y,$ and $Y \Vdash_n B,$ then $X \Vdash_n B.$

Let $\mathcal{L}_n^\#$ be a language and $V\mathcal{L}_n^\#$ the set of its admissible valuations. We shall think of the members of $V\mathcal{L}_n^\#$ as the points in an abstract space, the “valuation space” of $\mathcal{L}_n^\#$.Regions in that space are just sets of these points, that is, subsets of $V\mathcal{L}_n^\#$. An important kind of region is that usually designated as “elementary class.”

Definition 7.12. If A is a sentence of $\mathcal{L}_n^\#$ and $V\mathcal{L}_n^\#$ the set of admissible valuations of $\mathcal{L}^\#,$
$H_n^\#(A) = \{v \in V\mathcal{L}_n^\# : v^\#(A) = \mathbf{T}\};$ and a set of $X \subseteq V\mathcal{L}_n^\#$ is an elementary class iff there is a sentence A such that $X = H_n^\#(A). H_n^\#(A)$ may be called the truth set of $A;$ if we were to discuss several languages at once, we would obviously use expressions such as “$H_n^\#(A)$
in $\mathcal{L}_n^\#$.”

Definition 7.13. The valuation space of $\mathcal{L}_n^\#$ is $H^\# =$
$\left\langle V\mathcal{L}^\#, \{H(A) : A \text{ a sentence of } \mathcal{L}^\#\} \right\rangle.$

Definition 7.14. We call the members of $V\mathcal{L}_n^\#$ the points in $H^\#,$ and write $x \in H^\#$ when x
is such a point, or $X^\# \subseteq H^\#$ when $X^\#$ is a class of such points (region). So the valuation space consists of a set of points, plus a family of regions that are singled out for special
consideration. These regions, which we call the elementary classes, are also called “arithmetical classes” or “axiomatic model classes.” Sometimes infinite intersections $H_n^\#(X^\#) = \bigcap_{A \in X} H_n^\#(A) =$ the set of all admissible valuations that satisfy $X$are also called
elementary classes. We shall accept this shorthand notation, but we shall not extend the
term “elementary class” in this way.Note that $H_n^\#(\varnothing) = H_n^\#$ by the above definition, and restricting the range of our variables to $H_n^\#$.

The basic semantic concepts are easily expressed in terms of the valuation space:
(i) A is a valid sentence iff $H_n^\#(A) = H_n^\#$.
(ii) X is unassailable iff $\bigcup_{A \in X} H_n^\#(A) = H_n^\#$.
(iii) X is satisfiable iff $\bigcap_{A \in X} H_n^\#(A) \neq \varnothing$.
(iv) B semantically entails A iff $H_n^\#(B) \subseteq H_n^\#(A)$.
(v) X semantically entails A iff $H_n^\#(X) \subseteq H_n^\#(A)$.

Let us take as an examples of the classical and nonclassical bivalent propositional languages with a strong negation $\neg_{\mathbf{s}}$ and with a weak negation $\neg_w$ and with just two atomic sentences,p and q.

Example 7.1.Classical bivalent propositional language with a strong negation $\neg_{\mathbf{s}}$ and with just two atomic sentences,p and q.

This language has just four admissible valuations, which are partially depicted by the following truth table:

$$\begin{array}{ccccccc} * & p & q & \neg_{\mathbf{s}}p & \neg_{\mathbf{s}}q & p\wedge\neg_{\mathbf{s}}q & p\wedge\neg_{\mathbf{s}}p \\ v_1 & \mathbf{T} & \mathbf{T} \\ v_2 & \mathbf{T} & \mathbf{F} \\ v_3 & \mathbf{F} & \mathbf{T} \\ v_4 & \mathbf{F} & \mathbf{F} \end{array} \quad (7.1)$$

Here $H(p) = \{v_1,v_2\};H(\neg_{\mathbf{s}}p) = \{v_3,v_4\};H(q) = \{v_1,v_3\};H(\neg_{\mathbf{s}}q) = \{v_2,v_4\}$ $H(p\wedge q) = \{v_1\};H(p\wedge\neg_{\mathbf{s}}p) = \varnothing$.

We also say that $H = \{v_1,v_2,v_3,v_4\}$ - although this is clearly an inaccurate way of speaking-hence H and $\varnothing$function as the universal and null set here. Note that just as $\varnothing$ is the elementary class defined by a contradiction, so H is the elementary class defined by a tautology.

8.First order paraconsistent propositional logic with zero levels of a contradictions $\mathbf{LP}_0^{\#}$.

The postulates of propositional paraconsistent logic $\mathbf{LP}_0^{\#} \triangleq \mathbf{LP}_0^{\#}[\Delta_1,\Delta_2]$ are the following. Let $\mathcal{L}_0^{\#} = \mathcal{L}_0^{\#}(\mathbf{LP}_0^{\#})$ be the formal language corresponding to logic $\mathbf{LP}_0^{\#}$ and let $\mathcal{F}_0^{\#}$ be the set of the all wff's of $\mathcal{L}_0^{\#}$,where

$$\Delta_1,\Delta_2 \subsetneqq \mathcal{F}_0^{\#}. \quad (8.1)$$

The language $\mathcal{L}_0^{\#}$ of paraconsistent logic $\mathbf{LP}_0^{\#} \triangleq \mathbf{LP}_0^{\#}[\Delta_1,\Delta_2]$ has as primitive symbols:
(i) countable set of propositional variables (formulas that are not analyzed at the propositional level);(ii) the connectives

$$\neg_w,\neg_{\mathbf{s}},\wedge,\vee,\rightarrow \quad (8.2)$$

and (iii) the parentheses (,).
Formulas are defined as follows: (i) any propositional variable is a formula;
(ii) if α and β are formulas, then

$$\alpha\rightarrow\beta,\alpha\wedge\beta,\alpha\vee\beta,\neg_{\mathbf{s}}\alpha,\neg_w\alpha \quad (8.3)$$

are formulas;
(iii) the only formulas are those obtained from the preceding conditions (i) and (ii).

Definition 8.1. $\alpha\leftrightarrow\beta \triangleq (\alpha\rightarrow\beta)\wedge(\beta\rightarrow\alpha)$.

Remark 8.1.Note that in logic $\mathbf{LP}_0^{\#}$ we distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$:
(i) a strong negation $\neg_s$ is a classical negation, i.e. $\neg_{\mathbf{s}}A$ meant $A\vdash B$ if $\neg_{\mathbf{s}}A$ holds;
(ii) a weak negation $\neg_w$ is a nonclassical negation, i.e. $\neg_wA$ meant $A\vdash B$ if and only if $B\notin\Delta_1$ despite the fact that $\neg_wA\wedge A$ holds and therefore $\neg_wA\wedge A \nvdash_{RMP} B$.

Remark 8.2.Note that in contrast with a classical negation $\neg_s A$ which always meant the
absolute impossibility of the statement A the nonclassical negation $\neg_w A$ does not always
meant the absolute impossibility of the statement A. Thus there exists a set Δ
such that for any $A \in \Delta$ the statement $A \wedge \neg_w A$ does not trivialize the system $\mathbf{LP}_0^{\#}$ but
however the statement $A \wedge \neg_w A$ is excluded by the law of excluded 4-th, see (8.4).

I. **Logical postulates**:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,

(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9) $\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,

(10) $\neg_{\mathbf{s}}\mathbf{A} \nrightarrow \neg_w \mathbf{A}$,

(11) $\neg_{\mathbf{s}}\neg_w \mathbf{A} \nrightarrow \mathbf{A}$,

(12) $\neg_w\neg_w \mathbf{A} \nrightarrow \mathbf{A}$,

(13) $\mathbf{A} \to (\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{B})$,

(14) $\mathbf{A} \to (\neg_w \mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \mathbf{\Delta}_1$,

where $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathcal{F}_0^{\#}$.

(15) **The law of excluded 5-th**

$$[\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w \mathbf{A} \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w \mathbf{A})] \wedge \neg_{\mathbf{s}}(\mathbf{A} \wedge \neg_w \mathbf{A}). \tag{8.4}$$

Remark 8.3.Note that (7.4) obviously means that $\mathbf{A} \wedge \neg_w \mathbf{A}$ is not holds in $\mathbf{LP}_0^{\#}$ since
by the Restricted Modus Ponens rule: $\mathbf{A} \wedge \neg_w \mathbf{A}, \neg_{\mathbf{s}}(\mathbf{A} \wedge \neg_w \mathbf{A}) \vdash_{RMP} \top$.

II.**Rules of a conclusion**:

Restricted Modus Ponens rule RMP:

(i) $\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \mathbf{\Delta}_2$.

(ii) $\mathbf{A}, \neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}_0^{\#}$.

Modus Tollens rule for a strong negation:

$\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

9.First order paraconsistent quantificational logic with zero levels of a contradiction $\overline{\mathbf{LP}}_0^{\#}$.

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_{\mathbf{0}}^{\#}[\bar{\Delta}_1, \bar{\Delta}_2]$ we
construct the corresponding paraconsistent relevant first-order predicate calculus
$\overline{\mathbf{LP}}_0^{\#} = \overline{\mathbf{LP}}_{\mathbf{0}}^{\#}[\bar{\Delta}_1, \bar{\Delta}_2]$. Let $\mathcal{L}_0^{\#} = \mathcal{L}_0^{\#}\left(\overline{\mathbf{LP}}_0^{\#}\right)$ be the formal language corresponding to logic
$\mathbf{LP}_0^{\#}$ and let $\overline{\mathcal{F}}_0^{\#}$ be the set of the all wff's of $\overline{\mathcal{L}}_0^{\#}$, where

$$\bar{\Delta}_1, \bar{\Delta}_2 \subsetneqq \overline{\mathcal{F}}_0^{\#}. \tag{9.1}$$

Remark 9.1.Note that in contrast with a set Δ_1 and a set Δ_2 the set $\bar{\Delta}_1$ and the set $\bar{\Delta}_2$
are recursively undecidable.

The language of the paraconsistent predicate calculus $\overline{\mathbf{LP}}_0^\#$, denoted above by $\overline{\mathcal{L}}_0^\#$, is an

extension of the language $\mathcal{L}_0^\#$ introduced above, by adding:

(i) for every $m \in \mathbb{N}$, denumerable families of m-ary consistent (or strong) predicate symbols $\Sigma_{con} = \bar{\Sigma} = \bar{R}_1^m, \bar{R}_2^m, \ldots, \bar{R}_n^m, \ldots$ and m-ary consistent function symbols $\bar{f}_1^m, \bar{f}_2^m, \ldots, \bar{f}_n^m, \ldots,$ which depend only on classical consistent object (or consistent set) variables;

(ii) for every $m \in \mathbb{N}$, denumerable families of m-ary inconsistent predicate symbols $\breve{R}_1^m, \breve{R}_2^m, \ldots, \breve{R}_n^m, \ldots,$ and m-ary inconsistent function symbols $\breve{f}_1^m, \breve{f}_2^m, \ldots, \breve{f}_n^m, \ldots,$ which depend only on non classical inconsistent object (or inconsistent set) variables;

(iii) for every $m, l \in \mathbb{N}$, denumerable families of $m_1 + m_2$-ary mixed predicate symbols $\hat{R}_1^{m_1+m_2}, \hat{R}_2^{m_1+m_2}, \ldots, \hat{R}_n^{m_1+m_2}, \ldots,$ and $m_1 + m_2$-ary mixed function symbols $\hat{f}_1^{m_1+m_2}, \hat{f}_2^{m_1+m_2}, \ldots, \hat{f}_n^{m_1+m_2}, \ldots,$ which depend on classical consistent object variables and on non classical inconsistent object (or inconsistent set) variables;

(iv) the universal $\forall$ and existential $\exists$ quantifiers.

We assume throughout that: the language $\overline{\mathcal{L}}_0^\#$ contains also

(i) the classical numerals $\bar{0}, \bar{1}, \ldots;$

(ii) countable set $\bar{\Gamma}$ of the classical consistent object (or consistent set) variables $\Gamma_{\mathbf{Con}} = \bar{\Gamma} = \{\bar{x}, \bar{y}, \bar{z}, \ldots\} = \{x_{con}, y_{con}, z_{con} \ldots\};$

(iii) countable set $\breve{\Gamma}$ of the non classical inconsistent object (or inconsistent set) variables $\Gamma_{\mathbf{Inc}} = \breve{\Gamma} = \{\breve{x}, \breve{y}, \breve{z}, \ldots\} = \{x_{inc}, y_{inc}, z_{inc} \ldots\};$

(iv) countable set $\bar{\Theta}$ of the classical non-logical constants $\bar{\Theta} = \{\bar{a}, \bar{b}, \bar{c} \ldots\};$

(v) countable set $\breve{\Theta}$ of the non classical non-logical constants $\breve{\Theta} = \left\{\breve{a}, \breve{b}, \breve{c} \ldots\right\};$

(vi)The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi.

The postulates of $\overline{\mathbf{LP}}_0^\#$ are those of $\mathbf{LP}_0^\#$ (suitably adapted), i.e.

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A}),$

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C})),$

(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B}),$

(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A},$

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B},$

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B}),$

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B}),$

(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C})),$

(9) $\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A},$

(10) $\neg_{\mathbf{s}}\mathbf{A} \nrightarrow \neg_w\mathbf{A},$

(11) $\neg_{\mathbf{s}}\neg_w\mathbf{A} \nrightarrow \mathbf{A},$

(12) $\neg_w\neg_w\mathbf{A} \nrightarrow \mathbf{A},$

(13) $\mathbf{A} \to (\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{B}),$

(14) $\mathbf{A} \to (\neg_w\mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \bar{\mathbf{\Delta}}_1,$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \overline{\mathcal{F}}_0^\#.$

(15) **The law of excluded 5-th**:

$$[\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A} \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w\mathbf{A})] \wedge \neg_{\mathbf{s}}(\mathbf{A} \wedge \neg_w\mathbf{A}). \quad (9.2)$$

Remark 9.2.Note that (8.2) obviously means that $\mathbf{A} \wedge \neg_w \mathbf{A}$ is not holds in $\overline{\mathbf{LP}}_0^{\#}$since by the Restricted Modus Ponens rule: $\mathbf{A} \wedge \neg_w \mathbf{A}, \neg_{\mathbf{s}}(\mathbf{A} \wedge \neg_w \mathbf{A}) \vdash_{RMP} \top$.

Plus the following:

(1a) $\dfrac{\alpha \to \beta(\bar{x})}{\alpha \to \forall\bar{x}\beta(\bar{x})}$, (1b) $\dfrac{\alpha \to \beta(\check{x})}{\alpha \to \forall\check{x}\beta(\check{x})}$, (1c) $\dfrac{\alpha \to \beta(\bar{x},\check{y})}{\alpha \to \forall\bar{x}\forall\check{y}\beta(\bar{x},\check{y})}$,

(2a) $\forall\bar{x}\alpha(\bar{x}) \to \alpha(\bar{y})$, (2b) $\forall\check{x}\alpha(\check{x}) \to \alpha(\check{y})$, (2c) $\forall\bar{x}\forall\check{y}\alpha(\bar{x},\check{y}) \to \alpha(\bar{x},\check{y})$,

(3a) $\alpha(\bar{x}) \to \exists\bar{x}\alpha(\bar{x})$, (3b) $\alpha(\check{x}) \to \exists\check{x}\alpha(\check{x})$, (3c) $\alpha(\bar{x},\check{y}) \to \exists\bar{x}\exists\check{y}\alpha(\bar{x},\check{y})$,

(4a) $\dfrac{\alpha(\bar{x}) \to \beta}{\exists\bar{x}\alpha(\bar{x}) \to \beta}$, (4b) $\dfrac{\alpha(\check{x}) \to \beta}{\exists\check{x}\alpha(\check{x}) \to \beta}$, (4c) $\dfrac{\alpha(\bar{x},\check{y}) \to \beta}{\exists\bar{x}\exists\check{y}\alpha(\bar{x},\check{y}) \to \beta}$,

(5a) $\forall\bar{x}[(\alpha(\bar{x}))] \to (\exists\bar{x}\alpha(\bar{x}))$, (5b) $\forall\check{x}[(\alpha(\check{x}))^{\{0\}}] \to (\forall\check{x}\alpha(\check{x}))^{\{0\}}$,

(5c) $\forall\bar{x}\forall\check{y}[(\alpha(\bar{x},\check{y}))^{\{0\}}] \to (\forall\bar{x}\forall\check{y}\alpha(\bar{x},\check{y}))^{\{0\}}$,

where we have used the following definition.

Definition 9.2.$\alpha^{\{0\}} \triangleq \alpha \wedge \neg_{\mathbf{s}}(\alpha \wedge \neg_w \alpha)$.

Where the variables $\bar{x},\check{x},\bar{y},\check{y}$ and the formulas α and β satisfy the usual definition.

From the calculi $\overline{\mathbf{LP}}_0^{\#}$,one can construct the following predicate calculus with equality. This is done by adding to their languages the binary predicates symbol of a strong (consistent) equality ($\bullet =_{\mathbf{s}} \bullet$) and a weak equality($\bullet =_w \bullet$) with suitable modifications in the concept of formula, and by adding the following postulates:

(1) $\forall\bar{x}(\bar{x} =_s \bar{x})$,

(2) $\forall\bar{x}\forall\bar{y}[\bar{x} =_s \bar{y} \to (\alpha(\bar{x}) \leftrightarrow \alpha(\bar{y}))]$,

(3) $\forall\bar{x}\forall\bar{y}\forall\bar{z}[(\bar{x} =_s \bar{y}) \wedge (\bar{y} =_s \bar{z}) \to \bar{x} =_s \bar{z}]$,

(4) $\forall\check{x}[(\check{x} =_w \check{x})^{\{0\}}]$,

(5) $\forall\check{x}\forall\check{y}\left[(\check{x} =_w \check{y})^{\{0\}} \to ((\alpha(\check{x}))^{\{0\}} \leftrightarrow (\alpha(\check{y}))^{\{0\}})\right]$,

(6) $\forall\check{x}\forall\check{y}\forall\check{z}[(\check{x} =_w \check{y}) \wedge (\check{y} =_w \check{z}) \to \check{x} =_w \check{z}]$,

(7) $\forall\bar{y}\exists\check{x}(\bar{y} =_w x)^{\{0\}}$,

(8) $\forall\check{y}\exists\check{x}(y =_w x)^{\{0\}}$,

(9) $\forall\check{x}\forall\check{y}[[(\check{x} =_w \check{y}) \vee \neg_{\mathbf{s}}(\check{x} =_w \check{y}) \vee \neg_w(\check{x} =_w \check{y})] \wedge (\check{x} =_w \check{y})^{\{0\}}]$,

(10) $\forall\check{x}\forall\check{y}\forall z[(\check{x} =_w \check{y}) \wedge (\check{y} =_w \check{z}) \to \check{x} =_w \check{z}]$

(11) $\forall\check{x}\forall\check{y}\forall z\left[(\check{x} =_w \check{y})^{\{0\}} \wedge (\check{y} =_w \check{z})^{\{0\}} \to (\check{x} =_w \check{z})^{\{0\}}\right]$.

II.**Rules of a conclusion**:

Restricted Modus Ponens rule **RMP** :

(i) $\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \overline{\mathbf{\Delta}}_2$.

(ii) $\mathbf{A}, \neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}_0^{\#}$.

Modus Tollens rule for a strong negation:

$\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

10.First order paraconsistent propositional logic with one level of a contradiction $\mathbf{LP}_1^{\#}$.

The postulates of propositional paraconsistent logic $\mathbf{LP}_1^{\#} \triangleq \mathbf{LP}_1^{\#}[\Delta_1, \Delta_2]$ are the following. Let $\mathcal{L}_1^{\#} = \mathcal{L}_1^{\#}(\mathbf{LP}_1^{\#})$ be the formal language corresponding to logic $\mathbf{LP}_1^{\#}$ and let $\mathcal{F}_1^{\#}$ be the set of the all wff's of $\mathcal{L}_1^{\#}$,where

$$\Delta_1, \Delta_2 \subsetneqq \mathcal{F}_1^{\#}. \tag{10.1}$$

The language $\mathcal{L}_1^{\#}$ of paraconsistent logic $\mathbf{LP}_1^{\#} \triangleq \mathbf{LP}_1^{\#}[\Delta_1, \Delta_2]$ has as primitive symbols:

(i) countable set of propositional variables (formulas that are not analyzed at the propositional level);(ii) the connectives

$$\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \rightarrow \tag{10.2}$$

and (iii) the parentheses (,).

Formulas are defined as follows: (i) any propositional variable is a formula;

(ii) if α and β are formulas, then

$$\alpha \rightarrow \beta, \alpha \wedge \beta, \alpha \vee \beta, \neg_{\mathbf{s}}\alpha, \neg_w\alpha \tag{10.3}$$

are formulas;

(iii) the only formulas are those obtained from the preceding conditions (i) and (ii).

Definition 10.1. $\alpha \leftrightarrow \beta \triangleq (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

Remark 10.1.Note that in logic $\mathbf{LP}_1^{\#}$ we distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$:

(i) a strong negation $\neg_s$ is a classical negation, i.e. $A \wedge \neg_s A \vdash B$

(ii) a weak negation $\neg_w$ is a nonclassical negation, i.e. $\neg_w A$ meant $A \nvdash_{RMP} B$ if $B \notin \Delta_1$.

Remark 10.2.Note that in contrast with a classical negation $\neg_s A$ which always meant the

absolute impossibility of the statement A the nonclassical negation $\neg_w A$ does not always

meant the absolute impossibility of the statement A.Thus there exists a set Δ such that for any $A \in \Delta$ the statement $A \wedge \neg_w A$ does not trivialize the system $\mathbf{LP}_1^{\#}$.

I. **Logical postulates**:

(1) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A})$,

(2) $(\mathbf{A} \rightarrow \mathbf{B}) \rightarrow ((\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow (\mathbf{A} \rightarrow \mathbf{C}))$,

(3) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B}$,

(6) $\mathbf{A} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \rightarrow \mathbf{C}) \rightarrow ((\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{A} \vee \mathbf{B} \rightarrow \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \rightarrow \mathbf{A}$,

(10) $\neg_{\mathbf{s}}\neg_w\mathbf{A} \rightarrow \mathbf{A}$,

(11) $\underline{\neg_w\neg_w\mathbf{A} \nrightarrow \mathbf{A}}$,

(12) $\mathbf{A} \rightarrow (\neg_s\mathbf{A} \rightarrow \mathbf{B})$,

(13) $\mathbf{A} \rightarrow (\neg_w\mathbf{A} \rightarrow \mathbf{B})$ if $\mathbf{B} \notin \mathbf{\Delta}_1$

(14) $\mathbf{A} \wedge \neg_w\mathbf{A}$ if $\mathbf{A} \in \mathbf{\Delta}_2$,

(15) **The law of excluded 8-th**

$$\begin{gathered}\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A} \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_{\mathbf{w}}\mathbf{A}) \vee (\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \neg_w(\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \\ \neg_w[(\mathbf{A} \wedge \neg_w\mathbf{A}) \wedge \neg_w(\mathbf{A} \wedge \neg_w\mathbf{A})] \wedge \\ \wedge\{\neg_{\mathbf{s}}[(\mathbf{A} \wedge \neg_w\mathbf{A}) \wedge \neg_w(\mathbf{A} \wedge \neg_w\mathbf{A})]\},\end{gathered} \tag{10.4}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{F}_1^{\#}$.

II.**Rules of a conclusion**:

Restricted Modus Ponens rule RMP:

(i)$\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \rightarrow \mathbf{B} \notin \mathbf{\Delta}_2$.

(ii) $\mathbf{A},\neg_{\mathbf{s}}\mathbf{A}\vdash_{RMP}\mathbf{B}\in\mathcal{F}_0^{\#}$.
Modus Tollens rules: $\mathbf{P}\to\mathbf{Q},\neg_{\mathbf{s}}\mathbf{Q}\vdash\neg_{\mathbf{s}}\mathbf{P}$.

11.First order paraconsistent quantificational logic with one level of a contradiction $\overline{\mathbf{LP}}_1^{\#}$.

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_{\mathbf{1}}^{\#}[\bar{\Delta}_1,\bar{\Delta}_2]$ we construct the corresponding paraconsistent relevant first-order predicate calculus $\overline{\mathbf{LP}}_1^{\#}=\overline{\mathbf{LP}}_{\mathbf{1}}^{\#}[\bar{\Delta}_1,\bar{\Delta}_2]$. Let $\mathcal{L}_1^{\#}=\mathcal{L}_1^{\#}\left(\overline{\mathbf{LP}}_1^{\#}\right)$ be the formal language corresponding to logic $\mathbf{LP}_1^{\#}$ and let $\overline{\mathcal{F}}_1^{\#}$ be the set of the all wff's of $\overline{\mathcal{L}}_1^{\#}$, where

$$\overline{\Delta}_1,\overline{\Delta}_2\subsetneqq\overline{\mathcal{F}}_1^{\#}. \tag{11.1}$$

Remark 10.1.Note that in contrast with a set Δ_1 and a set Δ_2 the set $\overline{\Delta}_1$ and the set $\overline{\Delta}_2$ are recursively undecidable.

The language of the paraconsistent predicate calculus $\overline{\mathbf{LP}}_1^{\#}$, denoted above by $\overline{\mathcal{L}}_1^{\#}$, is an
extension of the language $\mathcal{L}_1^{\#}$ introduced above, by adding:

(i) for every $m\in\mathbb{N}$, denumerable families of m-ary consistent (or strong) predicate symbols $\Sigma_{con}=\bar{\Sigma}=\bar{R}_1^m,\bar{R}_2^m,\ldots,\bar{R}_n^m,\ldots$ and m-ary consistent function symbols $\bar{f}_1^m,\bar{f}_2^m,\ldots,\bar{f}_n^m,\ldots$, which depend only on classical consistent object (or consistent set) variables;

(ii) for every $m\in\mathbb{N}$, denumerable families of m-ary inconsistent predicate symbols $\breve{R}_1^m,\breve{R}_2^m,\ldots,\breve{R}_n^m,\ldots$, and m-ary inconsistent function symbols $\breve{f}_1^m,\breve{f}_2^m,\ldots,\breve{f}_n^m,\ldots$, which depend only on non classical inconsistent object (or inconsistent set) variables;

(iii) for every $m,l\in\mathbb{N}$, denumerable families of m_1+m_2-ary mixed predicate symbols $\hat{R}_1^{m_1+m_2},\hat{R}_2^{m_1+m_2},\ldots,\hat{R}_n^{m_1+m_2},\ldots$, and m_1+m_2-ary mixed function symbols $\hat{f}_1^{m_1+m_2},\hat{f}_2^{m_1+m_2},\ldots,\hat{f}_n^{m_1+m_2},\ldots$, which depend on classical consistent object variables and on non classical inconsistent object (or inconsistent set) variables;

(iv) the universal $\forall$ and existential $\exists$ quantifiers.

We assume throughout that: the language $\overline{\mathcal{L}}_1^{\#}$ contains also

(i) the classical numerals $\bar{0},\bar{1},\ldots$;

(ii) countable set $\bar{\Gamma}$ of the classical consistent object (or consistent set) variables $\Gamma_{\mathbf{Con}}=\bar{\Gamma}=\{\bar{x},\bar{y},\bar{z},\ldots\}=\{x_{con},y_{con},z_{con}\ldots\}$;

(iii) countable set $\breve{\Gamma}$ of the non classical inconsistent object (or inconsistent set) variables $\Gamma_{\mathbf{Inc}}=\breve{\Gamma}=\{\breve{x},\breve{y},\breve{z},\ldots\}=\{x_{inc},y_{inc},z_{inc}\ldots\}$;

(iv) countable set $\bar{\Theta}$ of the classical non-logical constants $\bar{\Theta}=\{\bar{a},\bar{b},\bar{c}\ldots\}$;

(v) countable set $\breve{\Theta}$ of the non classical non-logical constants $\breve{\Theta}=\left\{\breve{a},\breve{b},\breve{c}\ldots\right\}$;

(vi)The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi.

The postulates of $\overline{\mathbf{LP}}_1^{\#}$ are those of $\mathbf{LP}_1^{\#}$ (suitably adapted), i.e.

(1) $\mathbf{A}\to(\mathbf{B}\to\mathbf{A})$,

(2) $(\mathbf{A}\to\mathbf{B})\to((\mathbf{A}\to(\mathbf{B}\to\mathbf{C}))\to(\mathbf{A}\to\mathbf{C}))$,

(3) $\mathbf{A}\to(\mathbf{B}\to\mathbf{A}\wedge\mathbf{B})$,

(4) $\mathbf{A}\wedge\mathbf{B}\to\mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,
(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,
(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,
(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,

(10) $\neg_{\mathbf{s}}\neg_{w}\mathbf{A} \to \mathbf{A}$,
(11) $\neg_{w}\neg_{w}\mathbf{A} \nrightarrow \mathbf{A}$,
(12) $\mathbf{A} \to (\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{B})$,
(13) $\mathbf{A} \to (\neg_{w}\mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \overline{\boldsymbol{\Delta}}_1$
(14) $\mathbf{A} \wedge \neg_{w}\mathbf{A}$ if $\mathbf{A} \in \overline{\boldsymbol{\Delta}}_2$,
(15) **The law of excluded 8-th**:

$$\begin{gathered}\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_{w}\mathbf{A} \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_{\mathbf{w}}\mathbf{A}) \vee (\mathbf{A} \wedge \neg_{w}\mathbf{A}) \vee \neg_{w}(\mathbf{A} \wedge \neg_{w}\mathbf{A}) \vee \\ \neg_{w}[(\mathbf{A} \wedge \neg_{w}\mathbf{A}) \wedge \neg_{w}(\mathbf{A} \wedge \neg_{w}\mathbf{A})] \wedge \\ \wedge\{\neg_{\mathbf{s}}[(\mathbf{A} \wedge \neg_{w}\mathbf{A}) \wedge \neg_{w}(\mathbf{A} \wedge \neg_{w}\mathbf{A})]\},\end{gathered} \tag{11.2}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \overline{\mathcal{F}}_1^{\#}$.
Plus the following:
(1a) $\dfrac{\alpha \to \beta(\bar{x})}{\alpha \to \forall \bar{x}\beta(\bar{x})}$, (1b) $\dfrac{\alpha \to \beta(\breve{x})}{\alpha \to \forall \breve{x}\beta(\breve{x})}$, (1c) $\dfrac{\alpha \to \beta(\bar{x},\breve{y})}{\alpha \to \forall \bar{x}\forall \breve{y}\beta(\bar{x},\breve{y})}$,
(2a) $\forall \bar{x}\alpha(\bar{x}) \to \alpha(\bar{y})$, (2b) $\forall \breve{x}\alpha(\breve{x}) \to \alpha(\breve{y})$, (2c) $\forall \bar{x}\forall \breve{y}\alpha(\bar{x},\breve{y}) \to \alpha(\bar{x},\breve{y})$,

(3a) $\alpha(\bar{x}) \to \exists \bar{x}\alpha(\bar{x})$, (3b) $\alpha(\breve{x}) \to \exists \breve{x}\alpha(\breve{x})$, (3c) $\alpha(\bar{x},\breve{y}) \to \exists \bar{x}\exists \breve{y}\alpha(\bar{x},\breve{y})$,
(4a) $\dfrac{\alpha(\bar{x}) \to \beta}{\exists \bar{x}\alpha(\bar{x}) \to \beta}$, (4b) $\dfrac{\alpha(\breve{x}) \to \beta}{\exists \breve{x}\alpha(\breve{x}) \to \beta}$, (4c) $\dfrac{\alpha(\bar{x},\breve{y}) \to \beta}{\exists \bar{x}\exists \breve{y}\alpha(\bar{x},\breve{y}) \to \beta}$,
(5a) $\forall \bar{x}[(\alpha(\bar{x}))^{[1]}] \to (\exists \bar{x}\alpha(\bar{x}))^{[1]}$, (5b) $\forall \breve{x}[(\alpha(\breve{x}))^{[1]}] \to (\forall \breve{x}\alpha(\breve{x}))^{[1]}$,
(5c) $\forall \bar{x}\forall \breve{y}[(\alpha(\bar{x},\breve{y}))^{[1]}] \to (\forall \bar{x}\forall \breve{y}\alpha(\bar{x},\breve{y}))^{[1]}$,
(6a) $[\forall \bar{x}((\alpha(\bar{x}))]^{\{1\}} \to (\forall \bar{x}\alpha(\bar{x})) \wedge (\exists \bar{x}\neg_{w}\alpha(\bar{x}))$,
(6b) $[\forall \breve{x}((\alpha(\breve{x}))]^{\{1\}} \to (\forall \breve{x}\alpha(\breve{x})) \wedge (\exists \breve{x}\neg_{w}\alpha(\breve{x}))$,
(6c) $[\forall \bar{x}\forall \breve{y}((\alpha(\bar{x},\breve{y}))]^{\{1\}} \to (\forall \bar{x}\forall \breve{y}\alpha(\bar{x},\breve{y})) \wedge (\exists \bar{x}\exists \breve{y}\neg_{w}\alpha(\bar{x},\breve{y}))$,
where we have used the following definition.

