Inadequacy of Classical Logic in Classical Harmonic Oscillator and the Principle of Superposition

Abhishek Majhi *[1] and Ramkumar Radhakrishnan †[2]

[1]Indian Statistical Institute,
Plot No. 203, B. T. Road, Baranagar,
Kolkata 700108, West Bengal, India.
[2]Sardar Vallabhbhai National Institute of Technology,
Ichchhanath, Surat - Dumas Road,
Surat 395007, Gujarat, India.

Abstract

In course of the development of modern science, inadequacy of classical logic and Eastern philosophy have generally been associated only with quantum mechanics in particular, notably by Schroedinger, Finkelstein and Zeilinger among others. Our motive is to showcase a deviation from this prototypical association. So, we consider the equation of motion of a classical harmonic oscillator, and demonstrate how our habit of writing the general solution, by applying the principle of superposition, can not be explained by remaining within the bounds of classical logic. The law of identity gets violated. The law of non-contradiction and the law of excluded middle fail to hold strictly throughout the whole process of reasoning consequently leading to a decision problem where we can not decide whether these two 'laws' hold or not. We discuss how we, by habit, apply our intuition to write down the general solution. Such intuitive steps of reasoning, if formalized in terms of propositions, result in a manifestation of the inadequacy of classical logic. In view of our discussion, we conclude that the middle way (*Mulamadhyamakakarika*), a feature of Eastern philosophy, founds the basis of human reasoning. The essence of the middle way can be realized through self-inquiry (*Atmavichar*), another crucial feature of Eastern philosophy, which however is exemplified by our exposition of the concerned problem. From the Western point of view, our work showcases an example of Hilbert's axiomatic approach to deal with the principle of superposition in the context of the classical harmonic oscillator. In the process, it becomes a manifestation of Brouwer's views concerning the role of intuition in human reasoning and inadequacy of classical logic which were very much influenced by, if not founded upon, Eastern philosophy.

*abhishek.majhi@gmail.com
†rramkumar0808@gmail.com

Contents

1 Introduction: Inadequacy of classical logic in classical harmonic oscillator (CHO)

It is quantum mechanics which has always been *associated* with the inadequacy of classical logic[1, 2, 3], albeit associated with strong objections[4, 5](see, however, ref.[6])[1]. Such *association*, as understood in the standard practice, is rooted to the *principle of (linear) superposition* (PoS)[9] *and* the Heisenberg uncertainty(indeterminacy) principle (HUP)[7], together termed as the "quantum superposition principle" (QSP) in accord with Dirac's clarification on p 14 of ref.[8] that "quantum superposition principle demands indeterminacy" i.e. the word "quantum" is justified by the involvement of the HUP. The thought of such *association* germinates in a convincing way from some not-so-well-understood aspects of quantum mechanics such as Schroedinger's cat paradox[14], the double slit experiment interpreted in terms of photons[15], EPR paradox[27] and the associated debate concerning hidden variables[28], etc. As far as modern technology is concerned, such *association* has today given birth to the language of quantum computing where the relation between the QSP and quantum logic gets strongly manifested through the writing of linear combination of kets so as to explain the linear superposition of quantum states[21, 22] and realization of the HUP is through the quantum speed limits in optical quantum control[10]. Due to such *association* of inadequacy of classical logic with quantum mechanics, certain aspects of Eastern philosophy have been discussed with particular reference to quantum mechanics only, by Schroedinger[16], Finkelstein[17], Zeilinger[17] and many others[19].

