Absorption of spin and momentum of a circular polarized electromagnetic wave

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Within the framework of classical electrodynamics, it is shown that the absorption of the spin of a circular polarized plane wave is the same natural process as the absorption of the momentum of such a wave.
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1. Introduction

It has been known since the 19th century [1,2] that electromagnetic radiation of circular polarization contains the angular momentum density, regardless of whether the boundary of this radiation is taken into account or the boundary is not considered. In particular, the relation $G = E\lambda/2\pi$ is owned to Poynting [2]. Here E is the energy per unit volume, and G represents the angular momentum that passed through a unit area per unit time, that is, the torque acting on a unit area. It is this *distributed* torque that was considered in [3]. It is shown there that the distributed torque causes specific mechanical stresses described by a non-symmetrical stress tensor.

The relation $G = E\lambda/2\pi$ is well known. It is given, in particular, in the collection [4]. Crawford [5] emphasizes: "A traveling plane wave can transfer not only energy and linear momentum, but also angular momentum". Feynman tells [6] how a distributed torque arises on a wall which is going to absorb such a wave: "As time goes on, the electric field $\mathbf{E}$ rotates and the displacement of the electron $\mathbf{r}$ rotates with the same frequency. Now let's look at the work being done on this electron. The rate that energy is being put into this electron is v, its velocity, times the component $\mathbf{E}_t$ parallel to the velocity, $dW/dt = e\mathbf{E}_t v$. But look, there is angular momentum being poured into this electron, because there is always a torque about the origin. The torque is $e\mathbf{E}_t r$, which must be equal to the rate of change of angular momentum $dJ/dt = e\mathbf{E}_t r$. Remembering that $v = \omega r$, we have that $dJ/dW = 1/\omega$". Naturally, the angular momentum density of the radiation is proportional to the energy density.

Similarly, Beth [7] writes: "The moment of force or torque exerted on a doubly refracting medium by a light wave passing through it arises from the fact that the dielectric constant is a tensor. Consequently the electric intensity $\mathbf{E}$ is not parallel to the electric polarization $\mathbf{P}$ in the medium. The torque per unit volume produced by the action of the electric field on the polarization of the medium is $\tau_\wedge = \mathbf{P}\times\mathbf{E}$". The famous Beth experiment confirmed this concept [8]. Beth's idea to use the torque to calculate the impact of a wave on a dielectric or a magnet was used in [9].

In the 20th century, this radiation angular momentum density is described by the canonical spin density (in short, the spin tensor) [10-12]

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \mathsf{L}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda}F^{\mu]\nu}\,, \qquad (1.1)$$

where $\mathsf{L} = -F_{\mu\nu}F^{\mu\nu}/4$ is the free electromagnetic field Lagrangian, A^λ is vector potential, and $F_{\mu\nu}$ is the field strength tensor. The local meaning of the spin tensor $\Upsilon^{\lambda\mu\nu}$ is as follows. Spin of a 4-element dV_ν is $dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu}dV_\nu$. This means, for example, that the component $dS^{xy} = dS_z$ of the

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spin that has passed through the area da_z in time dt, is equal to $dS^{xy} = \Upsilon^{xyz} da_z dt$, i.e. Υ^{xyz} is the spin flux density in the z direction. This is what Poynting called G.

Weyssenhoff [13] defined a spin liquid as "a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as energy and the linear momentum – to the volume of the element". According to this definition, circularly polarized electromagnetic radiation is a spin liquid.

2. Electromagnetic wave of circular polarization

The electromagnetic fields $\mathbf{E}, \mathbf{B}$ of a circularly polarized wave propagating in the direction of the z axis are expressed by the components of the electromagnetic field strength tensor

$$F_{\mu\nu} = \{F_{tx} = 1,\ F_{ty} = i,\ F_{zy} = -i\breve{k} / \omega,\ F_{xz} = \breve{k} / \omega\} E_0 e^{i\breve{k}z - i\omega t} \tag{2.1}$$

The wave number $\breve{k} = k' + ik''$, is, generally speaking, complex. This is marked with a breve icon. Expression (2.1) is justified by the fact that it satisfies the first pair of Maxwell's equations $\partial_{[\lambda} F_{\mu\nu]} = 0$. The specific values of the wavenumber $\breve{k}$ and fields $\mathbf{D}, \mathbf{H}$, which are expressed by the components $F^{\alpha\beta}_{\wedge}$ of the tensor density, depend on the medium in which the wave propagates. In the general case, the medium is characterized by complex quantities $\breve{\mu}, \breve{\varepsilon}$ and the real electrical conductivity σ.