Definition 11.2.$\alpha^{\{0\}} \triangleq \alpha \wedge \neg_{\mathbf{s}}(\alpha \wedge \neg_{w}\alpha)$, $\alpha^{\{1\}} \triangleq \alpha \wedge \neg_{w}\alpha$ and $\alpha^{[1]} \triangleq \alpha^{\{0\}} \vee \alpha^{\{1\}}$.
Where the variables $\bar{x},\breve{x},\bar{y},\breve{y}$ and the formulas α and β satisfy the usual definition.

From the calculi $\overline{\mathbf{LP}}_{\mathbf{1}}^{\#}$,one can construct the following predicate calculus with equality. This is done by adding to their languages the binary predicates symbol of a strong (consistent) equality ($\cdot =_{\mathbf{s}} \cdot$) and a weak equality($\cdot =_{w} \cdot$) with suitable modifications in the concept of formula, and by adding the following postulates:
(1) $\forall \bar{x}(\bar{x} =_{s} \bar{x})$,
(2) $\forall \bar{x}\forall \bar{y}[\bar{x} =_{s} \bar{y} \to (\alpha(\bar{x}) \leftrightarrow \alpha(\bar{y}))]$,
(3) $\forall \bar{x}\forall \bar{y}\forall \bar{z}[(\bar{x} =_{s} \bar{y}) \wedge (\bar{y} =_{s} \bar{z}) \to \bar{x} =_{s} \bar{z}]$,
(4) $\forall \breve{x}[(\breve{x} =_{w} \breve{x})^{[1]}]$,
(5) $\forall \breve{x}\forall \breve{y}\left[(\breve{x} =_{w} \breve{y})^{[1]} \to ((\alpha(\breve{x}))^{[1]} \leftrightarrow (\alpha(\breve{y}))^{[1]})\right]$,
(6) $\forall \breve{x}\forall \breve{y}\forall \breve{z}[(\breve{x} =_{w} \breve{y}) \wedge (\breve{y} =_{w} \breve{z}) \to \breve{x} =_{w} \breve{z}]$,
(7) $\forall \bar{y}\exists \breve{x}(\bar{y} =_{w} x)$,

(8) $\forall\breve{y}\exists\breve{x}(y =_w x)^{\{1\}}$,

(9) $\forall\breve{x}\forall\breve{y}[(\breve{x} =_w \breve{y}) \vee \neg_s(\breve{x} =_w \breve{y}) \vee \neg_w(\breve{x} =_w \breve{y}) \vee (\breve{x} =_w \breve{y})^{[1]}]$,

(10) $\forall\breve{x}\forall\breve{y}\forall z[(\breve{x} =_w \breve{y}) \wedge (\breve{y} =_w \breve{z}) \to \breve{x} =_w \breve{z}]$

(11) $\forall\breve{x}\forall\breve{y}\forall z\left[(\breve{x} =_w \breve{y})^{[1]} \wedge (\breve{y} =_w \breve{z})^{[1]} \to (\breve{x} =_w \breve{z})^{[1]}\right]$.

II.**Rules of a conclusion**:

Restricted Modus Ponens rule $\mathbf{RMP}$:

(i) $\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \overline{\boldsymbol{\Delta}}_2$.

(ii) $\mathbf{A}, \neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}^{\#}_0$.

Remark 11.2.For example if we set $[(1 =_w y) \wedge (y =_w 0) \to 0 =_w 1] \in \overline{\boldsymbol{\Delta}}_2$

$(1 =_w y) \wedge (y =_w 0), (1 =_w y) \wedge (y =_w 0) \to 0 =_w 1 \nvdash_{\mathbf{RMP}} 0 =_w 1$.

Modus Tollens rules: $\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

Remark 11.3.Note that in contrast with classical rules of a conclusion the restricted modus Ponens rule $\mathbf{RMP}$ is not recursive rule of a conclusion, since the set $\overline{\Delta}_1$ and the set $\overline{\Delta}_2$ are recursively undecidable.

12. First order paraconsistent propositional logic with n levels of a contradiction $\mathbf{LP}^{\#}_n$.

The postulates of propositional paraconsistent logic $\mathbf{LP}^{\#}_n = \mathbf{LP}^{\#}_n\left[\boldsymbol{\Delta}^{\{n\}}_1, \boldsymbol{\Delta}^{\{n\}}_2\right]$ are the following.Let $\mathcal{L}^{\#}_n = \mathcal{L}^{\#}_n(\mathbf{LP}^{\#}_1)$ be the formal language corresponding to logic $\mathbf{LP}^{\#}_n$ and let $\mathcal{F}^{\#}_n$ be the set of the all wff's of $\mathcal{L}^{\#}_n$,where

$$\boldsymbol{\Delta}^{\{n\}}_1, \boldsymbol{\Delta}^{\{n\}}_2 \subsetneqq \mathcal{F}^{\#}_n. \tag{12.1}$$

The language $\mathcal{L}^{\#}_n$ of paraconsistent propositional logic $\mathbf{LP}^{\#}_n$ has as primitive symbols (i) countable set of a propositional variables, (ii) the connectives $\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \to$ and (iii) the parentheses (,).

$\mathbf{A}, \mathbf{B}, \mathbf{C}$, ... will be used as metalanguage variables which indicate formulas of $\mathbf{LP}^{\#}_n\left[\Delta^{\{n\}}_1, \Delta^{\{n\}}_2\right]$.

Definition 12.1. $\alpha^{\{0\}} \triangleq \alpha \wedge \neg_{\mathbf{s}}\alpha \wedge \neg_w\alpha, \alpha^{\{1\}} = (\alpha \wedge \neg_w\alpha)$.

$\alpha^{\{k+1\}}$ stands for $(\alpha^{\{k\}})^{\{1\}} = \alpha^{\{k\}} \wedge \neg_w\alpha^{\{k\}}, 1 \leq k \leq n$,

Definition 12.2. $\alpha^{[n]}$ stands for $\alpha^{[n]} = \alpha^{\{0\}} \vee \alpha^{\{1\}} \vee \ldots \vee \alpha^{\{n\}}, 1 \leq n$.

I. **Logical postulates**:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,

(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,

(10) $\neg_{\mathbf{s}}\neg_w\mathbf{A} \to \mathbf{A}$,

(11) $\neg_w\neg_w\mathbf{A} \nrightarrow \mathbf{A}$,

(12) $\mathbf{A} \to (\neg_s\mathbf{A} \to \mathbf{B})$,

(13) $\mathbf{A} \to (\neg_w \mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \Delta_1^{\{n\}}$,

(14) $\mathbf{A} \wedge \neg_w \mathbf{A}$ if $\mathbf{A} \in \Delta_2^{\{n\}}$,

(15) **The law of excluded** $(n+8)$**-th**:

$$\{(\mathbf{A} \vee \neg_{\mathbf{s}} \mathbf{A} \vee \neg_w \mathbf{A}) \vee (\neg_{\mathbf{s}} \mathbf{A} \wedge \neg_w \mathbf{A}) \vee (\mathbf{A} \wedge \neg_w \mathbf{A}) \vee \neg_w (\mathbf{A} \wedge \neg_w \mathbf{A}) \vee \mathbf{A}^{[2]} \vee \ldots \vee \mathbf{A}^{[k]} \vee \ldots \vee \mathbf{A}^{[n]}\} \wedge \neg_{\mathbf{s}} \mathbf{A}^{\{n+1\}}, \quad (12.2)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{F}_n^\#$.

II.Rules of a conclusion:

Restricted Modus Ponens rule RMP:

(i)$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \Delta_2^{\{n\}}$.

(ii) $\mathbf{A}, \neg_{\mathbf{s}} \mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}_0^\#$.

Modus Tollens rules: $\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}} \mathbf{Q} \vdash \neg_{\mathbf{s}} \mathbf{P}$.

13. First order paraconsistent quantificational logic with n levels of a contradiction $\overline{\mathbf{LP}}_n^\#$.

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_n^\#\left[\bar{\Delta}_1^{\{n\}}, \bar{\Delta}_2^{\{n\}}\right]$ we construct the corresponding paraconsistent relevant first-order predicate calculus $\overline{\mathbf{LP}}_n^\# = \mathbf{LP}_n^\#\left[\bar{\Delta}_1^{\{n\}}, \bar{\Delta}_2^{\{n\}}\right]$. Let $\mathcal{L}_n^\# = \mathcal{L}_n^\#\left(\overline{\mathbf{LP}}_n^\#\right)$ be the formal language corresponding to logic $\mathbf{LP}_n^\#$ and let $\overline{\mathcal{F}}_n^\#$ be the set of the all wff's of $\overline{\mathcal{L}}_n^\#$, where

$$\bar{\Delta}_1^{\{n\}}, \bar{\Delta}_2^{\{n\}} \subsetneqq \overline{\mathcal{F}}_n^\#. \quad (13.1)$$

Remark 13.1.Note that in contrast with a set $\Delta_1^{\{n\}}$ and a set $\Delta_2^{\{n\}}$ the set $\bar{\Delta}_1^{\{n\}}$ and the set $\bar{\Delta}_2^{\{n\}}$ are recursively undecidable.

The language of the paraconsistent predicate calculus $\overline{\mathbf{LP}}_n^\#$, denoted by $\overline{\mathcal{L}}_n^\#$, is an extension of the language $\mathcal{L}_n^\#$ introduced above, by adding:

(i) for every $m \in \mathbb{N}$, denumerable families of m-ary consistent (or strong) predicate symbols $\Sigma_{con} = \bar{\Sigma} = \bar{R}_1^m, \bar{R}_2^m, \ldots, \bar{R}_n^m, \ldots$ and m-ary consistent function symbols $\bar{f}_1^m, \bar{f}_2^m, \ldots, \bar{f}_n^m, \ldots$, which depend only on classical consistent object (or consistent set) variables;

(ii) for every $m \in \mathbb{N}$, denumerable families of m-ary inconsistent predicate symbols $\breve{R}_1^m, \breve{R}_2^m, \ldots, \breve{R}_n^m, \ldots$, and m-ary inconsistent function symbols $\breve{f}_1^m, \breve{f}_2^m, \ldots, \breve{f}_n^m, \ldots$, which depend only on non classical inconsistent object (or inconsistent set) variables;

(iii) for every $m, l \in \mathbb{N}$, denumerable families of $m_1 + m_2$-ary mixed predicate symbols $\hat{R}_1^{m_1+m_2}, \hat{R}_2^{m_1+m_2}, \ldots, \hat{R}_n^{m_1+m_2}, \ldots$, and $m_1 + m_2$-ary mixed function symbols $\hat{f}_1^{m_1+m_2}, \hat{f}_2^{m_1+m_2}, \ldots, \hat{f}_n^{m_1+m_2}, \ldots$, which depend on classical consistent object variables and on non classical inconsistent object (or inconsistent set) variables;

(iv) the universal $\forall$ and existential $\exists$ quantifiers.

We assume throughout that: the language $\overline{\mathcal{L}}_1^\#$ contains also

(i) the classical numerals $\bar{0}, \bar{1}, \ldots$;

(ii) countable set $\bar{\Gamma}$ of the classical consistent object (or consistent set) variables $\Gamma_{\mathbf{Con}} = \bar{\Gamma} = \{\bar{x}, \bar{y}, \bar{z}, \ldots\} = \{x_{con}, y_{con}, z_{con} \ldots\}$;

(iii) countable set $\breve{\Gamma}$ of the non classical inconsistent object (or inconsistent set) variables $\Gamma_{\mathbf{Inc}} = \breve{\Gamma} = \{\breve{x}, \breve{y}, \breve{z}, \ldots\} = \{x_{inc}, y_{inc}, z_{inc} \ldots\}$;

(iv) countable set $\bar{\Theta}$ of the classical non-logical constants $\bar{\Theta} = \{\bar{a},\bar{b},\bar{c}\ldots\}$;
(v) countable set $\breve{\Theta}$ of the non classical non-logical constants $\breve{\Theta} = \left\{\breve{a},\breve{b},\breve{c}\ldots\right\}$;
(vi)The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi.
The postulates of $\overline{\mathbf{LP}}_n^{\#}$ are those of $\mathbf{LP}_n^{\#}$ (suitably adapted), i.e.

I. Logical postulates:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,
(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,
(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,
(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,
(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,
(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,
(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,
(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,
(10) $\neg_{\mathbf{s}}\neg_{w}\mathbf{A} \to \mathbf{A}$,
(11) $\neg_{w}\neg_{w}\mathbf{A} \nrightarrow \mathbf{A}$,
(12) $\mathbf{A} \to (\neg_{s}\mathbf{A} \to \mathbf{B})$,
(13) $\mathbf{A} \to (\neg_{w}\mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \bar{\mathbf{\Delta}}_1^{\{n\}}$,
(14) $\mathbf{A} \wedge \neg_{w}\mathbf{A}$ if $\mathbf{A} \in \bar{\mathbf{\Delta}}_2^{\{n\}}$,
(15) **The law of excluded** $(n+8)$**-th**:

$$\{(\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_{w}\mathbf{A}) \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_{w}\mathbf{A}) \vee (\mathbf{A} \wedge \neg_{w}\mathbf{A}) \vee \neg_{w}(\mathbf{A} \wedge \neg_{w}\mathbf{A}) \vee \mathbf{A}^{[2]} \vee \ldots \vee \mathbf{A}^{[k]} \vee \ldots \vee \mathbf{A}^{[n]}\} \wedge \neg_{\mathbf{s}}\mathbf{A}^{\{n+1\}}, \tag{13.2}$$

where $\mathbf{A},\mathbf{B},\mathbf{C} \in \overline{\mathcal{F}}_n^{\#}$, plus the following:
(1a) $\dfrac{\alpha \to \beta(\bar{x})}{\alpha \to \forall\bar{x}\beta(\bar{x})}$, (1b) $\dfrac{\alpha \to \beta(\breve{x})}{\alpha \to \forall\breve{x}\beta(\breve{x})}$, (1c) $\dfrac{\alpha \to \beta(\bar{x},\breve{y})}{\alpha \to \forall\bar{x}\forall\breve{y}\beta(\bar{x},\breve{y})}$,
(2a) $\forall\bar{x}\alpha(\bar{x}) \to \alpha(\bar{y})$, (2b) $\forall\breve{x}\alpha(\breve{x}) \to \alpha(\breve{y})$, (2c) $\forall\bar{x}\forall\breve{y}\alpha(\bar{x},\breve{y}) \to \alpha(\bar{x},\breve{y})$,

(3a) $\alpha(\bar{x}) \to \exists\bar{x}\alpha(\bar{x})$, (3b) $\alpha(\breve{x}) \to \exists\breve{x}\alpha(\breve{x})$, (3c) $\alpha(\bar{x},\breve{y}) \to \exists\bar{x}\exists\breve{y}\alpha(\bar{x},\breve{y})$,
(4a) $\dfrac{\alpha(\bar{x}) \to \beta}{\exists\bar{x}\alpha(\bar{x}) \to \beta}$, (4b) $\dfrac{\alpha(\breve{x}) \to \beta}{\exists\breve{x}\alpha(\breve{x}) \to \beta}$, (4c) $\dfrac{\alpha(\bar{x},\breve{y}) \to \beta}{\exists\bar{x}\exists\breve{y}\alpha(\bar{x},\breve{y}) \to \beta}$,
(5a) $\forall\bar{x}[(\alpha(\bar{x}))^{[n]}] \to (\exists\bar{x}\alpha(\bar{x}))^{[n]}$, (5b) $\forall\breve{x}[(\alpha(\breve{x}))^{[n]}] \to (\forall\breve{x}\alpha(\breve{x}))^{[n]}$
(5c) $\forall\bar{x}\forall\breve{y}[(\alpha(\bar{x},\breve{y}))^{[n]}] \to (\forall\bar{x}\forall\breve{y}\alpha(\bar{x},\breve{y}))^{[n]}$,
(6a) $[\forall\bar{x}((\alpha(\bar{x}))]^{\{n\}} \to (\forall\bar{x}\alpha(\bar{x})) \wedge (\exists\bar{x}\neg_{w}\alpha(\bar{x}))$,
(6b) $[\forall\breve{x}((\alpha(\breve{x}))]^{\{n\}} \to (\forall\breve{x}\alpha(\breve{x})) \wedge (\exists\breve{x}\neg_{w}\alpha(\breve{x}))$,
(6c) $[\forall\bar{x}\forall\breve{y}((\alpha(\bar{x},\breve{y}))]^{\{n\}} \to (\forall\bar{x}\forall\breve{y}\alpha(\bar{x},\breve{y})) \wedge (\exists\bar{x}\exists\breve{y}\neg_{w}\alpha(\bar{x},\breve{y}))$,

II. Rules of a conclusion:

Restricted Modus Ponens rule *RMP*:
(i) $\mathbf{A},\mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \mathbf{\Delta}_2^{\{n\}}$.
(ii) $\mathbf{A},\neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP}\mathbf{B} \in \mathcal{F}_0^{\#}$.
Modus Tollens rules: $\mathbf{P} \to \mathbf{Q},\neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

III.**Inconsistent equality**

From the calculus $\overline{\mathbf{LP}}_n^{\#}$,we can construct the following predicate calculus with inconsistent equality.This is done by adding to their languages the binary predicates symbol of strong equality $(\bullet =_{\mathbf{s}} \bullet)$ and weak equality$(\bullet =_w \bullet)$ with suitable modifications in the concept of formula, and by adding the following postulates:

(1) $\forall\bar{x}(\bar{x} =_{\mathbf{s}} \bar{x})$,

(2) $\forall\bar{x}\Big[\ (\bar{x} =_{\mathbf{s}} \bar{x})^{\{1\}} \vdash \mathbf{B}\Big]$,

(3) $\forall\bar{x}\forall\bar{y}[\bar{x} =_{\mathbf{s}} \bar{y} \to (\alpha(\bar{x}) \leftrightarrow \alpha(\bar{y}))]$,

(4) $\forall\bar{x}\forall\bar{y}\forall\bar{z}[(\bar{x} =_s \bar{y}) \wedge (\bar{y} =_s \bar{z}) \to \bar{x} =_s \bar{z}]$,

(5) $\forall k(k \leq n)\exists\check{x}(\check{x} =_w \check{x})^{\{k\}}$,

(6) $\forall\check{y}\forall k(k \leq n)\exists\check{x}(\check{y} =_w \check{x})^{[k]}$,

(7) $\forall\check{x}\forall\check{y}\forall k(k \leq n)\Big[(\check{x} =_w \check{y})^{[k]} \to \forall\alpha(\circ)(\alpha^{[k]}(\check{x}) \leftrightarrow \alpha^{[k]}(\check{y}))\Big]$,

(8) $\forall\check{x}\forall\check{y}\forall k(k \leq n)\Big[(\check{x} =_w \check{y})^{\{k\}} \to \forall\alpha(\circ)(\alpha^{\{k\}}(\check{x}) \leftrightarrow \alpha^{\{k\}}(\check{y}))\Big]$,

(9) $\forall\check{x}\forall\check{y}\forall\check{z}\forall k(k \leq n)\Big[(\check{x} =_w \check{y})^{[k]} \wedge (\check{y} =_w z)^{[k]} \to (\check{x} =_w z)^{[k]}\Big]$,

(10) $\forall\check{x}\forall\check{y}\forall\check{z}\forall k(k \leq n)\Big[(\check{x} =_w \check{y})^{\{k\}} \wedge (\check{y} =_w z)^{\{k\}} \to (\check{x} =_w z)^{\{k\}}\Big]$.

14. First order paraconsistent propositional logic with countable levels of a contradictions $\mathbf{LP}_\omega^{\#}$.

The postulates (or their axioms schemata) of propositional paraconsistent logic $\mathbf{LP}_\omega^{\#} = \mathbf{LP}_\omega^{\#}[\Delta_1^\omega, \Delta_2^\omega]$ are the following. Let $\mathcal{L}_\omega^{\#} = \mathcal{L}_\omega^{\#}(\mathbf{LP}_\omega^{\#})$ be the formal language corresponding to logic $\mathbf{LP}_\omega^{\#}$ and let $\mathcal{F}_\omega^{\#}$ be the set of the all wff's of $\mathcal{L}_\omega^{\#}$,where

$$\Delta_1^\omega, \Delta_2^\omega \subsetneqq \mathcal{F}_\omega^{\#}. \qquad (14.1)$$

The language $\mathcal{L}_{\boldsymbol{\omega}}^{\#}$ of paraconsistent logic $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}$ has as primitive symbols (i) countable set

of a clalassical propositional variables, (ii) the connectives $\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \to$ and (iv) the parentheses (,), (iii) the letters $\mathbf{A}, \mathbf{B}, \mathbf{C}$, ... will be used as metalanguage variables which

indicate formulas of $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}[\Delta_1^\omega, \Delta_2^\omega]$.

Remark 14.1.We distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$.

The definition of formula is the usual. We denote the set of the all formulae of $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}[\Delta_1^\omega, \Delta_2^\omega]$ by $\mathcal{F}_{\boldsymbol{\omega}}^{\#}$ where Δ_1^ω and Δ_2^ω is a given by $\Delta_1^\omega = \bigcup_{n\in\mathbb{N}} \Delta_1^{\{n\}}, \Delta_2^\omega = \bigcup_{n\in\mathbb{N}} \Delta_2^{\{n\}}$.

I. **Logical postulates**:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,

(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,

(10) $\neg_{\mathbf{s}}\neg_w\mathbf{A} \to \mathbf{A}$,

(11) $\neg_w\neg_w\mathbf{A} \nrightarrow \mathbf{A}$,

(12) $\mathbf{A} \to (\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{B})$,

(13) $\mathbf{A} \to (\neg_{w}\mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \Delta_1^{\omega}$,

(14) $\mathbf{A} \wedge \neg_w \mathbf{A}$ if $\mathbf{A} \in \Delta_2^{\omega}$,

(15)**The law of non exclusion the conrradictions**

$$(\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A}) \vee (\mathbf{A} \wedge \neg_w\mathbf{A}) \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \mathbf{A}^{[2]} \vee \ldots \vee \mathbf{A}^{[k]} \vee \ldots \vee \mathbf{A}^{[n]} \vee \ldots \quad (14.2)$$

or

$$(\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A}) \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \bigvee_{1 \leq n < \omega} \mathbf{A}^{[n]}. \quad (14.3)$$

II.Rules of a conclusion:

Restricted Modus Ponens rule RMP:

(i)$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ **iff** $\mathbf{A} \notin \hat{\mathbf{V}}$.

(ii) $\mathbf{A}, \neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}_0^{\#}$.

Modus Tollens rule: $\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

15. First order paraconsistent quantificational logic with countable levels of a contradiction $\overline{\mathbf{LP}}_{\omega}^{\#}$.

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}[\Delta_1^{\omega}, \Delta_2^{\omega}]$ we construct the corresponding paraconsistent relevant first-order predicate calculus. These new calculus will be denoted by $\overline{\mathbf{LP}}_{\boldsymbol{\omega}}^{\#}[\bar{\Delta}_1^{\omega}, \bar{\Delta}_2^{\omega}]$. Let $\overline{\mathcal{L}}_{\omega}^{\#} = \mathcal{L}_{\omega}^{\#}\left(\overline{\mathbf{LP}}_{\omega}^{\#}\right)$ be the formal language corresponding to logic $\overline{\mathbf{LP}}_{\omega}^{\#}$ and let $\overline{\mathcal{F}}_{\omega}^{\#}$ be the set of the all wff's of $\overline{\mathcal{L}}_{\omega}^{\#}$, where

$$\bar{\Delta}_1^{\omega}, \bar{\Delta}_2^{\omega} \subsetneqq \overline{\mathcal{F}}_{\omega}^{\#}. \quad (15.1)$$

Remark 15.1.We distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$.

The definition of formula is the usual. We denote the set of the all formulae of $\overline{\mathbf{LP}}_{\boldsymbol{\omega}}^{\#}[\bar{\Delta}_1^{\omega}, \bar{\Delta}_2^{\omega}]$ by $\overline{\mathcal{F}}_{\boldsymbol{\omega}}^{\#}$ where $\bar{\Delta}_1^{\omega}$ and $\bar{\Delta}_2^{\omega}$ is a given by $\bar{\Delta}_1^{\omega} = \cup_{n \in \mathbb{N}} \bar{\Delta}_1^{\{n\}}, \bar{\Delta}_2^{\omega} = \cup_{n \in \mathbb{N}} \bar{\Delta}_2^{\{n\}}$.

The language $\overline{\mathcal{L}}_{\boldsymbol{\omega}}^{\#}$ of paraconsistent logic $\overline{\mathbf{LP}}_{\boldsymbol{\omega}}^{\#}$ has as primitive symbols (i) countable set of a clalassical propositional variables, (ii) the connectives $\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \to$ and (iv) the parentheses (,), (iii) the letters $\mathbf{A}, \mathbf{B}, \mathbf{C}$, ... will be used as metalanguage variables which indicate formulas of $\overline{\mathbf{LP}}_{\boldsymbol{\omega}}^{\#}[\bar{\Delta}_1^{\omega}, \bar{\Delta}_2^{\omega}]$.

Remark 15.2.Note that in contrast with a set Δ_1^{ω} and a set Δ_2^{ω} the set $\bar{\Delta}_1^{\{n\}}$ and the set $\bar{\Delta}_2^{\{n\}}$ are recursively undecidable.

The language of the paraconsistent predicate calculus $\overline{\mathbf{LP}}_{n}^{\#}$, denoted by $\overline{\mathcal{L}}_{n}^{\#}$, is an extension of the language $\mathcal{L}_n^{\#}$ introduced above, by adding:

(i) for every $m \in \mathbb{N}$, denumerable families of m-ary consistent (or strong) predicate symbols $\Sigma_{con} = \bar{\Sigma} = \bar{R}_1^m, \bar{R}_2^m, \ldots, \bar{R}_n^m, \ldots$ and m-ary consistent function symbols $\bar{f}_1^m, \bar{f}_2^m, \ldots, \bar{f}_n^m, \ldots$, which depend only on classical consistent object (or consistent set) variables;

(ii) for every $m \in \mathbb{N}$, denumerable families of m-ary inconsistent predicate symbols $\breve{R}_1^m, \breve{R}_2^m, \ldots, \breve{R}_n^m, \ldots$, and m-ary inconsistent function symbols $\breve{f}_1^m, \breve{f}_2^m, \ldots, \breve{f}_n^m, \ldots$, which depend only on non classical inconsistent object (or inconsistent set) variables;

(iii) for every $m, l \in \mathbb{N}$, denumerable families of $m_1 + m_2$-ary mixed predicate symbols

$\hat{R}_1^{m_1+m_2}, \hat{R}_2^{m_1+m_2}, \ldots, \hat{R}_n^{m_1+m_2}, \ldots,$ and $m_1 + m_2$-ary mixed function symbols $\hat{f}_1^{m_1+m_2}, \hat{f}_2^{m_1+m_2}, \ldots, \hat{f}_n^{m_1+m_2}, \ldots,$ which depend on classical consistent object variables and on non classical inconsistent object (or inconsistent set) variables;
(iv) the universal $\forall$ and existential $\exists$ quantifiers.
We assume throughout that: the language $\overline{\mathcal{L}}_\omega^\#$ contains also
(i) the classical numerals $\bar{0}, \bar{1}, \ldots;$
(ii) countable set $\bar{\Gamma}$ of the classical consistent object (or consistent set) variables $\Gamma_{\mathbf{Con}} = \bar{\Gamma} = \{\bar{x}, \bar{y}, \bar{z}, \ldots\} = \{x_{con}, y_{con}, z_{con} \ldots\};$
(iii) countable set $\breve{\Gamma}$ of the non classical inconsistent object (or inconsistent set) variables $\Gamma_{\mathbf{Inc}} = \breve{\Gamma} = \{\breve{x}, \breve{y}, \breve{z}, \ldots\} = \{x_{inc}, y_{inc}, z_{inc} \ldots\};$
(iv) countable set $\bar{\Theta}$ of the classical non-logical constants $\bar{\Theta} = \{\bar{a}, \bar{b}, \bar{c} \ldots\};$
(v) countable set $\breve{\Theta}$ of the non classical non-logical constants $\breve{\Theta} = \left\{\breve{a}, \breve{b}, \breve{c} \ldots\right\};$
(vi)The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi.
The postulates of $\overline{\mathbf{LP}}_n^\#$ are those of $\mathbf{LP}_n^\#$ (suitably adapted), i.e.