In this work we intend to show that the inadequacy of classical logic has a priori nothing to do with QSP and hence, quantum mechanics in particular, but it is the PoS that can not be explained by strictly adhering to the basic principles of classical logic; certain aspects of Eastern philosophy, that surpass the basic tenets of classical logic, become necessary for an understanding of the PoS. Needless to assert that PoS is a characteristic feature of a linear differential equation which has nothing to do with quantum mechanics in particular. Rather it is a part of the mathematical construction of general solutions of linear differential equations which also occur in classical mechanics and physics in general[9, 13]. In order to elucidate the inadequacy of classical logic in association with the PoS, we revisit the scenario of the classical harmonic oscillator (CHO)[26]. We analyze, in particular, that the three basic laws of classical logic fall short of explaining the steps of reasoning based on which we write the general solution of the CHO through the application of the PoS[2]. While such shortcoming of classical logic has its roots in elementary algebra as has been demonstrated in ref.[20], we show its explicit manifestation at the level of the solution of the differential equation of the CHO. Naively the situation can be glimpsed as follows. Given the equation of motion of the CHO,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{1}$$

[1] We must emphasize that "inadequacy of classical logic" does not necessarily mean "quantum logic".

[2] From an experimental point of view one can argue that the general solution of the CHO can not be observed in practice. However, that does not restrict us from writing down the general solution[25]. We are interested in analyzing the steps of reasoning that let us write the general solution through the application of the PoS.

we begin with an assumption regarding the form of the solution[13] i.e. we consider the following *proposition*:

$$P: \quad x \text{ is of the form } Ae^{kt}. \tag{2}$$

Then, we end up writing the general solution of the CHO, by applying the PoS, as $x = A_+e^{i\omega t} + A_-e^{-i\omega t}$ which can not be cast into the form Ae^{kt} i.e. we arrive at the following true statement which is the *classical negation of* P:

$$\neg P: \quad x \text{ is NOT of the form } Ae^{kt}. \tag{3}$$

Considering this whole process of writing down the general solution of the CHO, starting from the assumption until the application of the PoS, as one and *the same process of reasoning*[3], the situation can be interpreted as follows:

- On symbolic ground, $Ae^{kt} = x \neq x = A_+e^{i\omega t} + A_-e^{-i\omega t}$ i.e. the law of identity is violated.
- On propositional ground, one way to look at the situation is that *both* P *and* $\neg P$ *are true in the same process of reasoning* ($P \wedge \neg P$) i.e. the law of non-contradiction fails to hold (considering the classical negation i.e. $\neg\neg P \equiv P$). The symbol "$\wedge$" means "logical conjunction (AND)".
- On propositional ground, the other way to look at the situation is that the fact, *either* P *or* $\neg P$ *is true in the same process of reasoning*, is false ($\neg(P \veebar \neg P)$) i.e. the law of excluded middle fails to hold. The symbol "$\veebar$" means "logical exclusive disjunction (EXCLUSIVE OR)"[4]

Since $\neg(P \veebar \neg P) \equiv P \wedge \neg P$, the last two of the above items are equivalent in the present situation. So, the basic laws of classical logic are vividly inadequate to explain the PoS in case of the CHO. The subtlety associated with such naive looking arguments can only be provided through the details that we are going to discuss. Nevertheless, as a consequence of the above glimpse it becomes apparent that we are in a habit of overlooking such inadequacy of classical logic owing to the usefulness of the end results which we obtain by applying intuition alongside classical logic, where by the word "intuition" we mean the steps of reasoning which can not be formalized in terms of logic. From such an investigation we intend to showcase that what is lacking, in the current practice of doing mathematical science, is self-inquiry i.e. in standard practice we lack consciousness about our own steps of reasoning while doing basic mathematics by applying logic and intuition. With such demonstration of self-inquiry while doing basic mathematics and physics, we expect our investigation to open up an apparently new avenue to look at the same from a very unorthodox point of view that concerns Eastern philosophy, namely, the middle way[41, 42, 43, 44, 45] i.e. acceptance/inclusion of the middle that is otherwise unaccepted/excluded to declare a principle of classical logic – the law of excluded middle.