The transition from the tensor $F_{\mu\nu}$ to the tensor density $F^{\alpha\beta}_{\wedge}$ is carried out by the *conjugation* [14], which involves raising the coordinate indices and includes the factor $\sqrt{g}_{\wedge}$, where g is the determinant of the metric tensor of the coordinates used, and the factor $\sqrt{\varepsilon_0 / \mu_0}$, which provides the common change of units. Also, the electric field is multiplied by $\breve{\varepsilon}$, and the magnetic field is divided by $\breve{\mu}$. The electrical conductivity of the medium does not affect the conjugation.

We use coordinates with the expression for interval

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \quad \sqrt{g}_{\wedge} = c \tag{2.2}$$

Therefore

$$F^{\alpha\beta}_{\wedge} = \{F^{tx}_{\wedge} = -\breve{\varepsilon}\varepsilon_0,\ F^{ty}_{\wedge} = -i\breve{\varepsilon}\varepsilon_0,\ F^{zy}_{\wedge} = -i\breve{k} / \omega\breve{\mu}\mu_0,\ F^{xz}_{\wedge} = \breve{k} / \omega\breve{\mu}\mu_0\} E_0 e^{i\breve{k}z - i\omega t}. \tag{2.3}$$

To denote tensor densities, we use the icon $\wedge$ instead of the Gothic fonts.

The second pair of Maxwell's equations, $j^{\beta}_{\wedge} = \partial_\alpha F^{\alpha\beta}_{\wedge}$, gives the current density

$$j^{x}_{\wedge} = (i\omega\breve{\varepsilon}\varepsilon_0 - i\breve{k}^2 / \omega\breve{\mu}\mu_0) E_0 e^{i\breve{k}z - i\omega t}, \tag{2.4}$$

$$j^{y}_{\wedge} = (-\omega\breve{\varepsilon}\varepsilon_0 + \breve{k}^2 / \omega\breve{\mu}\mu_0) E_0 e^{i\breve{k}z - i\omega t}, \tag{2.5}$$

and Ohm's law, $\mathbf{j} = \sigma\mathbf{E}$, gives the value of the wave number

$$\breve{k}^2 = i\sigma\omega\breve{\mu}\mu_0 + \omega^2\breve{\varepsilon}\varepsilon_0\breve{\mu}\mu_0. \tag{2.6}$$

3. Spin tensor

In this article, to calculate the absorption of the spin and momentum of the wave, the forces acting on the medium are not considered, as was the case in [9]. Instead, spin and energy-momentum tensors are used. As the spin tensor, instead of the canonical tensor (1.1), we use the modified spin tensor [15,16]

$$\Upsilon^{\lambda\mu\nu}_{\wedge} = A^{\lambda}\partial^{\nu} A^{\mu}_{\wedge} - A^{\mu}\partial^{\nu} A^{\lambda}_{\wedge}. \tag{3.1}$$

The point is that the canonical tensor (1.1) correctly describes the spin flux in the direction of wave propagation. Indeed, the vector potential for the field (2.1) (without $E_0 e^{i\breve{k}z - i\omega t}$)

$$A_x = \int F_{tx} dt = i / \omega,\ A^x = -i / \omega, \quad A_y = \int F_{ty} dt = -1 / \omega,\ A^y = 1 / \omega \tag{3.2}$$

gives

$$\left\langle \underset{c}{\Upsilon}{}_{\wedge}^{xyz} \right\rangle = \Re\{-\bar{A}^x F_{\wedge}^{yz} + \bar{A}^y F_{\wedge}^{xz}\} E_0^2 / 2 = E_0^2 \sqrt{\varepsilon_0 / \mu_0} / \omega. \tag{3.3}$$