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,
(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,
(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,
(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,
(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,
(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,
(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9)$\neg_{\mathbf{s}}\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{A}$,
(10) $\neg_{\mathbf{s}}\neg_w\mathbf{A} \to \mathbf{A}$,
(11) $\neg_w\neg_w\mathbf{A} \nrightarrow \mathbf{A}$,
(12) $\mathbf{A} \to (\neg_{\mathbf{s}}\mathbf{A} \to \mathbf{B})$,
(13) $\mathbf{A} \to (\neg_w\mathbf{A} \to \mathbf{B})$ if $\mathbf{B} \notin \bar{\mathbf{\Delta}}_1^\omega$,
(14) $\mathbf{A} \wedge \neg_w\mathbf{A}$ if $\mathbf{A} \in \bar{\mathbf{\Delta}}_2^\omega$,
(15)**The law of non exclusion the contradictions**

$$(\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A}) \vee (\mathbf{A} \wedge \neg_w\mathbf{A}) \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \mathbf{A}^{[2]} \vee \ldots \vee \mathbf{A}^{[k]} \vee \ldots \vee \mathbf{A}^{[n]} \vee \ldots \quad (15.2)$$

or

$$(\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A} \vee \neg_w\mathbf{A}) \vee (\neg_{\mathbf{s}}\mathbf{A} \wedge \neg_w\mathbf{A}) \vee \bigvee_{1 \leq n < \omega} \mathbf{A}^{[n]}. \quad (15.3)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \overline{\mathcal{F}}_\omega^\#$, plus the following:

(1a) $\dfrac{\alpha \to \beta(\bar{x})}{\alpha \to \forall \bar{x}\beta(\bar{x})}$, (1b) $\dfrac{\alpha \to \beta(\breve{x})}{\alpha \to \forall \breve{x}\beta(\breve{x})}$, (1c) $\dfrac{\alpha \to \beta(\bar{x}, \breve{y})}{\alpha \to \forall \bar{x}\forall \breve{y}\beta(\bar{x}, \breve{y})}$,
(2a) $\forall \bar{x}\alpha(\bar{x}) \to \alpha(\bar{y})$, (2b) $\forall \breve{x}\alpha(\breve{x}) \to \alpha(\breve{y})$, (2c) $\forall \bar{x}\forall \breve{y}\alpha(\bar{x}, \breve{y}) \to \alpha(\bar{x}, \breve{y})$,

(3a) $\alpha(\bar{x}) \to \exists \bar{x}\alpha(\bar{x})$, (3b) $\alpha(\breve{x}) \to \exists \breve{x}\alpha(\breve{x})$, (3c) $\alpha(\bar{x}, \breve{y}) \to \exists \bar{x}\exists \breve{y}\alpha(\bar{x}, \breve{y})$, (4a) $\dfrac{\alpha(\bar{x}) \to \beta}{\exists\bar{x}\alpha(\bar{x}) \to \beta}$, (4b) $\dfrac{\alpha(\breve{x}) \to \beta}{\exists\breve{x}\alpha(\breve{x}) \to \beta}$, (4c) $\dfrac{\alpha(\bar{x},\breve{y}) \to \beta}{\exists\bar{x}\exists\breve{y}\alpha(\bar{x},\breve{y}) \to \beta}$,

(5a) $\forall\bar{x}[(\alpha(\bar{x}))^{[n]}] \to (\exists\bar{x}\alpha(\bar{x}))^{[n]}$, (5b) $\forall\breve{x}[(\alpha(\breve{x}))^{[n]}] \to (\forall\breve{x}\alpha(\breve{x}))^{[n]}$

(5c) $\forall\bar{x}\forall\breve{y}[(\alpha(\bar{x},\breve{y}))^{[n]}] \to (\forall\bar{x}\forall\breve{y}\alpha(\bar{x},\breve{y}))^{[n]}$,

(6a) $[\forall\bar{x}((\alpha(\bar{x}))]^{\{n\}} \to (\forall\bar{x}\alpha(\bar{x})) \wedge (\exists\bar{x}\neg_w\alpha(\bar{x}))$,

(6b) $[\forall\breve{x}((\alpha(\breve{x}))]^{\{n\}} \to (\forall\breve{x}\alpha(\breve{x})) \wedge (\exists\breve{x}\neg_w\alpha(\breve{x}))$,

(6c) $[\forall\bar{x}\forall\breve{y}((\alpha(\bar{x},\breve{y}))]^{\{n\}} \to (\forall\bar{x}\forall\breve{y}\alpha(\bar{x},\breve{y})) \wedge (\exists\bar{x}\exists\breve{y}\neg_w\alpha(\bar{x},\breve{y}))$,

II.**Rules of a conclusion**:

Restricted Modus Ponens rule *RMP*:

(i)$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash_{RMP} \mathbf{B}$ if $\mathbf{A} \to \mathbf{B} \notin \boldsymbol{\Delta}_2^{\{n\}}$.

(ii)$\mathbf{A}, \neg_{\mathbf{s}}\mathbf{A} \vdash_{RMP} \mathbf{B} \in \mathcal{F}_0^{\#}$.

Modus Tollens rules: $\mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}}\mathbf{Q} \vdash \neg_{\mathbf{s}}\mathbf{P}$.

III.**Inconsistent equality**

From the calculus $\overline{\mathbf{LP}}_n^{\#}$,we can construct the following predicate calculus with inconsistent equality.This is done by adding to their languages the binary predicates symbol of strong equality ($\cdot =_{\mathbf{s}} \cdot$) and weak equality($\cdot =_w \cdot$) with suitable modifications in the concept of formula, and by adding the following postulates:

(1) $\forall\bar{x}(\bar{x} =_{\mathbf{s}} \bar{x})$,

(2) $\forall\bar{x}\left[(\bar{x} =_{\mathbf{s}} \bar{x})^{\{1\}} \vdash \mathbf{B} \right]$,

(3) $\forall\bar{x}\forall\bar{y}[\bar{x} =_{\mathbf{s}} \bar{y} \to (\alpha(\bar{x}) \leftrightarrow \alpha(\bar{y}))]$,

(4) $\forall\bar{x}\forall\bar{y}\forall\bar{z}[(\bar{x} =_s \bar{y}) \wedge (\bar{y} =_s \bar{z}) \to \bar{x} =_s \bar{z}]$,

(5) $\forall k(k \leq n)\exists\breve{x}(\breve{x} =_w \breve{x})^{\{k\}}$,

(6) $\forall\breve{y}\forall k(k \leq n)\exists\breve{x}(\breve{y} =_w \breve{x})^{[k]}$,

(7) $\forall\breve{x}\forall\breve{y}\forall k(k \leq n)\left[(\breve{x} =_w \breve{y})^{[k]} \to \forall\alpha(\circ)(\alpha^{[k]}(\breve{x}) \leftrightarrow \alpha^{[k]}(\breve{y})) \right]$,

(8) $\forall\breve{x}\forall\breve{y}\forall k(k \leq n)\left[(\breve{x} =_w \breve{y})^{\{k\}} \to \forall\alpha(\circ)(\alpha^{\{k\}}(\breve{x}) \leftrightarrow \alpha^{\{k\}}(\breve{y})) \right]$,

(9) $\forall\breve{x}\forall\breve{y}\forall\breve{z}\forall k(k \leq n)\left[(\breve{x} =_w \breve{y})^{[k]} \wedge (\breve{y} =_w z)^{[k]} \to (\breve{x} =_w z)^{[k]} \right]$,

(10) $\forall\breve{x}\forall\breve{y}\forall\breve{z}\forall k(k \leq n)\left[(\breve{x} =_w \breve{y})^{\{k\}} \wedge (\breve{y} =_w z)^{\{k\}} \to (\breve{x} =_w z)^{\{k\}} \right]$.

16.Paraconsistent Set Theory $\mathbf{ZFC}_\omega^{\#}$.

In this section we distingvish: (i) classical von Neumann universe or von Neumann hierarchy of consistent sets, denoted $\mathbf{V}^{\mathbf{Con}}$, is the class of hereditary consistent well-founded sets. This consistent collection, which is formalized by Zermelo–Fraenkel set theory (ZFC), is often used to provide an interpretation or motivation of the axioms of *ZFC*;

(ii) nonclassical universe or hierarchy of inconsistent sets,denoted $\mathbf{V}^{\mathbf{Inc}}$.This inconsistent collection, which is formalized below by set theory $\mathbf{ZFC}_\omega^{\#}$.

The Axioms and Basic Properties of Inconsistent Sets.

Remark 16.**1**.In this section we distingvish:

(i) a classical consistent sets which is a members of von Neumann universe $\mathbf{V}^{\mathbf{Con}}$

(ii) a nonclassical inconsistent sets which is a members of non classical universe $\mathbf{V}^{\mathbf{Inc}}$.

(iii) a non classical mixed sets which is

Remark 16.**2**.In this section we distingvish:

(i) classical consistent set variables $\bar{x}, \bar{y}, \bar{z}, \ldots$;

(ii) non classical inconsistent set variables $\breve{x},\breve{y},\breve{z},\ldots$
(iii) a non classical mixed set variables $\hat{x},\hat{y},\hat{z},\ldots$.

Remark 16.3.In this section we distingvish:
(i) a strong membership predicate $(\bullet \bar{\in} \bullet)$ such that for any $\bar{x},\bar{y},\hat{z}$ only the following statements hold $\bar{x} \bar{\in} \bar{y},\bar{x} \bar{\in} \hat{z},$
(ii) a weak membership predicate $(\bullet \breve{\in} \bullet)$ such that for any $\breve{x},\breve{y},\hat{z}$ the following statements hold $\breve{x} \breve{\in} \breve{y},\breve{x} \breve{\in} \hat{z},\ldots,(\breve{x} \breve{\in} \breve{y})^{[n]},\ldots$

Designation 16.1.We denote: (i) a strong membership predicate $(\bullet \bar{\in} \bullet)$ by $\in_{\mathbf{s}}$,
(ii) a weak membership predicate $(\bullet \breve{\in} \bullet)$ by $\in_w$.

Definition 16.1. We shall say that:
(i) an well formed formula Φ of the set theory $\mathbf{ZFC}_{\boldsymbol{\omega}}^{\#}$ is a <u>classical</u> formula if formula Φ contains only consistent predicates $x =_{\mathbf{s}} y, x \in_{\mathbf{s}} y$ and contains only classical connectives $\neg_{\mathbf{s}},\wedge,\vee,\rightarrow$. We will be denoted such formula by $\bar{\Phi}$ or by Φ^s;
(ii) an well formed formula Φ of the set theory $\mathbf{ZFC}_{\boldsymbol{\omega}}^{\#}$ is a purely <u>non classical</u> formula if formula Φ contains only predicates $x =_w y, x \in_w y$ and only the following connectives $\neg_w,\wedge,\vee,\rightarrow$. We will be denoted such formula by $\breve{\Phi}$ or by Φ^w;
(iii) an well formed formula Φ of the set theory $\mathbf{ZFC}_{\boldsymbol{\omega}}^{\#}$ is a <u>mixed</u> formula if formula Φ contains predicates $x =_s y, x =_w y, x \in_s y, x \in_w y$ and the connectives $\neg_w,\neg_s \wedge,\vee,\rightarrow$. We will be denoted such formula by $\hat{\Phi}$ or by $\Phi^{s,w}$.

Abbreviation 15.1. Before introducing any set-theoretic axioms at all, we can introduce some important abbreviations. Let x,y and z be any classical sets, then
(i) $x \subseteq_s y$ abbreviates $\forall z(z \in_s x \rightarrow z \in_s y)$;
(ii) $x \subset_s y$ abbreviates $x \subseteq_s y \wedge x \neq_s y$;
(iii) $x \notin_s^s y$ abbreviates $\neg_s(x \in_s y)$;
(iv) $x \neq_s^s y$ abbreviates $\neg_s(x =_s y)$;
(v) $u =_s \ \cup_s x \triangleq \cup_s (x) \triangleq \forall z[z \in_s u \leftrightarrow (\exists y \in_s x)(z \in_s y)]$;
(vi) $u =_s \ \cap_s x \triangleq \cap_s (x) \triangleq \forall z[z \in_s u \leftrightarrow (\forall y \in_s x)(z \in_s y)]$;
(vii) $(\exists x \in_s y)\Phi$ abbreviates $\exists x(x \in_s y \wedge \Phi)$;
(viii) $(\forall x \in_s y)\Phi$ abbreviates $\forall x(x \in_s y \rightarrow \Phi)$;
(ix) $(\exists_s!x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \rightarrow x =_s y]$;
(x) $(\exists_w!x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \rightarrow x =_w y]$.

Abbreviation 15.2. (i) $\alpha^{\{k+1\}}$ stands for $(\alpha^{\{k\}})^{\{1\}} = \alpha^{\{n\}} \wedge \neg_w \alpha^{\{n\}}, 0 \leq k \leq n$ where $\alpha^{\{0\}} = \alpha \wedge \neg_s(\alpha \wedge \neg_w \alpha), \alpha^{\{1\}} = (\alpha \wedge \neg_w \alpha)$; (ii) $\alpha^{[n]}$ stands for

$$\alpha^{[n]} = \alpha^{\{0\}} \vee \alpha^{\{1\}} \vee \ldots \vee \alpha^{\{n\}}.$$

Abbreviation 15.3. For any terms $r,s,$ and $t,$ we make the following abbreviations of formulas.
(i) $(\forall x \in_s t)\Phi(x)$ for $\forall x(x \in_s t \rightarrow \Phi(x))$; (ii) $(\exists x \in_s t)\Phi(x)$ for $\exists x(x \in_s t \rightarrow \Phi(x))$;
(iii) $(\forall x \in_{w_{[n]}} t)\Phi(x)$ for $\forall x(x \in_{w_{[n]}} t \rightarrow \Phi(x))$; (iv) $(\exists x \in_{w_{[n]}} t)\Phi(x)$ for $\exists x(x \in_{w_{[n]}} t \rightarrow \Phi(x))$;
(v) $(\forall x \in_{w_{\{n\}}} t)\Phi(x)$ for $\forall x(x \in_{w_{\{n\}}} t \rightarrow \Phi(x))$; (vi) $(\exists x \in_{w_{\{n\}}} t)\Phi(x)$ for $\exists x(x \in_{w_{\{n\}}} t \rightarrow \Phi(x))$;

Abbreviation 15.4.We abbreviate:
(i) $x \in_{w_{\{n\}}} \breve{X}$ (or $x \in_{w\{n\}} \breve{X}$) instead $(x \in_w \breve{X})^{\{n\}}$;
(ii) $x \in_{w_{[n]}} \breve{X}$ (or $x \in_{w[n]} \breve{X}$) instead $(x \in_w \breve{X})^{[n]}$;

(iii) $x \in_{w_{\{\omega\}}} \breve{X}$ (or $x \in_{w\{\omega\}} \breve{X}$) instead $\bigwedge_{n\in\omega}(x \in_w \breve{X})^{\{n\}}$;

(iv) $x \in_{w_{[n]}} \breve{X}$ (or $x \in_{w[n]} \breve{X}$) instead $\bigvee_{n\in\omega}(x \in_w \breve{X})^{[n]}$.

Designation 15.2. We sometimes abbreviate

$$x \in_{w[n]} \breve{X} \text{ instead } x \in_{w_{[n]}} \breve{X}. \quad (15.1)$$

Definition 15.2. (i) If $x \in_w \breve{X}$ we call such x as w-element of the set $\breve{X}$.

(ii) If $x \in_{w_{\{n\}}} \breve{X}$ we call such x as $w_{\{n\}}$-element of the set $\breve{X}$.

(iii) If $x \in_{w_{[n]}} \breve{X}$ we call such x as $w_{[n]}$-element of the set $\breve{X}$.

(iv) If $x \in_{w_{\{n\}}} \hat{X}$ we call such x as $w_{\{n\}}$-element of the set $\hat{X}$.

(v) If $x \in_{w_{[n]}} \breve{X}$ we call such x as $w_{[n]}$-element of the set $\hat{X}$.

(vi) If $x \in_{w_{[0]}} \breve{X}$ we call such x as w-element of the set $\hat{X}$.

(vii) If $x \in_{w_{\{\omega\}}} \breve{X}$ we call such x as $w_{\{\omega\}}$-element of the set $\breve{X}$.

(viii) If $x \in_{w_{[\omega]}} \breve{X}$ we call such x as $w_{[\omega]}$-element of the set $\breve{X}$.

Designation 13.3.Let $\breve{x},\breve{y}$ and $\breve{z}$ be any nonclassical set, then

(i) $\breve{x} \neq^s_{w_{\{n\}}} \breve{y}$ abbreviates $\neg_s(\breve{x} =_{w_{\{n\}}} y)$;

(ii) $\breve{x} \subseteq_{w_{\{n\}}} y$ abbreviates $\forall z(z \in_{w_{\{n\}}} \breve{x} \to z \in_{w_{\{n\}}} y)$;

(iii) $\breve{x} \subset_{w_{\{n\}}} y$ abbreviates $\breve{x} \subseteq_{w_{\{n\}}} y \wedge \breve{x} \neq^s_{w_{\{n\}}} y$;

(iv) $x \notin^s_{w_{\{n\}}} y$ abbreviates $\neg_s(x \in_{w_{\{n\}}} y)$;

(v) $u =_{w_{\{n\}}} \cup_{w_{\{n\}}} x \triangleq_{w_{\{n\}}} \cup_{w_{\{n\}}} (x) \triangleq_{w_{\{n\}}} \forall z[z \in_{w_{\{n\}}} u \leftrightarrow (\exists y \in_{w_{\{n\}}} x)(z \in_{w_{\{n\}}} y)]$;

(vi) $u =_{w_{\{n\}}} \cap_{w_{\{n\}}} x \triangleq_{w_{\{n\}}} \cap_{w_{\{n\}}} (x) \triangleq_{w_{\{n\}}} \forall z[z \in_{w_{\{n\}}} u \leftrightarrow (\forall y \in_{w_{\{n\}}} x)(z \in_{w_{\{n\}}} y)]$;

(vii) $(\exists x \in_{w_{\{n\}}} y)\Phi$ abbreviates $\exists x(x \in_{w_{\{n\}}} y \wedge \Phi)$;

(viii) $(\forall x \in_{w_{\{n\}}} y)\Phi$ abbreviates $\forall x(x \in_{w_{\{n\}}} y \to \Phi)$;

(ix) $(\exists_{w_{\{n\}}} !x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \to x =_{w_{\{n\}}} y]$;

(x) $(\exists_w !x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \to x =_w y]$.

Designation 13.4.Let $\breve{x},\breve{y}$ and $\breve{z}$ be any nonclassical set, then

(i) $\breve{x} \neq^s_{w_{[n]}} \breve{y}$ abbreviates $\neg_s(\breve{x} =_{w_{[n]}} y)$;

(ii) $\breve{x} \subseteq_{w_{[n]}} y$ abbreviates $\forall z(z \in_{w_{[n]}} \breve{x} \to z \in_{w_{[n]}} y)$;

(iii) $\breve{x} \subset_{w_{[n]}} y$ abbreviates $\breve{x} \subseteq_{w_{[n]}} y \wedge \breve{x} \neq^s_{w_{[n]}} y$;

(iv) $x \notin^s_{w_{[n]}} y$ abbreviates $\neg_s(x \in_{w_{[n]}} y)$;

(v) $u =_{w_{[n]}} \cup_{w_{[n]}} x \triangleq_{w_{[n]}} \cup_{w_{[n]}} (x) \triangleq_{w_{[n]}} \forall z[z \in_{w_{[n]}} u \leftrightarrow (\exists y \in_{w_{[n]}} x)(z \in_{w_{[n]}} y)]$;

(vi) $u =_{w_{[n]}} \cap_{w_{[n]}} x \triangleq_{w_{[n]}} \cap_{w_{[n]}} (x) \triangleq_{w_{[n]}} \forall z[z \in_{w_{[n]}} u \leftrightarrow (\forall y \in_{w_{[n]}} x)(z \in_{w_{[n]}} y)]$;

(vii) $(\exists x \in_{w_{[n]}} y)\Phi$ abbreviates $\exists x(x \in_{w_{[n]}} y \wedge \Phi)$;

(viii) $(\forall x \in_{w_{[n]}} y)\Phi$ abbreviates $\forall x(x \in_{w_{[n]}} y \to \Phi)$;

(ix) $(\exists_{w_{[n]}} !x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \to x =_{w_{[n]}} y]$

Definition 13.3. (i) The set $\breve{X}$ is called $w_{\{n\}}$-set of the order inconsistency n, where $n = 0,1,\ldots$ if any w-element of the set $\breve{X}$ is a $w_{\{n\}}$-element of the set $\breve{X}$, i.e. $\forall x(x \in_w \breve{X} \Rightarrow x \in_{w_{\{n\}}} \breve{X})$ and there is no any $w_{\{n+1\}}$-element of the set $\breve{X}$.

(ii) The set $\breve{X}$ is called $w_{[n]}$-set of the order inconsistency n, where $n = 1,2,\ldots$ if there exists at least one $w_{\{n\}}$-element of the set $\breve{X}$ and there is no any $w_{\{n+1\}}$-element of the set $\breve{X}$.

(iii)The set $\hat{X}$ is called mixed $w_{\{n\}}$-set of the order inconsistency n, where $n = 0,1,\ldots$

if any w-element of the set $\hat{X}$ is a $w_{\{n\}}$-element of the set $\hat{X}$ and there is no any $w_{\{n+1\}}$-element of the set $\hat{X}$.

(iv)The set $\hat{X}$ is called mixed $w_{[n]}$-set of the order inconsistency n, where $n = 0, 1, \ldots$ if there exists at least one $w_{\{n\}}$-element of the set $\hat{X}$ and there is no any $w_{\{n+1\}}$-element of the set $\hat{X}$.

(vi) The set $\check{X}$ is called $w_{\{\omega\}}$-set of the order inconsistency ω if for any $n \in \mathbb{N}$ there exists at least one $w_{\{n\}}$-element of the set $\check{X}$.

(vii) The set $\check{X}$ is called $w_{[\omega]}$-set of the order inconsistency ω if for any $n \in \mathbb{N}$ there exists at least one $w_{[n]}$-element of the set $\check{X}$.

(viii) The set $\hat{X}$ is called mixed $w_{\{\omega\}}$-set of the order inconsistency ω if for any $n \in \mathbb{N}$ there exists at least one $w_{\{n\}}$-element of the set $\hat{X}$.

(ix)The set $\hat{X}$ is called mixed $w_{[\omega]}$-set of the order inconsistency ω if for any $n \in \mathbb{N}$ there exists at least one $w_{[n]}$-element of the set $\hat{X}$

Designation 13.5. (i) Let $\check{X}$ is a $w_{\{n\}}$-set of the order inconsistency n we denote such $w_{\{n\}}$-set $\check{X}$ by $\check{X}_{w_{\{n\}}}$ or $X_{w_{\{n\}}}$ or by $\check{X}_{w\{n\}}$.

(ii) Let $\check{X}$ is a $w_{[n]}$-set of the order inconsistency n we denote such $w_{[n]}$-set $\check{X}$ by $\check{X}_{w_{[n]}}$ or $X_{w_{[n]}}$ or by $\check{X}_{w[n]}$.

(iii) Let $\hat{X}$ is a mixed $w_{\{n\}}$-set of the order inconsistency n we denote such mixed $w_{\{n\}}$-set $\hat{X}$ by $\hat{X}_{w_{\{n\}}}$ or $X_{w_{\{n\}}}$ or by $\hat{X}_{w\{n\}}$.

(iv) Let $\hat{X}$ is a mixed $w_{[n]}$-set of the order inconsistency n we denote such mixed $w_{[n]}$-set $\check{X}$ by $\check{X}_{w_{[n]}}$ or $X_{w_{[n]}}$ or by $\check{X}_{w[n]}$.

(vi) Let $\check{X}$ is a $w_{\{\omega\}}$-set of the order inconsistency ω we denote such $w_{\{\omega\}}$-set $\check{X}$ by $\check{X}_{w_{\{\omega\}}}$ or $X_{w_{\{\omega\}}}$ or by $\check{X}_{w\{\omega\}}$.

(vii) Let $\check{X}$ is a $w_{[\omega]}$-set of the order inconsistency ω we denote such $w_{[\omega]}$-set $\check{X}$ by $\check{X}_{w_{[\omega]}}$ or $X_{w_{[\omega]}}$ or by $\check{X}_{w[\omega]}$.

(viii) Let $\hat{X}$ is a mixed $w_{\{\omega\}}$-set of the order inconsistency ω we denote such mixed $w_{\{\omega\}}$-set $\hat{X}$ by $\hat{X}_{w_{\{\omega\}}}$ or $X_{w_{\{\omega\}}}$ or by $\hat{X}_{w\{\omega\}}$.

(ix) Let $\hat{X}$ is a mixed $w_{[\omega]}$-set of the order inconsistency ω we denote such mixed $w_{[\omega]}$-set $\check{X}$ by $\check{X}_{w_{[\omega]}}$ or $X_{w_{[\omega]}}$ or by $\check{X}_{w[\omega]}$.

Definition 13.4. (i) The $w_{\{0\}}$-set $\check{X}$ of the order inconsistency zero is called $w_{\{0\}}$-set for short and we often denote such $w_{\{0\}}$-set $\check{X}$ by $\check{X}_{w_{\{0\}}}$ or simply by $X_{w_{\{0\}}}$.

Remark.13.1. Note that for any $w_{\{0\}}$-set $\check{X}$ the following statement holds by the non classical law of the excluded fourth (see sect.7,8)

$$\forall x\left[(x \in_w \check{X}) \vee \neg_{\mathbf{s}}(x \in_w \check{X}) \vee \neg_w(x \in_w \check{X})\right]. \tag{13.2}$$

It follows from (13.2) that the notion of the $w_{\{0\}}$-set is not equivalent the notion of the classical consistent sets, since for any classical set $\bar{X}$ the following statement holds by the

classical law of the excluded third

$$\forall x\left[(x \in_s \bar{X}) \vee \neg_{\mathbf{s}}(x \in_s \bar{X})\right]. \tag{13.3}$$

Definition 13.4. (i) The $w_{\{0\}}$-set $\check{X}_{w_{\{0\}}}$ is called <u>almost classical</u> $w_{\{0\}}$-set if only the following statement holds

$$\forall x\left[(x \in_w \check{X}_{w_{\{0\}}}) \vee \neg_{\mathbf{s}}(x \in_w \check{X}_{w_{\{0\}}})\right] \tag{13.4}$$

but not the full statement (3.1.2).

Remark 13.2. Note that the almost classical $w_{\{0\}}$-sets very similar to classical consistent sets, since the statement (3.1.4) says that the classical law of the excluded third holds.

Designation 13.6.(i) Let $\breve{X}_{w_{\{0\}}}$ is almost classical $w_{\{0\}}$-set then we denote such set by the symbol $\breve{X}_w^{\mathbf{cl}}$ or simply $X_w^{\mathbf{cl}}$. (ii) We shall often write

$$x \in_w^{\mathbf{cl}} \breve{X}_w^{\mathbf{cl}} \tag{13.5}$$

instead $x \in_w \breve{X}_w^{\mathbf{cl}}$ or simply $x \in_w X_w^{\mathbf{cl}}$ instead $x \in_w \breve{X}_w^{\mathbf{cl}}$.

Remark 13.3.Note that the almost classical w-sets look very similar to classical consistent sets, see Remark 13.2, however there exist fundamental differences in comparizon some properties of the almost classical w-sets with a properties of the classical consistent sets.These principal differences arises from the postulate of the Strong Separation 4.1.(i).

Designation 13.7.Let x,y and z be any almost classical w-set, then

(i) $x \neq_w^{\mathbf{cl}} y$ abbreviates $\neg_s(x =_w^{\mathbf{cl}} y)$;

(ii) $x \subseteq_w^{\mathbf{cl}} y$ abbreviates $\forall z(z \in_w^{\mathbf{cl}} x \to z \in_w^{\mathbf{cl}} y)$;

(iii) $x \subset_w^{\mathbf{cl}} y$ abbreviates $x \subseteq_w^{\mathbf{cl}} y \wedge x \neq_w^{\mathbf{cl}} y$;

(iv) $x \notin_w^{\mathbf{cl}} y$ abbreviates $\neg_s(x \in_w^{\mathbf{cl}} y)$;

(v) $u =_w^{\mathbf{cl}} \ \cup_w^{\mathbf{cl}} x \triangleq_w^{\mathbf{cl}} \ \cup_w^{\mathbf{cl}} (x) \triangleq_w^{\mathbf{cl}} \forall z[z \in_w^{\mathbf{cl}} u \leftrightarrow (\exists y \in_w^{\mathbf{cl}} x)(z \in_w^{\mathbf{cl}} y)]$;

(vi) $u =_w^{\mathbf{cl}} \ \cap_w^{\mathbf{cl}} x \triangleq_w^{\mathbf{cl}} \ \cap_w^{\mathbf{cl}} (x) \triangleq_w^{\mathbf{cl}} \forall z[z \in_w^{\mathbf{cl}} u \leftrightarrow (\forall y \in_w^{\mathbf{cl}} x)(z \in_w^{\mathbf{cl}} y)]$;

(vii) $(\exists x \in_w^{\mathbf{cl}} y)\Phi$ abbreviates $\exists x(x \in_w^{\mathbf{cl}} y \wedge \Phi)$;

(viii) $(\forall x \in_w^{\mathbf{cl}} y)\Phi$ abbreviates $\forall x(x \in_w^{\mathbf{cl}} y \to \Phi)$;

(ix) $(\exists_w^{\mathbf{cl}}!x)\Phi(x)$ abbreviates $(\exists x)\Phi(x) \wedge \forall x\forall y[\Phi(x) \wedge \Phi(y) \to x =_w^{\mathbf{cl}} y]$;

1.Axiom of Existence of the Universal set.

$$\exists \mathbf{V}^{\mathbf{Inc}} \forall \bar{x} \forall \breve{x} \exists n \forall \hat{x} \exists m \{(\bar{x} \in_s \mathbf{V}^{\mathbf{Inc}}) \wedge (\breve{x} \in_{w[n]} \mathbf{V}^{\mathbf{Inc}}) \wedge (\hat{x} \in_{w[m]} \mathbf{V}^{\mathbf{Inc}})\}. \tag{15.6}$$

2.Axioms of Existence of the empty set.