From the orthodox Western point of view this work is a *formalized* investigation of the standard process of reasoning, considering it as "*a logical system of thought*"(in Einstein's words[29]), that concerns the PoS in the context of the CHO. Considering Hilbert's point of view this work may be considered as a particular example of "*Mathematical treatment of the axioms of physics*"[30], the importance of which gets emphasized through Weierstrass's assertion, as quoted by Hilbert in ref.[30], that "*to make any progress in the sciences the study of particular problems is, of course, indispensable.*" The exposure of the role of intuition and shortcoming of the basic laws of classical logic in such an elementary problem in physics only showcases, as Brouwer might have written, that intuition "*subtilizes logic...and denounces logic as a source of truth*" which can be realized only through "*inner inquiry with revealing and liberating consequences*"[33]. In a nutshell, this work exemplifies the Brouwer-Hilbert debate concerning logic/formalism and intuition, albeit in the context of basic mathematics and

[3]We have borrowed the phrase "same process of reasoning" from Boole[12] as he, on page number 6 of ref.[12], put emphasis on the basic principles of reasoning with symbols as follows: "*..... first, that from the sense once conventionally established we never, in the same process of reasoning, depart; secondly, that the laws by which the process is conducted be founded exclusively upon the above fixed sense or meaning of the symbols employed.*"

[4]In electronics we call this logical operation as XOR. Now, let us further clarify that we have used the expression $\neg P \veebar P$, and not $\neg P \vee P$, as a direct symbolic derivative from the verbal expression "*either P is false or P is true*" (e.g. see ref.[34]) so as to remain as truthful as possible while making symbolic abbreviation of the concerned verbal statement. The reader may consult Appendix (A).

physics rather than the much more general and sophisticated topic of set theoretic foundations of mathematics which used to be case of debate for these gentlemen e.g. see ref.[35] and also see Chapter 5 of ref.[36]. However, we must declare emphatically that what we are doing is self-inquiry with basic mathematics in terms of classical logic. No knowledge of intuitionistic mathematics and intuitionistic logic are required to understand this work[5]. What we call "intuition" are the steps of our reasoning that we can not formalize, or, if we try to formalize then we end up jeopardizing the basic principles of classical logic.

2 Revisiting the standard steps of reasoning that lead to the general solution of the CHO

The equation of motion for the CHO is given by eq.(1), where x is a function of t (i.e. $x(t)$) and ω is a given number. Keeping aside the physical interpretation of the equation which is available in many standard textbooks of basic physics (e.g. see ref.[25]), we focus here on the logical structure of a particular method, discussed by Boole on pp. 194-195 in ref.[13], by which we solve eq.(1)[6]. The procedure consists broadly of the following steps:

1. We *assume* that the solution is of the following particular form:

$$x(t) = Ae^{kt} \ni A \neq 0. \tag{4}$$

2. Considering the *assumption* and eq.(1) we obtain $(k^2 + \omega^2)x = 0$, from which we conclude that

$$(k^2 + \omega^2) = 0, \tag{5}$$

   which we call as the auxiliary equation.

3. We *factorize* eq.(5), the auxiliary equation, to write

$$(k + i\omega)(k - i\omega) = 0. \tag{6}$$

   We conclude that there are two roots i.e. k can take two values viz. $i\omega, -i\omega$.

4. Finally we come to the following conclusion about the solutions of eq.(1).
   (a) Considering $k = i\omega$ and the assumption (4), we write $x(t) = Ae^{i\omega t} \ni A \neq 0$.
   (b) Considering $k = -i\omega$ and the assumption (4), we write $x(t) = Ae^{-i\omega t} \ni A \neq 0$.
   (c) We also write is $x(t) = A_+e^{i\omega t} + A_-e^{-i\omega t} \ni A_+ \neq 0 \neq A_-$. Using it on the left hand side of eq.(1) we calculate the result to be zero and claim that it is the *general solution* of eq.(1). We call this act of linearly superposing the two particular solutions, to write the general solution, as the PoS.

Henceforth, we shall refer to the above three steps of reasoning as 4(a), 4(b) and 4(c) respectively.

2.1 Some observations concerning the standard steps of reasoning

In the standard steps of reasoning, which we have just discussed in the previous section, the following issues are never clarified.