This corresponds to the power density $E_0^2 \sqrt{\varepsilon_0 / \mu_0}$. Therefore, the canonical tensor could be successfully used in [9,17–23]. However, this tensor gives the wrong result for directions perpendicular to the direction of propagation. For example, instead of zero, we have

$$\left\langle \underset{c}{\Upsilon}{}_{\wedge}^{zxy} \right\rangle = \Re\{\bar{A}^x F_{\wedge}^{zy}\} / 2 = \Re\{(i/\omega)(-ik/\omega\mu_0)\} E_0^2 / 2 = E_0^2 \sqrt{\varepsilon_0 / \mu_0} / 2\omega. \tag{3.4}$$

4. Spin absorption

To use the spin tensor (3.1), it is necessary to calculate the conjugate vector potentials. They are obtained from (3.2):

$$A_{\wedge}^{x} = A_x \sqrt{g}\,{}_{\wedge} g^{xx} \sqrt{\varepsilon_0 \mu_0} / \breve{\mu} = -i / \omega\breve{\mu}\mu_0, \; A_{\wedge}^{y} = 1 / \omega\breve{\mu}\mu_0. \tag{4.1}$$

Given that $\partial^i = -\partial_i$ due to the signature of the metric $(+---)$, we have the flux density of the spin angular momentum, that is, the torque per unit area

$$\begin{aligned} < \Upsilon_{\wedge}^{xyz} > &= \Re\{\bar{A}^x \partial^z A_{\wedge}^{y} - \bar{A}^y \partial^z A_{\wedge}^{x}\} / 2 \\ &= \Re\{(i/\omega)(-i\breve{k})(1/\omega\breve{\mu}\mu_0) - (1/\omega)(-i\breve{k})(-i/\omega\breve{\mu}\mu_0)\} E_0^2 e^{-2k''z} / 2 = \Re\{\breve{k}/\breve{\mu}\} E_0^2 e^{-2k''z} / \omega^2\mu_0. \end{aligned} \tag{4.2}$$

The volume density of the torque is obtained by the differentiation

$$\tau_{\wedge} = -\partial_z < \Upsilon_{\wedge}^{xyz} > = -\partial_z \Re\{\breve{k}/\breve{\mu}\} E_0^2 e^{-2k''z} / \omega^2\mu_0 = 2k'' \Re\{\breve{k}/\breve{\mu}\} E_0^2 e^{-2k''z} / \omega^2\mu_0. \tag{4.3}$$

In case of real ε, μ, $\Re\{\breve{k}/\breve{\mu}\} = k'/\mu$, and from (2.6)

$$k'^2 + 2ik'k'' - k''^2 = i\sigma\omega\mu\mu_0 + \omega^2 \varepsilon\varepsilon_0 \mu\mu_0, \tag{4.4}$$

$$2k'k'' = \sigma\omega\mu\mu_0. \tag{4.5}$$

However, $\sigma E_0 = j$ is a current density, and $E_0 / \omega = A$ is the vector potential. Therefore, the volume density of the torque is expressed simply: $\tau_{\wedge} = jA$. This expression can be given a vector meaning

$$\tau_{\wedge}^{\lambda\mu} = -\partial_\nu \Upsilon^{\lambda\mu\nu} = -\partial_\nu A^\lambda \partial^\nu A^\mu + \partial_\nu A^\mu \partial^\nu A^\lambda - A^\lambda \partial_\nu^\nu A^\mu + A^\mu \partial_\nu^\nu A^\lambda. \tag{4.6}$$

The first pair of terms cancels out due to antisymmetry

$$\partial_\nu A^\lambda \partial^\nu A^\mu - \partial_\nu A^\mu \partial^\nu A^\lambda = 2 g^{\nu\sigma} \partial_\nu A^{[\mu} \partial_\sigma A^{\lambda]} = 0. \tag{4.7}$$

And in the second pair there is a current density [24 (12.123)]: $\partial_\nu^\nu A^\mu = j^\mu$