(i) There exists almost classical $w_{\{0\}}$- set $\emptyset_{w_{\{0\}}}^{\mathbf{cl}}$ which has no $\mathbf{s}$-elements and which has no w-elements in a strong consistent sense

$$\exists \emptyset_{w_{\{0\}}}^{\mathbf{cl}} \forall x\left[\neg_{\mathbf{s}}\left(x \in_{\mathbf{s}} \emptyset_{w_{\{0\}}}^{\mathbf{cl}}\right) \wedge \neg_{\mathbf{s}}\left(x \in_w \emptyset_{w_{\{0\}}}^{\mathbf{cl}}\right)\right]. \tag{13.7}$$

(ii) There exists a $w_{\{0\}}$-set $\emptyset_{w_{\{0\}}}$ which has no s-elements in a strong consistent sense and which has no w-elements in a weak sense

$$\exists \emptyset_{w_{\{0\}}} \forall x[\neg_{\mathbf{s}}(x \in_{\mathbf{s}} \emptyset_{w_{\{0\}}}) \wedge \neg_w(x \in_w \emptyset_{w_{\{0\}}})]. \tag{13.8}$$

3. Axioms of a Strong Extensionality.

3.1.(i) Let $\breve{X}_w^{\mathbf{cl}}$ and $\breve{Y}_w^{\mathbf{cl}}$ be any almost classical w-sets.

$$\forall \breve{X}_w^{\mathbf{cl}} \forall \breve{Y}_w^{\mathbf{cl}}[\breve{X}_w^{\mathbf{cl}} =_w \breve{Y}_w^{\mathbf{cl}} \leftrightarrow \forall \breve{x}(\breve{x} \in_w \breve{X}_w^{\mathbf{cl}} \leftrightarrow \breve{x} \in_w \breve{Y}_w^{\mathbf{cl}})]. \tag{13.9}$$

(ii) Let $\hat{X}_w^{\mathbf{cl}}$ and $\hat{Y}_w^{\mathbf{cl}}$ be any mixed almost classical w-sets.

$$\forall \hat{X}_w^{\mathbf{cl}} \forall \hat{Y}_w^{\mathbf{cl}} \Big[\hat{X}_w^{\mathbf{cl}} =_w \hat{Y}_w^{\mathbf{cl}} \leftrightarrow \Big[\forall \bar{x} \Big(\bar{x} \in_s \hat{X}_w^{\mathbf{cl}} \leftrightarrow \bar{x} \in_s \hat{Y}_w^{\mathbf{cl}} \Big) \Big] \wedge \searrow$$
$$\wedge \Big[\forall \breve{x} \Big(\breve{x} \in_w \hat{X}_w^{\mathbf{cl}} \leftrightarrow \breve{x} \in_w \hat{Y}_w^{\mathbf{cl}} \Big) \Big] \Big]. \quad (13.10)$$

3.2.(i) Let $\breve{X}_w$ and $\breve{Y}_w$ be any w-sets of the order inconsistency zero.

$$\forall \breve{X}_w \forall \breve{Y}_w [\breve{X}_w =_w \breve{Y}_w \leftrightarrow \forall \breve{x} (\breve{x} \in_w \breve{X}_w \leftrightarrow \breve{x} \in_w \breve{Y}_w)]. \quad (13.11)$$

(ii) Let $\hat{X}_w$ and $\hat{Y}_w$ be any mixed w-sets of the order inconsistency zero.

$$\forall \hat{X}_w \forall \hat{Y}_w \Big[\hat{X}_w =_w \hat{Y}_w \leftrightarrow \Big[\forall \bar{x} \Big(\bar{x} \in_s \hat{X}_w \leftrightarrow \bar{x} \in_s \hat{Y}_w \Big) \Big] \wedge \searrow$$
$$\wedge \Big[\forall \breve{x} \Big(\breve{x} \in_w \hat{X}_w \leftrightarrow \breve{x} \in_w \hat{Y}_w \Big) \Big] \Big]. \quad (13.12)$$

3.3.(i) Let $\breve{X}_{w\{n\}}$ and $\breve{Y}_{w\{n\}}$ be a $w_{\{n\}}$-sets of the order inconsistency n.

$$\forall \breve{X}_{w\{n\}} \forall \breve{Y}_{w\{n\}} [\breve{X}_{w\{n\}} =_{w\{n\}} \breve{Y}_{w\{n\}} \leftrightarrow \searrow$$
$$\forall \breve{x} \forall m (m \leq n) (\breve{x} \in_{w\{m\}} \breve{X}_{w\{n\}} \leftrightarrow \breve{x} \in_{w\{m\}} \breve{Y}_{w\{n\}})]. \quad (13.13)$$

(ii) Let $\breve{X}_{w[n]}$ and $\breve{Y}_{w[n]}$ be a $w_{[n]}$-sets of the order inconsistency n.

$$\forall \breve{X}_{w[n]} \forall \breve{Y}_{w[n]} [\breve{X}_{w[n]} =_{w[n]} \breve{Y}_{w[n]} \leftrightarrow$$
$$\forall \breve{x} \forall m (m \leq n) (\breve{x} \in_{w[m]} \breve{X}_{w[n]} \leftrightarrow \breve{x} \in_{w[m]} \breve{Y}_{w[n]})]. \quad (13.14)$$

3.4.(i) Let $\hat{X}_{w\{n\}}$ and $\hat{Y}_{w\{n\}}$ be a mixed $w_{\{n\}}$-sets of the order inconsistency n.

$$\forall \hat{X}_{w\{n\}} \forall \hat{Y}_{w\{n\}} \Big[\hat{X}_{w\{n\}} =_{w\{n\}} \hat{Y}_{w\{n\}} \leftrightarrow$$
$$\Big[\forall \bar{x} \Big(\bar{x} \in_s \hat{X}_{w\{n\}} \leftrightarrow \bar{x} \in_s \hat{Y}_{w\{n\}} \Big) \Big] \wedge \searrow$$
$$\Big[\forall \breve{x} \forall m (m \leq n) \Big(\breve{x} \in_{w\{m\}} \hat{X}_{w\{n\}} \leftrightarrow \breve{x} \in_{w\{m\}} \hat{Y}_{w\{n\}} \Big) \Big] \Big]. \quad (13.15)$$

.(ii) Let $\hat{X}_{w[n]}$ and $\hat{Y}_{w[n]}$ be a mixed $w_{[n]}$-sets of the order inconsistency n.

$$\forall \hat{X}_{w[n]} \forall \hat{Y}_{w[n]} \Big[\hat{X}_{w[n]} =_{w[n]} \hat{Y}_{w[n]} \leftrightarrow$$
$$\Big[\forall \bar{x} \Big(\bar{x} \in_s \hat{X}_{w[n]} \leftrightarrow \bar{x} \in_s \hat{Y}_{w[n]} \Big) \Big] \wedge \searrow$$
$$\Big[\forall \breve{x} \forall m (m \leq n) \Big(\breve{x} \in_{w[n]} \hat{X}_{w[n]} \leftrightarrow \breve{x} \in_{w[n]} \hat{Y}_{w[n]} \Big) \Big] \Big]. \quad (13.16)$$

3.5.(i) Let $\breve{X}_{w\{\omega\}}$ and $\breve{Y}_{w\{\omega\}}$ be a $w_{\{\omega\}}$-sets of the order inconsistency ω.

$$\forall \breve{X}_{w\{n\}} \forall \breve{Y}_{w\{n\}} [\breve{X}_{w\{\omega\}} =_{w\{\omega\}} \breve{Y}_{w\{\omega\}} \leftrightarrow \searrow$$
$$\forall \breve{x} \forall m (m < \omega) (\breve{x} \in_{w\{\omega\}} \breve{X}_{w\{\omega\}} \leftrightarrow \breve{x} \in_{w\{\omega\}} \breve{Y}_{w\{\omega\}})]. \quad (3.16)$$

(ii) Let $\breve{X}_{w[\omega]}$ and $\breve{Y}_{w[\omega]}$ be a $w_{[\omega]}$-sets of the order inconsistency ω.

$$\forall \breve{X}_{w[\omega]} \forall \breve{Y}_{w[\omega]} [\breve{X}_{w[n]} =_{w[\omega]} \breve{Y}_{w[\omega]} \leftrightarrow$$
$$\forall \breve{x} \forall m (m < \omega) (\breve{x} \in_{w[m]} \breve{X}_{w[\omega]} \leftrightarrow \breve{x} \in_{w[m]} \breve{Y}_{w[\omega]})]. \quad (13.17)$$

4. Axioms of a Weak Extensionality.

4.1.(i) Let $\breve{X}_{w_{[n]}}$ be any $w_{[n]}$-set of the order inconsistency $0 \leq n \leq \omega$ and $\breve{Y}_{w_{[m]}}$ be any inconsistent $w_{[m]}$-set of the order inconsistency $0 \leq m \leq \omega$.

$$\forall \breve{X}_{w[n]} \forall \breve{Y}_{w[n]} [\breve{X}_{w[n]} =_{w[k]} \breve{Y}_{w[m]} \leftrightarrow \forall \breve{x} \forall l (l \leq n) \forall r (r \leq m) (\breve{x} \in_{w[l]} \breve{X}_{w[n]} \leftrightarrow \breve{x} \in_{w[r]} \breve{Y}_{w[m]})]. \quad (13.18)$$

where $k = \min\{n,m\}$

(ii) Let $\breve{X}_{w\{n\}}$ be a $w_{\{n\}}$-set of the order inconsistency $0 \leq n \leq \omega$ and and $\breve{Y}_{w\{m\}}$ be a $w_{\{m\}}$-set of the order inconsistency $0 \leq m \leq \omega$.

$$\forall \breve{X}_{w\{n\}} \forall \breve{Y}_{w\{m\}} [\breve{X}_{w\{n\}} =_{w\{k\}} \breve{Y}_{w\{m\}} \leftrightarrow \searrow \forall \breve{x} \forall l (l \leq n) \forall r (r \leq m) (\breve{x} \in_{w\{l\}} \breve{X}_{w\{n\}} \leftrightarrow \breve{x} \in_{w\{r\}} \breve{Y}_{w\{m\}})]. \quad (13.19)$$

where $k = \min\{n,m\}$.

4.2.(i) Let $\hat{X}_{w\{n\}}$ be a mixed $w_{\{n\}}$-set of the order inconsistency $0 \leq n \leq \omega$ and $\hat{\breve{Y}}_{w\{m\}}$ be a mixed $w_{\{m\}}$-set of the order inconsistency $0 \leq m \leq \omega$.

$$\forall \hat{X}_{w\{n\}} \forall \hat{Y}_{w\{n\}} \left[\hat{X}_{w\{n\}} =_{w\{n\}} \hat{Y}_{w\{n\}} \leftrightarrow \left[\forall \bar{x} \left(\bar{x} \in_s \hat{X}_{w\{n\}} \leftrightarrow \bar{x} \in_s \hat{Y}_{w\{n\}} \right) \right] \wedge \searrow \left[\forall \breve{x} \forall l (l \leq n) \forall r (r \leq m) \left(\breve{x} \in_{w\{m\}} \hat{X}_{w\{n\}} \leftrightarrow \breve{x} \in_{w\{m\}} \hat{Y}_{w\{n\}} \right) \right] \right]. \quad (13.20)$$

where $k = \min\{n,m\}$.

(ii) Let $\hat{X}_{w\{n\}}$ be a mixed $w_{\{n\}}$-set of the order inconsistency $0 \leq n \leq \omega$ and $\hat{\breve{Y}}_{w\{m\}}$ be a mixed $w_{\{m\}}$-set of the order inconsistency $0 \leq m \leq \omega$.

$$\forall \hat{X}_{w\{n\}} \forall \hat{Y}_{w\{n\}} \left[\hat{X}_{w\{n\}} =_{w\{n\}} \hat{Y}_{w\{n\}} \leftrightarrow \left[\forall \bar{x} \left(\bar{x} \in_s \hat{X}_{w\{n\}} \leftrightarrow \bar{x} \in_s \hat{Y}_{w\{n\}} \right) \right] \wedge \searrow \left[\forall \breve{x} \forall l (l \leq n) \forall r (r \leq m) \left(\breve{x} \in_{w\{m\}} \hat{X}_{w\{n\}} \leftrightarrow \breve{x} \in_{w\{m\}} \hat{Y}_{w\{n\}} \right) \right] \right]. \quad (13.21)$$

where $k = \min\{n,m\}$.

5.Axioms of separation.

4.1 Strong Separation Schemes.

(i) Let $\phi(u,p_1,\ldots,p_k)$ be a formula free from symbols $\notin_w^{\mathbf{cl}}$ $\notin_w^s$, $\notin_w^s$, $\notin_w^w$.

For any almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ and almost classical w-sets $P_{1,w}^{\mathbf{cl}},\ldots,P_{k,w}^{\mathbf{cl}}$, there exists almost classical w-set $\breve{Y}_w^{\mathbf{cl}}$:

$$\breve{Y}_w^{\mathbf{cl}} =_w \{u \in_w \breve{X}_w^{\mathbf{cl}} | \phi(u, P_{1,w}^{\mathbf{cl}}, \ldots, P_{k,w}^{\mathbf{cl}})\}_w^{\mathbf{cl}}, \quad (13.22)$$

i.e.

$$\forall \breve{X}_w^{\mathbf{cl}} \forall P_{1,w}^{\mathbf{cl}} \ldots \forall P_{k,w}^{\mathbf{cl}} \exists \breve{Y}_w \forall u [u \in_w \breve{Y}_w^{\mathbf{cl}} \leftrightarrow (u \in_w \breve{X}_w^{\mathbf{cl}}) \wedge \phi(u, P_{1,w}^{\mathbf{cl}}, \ldots, P_{k,w}^{\mathbf{cl}})] \quad (13.23)$$

(ii) Let $\phi(u,p_1,\ldots,p_k)$ be a formula free from symbols $\notin_w^{\mathbf{cl}}$ $\notin_w^s$, $\notin_{w_{\{n\}}}^s$, $\notin_w^w$.

For any w-set $\breve{X}_w$ of the order inconsistency zero and w-sets $P_{1,w},\ldots,P_{k,w}$ of the order inconsistency zero there exists a w-set of the order inconsistency zero $\breve{Y}_w$:

$$\breve{Y}_w =_w \{u \in_w \breve{X}_w | \phi(u, P_{1,w}, \ldots, P_{k,w})\}_w, \quad (13.24)$$

i.e.

$$\forall \breve{X}_w \forall P_{1,w} \ldots \forall P_{k,w} \exists \breve{Y}_w \forall u [u \in_w \breve{Y}_w \leftrightarrow (u \in_w \breve{X}_w) \wedge \phi(u, P_{1,w}, \ldots, P_{k,w})] \quad (13.25)$$

(iii) Let $\phi(u,p_1,\ldots,p_k)$ be a formula free from symbols $\notin^s_{w_{[n]}}, \notin^s_{w_{\{n\}}}, \notin^w_{w_{[n]}}, \notin^w_{w_{\{n\}}}$
For any $w_{[n]}$-set $\breve{X}_{w_{[n]}}$ of the order inconsistency n and $w_{[n]}$-sets $P_{1,w_{[n]}},\ldots,P_{k,w_{[n]}}$ of the order inconsistency n there exists a $w_{[n]}$-set of the order inconsistency n, $\breve{Y}_{w_{[n]}}$:

$$\breve{Y}_{w_{[n]}} =_{w_{[n]}} \{u \in_{w_{[n]}} \breve{X}_{w_{[n]}} | \phi(u,P_{1,w_{[n]}},\ldots,P_{k,w_{[n]}})\}_{w_{[n]}} \quad (13.26)$$

i.e.

$$\begin{gathered}\forall \breve{X}_{w_{[n]}} \forall P_{1,w_{[n]}} \ldots \forall P_{k,w_{[n]}} \searrow \\ \exists \breve{Y}_{w_{[n]}} \forall u[u \in_{w_{[n]}} Y \leftrightarrow (u \in_{w_{[n]}} X) \wedge \phi(u,P_{1,w_{[n]}},\ldots,P_{k,w_{[n]}})]\end{gathered} \quad (13.27)$$

(iv) Let $\phi(u,p_1,\ldots,p_k)$ be a formula free from symbols $\notin^s_{w_{[n]}}, \notin^s_{w_{\{n\}}}, \notin^w_{w_{[n]}}, \notin^w_{w_{\{n\}}}$
For any X and $p_1,\ldots,p_k$, there exists a set $Y =_{w_{\{n\}}} \{u \in_{w_{\{n\}}} X | \phi(u,p_1,\ldots,p_k)\}_{w_{\{n\}}}$, i.e.

$$\forall X \forall p \exists Y \forall u[u \in_{w_{\{n\}}} Y \leftrightarrow (u \in_{w_{\{n\}}} X) \wedge \phi(u,p_1,\ldots,p_k)] \quad (13.28)$$

(**2**) **Weak Separation Schemes**.
(i) Let $\phi(u,p_1,\ldots,p_k)$ be a stratified formula. For any X and $p_1,\ldots,p_k$, there exists a set $Y =_w \{u \in_w X | \phi(u,p_1,\ldots,p_k)\}_w$, i.e.

$$\forall X \forall p \exists Y \forall u[u \in_w Y \leftrightarrow (u \in_w X) \wedge \phi(u,p_1,\ldots,p_k)] \quad (13.29)$$

(ii) Let $\phi(u,p_1,\ldots,p_k)$ be a stratified formula. For any X and $p_1,\ldots,p_k$, there exists a set $Y =_{w_{[n]}} \{u \in_{w_{[n]}} X | \phi(u,p_1,\ldots,p_k)\}_{w_{[n]}}$, i.e.

$$\forall X \forall p \exists Y \forall u[u \in_{w_{[n]}} Y \leftrightarrow (u \in_{w_{[n]}} X) \wedge \phi(u,p_1,\ldots,p_k)] \quad (13.30)$$

(iii) Let $\phi(u,p_1,\ldots,p_k)$ be a stratified stratified formula. For any X and $p_1,\ldots,p_k$, there exists a set $Y =_{w_{\{n\}}} \{u \in_{w_{\{n\}}} X | \phi(u,p_1,\ldots,p_k)\}_{w_{\{n\}}}$, i.e.

$$\forall X \forall p \exists Y \forall u[u \in_{w_{\{n\}}} Y \leftrightarrow (u \in_{w_{\{n\}}} X) \wedge \phi(u,p_1,\ldots,p_k)] \quad (13.31)$$

5.Axioms of Inconsistent Pairing.

5.1.Axiom of almost classical Pairing.
(i) For any almost classical w-sets $\breve{A}^{\mathbf{cl}}_w$ and $\breve{B}^{\mathbf{cl}}_w$, there exists w-set $\breve{C}^{\mathbf{cl}}_w$ such that $\breve{x} \in^{\mathbf{cl}}_w \breve{C}^{\mathbf{cl}}_w$ if and only if $\breve{x} =_w \breve{A}^{\mathbf{cl}}_w$ or $\breve{x} =_w \breve{B}^{\mathbf{cl}}_w$.
5.2.Axiom of mixed Pairing.
(i) For any $\bar{A}$ and $\breve{B}$, there exists mixed w-set $\hat{C}_{s,w}$ such that
$x \in_s \hat{C}_{s,w}$ if and only if $x =_s \bar{A}$ and $x \in_w \hat{C}_{s,w}$ if and only if $x =_w \breve{B}$.
(ii) For any $\bar{A}$ and $\breve{B}$, there exists mixed $w_{\{n\}}$-set $\hat{C}_{s,w\{n\}}$ such that $x \in_s \hat{C}_{s,w\{n\}}$ if and only if $x =_s \bar{A}$ and $x \in_{w_{\{n\}}} \hat{C}_{s,w\{n\}}$ if and only if $x =_{w_{\{n\}}} \breve{B}$.
(iii) For any $\bar{A}$ and $\breve{B}$, there exists mixed $w_{[n]}$-set $\hat{C}_{s,w[n]}$ such that $x \in_s \hat{C}_{s,w[n]}$ if and only if $x =_s \bar{A}$ and $x \in_{w_{[n]}} \hat{C}_{s,w[n]}$ if and only if $x =_{w_{[n]}} \breve{B}$.
5.3.Axiom of inconsistent Pairing.
(i) For any w-sets $\breve{A}_w$ and $\breve{B}_w$, there exists w-set $\breve{C}_w$ such that $\breve{x} \in_w \breve{C}_w$ if and only if $\breve{x} =_w \breve{A}_w$ or $\breve{x} =_w \breve{B}_w$.
(ii) For any $w_{\{n\}}$-sets $\breve{A}_{w_{\{n\}}}$ and $\breve{B}_{w_{\{n\}}}$, there exists $w_{\{n\}}$-set $\hat{C}_{w\{n\}}$ such that $x \in_{w\{n\}} \hat{C}_{w\{n\}}$ if and only if $x =_{w\{n\}} \breve{A}_{w\{n\}}$ and $x \in_{w_{\{n\}}} \hat{C}_{w\{n\}}$ if and only if $\breve{x} =_{w_{\{n\}}} \breve{A}_{w_{\{n\}}}$ $x =_{w_{\{n\}}} \breve{B}_{w\{n\}}$.
(iii) For any $w_{[n]}$-sets $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$, there exists $w_{[n]}$-set $\hat{C}_{w[n]}$ such that $x \in_{w[n]} \hat{C}_{w[n]}$ if and only if $x =_{w[n]} \breve{A}_{w[n]}$ and $x \in_{w_{[n]}} \hat{C}_{w[n]}$ if and only if $\breve{x} =_{w_{[n]}} \breve{A}_{w_{\{n\}}}$ $x =_{w_{[n]}} \breve{B}_{w[n]}$.

Definition 13.5.(i) We define the mixed unordered pair $\hat{C}_{s,w}$ of $\bar{A}$ and $\breve{B}_w$ as the sw-set having exactly $\bar{A}$ and $\breve{B}_w$ as its s-element and w-element correspondingly use $\{\bar{A},\breve{B}_w\}_{s,w}$ to denote it.

Definition 13.6.(i) We define the unordered w-pair of $\breve{A}_w$ and $\breve{B}_w$ as the w-set having exactly $\breve{A}_w$ and $\breve{B}_w$ as its w-elements and use $\{\breve{A}_w,\breve{B}_w\}_w$ to denote it.

(ii) We define the unordered $w_{\{n\}}$-pair of $\breve{A}_{w_{\{n\}}}$ and $\breve{B}_{w_{\{n\}}}$ as the $w_{\{n\}}$-set having exactly $\breve{A}_{w_{\{n\}}}$ and $\breve{B}_{w_{\{n\}}}$ as its $w_{\{n\}}$-elements and use $\{\breve{A}_{w_{\{n\}}},\breve{B}_{w_{\{n\}}}\}_{w_{\{n\}}}$ to denote it.

(iii) We define the unordered $w_{[n]}$-pair of $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$ as the $w_{[n]}$-set having exactly $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$ as its $w_{[n]}$-elements and use $\{\breve{A}_{w_{[n]}},\breve{B}_{w_{[n]}}\}_{w_{[n]}}$ to denote it.

6.Axioms of union.

6.1. Axiom of w-union of w-set $\breve{\mathcal{F}}$

For any w-set $\breve{\mathcal{F}}$, there exists w-set $\breve{A}$ such that $x \in_w \breve{A}$ if and only if $x \in_w \breve{Y}$ for some $\breve{Y} \in_w \breve{\mathcal{F}}$:

$$\forall\breve{\mathcal{F}}\exists\breve{A}\forall\breve{Y}\forall x\left[x \in_w \breve{Y} \wedge \breve{Y} \in_w \breve{\mathcal{F}} \Rightarrow x \in_w \breve{A}\right] \tag{13.32}$$

Definition 13.7. We call the w-set $\breve{A}$ the w-union of w-set $\breve{\mathcal{F}}$ and denote it by w-$\bigcup\breve{\mathcal{F}}$ or by $\cup_w \breve{\mathcal{F}}$

Definition 13.8 We call w-set $\breve{A}$ a w-subset of $\breve{B}$ if every w-element of $\breve{A}$ is also an w-element of $\breve{B}$: $\forall z[z \in_w \breve{A} \Rightarrow z \in_w \breve{B}]$. We denote this by $\breve{A} \subseteq_w \breve{B}$.

6.2.Axiom of $w_{[n]}$-union of $w_{[n]}$-set $x_{w_{[n]}}$.

For any $w_{[n]}$-set $x_{w_{[n]}}$ there exists $w_{[n]}$-set $y_{w_{[n]}}$ such that the following holds

$$\forall x_{w_{[n]}}\exists y_{w_{[n]}}\forall t[t \in_{w_{[n]}} y_{w_{[n]}} \leftrightarrow \exists u(u \in_{w_{[n]}} x \wedge t \in_{w_{[n]}} u)]. \tag{13.33}$$

The set $y_{w_{[n]}}$ is denoted $\cup_{w_{[n]}} x$ or $w_{[n]}$-$\cup x_{w_{[n]}}$.

6.3.Axiom of $w_{\{n\}}$-union of $w_{\{n\}}$-set $x_{w_{\{n\}}}$.

$$\forall x_{\{n\}}\exists y_{\{n\}}\forall t[t \in_{w_{\{n\}}} y_{\{n\}} \leftrightarrow \exists u(u \in_{w_{\{n\}}} x \wedge t \in_{w_{\{n\}}} u)]. \tag{13.34}$$

The set $y_{\{n\}}$ is denoted $\cup_{w_{\{n\}}} x$ or $w_{\{n\}}$-$\cup x$.

7.Axioms of Power Set.

(i) Axiom of w-power set.

$$\forall X_w\exists Y_w\forall t[t \in_w Y_w \leftrightarrow \forall z(z \in_w t \rightarrow z \in_w X_w)] \tag{13.35}$$

For any w-set X_w, a w-set Y_w is denoted $\mathbf{P}_w(X_w)$.

(ii) Axiom of $w_{[n]}$-power set.

$$\forall X_{w_{[n]}}\exists Y_{w_{[n]}}\forall t[t \in_{w_{[n]}} Y_{w_{[n]}} \leftrightarrow \forall z(z \in_{w_{[n]}} t \rightarrow z \in_{w_{[n]}} X_{w_{[n]}})] \tag{13.36}$$

For any $w_{[n]}$-set $X_{w_{[n]}}$, a $w_{[n]}$-set $Y_{w_{[n]}}$ is denoted $\mathbf{P}_{w_{[n]}}(X_{w_{[n]}})$.

(iv) Axiom of $w_{\{n\}}$-power set.

$$\forall X_{w_{\{n\}}}\exists Y_{w_{\{n\}}}\forall t[t \in_{w_{\{n\}}} Y_{w_{\{n\}}} \leftrightarrow \forall z(z \in_{w_{\{n\}}} t \rightarrow z \in_{w_{\{n\}}} X_{w_{\{n\}}})] \tag{13.37}$$

For any $w_{\{n\}}$-set X, a $w_{\{n\}}$-set $Y_{w_{\{n\}}}$ is denoted $\mathbf{P}_{w_{\{n\}}}(X_{w_{\{n\}}})$.

Definition 13.9. (i) We call $\mathbf{P}_w(X_w)$ the w-power set of X_w.

(ii) We call $\mathbf{P}_{w_{[n]}}(X_{w_{[n]}})$ the $w_{[n]}$-power set of $X_{w_{[n]}}$.

(iii) We call $\mathbf{P}_{w_{\{n\}}}(X_{w_{\{n\}}})$ the $w_{\{n\}}$-power set of $X_{w_{\{n\}}}$.
(iii) We call $\mathbf{P}_{w_{\{n\}}}(X_{w_{\{n\}}})$ the $w_{\{n\}}$-power set of $X_{w_{\{n\}}}$.

10. Axiom of Foundation (or Regularity)

10.1. Axiom of Foundation (or Regularity) to classical sets.

$$\forall \bar{x}[\bar{x} \neq_s \varnothing_{s,w} \to (\exists \bar{y} \in \bar{x})(\bar{x} \cap_s \bar{y} =_s \varnothing_{s,w})]. \tag{13.38}$$

Let's investigate what this axiom says: suppose there were a non-empty $\bar{x}$ such that $\forall$ y(y $\in_s$ x) (x $\cap_s$ y $\neq_s$ $\varnothing_{s,w}$). For any $z_1 \in$ x we would be able to get $z_2 \in z_1 \cap_s$x. Since $z_2 \in$ x we would be able to get $z_3 \in z_2 \cap_s$ x. The process continues forever:

11. Axiom of Foundation (or Regularity) for a mixed nonclassical sets.

$$\forall \bar{x}[\bar{x} \neq_s \varnothing_{s,w} \to (\exists \bar{y} \in \bar{x})(\bar{x} \cap_s \bar{y} =_s \varnothing_{s,w})]. \tag{13.39}$$

12. Axiom of regularity.

Definition 3.1.10. Almost classical w-set $\breve{x}_w^{\mathbf{cl}}$ is regular (or well founded) if the following the regularity condition holds

$$\forall \breve{x}_w^{\mathbf{cl}}[\breve{x}_w^{\mathbf{cl}} \neq_w^w \varnothing_w \to (\exists \breve{y}_w^{\mathbf{cl}} \in \breve{x}_w^{\mathbf{cl}})(\breve{x}_w^{\mathbf{cl}} \cap_w^{\mathbf{cl}} \breve{y}_w^{\mathbf{cl}} =_w \varnothing_w)]. \tag{13.40}$$

The regularity condition for almost classical w-set $\breve{x}_w^{\mathbf{cl}}$ is abbreviated as

$$\mathbf{reg}(\breve{x}_w^{\mathbf{cl}}). \tag{13.41}$$

11. Axioms of $w_{[n]}$-infinity.

11.1.1. Strong Axiom of almost classical regular w-infinity.

Let $\mathbf{S}_w^{\mathbf{cl}}(\breve{y}_w^{\mathbf{cl}})$ is abbreviated as $\breve{y}_w^{\mathbf{cl}} \cup_w^{\mathbf{cl}} \{\breve{y}_w^{\mathbf{cl}}\}_w^{\mathbf{cl}}$. There exists at least one almost classical w-family $\breve{X}_w^{\mathbf{cl}}$ of the almost classical w-regular w-sets such that the following condition holds.

$$\begin{gathered}\exists \breve{X}_w^{\mathbf{cl}}\Big\{\Big(\breve{\emptyset}_w \in_w \breve{X}_w^{\mathbf{cl}}\Big) \wedge \forall \breve{y}_w(\breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(\breve{y}_w) \in_w \breve{X}_w^{\mathbf{cl}}) \wedge\searrow \\ \wedge \forall \breve{y}_w[\breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}} \to \mathbf{reg}(\breve{x}_w^{\mathbf{cl}})]\Big\}.\end{gathered} \tag{3.1.31}$$

Theorem 3.1.1.(Finite or weak almost classical regular w-induction)
There exists w-unique almost classical w-family of the almost classical w-sets $\mathbb{N}_w^{\mathbf{cl}}$ such that

(i) $\breve{\emptyset}_w \in_w \mathbb{N}_w^{\mathbf{cl}}$
(ii) $x_w^{\mathbf{cl}} \in_w \mathbb{N}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(x_w^{\mathbf{cl}}) \in_w \mathbb{N}_w^{\mathbf{cl}}$
(iii) if $\breve{K}_w^{\mathbf{cl}}$ satisfies (i) and (ii), then $\mathbb{N}_w^{\mathbf{cl}} \subset_w \breve{K}_w^{\mathbf{cl}}$.

8.1.1. Weak Axiom of almost classical w-Infinity.

Let $\mathbf{S}_w^{\mathbf{cl}}(y_w^{\mathbf{cl}})$ is abbreviated as $y_w^{\mathbf{cl}} \cup_w^{\mathbf{cl}} \{y_w^{\mathbf{cl}}\}_w^{\mathbf{cl}}$. There exists at least one almost classical w-family $\breve{X}_w^{\mathbf{cl}}$ of the almost classical w-sets such that the following condition holds.

$$\exists \breve{X}_w^{\mathbf{cl}} \Big\{ \big(\breve{\emptyset}_w \in_w \breve{X}_w^{\mathbf{cl}} \big) \wedge \exists \breve{x}_w^{\mathbf{cl}} [(\breve{x}_w^{\mathbf{cl}} \in \breve{X}_w^{\mathbf{cl}}) \wedge \neg \mathbf{reg}(\breve{x}_w^{\mathbf{cl}})] \wedge \searrow$$
$$\forall \breve{y}_w (\breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(\breve{y}_w) \in_w \breve{X}_w^{\mathbf{cl}}) \wedge \searrow \quad (3.1.31)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w [(\breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}}) \wedge (\breve{z}_w \in_w \breve{X}_w^{\mathbf{cl}}) \to (\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w)] \Big\}.$$

Theorem 3.1.1.(**Finite or weak almost classical nonregular** w-**induction**)
There exists
w-unique almost classical w-family of the almost classical w-sets ${}^1\mathbb{N}_w^{\mathbf{cl}}$ such that

(i) $\breve{\emptyset}_w \in_w \mathbb{N}_w^{\mathbf{cl}}$

(ii) $x_w^{\mathbf{cl}} \in_w \mathbb{N}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(x_w^{\mathbf{cl}}) \in_w \mathbb{N}_w^{\mathbf{cl}}$

(iii) if $\breve{K}_w^{\mathbf{cl}}$ satisfies (i) and (ii), then $\mathbb{N}_w^{\mathbf{cl}} \subset_w \breve{K}_w^{\mathbf{cl}}$.

Proof. It follows from the weak axiom of almost classical w-infinity 8.1.1 that there exists at least one almost classical w-family $\breve{X}_w^{\mathbf{cl}}$ satisfying conditions (i) and (ii).
Let $\mathcal{F}_w^{\mathbf{cl}}$ be the almost classical w-family of all those w-subsets of $\breve{X}_w^{\mathbf{cl}}$ which satisfy (i) and (ii):

$$\mathcal{F}_w^{\mathbf{cl}} = \Big\{ \breve{S}_w^{\mathbf{cl}} \subset_w \breve{X}_w^{\mathbf{cl}} | \breve{\emptyset}_w \in_w \breve{S}_w^{\mathbf{cl}} \wedge \forall \breve{y}_w \big(\breve{y}_w \in_w \breve{S}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(\breve{y}_w) \in_w \breve{S}_w^{\mathbf{cl}} \big) \wedge \searrow \quad (3.1.32)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w \Big[\big(\breve{y}_w \in_w \breve{S}_w^{\mathbf{cl}} \big) \wedge \big(\breve{z}_w \in_w \breve{S}_w^{\mathbf{cl}} \big) \to (\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w) \Big] \Big\}.$$

It is easy to show that w-$\bigcap \mathcal{F}_w^{\mathbf{cl}}$ (see Definition 3.2.3 (i)) is the required almost classical w-family.