- *Which value of k is implemented in 4(c) while applying the PoS?*

[5]Recently Gisin has put emphasis on Brouwer's views in ref.[37, 38], which however requires the knowledge of inuitionistic mathematics to be appreciated. Our work is, in the Brouwerian way of putting it, just an "inner inquiry" concerning usual (classical) mathematics that is practiced everyday in standard literature.

[6]We have chosen Boole's method as we think this is the easiest and the most used method of solving the CHO in the physics literature and also in the engineering literature where we find the practical use of the theories of physics[23, 24].

- *Can we consider the step 4(c) to be logical, even though the general solution can not be cast into the assumed form of the solution?*

Nevertheless, it is quite trivial to see that 4(c) is the implementation of the scenario where $i\omega = k \neq k = -i\omega$. That is, the general solution obtained through the application of the PoS is an implementation of *both* $k = i\omega$ *and* $k = -i\omega$ *in a single expression.* The basis of this reasoning is rooted to how we can reach the conclusion about the roots of the auxiliary equation (5), considering the arithmetic truth "$0.0 = 0$", as we explain schematically below in AEBOX (see ref.[20] for further details concerning quadratic equations in general).

AEBOX

1. $$\text{“}\underbrace{\underbrace{(k - i\omega)}_{=0}.\underbrace{(k + i\omega)}_{=n\neq 0}}_{\text{“}0.n = 0\text{”}} = 0\text{”} \quad \leftarrow \quad \text{“}(k - i\omega) = 0\text{”} \;\; \text{AND} \;\; \text{“}(k + i\omega) \neq 0\text{”} \quad \equiv \quad \text{“}k = i\omega\text{”} \;\; \text{AND} \;\; \text{“}k \neq -i\omega\text{”}.$$
2. $$\text{“}\underbrace{\underbrace{(k - i\omega)}_{=n\neq 0}.\underbrace{(k + i\omega)}_{=0}}_{\text{“}n.0 = 0\text{”}} = 0\text{”} \quad \leftarrow \quad \text{“}(k - i\omega) \neq 0\text{”} \;\; \text{AND} \;\; \text{“}(k + i\omega) = 0\text{”} \quad \equiv \quad \text{“}k \neq i\omega\text{”} \;\; \text{AND} \;\; \text{“}k = -i\omega\text{”}.$$
3. $$\text{“}\underbrace{\underbrace{(k - i\omega)}_{=0}.\underbrace{(k + i\omega)}_{=0}}_{\text{“}0.0 = 0\text{”}} = 0\text{”} \quad \leftarrow \quad \text{“}(k - i\omega) = 0\text{”} \;\; \text{AND} \;\; \text{“}(k + i\omega) = 0\text{”} \quad \equiv \quad \text{“}k = i\omega\text{”} \;\; \text{AND} \;\; \text{“}k = -i\omega\text{”}.$$

Now, it is simple to understand how we write 4(a), 4(b) and 4(c). We explain the steps below in GSBOX.

GSBOX

1. Option 1 of AEBOX and the assumption (4) together yield "$x = Ae^{i\omega t} \ni A \neq 0$" i.e. 4(a).
2. Option 2 of AEBOX and the assumption (4) together yield "$x = Ae^{-i\omega t} \ni A \neq 0$" i.e. 4(b).
3. Option 3 of AEBOX is implemented both part by part and as a whole as follows. "$k = i\omega$" and the assumption (4) are used together to write $A_+e^{i\omega t}$; "$k = -i\omega$" and the assumption (4) are used together to write $A_-e^{-i\omega t}$; both $k = i\omega$ and $k = -i\omega$ (i.e. $i\omega = k \neq k = -i\omega$) are implemented together by writing the linear combination as x, i.e. "$x = A_+e^{i\omega t} + A_-e^{-i\omega t}$", so that we have two slots to write two values of k simultaneously in a single expression. Using it on the left hand side of eq.(1) we obtain $A_+[(i\omega)^2 + \omega^2] + A_-[(-i\omega)^2 + \omega^2] = (A_+.0) + (A_-.0) = 0$. We claim that it is the general solution of eq.(1) i.e. 4(c).