Thus, an analogue of the Lorentz force is obtained

$$-\partial_\nu \Upsilon^{\lambda\mu\nu} = \tau_{\wedge}^{\lambda\mu} = 2 j^{[\lambda} A^{\mu]} \quad \text{or} \quad \tau_{\wedge} = \mathbf{j} \times \mathbf{A}. \tag{4.8}$$

5. Absorption of the linear momentum

For comparison, we present a similar calculation of momentum. The momentum flux density, i.e. pressure, is described by the component of the energy-momentum tensor [24]

$$T_{\wedge}^{zz} = -g^{zz}(F_{zx} F_{\wedge}^{zx} + F_{zy} F_{\wedge}^{zy}) + g^{zz} F_{\mu\nu} F_{\wedge}^{\mu\nu} / 4. \tag{5.1}$$

It means

$$< T_{\wedge}^{zz} > = \Re\{\bar{F}_{zx} F_{\wedge}^{zx} + \bar{F}_{zy} F_{\wedge}^{zy} - \bar{F}_{tx} F_{\wedge}^{tx} - \bar{F}_{ty} F_{\wedge}^{ty}\} / 4. \tag{5.2}$$

By the use of (2.1) and (2.3)

$$< T_{\wedge}^{zz} > = \Re\{\bar{k}\breve{k} / \omega^2 \breve{\mu}\mu_0 + \breve{\varepsilon}\varepsilon_0\} E_0 e^{ikz - i\omega t} / 2. \tag{5.3}$$

However, due to (2.6)

$$\breve{\varepsilon}\varepsilon_0 = \breve{k}^2 / \omega^2 \breve{\mu}\mu_0 - i\sigma / \omega. \tag{5.4}$$

Therefore

$$< T_{\wedge}^{zz} > = \Re\{(k^2 + \breve{k}^2) / \omega^2 \breve{\mu}\mu_0\} E_0 e^{ikz - i\omega t} / 2 = k' \Re\{\breve{k}/\breve{\mu}\} E_0^2 e^{-2k''z} / \omega^2\mu_0. \tag{5.5}$$

This expression differs from (4.2) just as the momentum of a photon p differs from spin $\hbar$

$$p = \hbar / \lambdabar \,. \tag{5.6}$$

Similarly to (4.3), we find the volume force density

$$f_{\wedge}^{z} = -\partial_z < T_{\wedge}^{zz} >= 2k''k'\Re\{\breve{k} / \breve{\mu}\} E_0^2 e^{-2k''z} / \omega^2 \mu_0 \,. \tag{5.7}$$

In case of real ε, μ, $2k'k'' = \sigma\omega\mu\mu_0$. However $\sigma E_0 = j$, and due to (2.1) $k'E_0 / \omega = B$. So what is the Lorentz force $f_{\wedge} = j_{\wedge} B$

This expression can be given a vector meaning

$$f_{\wedge}^{\mu} = -\partial_\nu T_{\wedge}^{\mu\nu} = g^{\mu\lambda}(\partial_\nu F_{\lambda\sigma} F_{\wedge}^{\nu\sigma} + F_{\lambda\sigma}\partial_\nu F_{\wedge}^{\nu\sigma} - \partial_\lambda F_{\rho\sigma} F_{\wedge}^{\rho\sigma} - F_{\rho\sigma}\partial_\lambda F_{\wedge}^{\rho\sigma}) \,. \tag{5.8}$$

The second term contains the current $j_{\wedge}^{\sigma} = \partial_\nu F_{\wedge}^{\nu\sigma}$. So this term gives the Lorentz force

$$f_{\wedge}^{\mu} = g^{\mu\lambda} j_{\wedge}^{\sigma} F_{\lambda\sigma}, \quad \mathbf{f} = \mathbf{j} \times \mathbf{B} \,. \tag{5.9}$$

It can be shown that the remaining three terms cancel each other out in the absence of a lossy dielectric and a lossy magnet.

6. Conclusion

The spin tensor, which describes the spin of electromagnetic radiation, is the same natural construction as the energy-momentum tensor, which describes energy and momentum..

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