8.1.2.Strong Axiom of almost classical w-**Infinity**.
Let $\mathbf{S}_w^{\mathbf{cl}}(y)$ abbreviate $y \cup_w^{\mathbf{cl}} \{y\}_w^{\mathbf{cl}}$. Let $[\breve{y}_w]$ be almost classical w-set such that $\forall \breve{x}_w [\breve{x}_w \in_w [\breve{y}_w] \leftrightarrow \breve{x}_w \in_w \breve{y}_w]$. There exists at least one almost classical w-family $\breve{X}_w^{\mathbf{cl}}$ of the almost classical w-sets such that the following condition holds.

$$\exists \breve{X}_w^{\mathbf{cl}} \Big\{ \big(\breve{\emptyset}_{s,w} \in_w \breve{X}_w^{\mathbf{cl}} \big) \wedge \forall \breve{y}_w [[\breve{y}_w] \subset_w^{\mathbf{cl}} \breve{X}_w^{\mathbf{cl}} \to \breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}}] \wedge \searrow \quad (3.1.33)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w [(\breve{y}_w \in_w \breve{X}_w^{\mathbf{cl}}) \wedge (\breve{z}_w \in_w \breve{X}_w^{\mathbf{cl}}) \to (\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w)] \Big\}.$$

Theorem 3.1.2.(**Complete or strong almost classical** w-**induction**).There exists w-unique almost classical w-family of the almost classical w-sets $\mathbb{N}_w^{\mathbf{cl}}$ such that:

(i) $\breve{\emptyset}_{s,w} \in_w \mathbb{N}_w^{\mathbf{cl}}$

(ii) $[\breve{x}_w^{\mathbf{cl}}] \subset_w^{\mathbf{cl}} \mathbb{N}_w^{\mathbf{cl}} \to \breve{x}_w^{\mathbf{cl}} \in_w \mathbb{N}_w^{\mathbf{cl}}$

(iii) if $\breve{K}_w^{\mathbf{cl}}$ satisfies (i) and (ii), then $\mathbb{N}_w^{\mathbf{cl}} \subset_w \breve{K}_w^{\mathbf{cl}}$.

Proof. It follows from the weak axiom of almost classical w-infinity that there exists at least one almost classical w-family $\breve{X}_w^{\mathbf{cl}}$ satisfying conditions (i) and (ii). Let $\mathcal{F}_w^{\mathbf{cl}}$ be the almost classical w-family of all those w-subsets of $\breve{X}_w^{\mathbf{cl}}$ which satisfy (i) and (ii):

$$\mathcal{F}_w^{\mathbf{cl}} = \Big\{ \breve{S}_w^{\mathbf{cl}} \subset_w \breve{X}_w^{\mathbf{cl}} | \breve{\emptyset}_{s,w} \in_w \breve{S}_w^{\mathbf{cl}} \wedge \forall \breve{y}_w \big([\breve{y}_w] \subset_w^{\mathbf{cl}} \breve{S}_w^{\mathbf{cl}} \to \breve{y}_w \in_w \breve{S}_w^{\mathbf{cl}} \big) \wedge \searrow \quad (3.1.34)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w \Big[\big(\breve{y}_w \in_w \breve{S}_w^{\mathbf{cl}} \big) \wedge \big(\breve{z}_w \in_w \breve{S}_w^{\mathbf{cl}} \big) \to (\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w) \Big] \Big\}.$$

It is easy to show that w-$\bigcap \mathcal{F}_w^{\mathbf{cl}}$ (see Definition 3.2.3 (i)) is the required almost classical w-family.

8.2.1.Weak Axiom of w-**Infinity of the order inconsistency zero**.

(1) Let $\mathbf{S}_w(y)$ abbreviate $y \cup_w \{y\}_w$. There exists at least one w-family $\breve{X}_w$ of the order inconsistency zero such that the following conditions hold.

$$\exists \breve{X}_w \Big\{ \Big(\breve{\emptyset}_{s,w} \in_w \breve{X}_w\Big) \wedge \Big(\breve{\emptyset}_w \in_w \breve{X}_w\Big) \wedge \forall \breve{y}_w (\breve{y}_w \in_w \breve{X}_w \to \mathbf{S}_w(\breve{y}_w) \in_w \breve{X}) \wedge \searrow$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w [(\breve{y}_w \in_w \breve{X}_w) \wedge (\breve{z}_w \in_w \breve{X}_w) \to \searrow \tag{3.1.35}$$
$$(\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w) \vee \neg_w(\breve{y}_w \in_w \breve{z}_w) \vee \neg_w(\breve{z}_w \in_w \breve{y}_w)]\Big\}.$$

(2) Let $\mathcal{F}^*_w$ be any w-family of all those w-subsets $\breve{S}_w$ of $\breve{X}_w$ such that

$$\mathcal{F}^*_w =_w$$
$$\Big\{ \breve{S}_w \subset_w \breve{X}_w | \Big(\breve{\emptyset}_{s,w} \in_w \breve{S}_w\Big) \wedge \Big(\breve{\emptyset}_w \in_w \breve{S}_w\Big) \wedge \forall \breve{y}_w \Big(\breve{y}_w \in_w \breve{S}_w \to \mathbf{S}_w(\breve{y}_w) \in_w \breve{S}_w\Big)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w \Big[\Big(\breve{y}_w \in_w \breve{S}_w\Big) \wedge \Big(\breve{z}_w \in_w \breve{S}_w\Big) \to \searrow \tag{3.1.33}$$
$$(\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w) \vee \neg_w(\breve{y}_w \in_w \breve{z}_w) \vee \neg_w(\breve{z}_w \in_w \breve{y}_w)]\Big\}.$$

Then

$$\breve{X}^{\mathbf{cl}}_w \subsetneqq^s_w w\text{-}\bigcap \mathcal{F}^*_w, \tag{3.1.34}$$

see Definition 3.2.3 (ii).

Theorem 3.1.3.(**Finite or weak induction**)There exists w-unique w-family of the order inconsistency zero $\mathbb{N}_w$ of w-sets of the order inconsistency zero such that the following conditions hold.

(i) $\breve{\emptyset}_{s,w} \in_w \mathbb{N}_w, \breve{\emptyset}_w \in_w \mathbb{N}_w$

(ii) $x_w \in_w \mathbb{N}_w \to \mathbf{S}_w(x_w) \in_w \mathbb{N}_w$

(iii) $\mathbb{N}^{\mathbf{cl}}_w \subsetneqq^s_w \mathbb{N}_w$

(iv) if $\breve{K}_w$ satisfies (i) and (ii), then $\mathbb{N}_w \subset_w \breve{K}_w$.

Proof. It follows from the strong axiom of w-infinity that there exists at least one w-family of the order inconsistency zero $\breve{X}_w$ satisfying conditions (i),(ii) and (iii).
Let $\widetilde{\mathcal{F}}_w$ be the w-family of all those w-subsets of $\breve{X}_w$ which satisfy (i),(ii) and (iii):

$$\widetilde{\mathcal{F}}_w =_w$$
$$\Big\{ \breve{S}_w \subset_w \breve{X}_w | \Big(\breve{\emptyset}_{s,w} \in_w \breve{S}_w\Big) \wedge \Big(\breve{\emptyset}_w \in_w \breve{S}_w\Big) \wedge \forall \breve{y}_w \Big(\breve{y}_w \in_w \breve{S}_w \to \mathbf{S}_w(\breve{y}_w) \in_w \breve{S}_w\Big)$$
$$\wedge \forall \breve{y}_w \forall \breve{z}_w \Big[\Big(\breve{y}_w \in_w \breve{S}_w\Big) \wedge \Big(\breve{z}_w \in_w \breve{S}_w\Big) \to \searrow \tag{3.1.35}$$
$$(\breve{y}_w \in_w \breve{z}_w) \vee (\breve{z}_w \in_w \breve{y}_w) \vee \neg_w(\breve{y}_w \in_w \breve{z}_w) \vee \neg_w(\breve{z}_w \in_w \breve{y}_w)]\Big\}.$$

It is easy to show that $w\text{-}\bigcap \widetilde{\mathcal{F}}_w$ (see Definition 3.2.3 (ii)) is the required w-family.

Remark.**3**.**1**.**3**. Note that by 8.2.3 it follows $\mathbb{N}^{\mathbf{cl}}_w \subsetneqq^s_w \mathbb{N}_w$.

The next theorem scheme justifies strong mathematical induction. For brevity we shall write $\widehat{w}$ for $w_1, \ldots, w_n$.

Theorem 3.**1**.**3**.(Strong induction) For all $\widehat{w}$, if

$$\forall n (n \in_w \mathbb{N}_w) \Big[\forall m \in_w \mathbb{N}_w \Big[\Phi\Big(m, \widehat{w}\Big) \Rightarrow \Phi\Big(n, \widehat{w}\Big) \Big] \Big] \tag{13}$$

then

$$\forall n (n \in_w \mathbb{N}_w) \Big[\Phi\Big(n, \widehat{w}\Big) \Big]. \tag{13}$$

8.3.Axiom of $w_{[1]}$-Infinity.

Let $\mathbf{S}_{w_{[1]}}(\breve{y})$ abbreviate $\breve{y}_{w_{[1]}} \cup_{w_{[1]}} \{\breve{y}_{w_{[1]}}\}_{w_{[1]}}$.

$$\exists \breve{X}_{w_{[n]}} \left\{ \left(\breve{\emptyset}_{s,w} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \right) \wedge \left(\breve{\emptyset}_{w} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \right) \wedge \searrow \right. \\ \left. \wedge \forall \breve{y}_{w_{[n]}} (\breve{y}_{w_{[n]}} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \to \mathbf{S}_{w_{[n]}}(\breve{y}_{w_{[n]}}) \in_{w_{[n]}} \breve{X}_{w_{[n]}}) \right\}. \quad (3.1.36)$$

8.4.Axiom of $w_{[n]}$-Infinity.

Let $\mathbf{S}_{w_{[n]}}(\breve{y})$ abbreviate $\breve{y}_{w_{[n]}} \cup_{w_{[n]}} \{\breve{y}_{w_{[n]}}\}_{w_{[n]}}$.

$$\exists \breve{X}_{w_{[n]}} \left\{ \left(\breve{\emptyset}_{s,w} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \right) \wedge \left(\breve{\emptyset}_{w} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \right) \wedge \searrow \right. \\ \left. \wedge \forall \breve{y}_{w_{[n]}} (\breve{y}_{w_{[n]}} \in_{w_{[n]}} \breve{X}_{w_{[n]}} \to \mathbf{S}_{w_{[n]}}(\breve{y}_{w_{[n]}}) \in_{w_{[n]}} \breve{X}_{w_{[n]}}) \right\}. \quad (3.1.34)$$

Theorem 3.1.3.There exists exactly one $w_{[n]}$-family of $w_{[n]}$-sets $\mathbb{N}_{w_{[n]}}$ such that

(i) $\breve{\emptyset}_{s,w} \in_{w_{[n]}} \mathbb{N}_{w_{[n]}}, \breve{\emptyset}_{w} \in_{w_{[n]}} \mathbb{N}_{w_{[n]}}$

(ii) $x_{w_{[n]}} \in_{w_{[n]}} \mathbb{N}_{w_{[n]}} \to \mathbf{S}_{w_{[n]}}(x_{w_{[n]}}) \in_{w_{[n]}} \mathbb{N}_{w_{[n]}}$

(iii) if $\breve{K}_{w_{[n]}}$ satisfies (i) and (ii), then $\mathbb{N}_{w_{[n]}} \subset_{w_{[n]}} \breve{K}_{w_{[n]}}$.

Proof. It follows from the strong axiom of $w_{[n]}$-infinity that there exists at least one $w_{[n]}$-family $\breve{X}_{w_{[n]}}$ satisfying conditions (i) and (ii). Let $\mathcal{F}_{w_{[n]}}$ be the $w_{[n]}$-family of all those $w_{[n]}$-subsets of $\breve{X}_{w_{[n]}}$ which satisfy (i) and (ii):

$\mathcal{F}_{w_{[n]}} =_{w_{[n]}} \left\{ \breve{S}_{w_{[n]}} \subset_{w} \breve{X}_{w_{[n]}} \middle| \left(\breve{\emptyset}_{s,w} \in_{w} \breve{S}_{w_{[n]}} \right) \wedge \left(\breve{\emptyset}_{w} \in_{w_{[n]}} \breve{S}_{w_{[n]}} \right) \wedge \searrow \right.$
$\left. \wedge \forall \breve{y}_{w_{[n]}} \left(\breve{y}_{w_{[n]}} \in_{w_{[n]}} \breve{S}_{w_{[n]}} \to \mathbf{S}_{w_{[n]}}(\breve{y}_{w_{[n]}}) \in_{w_{[n]}} \breve{S}_{w_{[n]}} \right) \right\}$.

It is easy to show that w-$\bigcap \mathcal{F}_{w_{[n]}}$ (see Definition 3.2.3 (iii)) is the required $w_{[n]}$-family.

8.4.Axiom of $w_{\{n\}}$-Infinity.

Let $\mathbf{S}_{w_{\{n\}}}(\breve{y})$ abbreviate $\breve{y}_{w_{\{n\}}} \cup_{w_{\{n\}}} \{\breve{y}_{w_{\{n\}}}\}_{w_{\{n\}}}$.

$$\exists \breve{X}_{w_{\{n\}}} \left\{ \left(\breve{\emptyset}_{s,w} \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}} \right) \wedge \left(\breve{\emptyset}_{w} \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}} \right) \wedge \searrow \right. \\ \left. \wedge \forall \breve{y}_{w_{\{n\}}} (\breve{y}_{w_{\{n\}}} \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}} \to \mathbf{S}_{w_{\{n\}}}(\breve{y}_{w_{\{n\}}}) \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}}) \right\}. \quad (3.1.35)$$

Theorem 3.1.4.There exists exactly one $w_{\{n\}}$-family of $w_{[n]}$-sets $\mathbb{N}_{w_{\{n\}}}$ such that

(i) $\breve{\emptyset}_{s,w} \in_{w_{\{n\}}} \mathbb{N}_{w}, \breve{\emptyset}_{w} \in_{w_{\{n\}}} \mathbb{N}_{w_{\{n\}}}$

(ii) $x_{w} \in_{w_{\{n\}}} \mathbb{N}_{w_{\{n\}}} \to \mathbf{S}_{w_{\{n\}}}(x_{w_{\{n\}}}) \in_{w_{\{n\}}} \mathbb{N}_{w_{\{n\}}}$

(iii) if $\breve{K}_{w_{\{n\}}}$ satisfies (i) and (ii), then $\mathbb{N}_{w_{\{n\}}} \subset_{w_{\{n\}}} \breve{K}_{w_{\{n\}}}$.

Proof. It follows from the strong axiom of $w_{\{n\}}$-infinity that there exists at least one $w_{\{n\}}$-family $\breve{X}_{w_{\{n\}}}$ satisfying conditions (i) and (ii). Let $\mathcal{F}_{w_{\{n\}}}$ be the $w_{\{n\}}$-family of all those $w_{\{n\}}$-subsets of $\breve{X}_{w_{\{n\}}}$ which satisfy (i) and (ii):

$\mathcal{F}_{w_{\{n\}}} =_{w_{\{n\}}} \left\{\breve{S}_{w_{\{n\}}} \subset_{w_{\{n\}}} \breve{X}_{w_{[n]}} | \left(\breve{\emptyset}_{s,w} \in_{w_{\{n\}}} \breve{S}_{w_{[n]}}\right) \wedge \left(\breve{\emptyset}_{w} \in_{w_{\{n\}}} \breve{S}_{w_{\{n\}}}\right) \wedge \searrow$
$\wedge \forall \breve{y}_{w_{\{n\}}} \left(\breve{y}_{w_{\{n\}}} \in_{w_{\{n\}}} \breve{S}_{w_{\{n\}}} \to \mathbf{S}_{w_{\{n\}}}(\breve{y}_{w_{\{n\}}}) \in_{w_{\{n\}}} \breve{S}_{w_{\{n\}}}\right)\right\}.$
It is easy to show that $w_{\{n\}}$-$\bigcap \mathcal{F}_{w_{\{n\}}}$ (see Definition 3.2.3 (iv)) is the required $w_{\{n\}}$-family.

12. Axioms of Replacement.

(**1**) **Strong Replacement Scheme**.

(i) Let $\phi(x,y,u)$ be a formula free from symbols $\notin_w^s$, $\notin_w^w$, then

$$\forall x \forall y \forall y' [\phi(x,y,u) \wedge \phi(x,y',u) \to y =_w y'] \to$$
$$\to \forall s \exists z \forall y [y \in_w z \leftrightarrow \exists x (x \in_w s) \phi(x,y,u)]. \quad (3.1.34)$$

The set z is denoted $\{y | \exists x \phi(x,y,u) \wedge (x \in_w s)\}_w$.

(ii) Let $\phi(x,y,u)$ be a formula free from symbols $\notin_{w_{[n]}}^s$, $\notin_{w_{\{n\}}}^s$, then for any $u = (p_1,\ldots,p_k), \quad n = 1,2,\ldots.$

$$\forall x \forall y \forall y' [\phi(x,y,u) \wedge \phi(x,y',u) \to y =_{w_{[n]}} y'] \to$$
$$\to \forall s \exists z \forall y [y \in_{w_{[n]}} z \leftrightarrow \exists x (x \in_{w_{[n]}} s) \phi(x,y,u)]. \quad (3.1.35)$$

The set z is denoted $\{y | \exists x \phi(x,y,u) \wedge (x \in_{w_{[n]}} s)\}_{w_{[n]}}$.

(iii) Let $\phi(x,y,u)$ be a formula free from symbols $\notin_{w_{[n]}}^s$, $\notin_{w_{\{n\}}}^s$, then for any $u = (p_1,\ldots,p_k), \quad n = 1,2,\ldots.$

$$\forall x \forall y \forall y' [\phi(x,y,u) \wedge \phi(x,y',u) \to y =_{w_{\{n\}}} y'] \to$$
$$\to \forall s \exists z \forall y [y \in_{w_{\{n\}}} z \leftrightarrow \exists x (x \in_{w_{\{n\}}} s) \phi(x,y,u)]. \quad (3.1.36)$$

The set z is denoted $\{y | \exists x \phi(x,y,u) \wedge (x \in_{w_{\{n\}}} s)\}_{w_{\{n\}}}$.

(**2**) **Weak Replacement Scheme**.

(i) Let $\phi(x,y,u)$ be a stratified formula, then for any $u = (p_1,\ldots,p_k),\ n = 1,2,\ldots$

$$\forall x \forall y \forall y' [\phi(x,y,u) \wedge \phi(x,y',u) \Rightarrow_w y =_w y'] \Rightarrow_w$$
$$\Rightarrow_w \forall s \exists z \forall y [y \in_w z \Leftrightarrow_w \exists x (x \in_w s) \phi(x,y,u)]. \quad (3.1.37)$$

The set z is denoted $\{y | \exists x \phi(x,y,u) \wedge (x \in_w s)\}_w$.

(ii) Let $\phi(x,y,u)$ be a stratified formula, then for any $u = (p_1,\ldots,p_k),\ n = 1,2,\ldots$

$$\forall x \forall y \forall y' [\phi(x,y,u) \wedge \phi(x,y',u) \to y =_{w_{[n]}} y'] \to$$
$$\to \forall s \exists z \forall y [y \in_{w_{[n]}} z \leftrightarrow \exists x (x \in_{w_{[n]}} s) \phi(x,y,u)]. \quad (3.1.38)$$

The set z is denoted $\{y|\exists x\phi(x,y,u) \wedge (x \in_{w_{[n]}} s)\}_{w_{[n]}}$.

(iii) Let $\phi(x,y,u)$ be a stratified formula,then for any $u = (p_1,\ldots,p_k), n = 1,2,\ldots$

$$\forall x\forall y\forall y'[\phi(x,y,u) \wedge \phi(x,y',u) \rightarrow y =_{w_{\{n\}}} y'] \rightarrow \\ \rightarrow \forall s\exists z\forall y[y \in_{w_{\{n\}}} z \leftrightarrow \exists x(x \in_{w_{\{n\}}} s)\phi(x,y,u)]. \quad (3.1.39)$$

The set z is denoted $\{y|\exists x\phi(x,y,u) \wedge (x \in_{w_{\{n\}}} s)\}_{w_{\{n\}}}$.

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11. Axioms of inconsistent choice

11.1. Weak Axiom of w-Choice

$$\forall \breve{X}_w^{\mathbf{cl}}[(\forall x \in_w \breve{X}_w^{\mathbf{cl}} \forall y \in_w \breve{X}_w^{\mathbf{cl}}(x =_w^{\mathbf{cl}} y \leftrightarrow x \cap_w^{\mathbf{cl}} y \neq_w^{\mathbf{cl}} \varnothing_{s,w})) \rightarrow\searrow \\ \rightarrow \exists z_w^{\mathbf{cl}}(\forall x(x \in_w X_w^{\mathbf{cl}})\exists_w^{\mathbf{cl}}!y_w^{\mathbf{cl}}(y \in_w x \cap_w^{\mathbf{cl}} z_w^{\mathbf{cl}}))]. \quad (3.1.39)$$

Remark.3.1.3. Note that in non formal language, the Weak Axiom of Choice says that if you have almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ of pairwise w-disjoint non-empty almost classical w-sets, then you get almost classical w-set $z_w^{\mathbf{cl}}$ which contains one w-element from each set in the collection. Although the axiom gives the existence of some almost classical “choice w-set” $z_w^{\mathbf{cl}}$, there is no mention of w-uniqueness-there are quite likely many possible sets $z_w^{\mathbf{cl}}$ which satisfy the axiom and we are given no formula which would single out any one particular $z_w^{\mathbf{cl}}$.

Theorem 5. almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ there is a w-choice almost classical w-function on

any almost classical w-set of non-empty almost classical w-sets; i.e.,

$$\forall X_w^{\mathbf{cl}}[\varnothing_{s,w} \neq_w^w X_w^{\mathbf{cl}} \rightarrow (\exists \breve{f}_w^{\mathbf{cl}})(\breve{f}_w^{\mathbf{cl}} : X_w^{\mathbf{cl}} \rightarrow [X_w^{\mathbf{cl}} \wedge (\forall x \in_w X_w^{\mathbf{cl}})(f_w^{\mathbf{cl}}(x) \in_w x))]. \quad (3.1.39)$$

Proof. Given such an X, by Replacement there is a set Y = {{x} × x : x ∈ X}
which satisfies the hypothesis of the Weak Axiom of w-Choice. So, ∃z ∀y ∈ Y ∃!p p ∈ y ∩ z. Let f = z ∩ (S Y). Then f : X → S X and each f(x) ∈ x.

12. The $w_{[n]}$-union and $w_{[n]}$-intersection.

Definition 3.2.1. (i) The w-union of $\breve{A}_w^{\mathbf{cl}}$ and $\breve{B}_w^{\mathbf{cl}}$ is the almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ such that

$$\forall x[x \in_w \breve{X}_w^{\mathbf{cl}} \leftrightarrow (x \in_w \breve{A}_w^{\mathbf{cl}}) \vee (x \in_w \breve{B}_w^{\mathbf{cl}})]. \quad (3.2.1)$$

We denote it by $\breve{A}_w^{\mathbf{cl}} \cup_w \breve{B}_w^{\mathbf{cl}}$.

(ii) The w-union of $\breve{A}_w$ and $\breve{B}_w$ is the w-set $\breve{X}_w$ such that

$$\forall x[x \in_w \breve{X}_w \leftrightarrow (x \in_w \breve{A}_w) \vee (x \in_w \breve{B}_w)]. \quad (3.2.2)$$

We denote it by $\breve{A}_w \cup_w \breve{B}_w$.

(iii) The $w_{\{n\}}$-union of $\breve{A}_w$ and $\breve{B}_{w_{\{n\}}}$ is the $w_{\{n\}}$-set $\breve{X}_{w_{\{n\}}}$ such that

$$\forall x[x \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}} \leftrightarrow (x \in_{w_{\{n\}}} \breve{A}_{w_{\{n\}}}) \vee (x \in_{w_{\{n\}}} \breve{B}_{w_{\{n\}}})]. \quad (3.2.3)$$

We denote it by $\breve{A}_{w_{\{n\}}} \cup_{w_{\{n\}}} \breve{B}_{w_{\{n\}}}$.

(iv) The $w_{[n]}$-union of $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$ is the $w_{[n]}$-set $\breve{X}_{w_{[n]}}$ such that

$$\forall x[x \in_{w_{[n]}} \breve{X}_{w_{[n]}} \leftrightarrow (x \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \vee (x \in_{w_{[n]}} \breve{B}_{w_{[n]}})]. \quad (3.2.4)$$

We denote it by $\breve{A}_{w_{[n]}} \cup_{w_{[n]}} \breve{B}_{w_{[n]}}$.

Definition 3.2.2. (i) The w-intersection of $\breve{A}_w^{\mathbf{cl}}$ and $\breve{B}_w^{\mathbf{cl}}$ is the almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ such that

$$\forall x[x \in_w \breve{X}_w^{\mathbf{cl}} \leftrightarrow (x \in_w \breve{A}_w^{\mathbf{cl}}) \wedge (x \in_w \breve{B}_w^{\mathbf{cl}})]. \quad (3.2.5)$$

We denote it by $\breve{A}_w^{\mathbf{cl}} \cap_w \breve{B}_w^{\mathbf{cl}}$.

(ii)The w-intersection of $\breve{A}_w$ and $\breve{B}_w$ is the w-set $\breve{X}_w$ such that

$$\forall x[x \in_w \breve{X}_w \leftrightarrow (x \in_w \breve{A}_w) \wedge (x \in_w \breve{B}_w)]. \quad (3.2.6)$$

We denote it by $\breve{A}_w \cap_w \breve{B}_w$.

(iii) The $w_{\{n\}}$-intersection of $\breve{A}_w$ and $\breve{B}_{w_{\{n\}}}$ is the $w_{\{n\}}$-set $\breve{X}_{w_{\{n\}}}$ such that

$$\forall x[x \in_{w_{\{n\}}} \breve{X}_{w_{\{n\}}} \leftrightarrow (x \in_{w_{\{n\}}} \breve{A}_{w_{\{n\}}}) \wedge (x \in_{w_{\{n\}}} \breve{B}_{w_{\{n\}}})]. \quad (3.2.7)$$

We denote it by $\breve{A}_{w_{\{n\}}} \cap_{w_{\{n\}}} \breve{B}_{w_{\{n\}}}$.

(iv) The $w_{[n]}$-intersection of $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$ is the $w_{[n]}$-set $\breve{X}_{w_{[n]}}$ such that

$$\forall x[x \in_{w_{[n]}} \breve{X}_{w_{[n]}} \leftrightarrow (x \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (x \in_{w_{[n]}} \breve{B}_{w_{[n]}})]. \quad (3.2.8)$$

We denote it by $\breve{A}_{w_{[n]}} \cap_{w_{[n]}} \breve{B}_{w_{[n]}}$.

Definition 3.2.3.(i) For any almost classical w-set $\breve{\mathcal{F}}_w^{\mathbf{cl}}$, there exists almost classical w-set $\breve{A}_w^{\mathbf{cl}}$ such that $x \in_w \breve{A}_w^{\mathbf{cl}}$ if and only if $x \in_w \breve{Y}_w^{\mathbf{cl}}$ for any $\breve{Y}_w^{\mathbf{cl}} \in_w \breve{\mathcal{F}}_w^{\mathbf{cl}}$. We call the w-set $\breve{A}_w^{\mathbf{cl}}$ the w-intersection of $\breve{\mathcal{F}}_w^{\mathbf{cl}}$ and denote it by

$$w\text{-}\bigcap \breve{\mathcal{F}}_w^{\mathbf{cl}} \text{ or } \bigcap_w \breve{\mathcal{F}}_w^{\mathbf{cl}}. \quad (3.2.9)$$

(ii) For any w-set $\breve{\mathcal{F}}_w$, there exists $\breve{A}_w$ such that $x \in_w \breve{A}_w$ if and only if $x \in_w \breve{Y}_w$ for any $\breve{Y} \in_w \breve{\mathcal{F}}_w$. We call the w-set $\breve{A}_w$ the w-intersection of $\breve{\mathcal{F}}_w$ and denote it by

$$w\text{-}\bigcap \breve{\mathcal{F}}_w \text{ or } \bigcap_w \breve{\mathcal{F}}_w. \quad (3.2.10)$$

(iii) For any $w_{\{n\}}$-set $\breve{\mathcal{F}}_{w_{\{n\}}}$, there exists $\breve{A}_{w_{\{n\}}}$ such that $x \in_{w_{\{n\}}} \breve{A}_{w_{\{n\}}}$ if and only if

$x \in_{w_{\{n\}}} \breve{Y}_{w_{\{n\}}}$ for any $\breve{Y} \in_{w_{\{n\}}} \breve{\mathcal{F}}_{w_{\{n\}}}$. We call the $w_{\{n\}}$-set $\breve{A}_w$ the w-intersection of $\breve{\mathcal{F}}_{w_{\{n\}}}$ and

denote it by

$$w_{\{n\}}\text{-}\bigcap \breve{\mathcal{F}}_{w_{\{n\}}} \text{ or } \bigcap_{w_{\{n\}}} \breve{\mathcal{F}}_{w_{\{n\}}}. \quad (3.2.11)$$

17.The $w_{[n]}$-difference

Definition 3.3.1.(i) The w-difference of $\breve{A}_w^{\mathbf{cl}}$ and $\breve{B}_w^{\mathbf{cl}}$ is the w-set $\breve{X}_w^{\mathbf{cl}}$ of all $x \in_w \breve{A}_w^{\mathbf{cl}}$ such that $\neg_w(x \in_w B_w^{\mathbf{cl}})$:

$$\forall x[x \in_w \breve{X}_w^{\mathbf{cl}} \leftrightarrow (x \in_w \breve{A}_w^{\mathbf{cl}}) \wedge \neg_w(x \in_w B_w^{\mathbf{cl}})] \quad (3.3.1)$$

and we denote it by

$$\breve{A}_w^{\mathbf{cl}} \backslash_w \breve{B}_w^{\mathbf{cl}} \text{ or } \breve{A}_w^{\mathbf{cl}} -_w \breve{B}_w^{\mathbf{cl}}. \tag{3.3.2}$$

(ii) The w-difference of $\breve{A}_w$ and $\breve{B}_w$ is the w-set $\breve{X}_w$ of all $x \in_w \breve{A}$ s
uch that $\neg_w(x \in_w B_w)$:

$$\forall x[x \in_w \breve{X}_w \leftrightarrow (x \in_w \breve{A}_w) \wedge \neg_w(x \in_w B_w)] \tag{3.3.3}$$

and we denote it by

$$\breve{A}_w \backslash_w \breve{B}_w \text{ or } \breve{A}_w -_w \breve{B}_w. \tag{3.3.4}$$

Definition 3.3.2. The s-w-difference (strong w-difference) of $\breve{A}_w$ and $\breve{B}_w$ is the w-set $\breve{X}_w$ of all $x \in_w \breve{A}_w$ such that $\neg_s(x \in_w \breve{B}_w)$: $\forall x[x \in_w \breve{X}_w \leftrightarrow (x \in_w \breve{A}_w) \wedge \neg_s(x \in_w \breve{B}_w)]$.
We denote it by $\breve{A}_w -_{s,w} \breve{B}_w$ or by $A_w \backslash_{s,w} B_w$. If $\breve{B}_w \subset_w \breve{A}_w$ we say that $\breve{A} \backslash_{s,w} \breve{B}$ is a s-w-complement $\breve{B}_w$ in $\breve{A}_w$ or $\breve{A}_w \backslash_{s,w} \breve{B}_w$ is a strong w-complement $\breve{B}_w$ in $\breve{A}_w$.

16.Inconsistent w-relations and w-functions of the order inconsistency zero.

Definition 16.1. (i) Almost classical w-ordered pair $(a,b)_w^{\mathbf{cl}}$ is defined to be

$$(a,b)_w^{\mathbf{cl}} =_w \{\{a\}_w^{\mathbf{cl}}, \{a,b\}_w^{\mathbf{cl}}\}_w^{\mathbf{cl}}. \tag{16.1}$$

We further define almost classical w-ordered triples

$$(a,b,c)_w^{\mathbf{cl}} =_w ((a,b)_w^{\mathbf{cl}}, c)_w^{\mathbf{cl}} =_w \{\{\{a\}_w^{\mathbf{cl}}, \{a,b\}_w^{\mathbf{cl}}\}_w, \{\{\{a\}_w^{\mathbf{cl}}, \{a,b\}_w^{\mathbf{cl}}\}, c\}_w^{\mathbf{cl}}\}_w^{\mathbf{cl}}, \tag{16.2}$$

almost classical w-ordered quadruples . . .almost classical w-ordered n-tuples etc.
(ii) An w-ordered pair $(a,b)_w$ is defined to be $\{\{a\}_w, \{a,b\}_w\}_w$.

$$(a,b)_w =_w \{\{a\}_w, \{a,b\}_w\}_w. \tag{16.3}$$

We further define w-ordered triples

$$(a,b,c)_w =_w ((a,b)_w, c)_w =_w \{\{\{a\}_w, \{a,b\}_w\}_w, \{\{\{a\}_w, \{a,b\}_w\}, c\}_w\}_w, \tag{16.4}$$

w-ordered quadruples . . .almost classical w-ordered n-tuples etc.

Definition 16.2. (i) Almost classical w-set $\breve{R}_w^{\mathbf{cl}}$ is an almost classical binary w-relation if all w-elements of $\breve{R}_w^{\mathbf{cl}}$ are almost classical w-ordered pairs, i.e. for $z \in_w \breve{R}_w^{\mathbf{cl}}$ there exists x and y such that $z =_w (x,y)_w^{\mathbf{cl}}$. We can also denote $(x,y)_w \in_w \breve{R}_w^{\mathbf{cl}}$ as $x\breve{R}_w^{\mathbf{cl}}y$, and say that x is in relation $\breve{R}_w^{\mathbf{cl}}$ with y if $x\breve{R}_w^{\mathbf{cl}}y$ holds.
(ii) A w-set $\breve{R}_w$ is a binary w-relation if all w-elements of $\breve{R}_w$ are w-ordered pairs, i.e. for $z \in_w \breve{R}_w$ there exists x and y such that $z =_w (x,y)_w$. We can also denote $(x,y)_w \in_w \breve{R}_w$ as $x\breve{R}_w y$, and say that x is in w-relation $\breve{R}_w$ with y if $x\breve{R}_w y$ holds.

Definition 16.3. (i) The almost classical w-membership w-relation on $\breve{A}_w^{\mathbf{cl}}$ is defined by

$$\in_{w,\breve{A}_w^{\mathbf{cl}}}^{\mathbf{cl}} =_w \{(a,b)_w^{\mathbf{cl}} | (a \in_w^{\mathbf{cl}} \breve{A}_w^{\mathbf{cl}}) \wedge (b \in_w^{\mathbf{cl}} \breve{A}_w^{\mathbf{cl}}) \wedge (a \in_w^{\mathbf{cl}} b)\}. \tag{16.5}$$

The almost classical w-identity w-relation on $\breve{A}_w^{\mathbf{cl}}$ is defined by

$$\mathbf{Id}_{w,\breve{A}_w^{\mathbf{cl}}}^{\mathbf{cl}} =_w \{(a,b)_w^{\mathbf{cl}} | (a \in_w^{\mathbf{cl}} \breve{A}_w^{\mathbf{cl}}) \wedge (b \in_w^{\mathbf{cl}} \breve{A}_w^{\mathbf{cl}}) \wedge (a =_w b)\}. \tag{16.6}$$

(ii) The w-membership w-relation on $\breve{A}_w$ is defined by

$$\in_{w,\breve{A}_w} =_w \{(a,b)_w | (a \in_w \breve{A}_w) \wedge (b \in_w \breve{A}_w) \wedge (a \in_w b)\}. \tag{16.7}$$

The w-identity w-relation on $\breve{A}_w$ is defined by

$$\mathbf{Id}_{w,\breve{A}_w} = \{(a,b)_w | (a \in_w \breve{A}_w) \wedge (b \in_w \breve{A}_w) \wedge (a =_w b)\}. \tag{16.8}$$

Definition 16.4.(i) Let $\breve{A}_w$ be w-set and $\bar{B}$ be a classical set. The cartesian ws-product of $\breve{A}$ and $\bar{B}$ is

$$\breve{A}_w \times_w \bar{B} =_{ws} \{(a,b)_{ws} | (a \in_w \breve{A}_w) \wedge (b \in_s \bar{B})\}. \quad (16.9)$$

(ii) Let $\bar{A}$ be a classical set and $\breve{B}$ be a w-set. The cartesian sw-product of $\bar{A}$ and $\breve{B}$ is

$$\bar{A} \times_w \breve{B} =_{ws} \{(a,b)_{sw} | (a \in_s \bar{A}) \wedge (b \in_w \breve{B})\}. \quad (16.10)$$

(iii) Let $\breve{A}, \breve{B}$ be w-sets. The cartesian w-product of $\breve{A}$ and $\breve{B}$ is defined by

$$\breve{A} \times_w \breve{B} = \{(a,b)_w | (a \in_w \breve{A}) \wedge (b \in_w \breve{B})\}. \quad (16.11)$$

Definition 16.5.(i) A binary w-relation $\breve{F}_w$ is called a w-function if $a\breve{F}_w b_1$ and $a\breve{F}_w b_2$ imply $b_1 =_w b_2$ for any $a, b_1,$ and b_2. This w-unique b is the value of $\breve{F}_w$ at a and is denoted $\breve{F}_w(a)$ or $\breve{F}_w a$. If $\mathbf{dom}(\breve{F}_w) =_w \breve{A}_w$ and ran $\breve{F}_w \subseteq_w \breve{B}_w$, we can denote $\breve{F}_w$ by $\breve{F}_w : \breve{A}_w \to \breve{B}_w$, $\langle \breve{F}_w(a) | a \in_w A_w \rangle_w, \langle \breve{F}_{w,a} | a \in_w \breve{A} \rangle_w$, or $\langle \breve{F}_{w,a} \rangle_{a \in_w \breve{A}_w}$.

Definition 16.6.(i) Let $\breve{f}_w : \breve{A}_w \to_w \breve{B}_w$ be a w-function.

1) $\breve{f}_w$ is w-injective if for $a_1 \in_w \breve{A}_w$ and $a_2 \in_w \breve{A}_w, \breve{f}_w(a_1) =_w \breve{f}_w(a_2)$ if and only if $a_1 =_w a_2$. We call $\breve{f}_w$ a w-injection.

2) $\breve{f}_w$ is w-surjective if for every $b \in_w \breve{B}_w$, there exists some $a \in_w \breve{A}_w$ such that $\breve{f}_w(a) =_w b$. We call $\breve{f}_w$ a w-surjection.

3) $\breve{f}_w$ is w-bijective if it is both w-injective and w-surjective. We call $\breve{f}_w$ a w-bijection.

(ii) Let $f_{s,w} : \bar{A} \to_{sw} \breve{B}_w$ be a sw-function.

1) $f_{s,w}$ is sw-injective if for $a_1 \in_s \bar{A}$ and $a_2 \in_s A, f_{s,w}(a_1) =_w f_{s,w}(a_2)$ if and only if $a_1 =_s a_2$. We call $f_{s,w}$ a sw-injection.

2) $f_{s,w}$ is sw-surjective if for every $b \in_w \breve{B}_w$, there exists some $a \in_s \bar{A}$ such that $f_{s,w}(a) =_w b$. We call $f_{s,w}$ a sw-surjection.

3) $f_{s,w}$ is sw-bijective if it is both sw-injective and sw-surjective. We call $f_{s,w}$ a sw-bijection.

(iii) Let $f_{w,s} : \breve{A}_w \to_{w,s} \bar{B}$ be a ws-function.

1) $f_{w,s}$ is ws-injective if for $a_1 \in_w \breve{A}_w$ and $a_2 \in_w \breve{A}_w, f_{w,s}(a_1) =_s f_{w,s}(a_2)$ if and only if $a_1 =_w a_2$. We call $f_{w,s}$ an ws-injection.

2) $f_{w,s}$ is ws-surjective if for every $b \in_s \bar{B}$, there exists some $a \in_w \breve{A}_w$ such that $f_{w,s}(a) =_s b$. We call $f_{w,s}$ a ws-surjection.

3) $f_{w,s}$ is ws-bijective if it is both ws-injective and ws-surjective. We call f_{ws} a ws-bijection.

Definition 16.7.(i) (a) w-functions f_w and g_w are called w-compatible if $f(x) =_w g(x)$ for all $x \in_w \mathbf{dom}(f_w) \cap_w \mathbf{dom}(g_w)$.

(b) A w-set of w-functions F_w is called a w-compatible system of w-functions if any two w-functions f_w and g_w from F_w are w-compatible.

Theorem 16.1. If F_w is a w-compatible system of w-functions, then w-$\bigcup F_w$ is a w-function with $\mathbf{dom}\left(w\text{-}\bigcup F_w\right) =_w w\text{-}\bigcup \{\mathbf{dom}(f_w) | f_w \in_w F_w\}_w$. The w-function w-$\bigcup F_w$ extends all $f_w \in_w F_w$.

Proof. We need to show that:
(1) $w\text{-}\bigcup F_w$ is a function and
(2) $\mathbf{dom}\left(w\text{-}\bigcup F_w\right) = w\text{-}\bigcup\{\mathbf{dom}(f_w)|f_w \in_w F_w\}_w$.
(1) Suppose there exists $(a,b_1)_w \in_w w\text{-}\bigcup F_w$ and $(a,b_2)_w \in_w w\text{-}\bigcup F_w$.
Then there exists functions $f_{w,1}, f_{w,2} \in_w F_w$ such that $f_{w,1}(a) =_w b_1$ and $f_{w,2}(a) =_w b_2$.
But since $f_{w,1}$ and $f_{w,2}$ are compatible and $a \in_w \mathbf{dom}(f_{w,1}) \cap_w \mathbf{dom}(f_{w,2})$, therefore $b_1 =_w f_{w,1}(a) =_w f_{w,2}(a) =_w b_2$.
This shows that $w\text{-}\bigcup F_w$ is a w-function.
(2) Suppose $x \in_w \mathbf{dom}\left(w\text{-}\bigcup F_w\right)$. Then $x \in_w \mathbf{dom}(f)$ for some $f_w \in_w F_w$.
Suppose $y \in_w \mathbf{dom}(f_w)$ for some $f_w \in_w F_w$. Then $x \in_w \mathbf{dom}\left(w\text{-}\bigcup F_w\right)$.
Therefore $\mathbf{dom}\left(w\text{-}\bigcup F_w\right) =_w w\text{-}\bigcup \{\mathbf{dom}(f_w)|f_w \in_w F_w\}_w$.
Definition 16.8.(i) Let $\breve{A}$ and $\breve{B}$ be w-sets. The set of all w-functions on $\breve{A}$ into $\breve{B}$ is denoted $w\text{-}\breve{B}^{\breve{A}}$.
(ii) Let $\bar{A}$ be a classical set and let $\breve{B}_w$ be w-set. The w-set of all sw-functions on $\bar{A}$ into $\breve{B}$ is denoted $w\text{-}\breve{B}^{\bar{A}}$.
(iii) Let $\breve{A}_w$ be w-set and let $\bar{B}$ a classical set. The w-set of all ws-functions on $\breve{A}$ into $\bar{B}$ is denoted $w\text{-}\bar{B}^{\breve{A}}$.

17. Inconsistent $w_{[n]}$-Relations and $w_{[n]}$-Functions of the order inconsistency $n \geq 1$.

Definition 17.1. (i) An $w_{\{n\}}$-ordered pair $(a,b)_{w\{n\}}$ is defined to be $\{\{a\}_{w\{n\}}, \{a,b\}_{w\{n\}}\}_{w\{n\}}$:

$$(a,b)_{w\{n\}} \triangleq \{\{a\}_{w\{n\}}, \{a,b\}_{w\{n\}}\}_{w\{n\}}. \tag{17.1}$$

We further define $w_{\{n\}}$-ordered triples

$$(a,b,c)_{w\{n\}} \triangleq ((a,b)_{w\{n\}}, c)_{w\{n\}} =_{w\{n\}} \searrow \\ \{\{\{a\}_{w\{n\}}, \{a,b\}_{w\{n\}}\}_{w\{n\}}, \{\{\{a\}_{w\{n\}}, \{a,b\}_{w\{n\}}\}, c\}_{w\{n\}}\}_{w\{n\}}, \tag{17.2}$$

$w_{\{n\}}$-ordered quadruples . . . $w_{\{n\}}$-ordered n-tuples etc.
(iv) An $w_{[n]}$-ordered pair $(a,b)_{w[n]}$ is defined to be $\{\{a\}_{w[n]}, \{a,b\}_{w[n]}\}_{w[n]}$.

$$(a,b)_{w[n]} \triangleq \{\{a\}_{w[n]}, \{a,b\}_{w[n]}\}_{w[n]}. \tag{17.3}$$

We further define $w_{[n]}$-ordered triples

$$(a,b,c)_{w[n]} \triangleq ((a,b)_{w[n]}, c)_{w[n]} =_{w[n]} \\ \{\{\{a\}_{w[n]}, \{a,b\}_{w[n]}\}_{w[n]}, \{\{\{a\}_{w[n]}, \{a,b\}_{w[n]}\}, c\}_{w[n]}\}_{w[n]}, \tag{17.4}$$

$w_{[n]}$-ordered quadruples . . . $w_{[n]}$-ordered n-tuples etc.
Definition 17.2.(i) A $w_{[n]}$-set $\breve{R}_{w_{[n]}}$ is a binary $w_{[n]}$-relation if all $w_{[n]}$-elements of $\breve{R}_{w_{[n]}}$ are $w_{[n]}$-ordered pairs, i.e. for $z \in_{w_{[n]}} \breve{R}_{w_{[n]}}$ there exists x and y such that $z =_{w_{[n]}} (x,y)_{w_{[n]}}$. We can
also denote $(x,y)_{w_{[n]}} \in_{w_{[n]}} \breve{R}_{w_{[n]}}$ as $x\breve{R}_{w_{[n]}}y$, and say that x is in $w_{[n]}$-relation $\breve{R}_{w_{[n]}}$ with y if $x\breve{R}_{w_{[n]}}y$ holds.
Definition 17.3.(i) The $w_{[n]}$-membership $w_{[n]}$-relation on $\breve{A}_{w_{[n]}}$ is defined by

$$\in_{w_{[n]},\breve{A}_{w_{[n]}}} =_{w_{[n]}} \{(a,b)_{w_{[n]}} | (a \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (b \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (a \in_{w_{[n]}} b)\}. \quad (17.5)$$

The $w_{[n]}$-identity $w_{[n]}$-relation on $\breve{A}_{w_{[n]}}$ is defined by

$$\mathbf{Id}_{w_{[n]},\breve{A}_{w_{[n]}}} = \{(a,b)_{w_{[n]}} | (a \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (b \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (a =_{w_{[n]}} b)\}. \quad (17.6)$$

Definition 17.4.(i) Let $\breve{A}_{w_{[n]}}$ be $w_{[n]}$-set and $\bar{B}$ be a classical set. The cartesian $w_{[n]}s$-product of $\breve{A}_{w_{[n]}}$ and $\bar{B}$ is

$$\breve{A}_{w_{[n]}} \times_{w_{[n]}} \bar{B} =_{w_{[n]},s} \{(a,b)_{w_{[n]}s} | (a \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (b \in_s \bar{B})\}. \quad (17.7)$$

(ii) Let $\bar{A}$ be a classical set and $\breve{B}_{w_{[n]}}$ be a $w_{[n]}$-set. The cartesian $sw_{[n]}$-product of $\bar{A}$ and $\breve{B}_{w_{[n]}}$ is

$$\bar{A} \times_{w_{[n]}} \breve{B}_{w_{[n]}} =_{s,w_{[n]}} \{(a,b)_{sw_{[n]}} | (a \in_s \bar{A}) \wedge (b \in_{w_{[n]}} \breve{B})\}. \quad (17.8)$$

(iii) Let $\breve{A}_{w_{[n]}}, \breve{B}_{w_{[n]}}$ be $w_{[n]}$-sets. The cartesian $w_{[n]}$-product of $\breve{A}_{w_{[n]}}$ and $\breve{B}_{w_{[n]}}$ is

$$\breve{A}_{w_{[n]}} \times_{w_{[n]}} \breve{B}_{w_{[n]}} =_{w_{[n]}} \{(a,b)_{w_{[n]}} | (a \in_{w_{[n]}} \breve{A}_{w_{[n]}}) \wedge (b \in_{w_{[n]}} \breve{B}_{w_{[n]}})\}. \quad (17.9)$$

Definition 17.5.(i) A binary $w_{[n]}$-relation $\breve{F}_{w_{[n]}}$ is called a $w_{[n]}$-function if $a\breve{F}_{w_{[n]}}b_1$ and $a\breve{F}_{w_{[n]}}b_2$ imply $b_1 =_{w_{[n]}} b_2$ for any $a, b_1,$ and b_2. This $w_{[n]}$-unique b is the value of $\breve{F}_{w_{[n]}}$ at a and is denoted $\breve{F}_{w_{[n]}}(a)$
or $\breve{F}_{w_{[n]}}a$. If $\mathbf{dom}(\breve{F}_{w_{[n]}}) =_{w_{[n]}} \breve{A}_{w_{[n]}}$ and $\mathbf{ran}(\breve{F}_w) \subseteq_w \breve{B}_w$, we can denote $\breve{F}_w$ by $\breve{F}_{w_{[n]}} : \breve{A}_{w_{[n]}} \to \breve{B}_{w_{[n]}}, \langle \breve{F}_{w_{[n]}}(a) | a \in_{w_{[n]}} A_{w_{[n]}} \rangle_{w_{[n]}}, \langle \breve{F}_{w_{[n]},a} | a \in_{w_{[n]}} \breve{A}_{w_{[n]}} \rangle_{w_{[n]}}$, or $\langle \breve{F}_{w_{[n]},a} \rangle_{a \in_{w_{[n]}} \breve{A}_{w_{[n]}}}$.

Definition 15.6.(i) Let $\breve{f}_{w_{[n]}} : \breve{A}_{w_{[n]}} \to_{w_{[n]}} \breve{B}_{w_{[n]}}$ be a $w_{[n]}$-function.

1) $\breve{f}_{w_{[n]}}$ is $w_{[n]}$-injective if for $a_1 \in_{w_{[n]}} \breve{A}_{w_{[n]}}$ and $a_2 \in_{w_{[n]}} \breve{A}_{w_{[n]}}, \breve{f}_{w_{[n]}}(a_1) =_{w_{[n]}} \breve{f}_w(a_2)$ if and only
if $a_1 =_{w_{[n]}} a_2$. We call $\breve{f}_{w_{[n]}}$ a $w_{[n]}$-injection.

2) $\breve{f}_{w_{[n]}}$ is $w_{[n]}$-surjective if for every $b \in_{w_{[n]}} \breve{B}_{w_{[n]}}$, there exists some $a \in_{w_{[n]}} \breve{A}_{w_{[n]}}$ such that
$\breve{f}_{w_{[n]}}(a) =_{w_{[n]}} b$. We call $\breve{f}_{w_{[n]}}$ a $w_{[n]}$-surjection.

3) $\breve{f}_{w_{[n]}}$ is $w_{[n]}$-bijective if it is both $w_{[n]}$-injective and $w_{[n]}$-surjective. We call $\breve{f}_{w_{[n]}}$ a $w_{[n]}$-bijection.

(ii) Let $f_{s,w_{[n]}} : \bar{A} \to_{sw_{[n]}} \breve{B}_{w_{[n]}}$ be a $sw_{[n]}$-function.

1) $f_{s,w_{[n]}}$ is $sw_{[n]}$-injective if for $a_1 \in_s \bar{A}$ and $a_2 \in_s A, f_{s,w_{[n]}}(a_1) =_{w_{[n]}} f_{s,w_{[n]}}(a_2)$ if and only if $a_1 =_s a_2$. We call $f_{s,w_{[n]}}$ a $sw_{[n]}$-injection.

2) $f_{s,w_{[n]}}$ is $sw_{[n]}$-surjective if for every $b \in_w \breve{B}_w$, there exists some $a \in_s \bar{A}$ such that $f_{s,w_{[n]}}(a) =_{w_{[n]}} b$. We call $f_{s,w_{[n]}}$ a $sw_{[n]}$-surjection.

3) $f_{s,w_{[n]}}$ is $sw_{[n]}$-bijective if it is both $sw_{[n]}$-injective and $sw_{[n]}$-surjective. We call $f_{s,w_{[n]}}$ a $sw_{[n]}$-bijection.

(iii) Let $f_{w_{[n]},s} : \breve{A}_{w_{[n]}} \to_{w_{[n]},s} \bar{B}$ be a $w_{[n]}s$-function.

1) $f_{w_{[n]},s}$ is $w_{[n]}$-injective if for $a_1 \in_{w_{[n]}} \breve{A}_{w_{[n]}}$ and $a_2 \in_{w_{[n]}} \breve{A}_{w_{[n]}}, f_{w_{[n]},s}(a_1) =_s f_{w_{[n]},s}(a_2)$ if and only if $a_1 =_{w_{[n]}} a_2$. We call $f_{w_{[n]},s}$ an $w_{[n]}s$-injection.

2) $f_{w,s}$ is $w_{[n]}s$-surjective if for every $b \in_s \bar{B}$, there exists some $a \in_w \breve{A}_w$ such that $f_{w_{[n]},s}(a) =_s b$. We call $f_{w_{[n]},s}$ a $w_{[n]}s$-surjection.

3) $f_{w_{[n]},s}$ is $w_{[n]}s$-bijective if it is both $w_{[n]}s$-injective and $w_{[n]}s$-surjective. We call $f_{w_{[n]}s}$ a $w_{[n]}s$-bijection.

Definition 17.7.(i) (a) $w_{[n]}$-functions $f_{w_{[n]}}$ and $g_{w_{[n]}}$ are called $w_{[n]}$-compatible if $f_{w_{[n]}}(x) =_{w_{[n]}} g_{w_{[n]}}(x)$ for all $x \in_{w_{[n]}} \mathbf{dom}(f_{w_{[n]}}) \cap_{w_{[n]}} \mathbf{dom}(g_{w_{[n]}})$.

(b) A $w_{[n]}$-set of $w_{[n]}$-functions $F_{w_{[n]}}$ is called a $w_{[n]}$-compatible system of $w_{[n]}$-functions if

any two $w_{[n]}$-functions $f_{w_{[n]}}$ and $g_{w_{[n]}}$ from $F_{w_{[n]}}$ are $w_{[n]}$-compatible.

Theorem 17.8. If $F_{w_{[n]}}$ is a $w_{[n]}$-compatible system of w-functions, then $w_{[n]}$-$\bigcup F_{w_{[n]}}$ is a $w_{[n]}$-function with

$\mathbf{dom}\left(w_{[n]}\text{-}\bigcup F_{w_{[n]}}\right) =_{w_{[n]}} w_{[n]}\text{-}\bigcup\{\mathbf{dom}(f_{w_{[n]}}) | f_{w_{[n]}} \in_{w_{[n]}} F_{w_{[n]}}\}_{w_{[n]}}$.

The $w_{[n]}$-function $w_{[n]}$-$\bigcup F_{w_{[n]}}$ extends all $f_{w_{[n]}} \in_{w_{[n]}} F_{w_{[n]}}$.

Proof. We need to show that:

(1) $w_{[n]}$-$\bigcup F_{w_{[n]}}$ is a function and

(2) $\mathbf{dom}\left(w_{[n]}\text{-}\bigcup F_{w_{[n]}}\right) =_{w_{[n]}} w_{[n]}\text{-}\bigcup\{\mathbf{dom}(f_{w_{[n]}}) | f_{w_{[n]}} \in_{w_{[n]}} F_{w_{[n]}}\}_{w_{[n]}}$.

(1) Suppose there exists $(a,b_1)_w \in_w w\text{-}\bigcup F_w$ and $(a,b_2)_w \in_w w\text{-}\bigcup F_w$.

Then there exists functions $f_{w,1}, f_{w,2} \in_w F_w$ such that $f_{w,1}(a) =_w b_1$ and $f_{w,2}(a) =_w b_2$.

But since $f_{w,1}$ and $f_{w,2}$ are compatible and $a \in_w \mathbf{dom}(f_{w,1}) \cap_w \mathbf{dom}(f_{w,2})$, therefore $b_1 =_w f_{w,1}(a) =_w f_{w,2}(a) =_w b_2$.

This shows that w-$\bigcup F_w$ is a w-function.

(2) Suppose $x \in_w \mathbf{dom}\left(w\text{-}\bigcup F_w\right)$. Then $x \in_w \mathbf{dom}(f)$ for some $f_w \in_w F_w$.

Suppose $y \in_w \mathbf{dom}(f_w)$ for some $f_w \in_w F_w$. Then $x \in_w \mathbf{dom}\left(w\text{-}\bigcup F_w\right)$.

Therefore $\mathbf{dom}\left(w\text{-}\bigcup F_w\right) =_w w\text{-}\bigcup \{\mathbf{dom}(f_w) | f_w \in_w F_w\}_{w_{[n]}}$.

Definition 17.9.(i) Let $\breve{A}$ and $\breve{B}$ be w-sets. The set of all w-functions on $\breve{A}$ into $\breve{B}$ is denoted w-$\breve{B}^{\breve{A}}$.

(ii) Let $\bar{A}$ be a classical set and let $\breve{B}_w$ be w-set. The w-set of all sw-functions on $\bar{A}$ into $\breve{B}$ is denoted w-$\breve{B}^{\bar{A}}$.

(iii) Let $\breve{A}_{w_{[n]}}$ be w-set and let $\bar{B}$ a classical set. The $w_{[n]}$-set of all $w_{[n]}s$-functions on $\breve{A}_{w_{[n]}}$ into $\bar{B}$ is denoted $w_{[n]}$-$\bar{B}^{\breve{A}_{w_{[n]}}}$.

Inconsistent Equivalences and Orderings.

18.Inconsistent w-Equivalences and w-Orderings of the order inconsistency zero.

In these subsections, we will finish defining a few important types of inconsistent relations that will help in defining inconsistent natural and inconsistent real numbers in set theory $ZFC^{\#}_{\omega}$.

Definition 18.1. (i) Let $\breve{R}^{\mathbf{cl}}_w$ be almost classical binary w-relation in w-set $\breve{A}^{\mathbf{cl}}_w$.

(a) $\breve{R}^{\mathbf{cl}}_w$ is w-reflexive in $\breve{A}^{\mathbf{cl}}_w$ if for all $a \in_w \breve{A}^{\mathbf{cl}}_w, a\breve{R}^{\mathbf{cl}}_w a$.

(b) $\breve{R}^{\mathbf{cl}}_w$ is w-symmetric in $\breve{A}^{\mathbf{cl}}_w$ if for all $a,b \in_w \breve{A}^{\mathbf{cl}}_w, a\breve{R}^{\mathbf{cl}}_w b$ implies $b\breve{R}^{\mathbf{cl}}_w a$.

(c) $\breve{R}^{\mathbf{cl}}_w$ is w-antisymmetric in $\breve{A}^{\mathbf{cl}}_w$ if for all $a,b \in_w \breve{A}^{\mathbf{cl}}_w, a\breve{R}^{\mathbf{cl}}_w b$ and $b\breve{R}^{\mathbf{cl}}_w a$ imply $a =_w b$.

(d) $\breve{R}^{\mathbf{cl}}_w$ is w-asymmetric in $\breve{A}^{\mathbf{cl}}_w$ if for all $a,b \in_w \breve{A}^{\mathbf{cl}}_w, a\breve{R}^{\mathbf{cl}}_w b$ implies that $\neg_{\mathbf{s}}(b\breve{R}^{\mathbf{cl}}_w a)$.

i.e. $a\breve{R}^{\mathbf{cl}}_w b$ and $b\breve{R}^{\mathbf{cl}}_w a$ cannot both be true.

(e) $\breve{R}^{\mathbf{cl}}_w$ is w-transitive in $\breve{A}^{\mathbf{cl}}_w$ if for all $a,b,c \in_w \breve{A}^{\mathbf{cl}}_w$, $a\breve{R}^{\mathbf{cl}}_w b$ and $b\breve{R}^{\mathbf{cl}}_w c$ imply $a\breve{R}^{\mathbf{cl}}_w c$.

(ii) Let $\breve{R}_w$ be a binary w-relation in w-set $\breve{A}_w$.

(a) $\breve{R}_w$ is w-reflexive in $\breve{A}_w$ if for all $a \in_w \breve{A}_w, a\breve{R}_w a$.

(b) $\breve{R}_w$ is w-symmetric in $\breve{A}_w$ if for all $a,b \in_w \breve{A}_w, a\breve{R}_w b$ implies $b\breve{R}_w a$.
(c) $\breve{R}_w$ is w-antisymmetric in $\breve{A}_w$ if for all $a,b \in_w \breve{A}_w, a\breve{R}_w b$ and $b\breve{R}_w a$ imply $a =_w b$.
(d) $\breve{R}_w$ is w-asymmetric in $\breve{A}_w$ if for all $a,b \in_w \breve{A}_w, a\breve{R}_w b$ implies that $\neg_s(b\breve{R}_w a)$.
i.e. $a\breve{R}_w b$ and $b\breve{R}_w a$ cannot both be true.
(e) $\breve{R}_w$ is w-transitive in $\breve{A}_w$ if for all $a,b,c \in_w \breve{A}_w$, $a\breve{R}_w b$ and $b\breve{R}_w c$ imply $a\breve{R}_w c$.

Remark.**18**.**1**. Note that if $\breve{R}_w$ ia binary w-relation in w-set $\breve{A}_w$ then by the non classical law of the excluded fourth (see sect. 2.1)

$$a\breve{R}_w b \vee \neg_s(a\breve{R}_w b) \vee \neg_w(a\breve{R}_w b) \qquad (18.1)$$

Definition 18.**2**.Let $\breve{R}_w$ be a binary w-relation in $\breve{A}_w$.
(a) $\breve{R}_w$ is an w-equivalence on $\breve{A}_w$ if it is w-reflexive, w-symmetric, and w-transitive in $\breve{A}_w$.
(b) $\breve{R}_w$ is a w-ordering of $\breve{A}_w$ if it is w-reflexive, w-antisymmetric, and w-transitive in $\breve{A}_w$.
The pair $(\breve{A}_w, \breve{R}_w)$ is called an w-ordered w-set.
(c) $\breve{R}_w$ is a strict w-ordering of $\breve{A}_w$ if it is w-asymmetric and w-transitive in $\breve{A}_w$.

Remark 18.**2**. Now that we have established the definition of w-orderings and strict w-orderings, we can use $\leq_w$ and to denote w-orderings and $<_w$ and $\prec_w$ to denote strict w-orderings.Thus $(\breve{A}_w, \leq_w)$ is an pair consisting of a set $\breve{A}_w$ and an w-ordering $\leq_w$, and $(\breve{B}_w, \prec_w)$ is a pair consisting of a set $\breve{B}_w$ and a strict w-ordering $\prec_w$.
There is a close relationship between w-orderings and strict w-orderings as we will see in the next theorem.

Theorem 18.**1**.(a) Let $\breve{R}_w$ be an w-ordering of $\breve{A}_w$. Then the w-relation $\breve{S}_w$ in $\breve{A}_w$ defined by $a\breve{S}_w b$ if and only if a$\breve{R}_w$b and $\neg_s(a =_w b)$ is a strict w-ordering of $\breve{A}_w$.
(b) Let $\breve{S}_w$ be a strict w-ordering of $\breve{A}_w$. Then the w-relation $\breve{R}_w$ in $\breve{A}_w$ defined by $a\breve{R}_w b$ if and only if $a\breve{S}_w b$ or $a =_w b$ is an w-ordering of $\breve{A}_w$.

Proof. (a) We need to show that $\breve{S}_w$ is w-asymmetric. Suppose $a\breve{S}_w b$ and $b\breve{S}_w a$ both hold
for some $a,b \in_w \breve{A}_w$. Then $a\breve{R}_w b$ and $b\breve{R}_w a$ both also hold. It follows that $a =_w b$ because
$\breve{R}_w$is w-antisymmetric. This is a contradiction since $\neg_s(a =_w b)$. Therefore $\breve{S}_w$ is w-asymmetric.
(b) We need to show that $\breve{R}_w$ is w-antisymmetric. Suppose $a\breve{R}_w b$ and $b\breve{R}_w a$ both hold for
some $a,b \in_w \breve{A}_w$.Suppose that $\neg_s(a =_w b)$. Then $a\breve{S}_w b$ and $b\breve{S}_w a$ both hold. This is a contradiction since $\breve{S}_w$ is w-asymmetric.Therefore $a =_w b$, showing that $\breve{R}_w$ is w-antisymmetric.

Definition 18.**3**.An w-ordering $<$ of $\breve{A}_w$ is called strong w-linear w-ordering if any two w-elements of $\breve{A}_w$ are comparable in the w-ordering $<_w$ in classical sense i.e. in accordance with classical law of the excluded third (see sect. 2.1) i.e. for any $a,b \in_w \breve{A}_w$, either

$$a <_w b, b <_w a, \text{or } a =_w b. \qquad (18.2)$$

The pair $(\breve{A}_w, <_w)$ is called a strongly w-linearly w-ordered w-set.

Definition 18.**4**.An w-ordering $<_w$ of $\breve{A}_w$ is called weak w-linear if any two w-elements of $\breve{A}_w$ are comparable in the w-ordering $<_w$ i.e. for any $a,b \in_w \breve{A}_w$, either

$$a <_w b, b <_w a, \neg_w(a <_w b), \neg_w(b <_w a), \text{ or } a =_w b. \qquad (18.3)$$

The pair ($\breve{A}_w$, $<_w$) is called weakly w-linearly w-ordered w-set.

Definition 18.5. Let $<_w$ be a w-linear w-ordering $<_w$ of a w-set $\breve{A}_w$.

(i) The condition that a w-set $\breve{X}_w \subseteq_w \breve{A}_w$ has a strong $<_w$ -least w-element x reads

$$\exists x(x \in_w \breve{X})[\forall y \in_w \breve{X}(x \leq_w y)]. \qquad (18.4)$$

(ii) We assume now that a set $\breve{X} \subseteq_w \breve{A}_w$ has no a strong $<_w$ -least element

The condition that a set $\breve{X} \subseteq_w \breve{A}$ has a weak $<_w$ -least element x reads

$$\exists x(x \in_w \breve{X})[\forall y \in_w \breve{X}[(x \leq_w y) \vee \neg_w(y <_w x)]] \qquad (18.5)$$

Remark.18.3.Note that the conditions (i) and (ii) are not equivalent since (3.5.4) and (3.5.5) are not equivalent by the non classical law of the excluded fourth (see sect. 2.1)

Definition 18.6.A w-linear w-ordering $<_w$ of a w-set $\breve{A}_w$ is a weak well w-ordering if every

nonempty w-subset $\breve{X}$ of $\breve{A}_w$ has at least a weak $<_w$ -least w-element. The structure ($\breve{A}_w$,$<_w$) is called a weakly well w-ordered w-set.

19. Inconsistent $w_{[n]}$-Equivalences and $w_{[n]}$-Orderings of the order inconsistency $n \geq 1$.

Definition 19.1.Let $\breve{R}_{w_{[n]}}$ be a binary $w_{[n]}$-relation in $w_{[n]}$-set $\breve{A}_{w_{[n]}}$.

(a) $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-reflexive in $\breve{A}_{w_{[n]}}$ if for all $a \in_{w_{[n]}} \breve{A}_{w_{[n]}}, a\breve{R}_{w_{[n]}}a$.

(b) $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-symmetric in $\breve{A}_{w_{[n]}}$ if for all $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}, a\breve{R}_{w_{[n]}}b$ implies $b\breve{R}_{w_{[n]}}a$.

(c) $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-antisymmetric in $\breve{A}_{w_{[n]}}$ if for all $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}, a\breve{R}_{w_{[n]}}b$ and $b\breve{R}_{w_{[n]}}a$ imply $a =_{w_{[n]}} b$.

(d) $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-asymmetric in $\breve{A}_w$ if for all $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}, a\breve{R}_{w_{[n]}}b$ implies that $\neg_s(b\breve{R}_{w_{[n]}}a)$.i.e. $a\breve{R}_{w_{[n]}}b$ and $b\breve{R}_{w_{[n]}}a$ cannot both be true.

(e) $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-transitive in $\breve{A}_{w_{[n]}}$ if for all $a,b,c \in_{w_{[n]}} \breve{A}_{w_{[n]}}$, $a\breve{R}_{w_{[n]}}b$ and $b\breve{R}_{w_{[n]}}c$ imply $a\breve{R}_{w_{[n]}}c$.

Definition 19.2.Let $\breve{R}_{w_{[n]}}$ be a binary $w_{[n]}$-relation in $\breve{A}_{w_{[n]}}$.

(a) $\breve{R}_{w_{[n]}}$ is an $w_{[n]}$-equivalence on $\breve{A}_w$ if it is $w_{[n]}$-reflexive, $w_{[n]}$-symmetric, and $w_{[n]}$-transitive in $\breve{A}_{w_{[n]}}$.

(b) $\breve{R}_{w_{[n]}}$ is a $w_{[n]}$-ordering of $\breve{A}_{w_{[n]}}$ if it is $w_{[n]}$-reflexive, $w_{[n]}$-antisymmetric, and $w_{[n]}$-transitive in $\breve{A}_{w_{[n]}}$. The pair ($\breve{A}_{w_{[n]}}$,$\breve{R}_{w_{[n]}}$) is called an $w_{[n]}$-ordered $w_{[n]}$-set.

(c) $\breve{R}_{w_{[n]}}$ is a strict w-ordering of $\breve{A}_{w_{[n]}}$ if it is w-asymmetric and $w_{[n]}$-transitive in $\breve{A}_{w_{[n]}}$.

Remark 19.1. Now that we have established the definition of $w_{[n]}$-orderings and strict $w_{[n]}$-orderings, we can use $\leq_{w_{[n]}}$ and to denote $w_{[n]}$-orderings and $<_{w_{[n]}}$ and $\prec_{w_{[n]}}$ to denote strict $w_{[n]}$-orderings.Thus ($\breve{A}_{w_{[n]}}$,$\leq_{w_{[n]}}$) is an pair consisting of a set $\breve{A}_{w_{[n]}}$ and an $w_{[n]}$-ordering $\leq_{w_{[n]}}$,and ($\breve{B}_{w_{[n]}}$,$\prec_{w_{[n]}}$) is a pair consisting of a set $\breve{B}_{w_{[n]}}$ and a strict $w_{[n]}$-ordering $\prec_{w_{[n]}}$.

There is a close relationship between $w_{[n]}$-orderings and strict $w_{[n]}$-orderings as we will see in the next theorem.

Theorem 19.1.(a) Let $\breve{R}_{w_{[n]}}$ be an $w_{[n]}$-ordering of $\breve{A}_{w_{[n]}}$. Then the $w_{[n]}$-relation $\breve{S}_{w_{[n]}}$ in $\breve{A}_{w_{[n]}}$

defined by $a\breve{S}_{w_{[n]}}b$ if and only if $a\breve{R}_{w_{[n]}}b$ and $\neg_s(a =_{w_{[n]}} b)$ is a strict $w_{[n]}$-ordering of $\breve{A}_{w_{[n]}}$.

(b) Let $\breve{S}_{w_{[n]}}$ be a strict $w_{[n]}$-ordering of $\breve{A}_{w_{[n]}}$. Then the w-relation $\breve{R}_{w_{[n]}}$ in $\breve{A}_{w_{[n]}}$ defined by $a\breve{R}_{w_{[n]}}b$ if and only if $a\breve{S}_{w_{[n]}}b$ or $a =_{w_{[n]}} b$ is an $w_{[n]}$-ordering of $\breve{A}_{w_{[n]}}$.

Proof. (a) We need to show that $\breve{S}_{w_{[n]}}$ is $w_{[n]}$-asymmetric. Suppose $a\breve{S}_{w_{[n]}}b$ and $b\breve{S}_{w_{[n]}}a$ both hold for some $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}$. Then $a\breve{R}_{w_{[n]}}b$ and $b\breve{R}_{w_{[n]}}a$ both also hold. It follows that $a =_{w_{[n]}} b$ because $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-antisymmetric. This is a contradiction since $\neg_s(a =_{w_{[n]}} b)$. Therefore $\breve{S}_{w_{[n]}}$ is $w_{[n]}$-asymmetric.

(b) We need to show that $\breve{R}_{w_{[n]}}$ is $w_{[n]}$-antisymmetric. Suppose $a\breve{R}_{w_{[n]}}b$ and $b\breve{R}_{w_{[n]}}a$ both hold for some $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}$. Suppose that $\neg_s(a =_{w_{[n]}} b)$. Then $a\breve{S}_{w_{[n]}}b$ and $b\breve{S}_{w_{[n]}}a$ both hold. This is a contradiction since $\breve{S}_{w_{[n]}}$ is $w_{[n]}$-asymmetric.Therefore $a =_{w_{[n]}} b$, showing that $\breve{R}_{w_{[n]}}$is $w_{[n]}$-antisymmetric.

Definition 19.3.An $w_{[n]}$-ordering $<$ of $\breve{A}_{w_{[n]}}$ is called strong $w_{[n]}$-linear $w_{[n]}$-ordering or strong total $w_{[n]}$-ordering if any two elements of $\breve{A}_{w_{[n]}}$ are comparable in the $w_{[n]}$-ordering $<_{w_{[n]}}$ i.e. for any $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}$, either

$$a <_{w_{[n]}} b, b <_{w_{[n]}} a, \text{or } a =_{w_{[n]}} b. \qquad (19.1)$$

The pair $(\breve{A}_{w_{[n]}}, <_{w_{[n]}})$ is called a strongly $w_{[n]}$-linearly $w_{[n]}$-ordered $w_{[n]}$-set.

Definition 19.4.An $w_{[n]}$-ordering $<_{w_{[n]}}$ of $\breve{A}_{w_{[n]}}$ is called weak $w_{[n]}$-linear if any two $w_{[n]}$-elements of $\breve{A}_{w_{[n]}}$ are comparable in the $w_{[n]}$-ordering $<_{w_{[n]}}$ i.e. for any $a,b \in_{w_{[n]}} \breve{A}_{w_{[n]}}$, either

$$a <_{w_{[n]}} b, b <_{w_{[n]}} a, \neg_w(a <_{w_{[n]}} b), \neg_w(b <_{w_{[n]}} a), \text{ or } a =_{w_{[n]}} b. \qquad (19.2)$$

The pair $(\breve{A}_{w_{[n]}}, <_{w_{[n]}})$ is called weakly w-linearly $w_{[n]}$-ordered $w_{[n]}$-set.

Definition 19.5. Let $<_{w_{[n]}}$ be a $w_{[n]}$-linear $w_{[n]}$-ordering $<_{w_{[n]}}$ of a $w_{[n]}$-set $\breve{A}_{w_{[n]}}$.

(i) The condition that a $w_{[n]}$-set $\breve{X}_{w_{[n]}} \subseteq_{w_{[n]}} \breve{A}_{w_{[n]}}$ has a strong $<_{w_{[n]}}$ -least $w_{[n]}$-element x reads

$$\exists x(x \in_{w_{[n]}} \breve{X}_{w_{[n]}})[\forall y \in_{w_{[n]}} \breve{X}_{w_{[n]}}(x \leq_{w_{[n]}} y)]. \qquad (19.3)$$

(ii) We assume now that a set $\breve{X}_{w_{[n]}} \subseteq_{w_{[n]}} \breve{A}_{w_{[n]}}$ has no a strong $<_{w_{[n]}}$ -least $w_{[n]}$-element The condition that a $w_{[n]}$-set $\breve{X}_{w_{[n]}} \subseteq_{w_{[n]}} \breve{A}_{w_{[n]}}$ has a weak $<_{w_{[n]}}$ -least $w_{[n]}$-element x reads

$$\exists x(x \in_{w_{[n]}} \breve{X}_{w_{[n]}})[\forall y \in_{w_{[n]}} \breve{X}_{w_{[n]}}[(x \leq_{w_{[n]}} y) \vee \neg_w(y <_{w_{[n]}} x)]] \qquad (19.4)$$

Remark.**19**.**2**.Note that the conditions (i) and (ii) are not equivalent since (3.5.4) and (3.5.5) are not equivalent by the non classical law of the excluded $(n+1)$-th (see sect. 2.2)

Definition 19.6.A $w_{[n]}$-linear $w_{[n]}$-ordering $<_{w_{[n]}}$ of a $w_{[n]}$-set $\breve{A}_{w_{[n]}}$ is a weak well $w_{[n]}$-ordering if every nonempty $w_{[n]}$-subset $\breve{X}_{w_{[n]}}$ of $\breve{A}_{w_{[n]}}$ has at least a weak $<_{w_{[n]}}$ -least $w_{[n]}$-element. The structure $(\breve{A}_{w_{[n]}}, <_{w_{[n]}})$ is called a weakly well $w_{[n]}$-ordered $w_{[n]}$-set.

Inconsistent natural numbers $\mathbb{N}_{w_{[n]}}$.

20. Almost classical w-natural numbers $\mathbb{N}_w^{\mathbf{cl}}$.

In defining the almost classical w-natural numbers (or a.cl. w-natural) we begin by examining the most fundamental set, the empty almost classical set $\breve{\emptyset}_{s,w}$.We can very easily create a pattern that is a prime candidate for the definition of the almost classical w-natural numbers:

$0_w^{\mathbf{cl}} = \breve{\emptyset}_{s,w},$

$1_w^{\mathbf{cl}} =_w \{0_w^{\mathbf{cl}}\}_w =_w 0_w^{\mathbf{cl}} \cup_w \{0_w^{\mathbf{cl}}\}_w =_w \{\breve{\emptyset}_{s,w}^{\mathbf{cl}}\}_w,$

$2_w^{\mathbf{cl}} = \{0_w^{\mathbf{cl}}, 1_w^{\mathbf{cl}}\}_w =_w 1_w^{\mathbf{cl}} \cup_w \{1_w^{\mathbf{cl}}\}_w =_w \{\breve{\emptyset}_{s,w}, \{\breve{\emptyset}_{s,w}\}_w\}_w$

$3_w^{\mathbf{cl}} =_w \{0_w^{\mathbf{cl}}, 1_w^{\mathbf{cl}}, 2_w^{\mathbf{cl}}\}_w =_w 2_w^{\mathbf{cl}} \cup_w \{2_w^{\mathbf{cl}}\}_w =_w \{\breve{\emptyset}_{s,w}, \{\breve{\emptyset}_{s,w}\}_w, \{\breve{\emptyset}_{s,w}, \{\breve{\emptyset}_{s,w}\}_w\}_w\}_w$, etc.

Definition 20.1. Let $\mathbf{S}_w^{\mathbf{cl}}(y)$ abbreviate $y \cup_w^{\mathbf{cl}} \{y\}_w^{\mathbf{cl}}$. Almost classical w-set $\breve{X}_w^{\mathbf{cl}}$ is called w-inductive if

$$\exists \breve{X}_w^{\mathbf{cl}}\left[\breve{\emptyset}_{s,w} \in_w \breve{X}_w^{\mathbf{cl}} \wedge \forall y(y \in_w \breve{X}_w^{\mathbf{cl}} \Rightarrow \mathbf{S}_w^{\mathbf{cl}}(y) \in_w \breve{X}_w^{\mathbf{cl}})\right]. \quad (20.1)$$

Definition 20.2. (i) The set of all almost classical w-natural numbers is defined by $\mathbb{N}_w^{\mathbf{cl}} \triangleq \left\{y | y \in_w \breve{X}_w^{\mathbf{cl}} \text{ for any almost classical } w\text{-inductive } w\text{-set } \breve{X}_w^{\mathbf{cl}}\right\}_w$.

(ii) If $n \in_w \mathbb{N}_w^{\mathbf{cl}}$, then $n +_w^{\mathbf{cl}} 1_w^{\mathbf{cl}} \in_w \mathbb{N}_w^{\mathbf{cl}}$, where $n +_w^{\mathbf{cl}} 1_w^{\mathbf{cl}}$ denotes the w-successor to n.

Theorem 20.1.Almost classical w-induction.

$$\forall \breve{X}_w^{\mathbf{cl}}(\breve{X}_w^{\mathbf{cl}} \subset_w \mathbb{N}_w^{\mathbf{cl}})\left[\left[\breve{\emptyset}_{s,w} \in_w \breve{X}_w^{\mathbf{cl}} \wedge \forall x[x \in_w \breve{X}_w^{\mathbf{cl}} \to \mathbf{S}_w^{\mathbf{cl}}(x) \in_w \breve{X}_w^{\mathbf{cl}}]\right] \to \breve{X}_w^{\mathbf{cl}} =_w \mathbb{N}_w^{\mathbf{cl}}\right]. \quad (20.2)$$

Proof.Immediately from theorem 3.4.1.

Definition 20.3. An almost classical w-ordering $<_w^{\mathbf{cl}}$ of $\mathbb{N}_w^{\mathbf{cl}}$ is called <u>almost classical w-linear w-ordering</u> if any two w-elements of $\mathbb{N}_w^{\mathbf{cl}}$ are comparable in the w-ordering $<_w^{\mathbf{cl}}$, i.e. for any $a, b \in_w \mathbb{N}_w^{\mathbf{cl}}$, either $a <_w^{\mathbf{cl}} b, b <_w^{\mathbf{cl}} a$ or $a =_w^{\mathbf{cl}} b$.

Definition 20.4.We define now the almost classical w-relations:

(i) $<_w^{\mathbf{cl}}$ on $\mathbb{N}_w^{\mathbf{cl}}$ by: for all $m, n \in_w \mathbb{N}_w^{\mathbf{cl}}, m <_w^{\mathbf{cl}} n$ if and only if $m \in_w n$,

(ii) $\leq_w^{\mathbf{cl}}$ on $\mathbb{N}_w$ by: for all $m, n \in_w \mathbb{N}_w, m \leq_w^{\mathbf{cl}} n$ if and only if $m \in_w n$ or $m =_w n$,

Theorem 20.2. $(\mathbb{N}_w^{\mathbf{cl}}, <_w^{\mathbf{cl}})$ is a linearly ordered almost classical w-set.

Proof. We need to show (**I**) The relation $<_w^{\mathbf{cl}}$ is an almost classical w-ordering of $\mathbb{N}_w^{\mathbf{cl}}$ and (**II**) Any two elements in $\mathbb{N}_w^{\mathbf{cl}}$ are comparable. We will do this by induction.

(**I**) We need to show (**A**) $<_w^{\mathbf{cl}}$ is w-transitive on $\mathbb{N}_w^{\mathbf{cl}}$ and

(**B**) $<_w^{\mathbf{cl}}$ is w-asymmetric on $\mathbb{N}_w^{\mathbf{cl}}$.

(**I.A.**) Consider the property $\mathbf{P}_w^{\mathbf{cl}}(n)$: for all $k, m \in_w \mathbb{N}_w^{\mathbf{cl}}$, if $k <_w^{\mathbf{cl}} m$ and $m <_w^{\mathbf{cl}} n$, then $k <_w^{\mathbf{cl}} n$.

We need to show this holds for all $n \in_w \mathbb{N}_w^{\mathbf{cl}}$.

(i) Base case: Consider $\mathbf{P}_w^{\mathbf{cl}}(0)$.

Since there does not exist an $m \in \mathbb{N}_w^{\mathbf{cl}}$ such that $m < 0$, $P_w^{\mathbf{cl}}(0)$ is trivially true.

(ii) Induction hypothesis: Suppose $P_w^{\mathbf{cl}}(n)$ holds. Consider $P_w^{\mathbf{cl}}(n+1)$.

Suppose $k < m$ and $m < n + 1$ both hold. This implies $m < n$ or $m = n$.

Case 1) $m < n$. Then $k < n$ by induction hypothesis.

Case 2) $m = n$. Then since $k < m$, $k < n$ is trivial.

Thus $P(n)$ holds for all $n \in \mathbb{N}_w^{\mathbf{cl}}$.

Therefore $<$ is transitive on $\mathbb{N}_w^{\mathbf{cl}}$.

(**I.B.**) Suppose have $n <_w^{\mathbf{cl}} m$ and $m <_w^{\mathbf{cl}} n$. Then by w-transitivity $n <_w^{\mathbf{cl}} n$.

Consider the property $\mathbf{Q}_w^{\mathbf{cl}}(n)$: $n \nless_w^{\mathbf{cl}} n$, where $n \nless_w^{\mathbf{cl}} n$ abbreviate $\neg_s(n <_w^{\mathbf{cl}} n)$.

We need to show this holds for all $n \in_w \mathbb{N}_w^{\mathbf{cl}}$.

(i) Base case: Consider $\mathbf{Q}_w^{\mathbf{cl}}(0_w^{\mathbf{cl}})$.

Suppose $\mathbf{Q}_w^{\mathbf{cl}}(0_w^{\mathbf{cl}})$ does not hold. Then we have $0_w^{\mathbf{cl}} <_w^{\mathbf{cl}} 0_w^{\mathbf{cl}}$, which by definition is $\breve{\emptyset}_{s,w} \in_w \breve{\emptyset}_{s,w}$, which is a contradiction to the defnition of $\breve{\emptyset}_{s,w}$.
(ii) Induction hypothesis: Suppose $Q_w^{\mathbf{cl}}$(n) holds. Consider $Q_w^{\mathbf{cl}}$(n + ${}_w^{\mathbf{cl}} 1_w^{\mathbf{cl}}$).
Suppose $Q_w^{\mathbf{cl}}$(n $+_w^{\mathbf{cl}}$ $1_w^{\mathbf{cl}}$) does not hold. Then n + 1 < n + 1, by definition, is n + 1 ∈ n + 1.
We know n + 1 = n ∪ {n}, which implies that n + 1 ∈ n or n + 1 = n.
Case 1) n + 1 ∈ n. Thus n + 1 < n. But since n < n + 1, by transitivity we have n < n, which contradicts the induction hypothesis.
Case 2) n + 1 = n. This is obviously a contradiction.
Thus Q(n) holds for all n ∈ $\mathbb{N}_w^{\mathbf{cl}}$.
Therefore < is asymmetric on $\mathbb{N}_w^{\mathbf{cl}}$.
(II) We need to show any two elements in $\mathbb{N}_w^{\mathbf{cl}}$ are comparable in <.
Consider the property R(n) : ∀m ∈ N, either m < n, n < m, or m = n. We need to show this holds for all n ∈ $\mathbb{N}_w^{\mathbf{cl}}$.
(i) Base case: Consider R(0).
0 ≤ m for all m ∈ $\mathbb{N}_w^{\mathbf{cl}}$, so 0 < m or m = m. Thus R(0) holds.
(ii) Induction hypothesis: Suppose R(n) holds. Consider R(n + 1).
Consider an arbitrary m ∈ $\mathbb{N}_w^{\mathbf{cl}}$. Since R(n) holds, n < m, m < n, or m = n.
Case 1) m < n. Then since n < n + 1, by transitivity m < n + 1.
Case 2) m = n. Then since n < n + 1, m < n + 1 is trivial.
Case 3) n < m. We need to show m = n + 1 or n + 1 < m.
Apply induction on m. Consider the property $S(m)$: for all n ∈ $\mathbb{N}_w^{\mathbf{cl}}$ if n $<_w^{\mathbf{cl}}$ m, then n + 1 $\leq_w^{\mathbf{cl}}$ m. Need to show this holds for all $m \in_w \mathbb{N}_w^{\mathbf{cl}}$.
a) Base case: Consider $S(0)$.
$S(0)$ holds since there is no $n <_w^{\mathbf{cl}} 0$.
b) Induction hypothesis: Suppose S(m) holds. Consider S(m + 1).
Assume n $<_w^{\mathbf{cl}}$ m + 1 ⇒ n $<_w^{\mathbf{cl}}$ m or m = n.
Case i) n < m. Thus n + 1 ≤ m by induction hypothesis.
m < m + 1 implies n + 1 < m + 1. Thus n + 1 ≤ m + 1.
Case ii) n = m. Thus n + 1 = m + 1 implies n + 1 ≤ m + 1.
∴ S(m) holds for all m ∈ N.
Thus R(n) holds for all n ∈ N.
Therefore any two elements in N are comparable in <.
Therefore (N, <) is a linearly ordered set.

21.Inconsistent $w_{\{0\}}$-natural numbers of the order inconsistency zero $\mathbb{N}_{w_{\{0\}}}$.

Definition 21.1. Let $S_w(y)$ abbreviate $y \cup_w \{y\}_w$.A w-set $\breve{X}$ is called w-inductive if

$$\exists \breve{X}\left[\breve{\emptyset}_w \in_w \breve{X} \wedge \forall y(y \in_w \breve{X} \Rightarrow S_w(y) \in_w \breve{X})\right]. \qquad (21.1)$$

Definition 21.2. The set of all w-natural numbers is defined by
$\mathbb{N}_w \triangleq \left\{y | y \in_w \breve{X} \text{ for any } w\text{-inductive } w\text{-set } \breve{X}\right\}$
We denote this w-set by $\mathbb{N}_w$.

Theorem 21.1. w**-Induction principle**.

$$\forall \breve{X}(\breve{X} \subset_w \mathbb{N}_w)\Big[\Big[\breve{\emptyset}_w \in_w \breve{X} \wedge \forall x[x \in_w \breve{X} \Rightarrow S_w(x) \in_w \breve{X}]\Big] \Rightarrow \breve{X} =_w \mathbb{N}_w\Big]. \qquad (21.2)$$

Proof.Immediately from theorem 3.4.2.

Definition 21.3. An $w_{\{0\}}$-ordering $<_{w_{\{0\}}}$ of $\mathbb{N}_{w_{\{0\}}}$ is called $w_{\{0\}}$-linear if any two $w_{\{0\}}$-elements of $\mathbb{N}_{w_{\{0\}}}$ are comparable in the $w_{\{0\}}$-ordering $<_{w_{\{0\}}}$, i.e. for any $a,b \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}$, either $a <_{w_{\{0\}}} b, a \nless^w_{w_{\{0\}}} b, b <_w a, b \nless^w_{w_{\{0\}}} a$, or $a =_w b$.

Definition 21.4. (i) The $w_{\{0\}}$-relation $<^*_{w_{\{0\}}}$ on $\mathbb{N}_{w_{\{0\}}}$ is defined by: for all $m,n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}$ such that $m \neq^w_{w_{\{0\}}} n$: $m <_{w_{\{0\}}} n$ if and only if $m \in_{w_{\{0\}}} n$ or $n \notin^w_{w_{\{0\}}} m$

(ii) The relation $\leq^*_w$ on $\mathbb{N}_w$ is defined by: for all $m,n \in_w \mathbb{N}_w, m \leq_w n$ if and only if $m \in_w n$ or $n \notin^w_{w_{\{0\}}} m$ or $m =_w n$.

Definition 21.5.We define now the $w_{\{0\}}$-relations:

(i) $<_{w_{\{0\}}}$ on $\mathbb{N}_{w_{\{0\}}}$ by: for all $m,n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}, m <_{w_{\{0\}}} n$ if and only if $m \in_{w_{\{0\}}} n$,

(ii) $\prec_{w_{\{0\}}}$ on $\mathbb{N}_{w_{\{0\}}}$ by: for all $m,n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}, m \prec_{w_{\{0\}}} n$ if and only if $n \notin^w_{w_{\{0\}}} m$,

(iii) $\leq_{w_{\{0\}}}$ on $\mathbb{N}_{w_{\{0\}}}$ by: for all $m,n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}, m \leq_{w_{\{0\}}} n$ if and only if $m \in_{w_{\{0\}}} n$ or $m =_{w_{\{0\}}} n$,

(iv) $\preceq_{w_{\{0\}}}$ on $\mathbb{N}_w$ by: for all $m,n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}, m \preceq_{w_{\{0\}}} n$ if and only if $n \notin^w_{w_{\{0\}}} m$ or $m =_{w_{\{0\}}} n$.

Theorem 21.2. $(\mathbb{N}_{w_{\{0\}}}, <_{w_{\{0\}}})$ is a $w_{\{0\}}$-linearly $w_{\{0\}}$-ordered $w_{\{0\}}$-set.

Proof. We need to show:

(**I**) The relation $<_{w_{\{0\}}}$ is an $w_{\{0\}}$-ordering of $\mathbb{N}_{w_{\{0\}}}$ and

(**II**) Any two $w_{\{0\}}$-elements in $\mathbb{N}_{w_{\{0\}}}$ are comparable in the $w_{\{0\}}$-ordering $<_{w_{\{0\}}}$.

We will do this by induction.

(**I**) We need to show:

(**A**) $<_{w_{\{0\}}}$ is $w_{\{0\}}$-transitive on $\mathbb{N}_{w_{\{0\}}}$ and

(**B**) $<_{w_{\{0\}}}$ is $w_{\{0\}}$-asymmetric on $\mathbb{N}_{w_{\{0\}}}$.

(**I**.**A**.) Consider the property $\mathbf{P}(n)$: for all $k,m \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}$, if $k <_{w_{\{0\}}} m$ and $m <_{w_{\{0\}}} n$, then $k <_{w_{\{0\}}} n$.

We need to show this holds for all $n \in_{w_{\{0\}}} \mathbb{N}_{w_{\{0\}}}$.

(i) Base case: Consider $\mathbf{P}(0_{w_{\{0\}}})$.

Since there does not exist an $m \in_w \mathbb{N}_w$ such that $m <_w \breve{\emptyset}, \mathbf{P}(\breve{\emptyset})$ is trivially true.

(ii) Induction hypothesis: Suppose $\mathbf{P}(n)$ holds. Consider $\mathbf{P}(n +_w 1)$.

Suppose $k <_w m$ and $m <_w n +_w 1_w$ both hold. This implies $m <_w n$ or $m =_w n$.

Case 1) $m <_w n$. Then $k <_w n$ by induction hypothesis.

Case 2) $m =_w n$. Then since $k <_w m, k <_w n$ is trivial.

Thus $\mathbf{P}(n)$ holds for all $n \in_w \mathbb{N}_w$.

Therefore $<_w$ is transitive on $\mathbb{N}_w$.

(**I**.**B**.) Suppose have $n <_w m$ and $m <_w n$. Then by transitivity $n <_w n$.

Consider the property $\mathbf{Q}(n)$:

$\neg_s(n <^1_w n)$. We need to show this holds for all $n \in_w \mathbb{N}_w$.

(i) Base case: Consider $\mathbf{Q}(0_w)$.

Suppose $\mathbf{Q}(0_w)$ does not hold. Then we have $0_w <^1_w 0_w$, which by definition is $\breve{\emptyset} \in_w \breve{\emptyset}$, which is a contradiction to the defnition of $\breve{\emptyset}_w$.

(ii) Induction hypothesis: Suppose $\mathbf{Q}(n)$ holds. Consider $\mathbf{Q}(n +_w 1_w)$.

Suppose $\mathbf{Q}(n +_w 1_w)$ does not hold. Then $n +_w 1_w <_w^1 n +_w 1_w$, by definition, is $n +_w 1_w \in_w n +_w 1_w$.
We know $n + 1_w =_w n \cup_w \{n\}_w$, which implies that $n + 1 \in_w n$ or $n +_w 1 =_w n$.
Case 1) $n +_w 1_w \in_w n$. Thus $n +_w 1_w <_w n$. But since $n <_w n +_w 1_w$, by transitivity we have $n <_w n$, which contradicts the induction hypothesis.
Case 2) $n +_w 1_w =_w n$. This is obviously a contradiction.
Thus $\mathbf{Q}(n)$ holds for all $n \in_w \mathbb{N}_w$.
Therefore $<$ is asymmetric on $\mathbb{N}_w$.
(**II**) We need to show any two elements in $\mathbb{N}_w$ are comparable in $(\cdot <_w \cdot)$
Consider the property
$\mathbf{R}(n)$: $\forall m \in_w \mathbb{N}_w$, either $m <_w n, n <_w m$, or $m =_w n$. We need to show this holds for all $n \in_w \mathbb{N}_w$.
(i) Base case: Consider $\mathbf{R}(0_w)$.
$0_w \leq_w m$ for all $m \in_w \mathbb{N}_w$, so $0_w <_w m$ or $m =_w m$. Thus $\mathbf{R}(0_w)$ holds.
(ii) Induction hypothesis: Suppose $\mathbf{R}(n)$ holds. Consider $\mathbf{R}(n +_w 1_w)$.
Consider an arbitrary $m \in_w \mathbb{N}_w$. Since $\mathbf{R}(n)$ holds, $n <_w m, m <_w n$, or $m =_w n$.
Case 1) $m <_w n$. Then since $n <_w n +_w 1_w$, by transitivity $m <_w n +_w 1_w$.
Case 2) $m =_w n$. Then since $n <_w n +_w 1_w, m <_w n +_w 1_w$ is trivial.
Case 3) $n <_w m$. We need to show $m =_w n +_w 1_w$ or $n +_w 1_w <_w m$.
Apply induction on m. Consider the property
$\mathbf{S}(m)$:for all $n \in_w \mathbb{N}_w$ if $n <_w m$, then $n +_w 1_w \leq_w m$.
Need to show this holds for all $m \in_w \mathbb{N}_w$.
a) Base case: Consider $\mathbf{S}(0_w)$. $\mathbf{S}(0_w)$ holds since there is no $n <_w 0_w$.
b) Induction hypothesis: Suppose $\mathbf{S}(m)$ holds. Consider $\mathbf{S}(m +_w 1_w)$.
Assume $n <_w m +_w 1_w \to n <_w m$ or $m =_w n$.
Case i) $n <_w m$. Thus $n +_w 1_w \leq_w m$ by induction hypothesis.
$m <_w m +_w 1_w$ implies $n +_w 1_w <_w m +_w 1_w$. Thus $n +_w 1_w \leq_w m +_w 1_w$.
Case ii) $n =_w m$. Thus $n +_w 1_w =_w m +_w 1_w$ implies $n +_w 1_w \leq_w m +_w 1_w$.
$\therefore$ $\mathbf{S}(m)$ holds for all $m \in_w \mathbb{N}_w$.
Thus $\mathbf{R}(n)$ holds for all $n \in_w \mathbb{N}_w$.
Therefore any two elements in $\mathbb{N}_w$ are comparable in $<_w$.
Therefore $(\mathbb{N}_w, <_w)$ is a w-linearly w-ordered set.
Definition 21.6. Let $<_w$ be a w-linear w-ordering $<_w$ of a set $\mathbb{N}_w$.
(i) The condition that a set $\breve{X} \subseteq_w \mathbb{N}_w$ has a <u>strong</u> $<_w$ -least element x reads

$$\exists x(x \in_w \breve{X})[\forall y \in_w \breve{X}(x \leq_w y)]. \tag{21.3}$$

(ii) We assume now that a set $\breve{X} \subseteq_w \mathbb{N}_w$ has no <u>strong</u> $<_w$ -least element
The condition that a set $\breve{X} \subseteq_w \breve{A}$ has a <u>weak</u> $<_w$ -least element x reads

$$\exists x(x \in_w \breve{X})[\forall y \in_w \breve{X}[(x \leq_w y) \vee \neg_w(y <_w x)]] \tag{21.4}$$

or in the following equivalent form

$$\exists x(x \in_w \breve{X})[\forall y \in_w \breve{X}(x \leq_w^2 y)]. \tag{21.5}$$

Remark.21.2. Note that the conditions (i) and (ii) are not equivalent since (3.6.1) and (3.6.3) are not equivalent by **Theorem 3.6.2**.
Definition 21.7.A w-linear w-ordering $<_w$ of a w-set $\mathbb{N}_w$ is a <u>weak</u> well w-ordering if every

nonempty w-subset $\breve{X}$ of $\mathbb{N}_w$ has at least a weak $<_w$ -least w-element. The structure $(\mathbb{N}_w, <_w)$ is called a weakly well w-ordered w-set.

Theorem 21.3. $(\mathbb{N}_w, <_w)$ is a weakly well w-ordered w-set.

Proof. We will prove by using induction.

(**I**) Let X be a nonempty w-subset of $\mathbb{N}_w$ and there exists strong complement $\mathbb{N}_w \backslash_{s,w} X$. Suppose that:

(i) X does not have a strong $<_w$ -least w-element and

(ii) X does not have a weak $<_w$ -least w-element.

Then consider the set $\mathbb{N}_w \backslash_{s,w} X$.

Case 1) $\mathbb{N}_w \backslash_{s,w} X =_w \breve{\emptyset}_w$. Then $X =_w \mathbb{N}_w$ and so 0_w is a strong $<_w$ -least element. But this is a contradiction.

Case 2) $\neg_s \left(\mathbb{N}_w \backslash_{s,w} X =_w \breve{\emptyset}_w\right)$. There exists an $n \in_w \mathbb{N}_w \backslash_{s,w} X$ such that for all k such that $(k <_w^1 n) \wedge (k <_w^1 n)$ the following hold $k \in_w \mathbb{N}_w \backslash_{s,w} X$ (n necessarily exists because $0_w \in_w \mathbb{N}_w \backslash_{s,w} X$, else $0_w \in_w X$ and would be a strong $<_w$ -least element of X.) Since we have supposed that:(i) X does not have a strong $<_w$ -least least element and (ii) X does not have a weak $<_w$ -least element, thus $\neg_s(n \in_w X)$ and therefore $n \in_w \mathbb{N}_w \backslash_{s,w} X$. Thus we see that if for all k such that $(k <_w^1 n) \wedge (k <_w^1 n), k \in_w \mathbb{N}_w \backslash_{s,w} X$ the following hold $n \in_w \mathbb{N}_w \backslash_{s,w} X$.

Using now strong induction we can conclude that $n \in_w \mathbb{N}_w \backslash_{s,w} X$ for all $n \in_w \mathbb{N}_w$. Thus $n \in_w \mathbb{N}_w \backslash_{s,w} X =_w \mathbb{N}_w$ implies $X =_w \breve{\emptyset}_w$. This is a contradiction to X being a nonempty subset of $\mathbb{N}_w$ in consistent sense.

22.Recursion and the addition operation in $\mathbb{N}_{w_{\{0\}}}$.

Definition 22.1. A w-sequence is a w-function whose domain is a w-natural number or $\mathbb{N}_w$. A w-sequence whose domain is some w-natural number $n_w \in_w \mathbb{N}_w$ is called a w-finite w-sequence of length n_w and is denoted $\langle a_i | i <_w n \rangle_w$.

Definition 22.1. A w-function $\breve{t}_w : (m +_w 1_w) \to \breve{A}_w$ is called an m-step w-computation based on a and $\breve{g}_w$ if $\breve{t}_{w,0_w} =_w a,$ and for all k such that $0_w \leq_w k <_w m, \breve{t}_{w,k+_w 1_w} =_w \breve{g}_w(\breve{t}_{w,k}, k)$.

Theorem 22.1. **The w-Recursion Theorem**.

For any w-set $\breve{A}_w$, any $a \in_w \breve{A}_w$, and any function $\breve{g}_w : \breve{A}_w \times_w \mathbb{N}_w \to \breve{A}_w$, there exists a w-unique w-sequence $\breve{f}_w : \mathbb{N}_w \to \breve{A}_w$ such that

(a) $\breve{f}_{w,0_w} =_w a$ and (b) $\breve{f}_{w,n_w +_w 1_w} =_w \breve{g}_w(f_{w,n}, n_w) \forall n_w \in_w \mathbb{N}_w$.

Proof. (The existence of $\breve{f}_w$)

Let $a \in_w \breve{A}_w$ and $\breve{g}_w : \mathbb{N}_w \times_w \breve{A}_w \to \breve{A}_w$.

Let $\breve{F}_w =_w \{\breve{t}_w \in_w \mathbf{P}_w(\mathbb{N}_w \times_w \breve{A}_w) \mid \breve{t}_w$ is an m-step w-computation on a and $\breve{g}_w$ for some $m \in_w \mathbb{N}_w\}$.

Let $\breve{f}_w =_w \cup_w \breve{F}_w$.

Claim 1: $\breve{f}_w$ is a w-function.

By Theorem 3.4.1, it is enough to show that F_w is a system of w-compatible w-functions.

Let $\breve{t}_w, \breve{u}_w \in \breve{F}_w, \mathbf{dom}(\breve{t}_w) =_w n \in_w \mathbb{N}_w, \mathbf{dom}(u) =_w m \in_w \mathbb{N}_w$.

We can assume without loss of generality that $n \leq_w m$. We will use w-induction principle (**Theorem 3.6**)

to prove $\forall k <_w n\ (t_k =_w u_k)$.
(a) Base case: $k =_w 0_w$.
We know $\breve{t}_w$ and $\breve{u}_w$ are w-computations based on a and $\breve{g}_w$.
Thus $\breve{t}_{w,0_w} =_w a =_w \breve{u}_{w,0_w}$is trivial.
(b) w-Induction hypothesis: Let k be such that $k +_w 1_w \leq_w n$.
Suppose $\breve{t}_{w,k} =_w \breve{u}_{w,k}$. Then $\breve{t}_{w,k+1} =_w \breve{g}_w(\breve{t}_k, k) =_w \breve{g}_w(\breve{u}_{w,k}, k) =_w \breve{u}_{w,k+_w 1_w}$.
Therefore F_w is a system of w-compatible w-functions.
Therefore $\breve{f}_w$ is a w-function.
Claim 2: $\mathbf{dom}(\breve{f}_w) =_w \mathbb{N}_w$ and ran $\breve{f}_w \subseteq_w \breve{A}_w$.
(It is obvious that dom$(\breve{f}_w) \subseteq_w \mathbb{N}_w$ and that ran $\breve{f}_w \subseteq_w \breve{A}_w$.
We then need to show that $\mathbb{N}_w \subseteq_w \mathbf{dom}(\breve{f}_w)$ to
prove $\mathbf{dom}(\breve{f}_w) =_w \mathbb{N}_w$. We will prove with w-induction.)
(a) Base case: Clearly $\breve{t}_w =_w \{(0_w, a)\}$ is a 0_w-step w-computation.
Thus $0_w \in_w \mathbf{dom}(\breve{f}_w)$.
(b) Induction hypothesis: Suppose t is an n-step computation, where
$n \in_w \mathbf{dom}(\breve{f}_w)$.
Define $\breve{t}'_w$ on $(n +_w 1_w) +_w 1_w$ by $\breve{t}'_{w,k} =_w t_{w,k}$ if $k \leq_w n, t'_{w,n+_w 1_w} =_w \breve{g}_w(\breve{t}_{w,n}, n)$.
We can see that $\breve{t}'_w$ is an n+$_w$1$_w$ step w-computation.
Thus $(n +_w 1_w) \in_w \mathbf{dom}(\breve{f}_w)$. Therefore $\mathbf{dom}(\breve{f}_w) =_w \mathbb{N}_w$.
Claim 3: $\breve{f}_w$ satisfies conditions (a) and (b)
(a) Clearly $\breve{f}_{w,0} =_w a$ since $\breve{t}_{w,0} =_w a$ for all $\breve{t}_w \in_w \breve{F}_w$. Thus satisfying (a).
(b) Let t be an (n+1) step computation. Then $\breve{f}_{w,k} =_w \breve{t}_{w,k}$ for all $k \in_w \mathbf{dom}(\breve{t}_w)$.
This implies $\breve{f}_{w,n+1} =_w \breve{t}_{w,n+1} =_w \breve{g}_w(\breve{t}_{w,n}, n) =_w \breve{g}_w(\breve{f}_{w,n}, n)$. Thus satisfying (b).
Therefore the existence of a function $\breve{f}_w$ satisfying the properties required by the Recursion Theorem follows from Claims 1,2,3.
(The uniqueness of $\breve{f}_w$)
Let $\breve{h}_w : \mathbb{N}_w \to \breve{A}_w$ satisfy (a) and (b). We will show $\breve{f}_{w,n} =_w \breve{h}_{w,n}$ for all $n \in_w \mathbb{N}_w$ by w-induction.
(a) Base case: $\breve{f}_{w,0} =_w a =_w \breve{h}_{w,0}$ is trivial.
(b) Induction hypothesis: Suppose $\breve{f}_{w,n} =_w \breve{h}_{w,n}$.
Then $\breve{f}_{w,n+1} =_w \breve{g}_w(\breve{f}_{w,n}, n) =_w \breve{g}_w(h_{w,n}, n) =_w \breve{h}_{w,n+1}$.
Therefore $\breve{h}_w =_w \breve{f}_w$.

Theorem 20.2. The Parametric w-Recursion Theorem
Let $\breve{a}_w : \breve{P}_w \to \breve{A}_w$ and $\breve{g}_w : \breve{P}_w \times_w \breve{A}_w \times_w \mathbb{N}_w \to \breve{A}_w$ be w-functions. There exists a unique w-function $\breve{f}_w : \breve{P}_w \times_w \mathbb{N}_w \to \breve{A}_w$ such that
(a) $\breve{f}_w(p, 0_w) =_w \breve{a}_w(p)$ for all $p \in_w \breve{P}_w$
(b) $\breve{f}_w(p, n +_w 1_w) =_w \breve{g}_w(p, \breve{f}_w(p, n), n)$ for all $n \in_w \mathbb{N}_w$ and $p \in_w \breve{P}_w$.
Proof. Define a parametric m-step computation to be a w-function
$\breve{t}_w : \breve{P}_w \times_w (m +_w 1_w) \to \breve{A}_w$ such
that, for all $p \in_w \breve{P}_w, \breve{t}_w(p, 0_w) =_w \breve{a}_w(p)$ and $\breve{t}_w(p, k +_w 1_w) =_w \breve{g}_w(p, \breve{t}_w(p, k), k)$
for all k such that $0_w \leq_w k <_w m$. The rest of the proof is similar to the proof of the recursive theorem with the additional task of carrying p along and so will be omitted.
Notice that the parametric version takes into account an additional variable of p. This allows us to define addition of w-natural numbers because addition is binary w-operation.

Theorem 20.3. **Addition Operation of w-Natural Numbers**.
There is a unique binary w-operation $(\cdot +_w \ \cdot) : \mathbb{N}_w \times \mathbb{N}_w \to \mathbb{N}_w$ such that
(a) $m +_w 0_w =_w m$ for all $m \in_w \mathbb{N}_w$,
(b) $m +_w (n +_w 1_w) =_w (m +_w n) + 1_w$ for all $m,n \in_w \mathbb{N}_w$.
Proof. This is the exact same proof as the parametric version of the w-recursion theorem.
Let $\breve{A}_w =_w \breve{P}_w =_w \mathbb{N}_w, \breve{a}_w(p) =_w p$ for all $p \in_w \breve{P}_w$, and $\breve{g}_w(p,x,n) =_w x +_w 1_w$ for all $p,x,n \in_w \mathbb{N}_w$.
This definition satisfies all properties of addition such as
(i) $a +_w 0_w =_w a$, (ii) $a +_w b =_w b +_w a$, (iii) $a +_w (b +_w c) = (a +_w b) +_w c$.

Inconsistent $w[n]$-Integers $\mathbb{Z}_{w[n]}$, $w[n]$-Rationals $\mathbb{Q}_{w[n]}$, and $w[n]$-Reals $\mathbb{R}_{w[n]}$.

21.Inconsistent w-Integers $\mathbb{Z}_w$, w-Rationals $\mathbb{Q}_w$ and w-Reals $\mathbb{R}_w$ of the order inconsistency zero $\mathbb{N}_w$.

Now that we have the inconsistent natural numbers, defining inconsistent integers and inconsistent rational numbers is well within reach.

Definition 21.1. (i) Let $\mathbb{Z}_w^{\mathbf{cl}\dagger} =_w \mathbb{N}_w^{\mathbf{cl}} \times_w \mathbb{N}_w^{\mathbf{cl}}$. We can define an w-equivalence relation $\approx_w$ on $Z_w^{\mathbf{cl}\dagger}$ by $(a,b)_w \approx_w (c,d)_w$ if and only if $a +_w d =_w b +_w c$. Then we denote the w-set
of all almost classical w-integers by $\mathbb{Z}_w^{\mathbf{cl}} =_w \mathbb{Z}_w^{\mathbf{cl}\dagger}/\approx_w$ (The set of all w-equivalence classes
of $\mathbb{Z}_w^{\mathbf{cl}\dagger}$ modulo $\approx_w$).
(ii) Let $\mathbb{Z}_w^{\dagger} =_w \mathbb{N}_w \times_w \mathbb{N}_w$. We can define an w-equivalence relation $\approx_w$ on $Z_w^{\dagger}$ by $(a,b)_w \approx_w (c,d)_w$ if and only if $a +_w d =_w b +_w c$. Then we denote the w-set of all w-integers by $\mathbb{Z}_w =_w \mathbb{Z}_w^{\dagger}/\approx_w$ (The set of all w-equivalence classes of $\mathbb{Z}_w^{\dagger}$ modulo $\approx_w$).

Definition 21.2. (i) Let $\mathbb{Q}_w^{\mathbf{cl}\dagger} =_w \mathbb{Z}_w^{\mathbf{cl}} \times_w (\mathbb{Z}_w^{\mathbf{cl}} \backslash_w \{0_w^{\mathbf{cl}}\}_w) =_w \{(a,b)_w \in_w \mathbb{Z}_w^{\mathbf{cl}} \times_w \mathbb{Z}_w^{\mathbf{cl}} | b \neq_s 0_w^{\mathbf{cl}}\}_w$.
We can define an w-equivalence relation $\approx_w$ on $\mathbb{Q}_w^{\mathbf{cl}\dagger}$ by $(a,b)_w \approx_w (c,d)_w$ if and only if $a \times_w d =_w b \times_w c$. Then we denote the w-set of all almost classical rational w-numbers by
$\mathbb{Q}_w^{\mathbf{cl}} =_w \mathbb{Q}_w^{\mathbf{cl}\dagger}/\approx_w$ i.e.the almost classical w-set of all equivalence classes of $\mathbb{Q}_w^{\mathbf{cl}\dagger}$ modulo $\approx_w$.
(ii) Let $\mathbb{Q}_w^{\dagger} =_w \mathbb{Z}_w \times_w (\mathbb{Z}_w \backslash_w \{0_w\}_w) =_w \{(a,b)_w \in_w \mathbb{Z}_w \times_w \mathbb{Z}_w | b \neq_s 0_w\}_w$.
We can define an w-equivalence relation $\approx_w$ on $\mathbb{Q}_w^{\dagger}$ by $(a,b)_w \approx_w (c,d)_w$ if and only if $a \times_w d =_w b \times_w c$. Then we denote the w-set of all inconsistent rational w-numbers by $\mathbb{Q}_w =_w \mathbb{Q}_w^{\dagger}/\approx_w$ i.e.the w-set of all equivalence classes of $\mathbb{Q}_w^{\dagger}$ modulo $\approx_w$.

Definition 21.3. A w-linearly w-ordered w-set $(\breve{P}_w, <_w)$ is called w-dense if for any $a,b \in_w \breve{P}$ such that $a <_w b$, there exists $z \in_w \breve{P}_w$ such that $a <_w z <_w b$.

Lemma 21.1. $(\mathbb{Q}_w, <_w)$ is w-dense.

Proof. Let $x = (a,b), y = (c,d) \in \mathbb{Q}_w$ be such that $x <_w y$.
Consider $z = (a \times_w d +_w b \times_w c, 2_w \times_w b \times_w d) \in \mathbb{Q}_w$. It is easily shown that $x <_w z <_w y$.
Before we can define the real numbers, we will need a few more concepts.

Definition 21.4. Let $(\breve{P}, <_w)$ be a linearly w-ordered set.

A pair of w-sets $(\breve{A},\breve{B})$ is called a w-cut if
(a) $\breve{A}$ and $\breve{B}$ are nonempty w-disjoint subsets of $\breve{P}$ and $\breve{A} \cup_w \breve{B} =_w \breve{P}$.
(b) If $a \in_w \breve{A}$ and $b \in_w \breve{B}$, then $a <_w b$.
Definition 21.5. $(\breve{A}_w,\breve{B}_w)$ is called a strong Dedekind w-cut if additionally
(a) $\breve{A}$ does not have a strong $<_w$ -greatest w-element.
$(\breve{A},\breve{B})$ is called a strong w-gap if additionally
(b) $\breve{B}$ does not have a strong $<_w$ -least w-element.
Definition 21.6. $(\breve{A},\breve{B})$ is called a weak Dedekind w-cut if additionally
(a) $\breve{A}$ does not have even a weak $<_w$ -greatest w-element.
$(\breve{A},\breve{B})$ is called a weak w-gap if additionally
(b) $\breve{B}$ does not have even a weak $<_w$ -least w-element.
Remark 21.1. We have two kinds of a strong Dedekind w-cuts
1) Ones where $B =_w \{x \in_w P | x_w \geq p$ for some $p \in_w P\}$,
2) strong w-gaps.

Remark 21.2. We have two kinds of a strong Dedekind w-cuts

This distinction will be needed later in the proof of completion.
We see even though rational numbers are dense, they clearly have gaps. Take for example the two sets
1) $\breve{A} =_w \{x \in_{w_{[n]}} \mathbb{Q}_w | x_w > 0_w$ and $x \times_w x_w > 2\}$
2) $\breve{B} =_w \{x \in_w \mathbb{Q}_w | \neg_w (x \in_w A)\}$
Clearly $(\breve{A},\breve{B})$ is a gap in $\mathbb{Q}_w$.
Intuitively, we know that the w-real numbers cannot have w-gaps,and so our next step is to explore how to close gaps. We notice that the existence of w-gaps is closely related to the existence of w-suprema of w-bounded w-sets.
Definition 21.5. Let $(\breve{P}_w,<_w)$ be a w-dense linearly w-ordered w-set.
(i) $\breve{P}_w$ is a strongly w-complete if every nonempty $\breve{S} \subseteq_w \breve{P}$ bounded above has a strong w-supremum.
(i.e. $(P_w,<_w)$ does not have any w-gaps.)
(ii)
There is a close relationship between dense linearly ordered w-sets and complete linearly w-ordered w-sets as we will show. This close relationship is what will allow us to define the w-real numbers.
Theorem 21.1. Let $(\breve{P}_w,<_w)$ be a dense linearly w-ordered w-set without endpoints. Then there exists a w-complete linearly w-ordered w-set $(\breve{C}_w,\prec_w)$ such that
(a) $\breve{P}_w \subseteq_w \breve{C}_w$.
(b) If $p,q \in_w P_w$, then $p <_w q$ if and only if $p \prec_w q$.
(c) P_w is w-dense in C_w.
(d) C_w does not have w-endpoints.
Furthermore, $(C_w,\prec_w)$ is unique up to an isomorphism over P_w. The w-linearly w-ordered
w-set $(C_w,\prec_w)$ is called the w-completion of $(P_w,<_w)$.
Proof. Part 1: (The existence of w-completion)
We reference the two kinds of Dedekind cuts from remark 7.6.

We will denote those of the first kind by

$[p]_w = (A_w, B_w)$ where $B_w =_w \{x \in_w P_w | x_w \geq p$ for some $p \in_w P_w\}_w$.

We can then define the w-set

$P_w =_w \{[p]_w | p \in_w P_w\}_w$

$C_w =_w \{(A_w, B_w) | (A_w, B_w)$ is a Dedekind w-cut in $(P_w, <_w)\}$.

Furthermore, we can order C_w and P_w

by $(A_w, B_w) \prec (A'_w, B'_w)$ if and only if $A_w \subset A'_w$.

Claim 1: $(P_w, \prec_w)$ is isomorphic to $(P_w, <_w)$.

Let $p, q \in'_w P'_w$ and the corresponding $[p]_w =_w (A_w, B_w), [q]_w =_w (A'_w, B'_w) \in_w P'_w$

where $A_w =_w \{x \in_w P_w | x <_w p\}_w$ and $A'_w = \{x \in P | x <_w q\}_w$.

Suppose $p <_w q$. Then it follows that $A_w \subset_w A'_w$.

So $[p]_w \prec_w [q]_w$, which proves the claim.

Claim 2: $(C_w, \prec_w)$ is a w-linearly w-ordered w-set.

a) Let $[r]_w =_w (A_w, B_w), [s]_w = (A'_w, B'_w)$, and $[t]_w =_w (A''_w, B''_w) \in_w C_w$

where $A_w =_w \{x \in_w P_w | x <_w r\}_w$,

$A'_w =_w \{x \in_w P_w | x <_w s\}_w$, and $A''_w =_w \{x \in_w P_w | x <_w t\}_w$.

Suppose $[r]_w \prec_w [s]_w$ and $[s]_w \prec_w [t]_w$. Then $A \subset_w A'_w$

and $A'_w \subset A''_w \Rightarrow A \subset A''_w \to [r] \prec_w [t]$. Therefore $(C_w, \prec_w)$ is w-transitive.

b) Suppose $[r]_w <_w [s]_w$ and $[s]_w <_w [r]_w$. Then $A \subset_w A'_w$ and $A'_w \subset_w A_w$ which is a contradiction. Therefore $(C_w, \prec_w)$ is w-asymmetric.

c) Take $[s]_w$ and $[t]_w$. Since these sets are defined based on s and $t \in_w P_w$,

one and only one of three cases

can occur: $s <_w t, t <_w s$, or $s =_w t$. It follows that $A \prec A'_w, A'_w \prec A$, or $A = A'_w$.

Thus $[s]_w \prec_w [t]_w, [t]_w \prec_w [s]_w$, or $[t]_w =_w [s]_w$. Therefore $(C_w, \prec_w)$ is w-comparable.

Therefore $(C_w, \prec_w)$ is a w-linearly w-ordered w-set.

Claim 3: $(C_w, \prec_w)$ satisfies (a)-(d) from the theorem.

(a) By definition, P'_w is a w-set of Dedekind w-cuts of P_w. Therefore $P'_w \subseteq_w C_w$

is trivial.

(b) Let $[p]_w =_w (A_w, B_w), [q]_w =_w (A'_w, B'_w) \in_w P'_w$

where $A_w =_w \{x \in_w P_w | x <_w p\}_w$ and

$A'_w =_w \{x \in_w P_w | x <_w q\}_w$. Suppose $[p]_w \prec_w [q]_w$

(where $\prec_w$ denotes the relation in P_w).

It follows that $A \subset_w A'_w$.

We know also that $[p]_w, [q]_w \in_w C_w$. $\therefore$ $[p]_w \prec_w [q]_w$

(where $\prec_w$ denotes the relation in C_w). The

converse is similarly trivial. This shows that $\prec_w$ in P_w coincides with $\prec_w$ in C_w.

(c) Let $[p]_w =_w (A_w, B_w), [q] =_w (A'_w, B'_w) \in_w P'_w$

where A $=_w$ {x $\in_w$ P_w | x $<_w$ p}$_w$ and

A'_w= {x $\in_w$ P_w | x $<_w$ q}$_w$. Suppose $[p]_w \prec_w [q]_w$. Thus $p <_w q$ and A $\subset_w$ A. Consider

z $\in$ A \A. Then

p < z < q and [p] $\prec$ [z] $\prec$ [q]. Since [z] $\in$ P, we can conlude that P is dense in (C, $\prec$).

(d) Let [p] = (A, B) where A = {x $\in$ P | x < p}. Since (P, <) does not have endpoints, there

exists z > p. It follows that there exists [z] such that [p] $\prec$ [z]. Therefore C does not have

w-endpoints.

Claim 4: (C_w, $\prec_w$) is w-complete.
Let S be a nonempty w-subset of C that is w-bounded above.
Let A_s = S {A | (A, B) ∈ S} and B_s = P - A_s = T {B | (A, B) ∈ S}.
We can see that (A_s , B_s) is a dedekind w-cut and is an upper bound of S.
(We need to show that (A_s , B_s) is the supremum of S.)
Suppose (A_0 , B_0) is an upper bound of S. Then A ⊆ A_0 ∀(A, B) ∈ S. It follows that $A_s \subseteq A_0$. This shows that (A_s , B_s) (A_0 , B_0). Therefore (A_s, B_s) is the supremum of S and (C_w, $\prec_w$) is w-complete.
We can see that (A_s, B_s) is a dedekind cut and is an w-upper bound of S.

(We need to show that (A_s , B_s) is the w-supremum of S.)
Suppose (A_0 , B_0) is an w-upper bound of S. Then A ⊆ A_0 ∀(A, B) ∈ S.
It follows that $A_s \subseteq A_0$. This
shows that (A_s, B_s) (A_0 , B_0). Therefore (A_s, B_s) is the w-supremum of S and (C, ≺) is w-complete. Therefore (C, ≺) is the w-completion of (P, <).
Part 2: (Uniqueness of w-completion up to an isomorphism)
Let (C, ≺) and (C^* $\prec^*$) be two w-complete w-linearly w-ordered w-sets satisfying (a)-(d).
We need to show there exists an isomorphism between the two.
If $c \in_w C$, then let $S_c =_w \{p \in P_w | p \leq_w c\}_w$.
If $c^* \in_w C$, then let $S_c^* =_w \{p \in_w P_w | p \leq_w^* c^*\}_w$.
We define the w-mapping h_w : $C_w \to C_w^*$ as follows: $h_w(c) =_w w\text{-}\sup^* S_c$.
We now need to prove that h is onto, preserves w-orderings, and $h_w(x) =_w x \forall x \in_w P_w$.
(1) Let $c^* \in_w C^*$. Then $c^* =_w w\text{-}\sup^*(S_c)$, so we can choose $c =_w {}_w w\text{-}\sup S_{c^*}$.
We see that
$S_c =_w S_{c^*}$ and $h_w(c) =_w c^*$, therefore showing that h is onto.
(2) Let $c \prec_w d$. Then there exists $p \in_w P_w$ such that $c \prec_w p \prec_w d$ because P_w is dense.
We see that $w\text{-}\sup^* S_c \prec_w^* p \prec_w^* w\text{-}\sup^* S_d$, showing that $h(c) \prec_w^* h(d)$.
(3) Let $x \in_w P_w$. Then $w\text{-}\sup(S_x) =_w w\text{-}\sup^*(S_x) =_w x$, so $h(x) =_w x$.

REFERENCES

[1] J.Foukzon,Paraconsistent First-Order Logic with infinite hierarchy levels of a contradiction $\mathbf{LP}^{\#}_{\omega}$. Axiomatical system $\mathbf{HST}^{\#}_{\omega}$, as paraconsistent generalization of Hrbacek set theory $\mathbf{HST}$, arXiv:0805.1481 [math.LO]
https://arxiv.org/abs/0805.1481

[2] J.Foukzon, Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series 4ECM Stockholm
2004 Contributed papers. http://www.math.kth.se/4ecm/poster.list.html

[3] J.Foukzon,Foundation of Paralogical Nonstandard Analysis and its Application to Some Famous Problems of Trigonometrical and Orthogonal Series (February 2004).
Mathematics Preprint Archive Vol. 2004, Issue 2. Available at SSRN:
https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3177547

[4] J.Foukzon,Foundation of Paralogical Nonstandard Analysis and its Application to Some Famous Problems of Trigonometrical and Orthogonal Series Part II.

(February 2004).Mathematics Preprint Archive Vol. 2004,Issue 2,pp. 236-257. Available at SSRN: https://ssrn.com/abstract=3177547

[5] C.E. Mortensen,Inconsistent Number Systems,Notre Dame Journal of Formal Logic Volume 29, Number 1, Winter 1988

[6] C.E. Mortensen,Inconsistent Mathematics,Springer Science & Business Media, 14 Mar 2013 - Mathematics - 158 pages

[7] C.E. Mortensen,Inconsistent Nonstandard Arithmetic, J. Symbolic Logic Volume 52,
Issue 2 (1987), 512-518.

[8] A.I. Arruda,Remarks In Da Costa's Paraconsistent Set Theories,Revista colombiana de matematicas.Volume 19 / 1985 / Article.