Option 3 of AEBOX, that is, "$i\omega = k \neq k = -i\omega$" is the violation of the law of identity (see ref.[20] for more details). So, it becomes clear that the violation of the law of identity that occurs in course of solving the auxiliary equation gets manifested through the way we write the general solution, by applying the PoS, because it can not be cast into the exponential form that we assumed to begin with in (4).

2.2 Split-identity, quadratic equation, the principle of superposition in the CHO and intuition

Now, let us discuss why the PoS is a manifestation of our intuition and can not be explained by remaining within the bounds of classical logic. To have this clarity of reasoning we need to begin our analysis from AEBOX. The three cases of AEBOX are mutually exclusive processes of reasoning because each of them follows from an arithmetic truth independent of the other viz. option 1 from "$n.0 = 0 \ni n \neq 0$", option 2 from "$0.n = 0 \ni n \neq 0$" and option 3 from "$0.0 = 0$". In each of the first two options of AEBOX k has one value at a time[7], namely, $k = i\omega$ in option 1 of AEBOX and $k = -i\omega$ in option 2 of AEBOX. In option 3 of AEBOX k has two values at the same time i.e. $k = i\omega$ and $k = -i\omega$. So, strictly speaking on symbolic logical grounds, k should be replaced by a collection of two different symbols k_+ and k_-, say, as if the identity of k has split into two identities k_+ and k_- so as to avoid writing "$i\omega = k \neq k = -i\omega$". Then we can write $k_+ = i\omega$ and $k_- = -i\omega$[8]. However, if we do that then there is no way we can get back the equations (5) and (6) because we end up writing an equation

[7] The phrases "at a time" and "in the same process of reasoning" should be considered to convey the same meaning.

[8] One can write a shorthand as follows: $k_\pm = \pm i\omega$.

like $(k_+ - i\omega)(k_- + i\omega) = 0$. So, to reverse the path of reasoning we have to remove the subscripts '+' and '−' by hand. We call this process of replacing the symbol k by the collection of symbols (k_+, k_-) as an *intuitive jump* as it can not be explained logically, particularly because we can not attach "a fixed sense" to the symbol k throughout one and "the same process of reasoning" through which we solve the auxiliary equation (5) (using, in quotes, Boole's words from footnote (3)). Since it appears as if the one identity that we associate with k has split into two identities that we associate with (k_+, k_-), we can call this as a case of split-identity or splitting of identity. We denote this particular step as follows:

$$k \dashrightarrow (k_+, k_-). \tag{7}$$

However, what we also need, as a consequence, to write the general solution is the following:

$$A \dashrightarrow (A_+, A_-), \tag{8}$$

of which only a special case is when one writes $A_+ = A_-$. The symbol "$\dashrightarrow$" stands for an *intuitive jump*.

Now, as par currently accepted practice in mathematical science, the general belief is that a variable can only take one value at a time which is nevertheless an informal translation of what we consider formally as the law of identity. Option 3 of AEBOX is ignored by choice (possibly by being unaware of such a choice). However, from the above discussion it is now clear that there is no illegitimate mathematics that leads us to the violation of the law of identity in the case of solving a quadratic equation[20]. Rather, the problem is logical. So, from the mathematical perspective it is a question of how we can use the split-identity, or both the identities, in a single expression and do calculations. The answer to this question is the PoS. We must note that the quadratic equation (5) is not any equation considered in isolation, but it is the auxiliary equation that occurs in the context of the CHO. Therefore, it is the context that provides ground to use the split-identity.

In view of above clarifications concerning the symbolic intricacy associated with the intuitive jump and the application of the split-identity, we rewrite GSBOX in a symbolically refined way in RGSBOX below, where we explain a distinctive feature of the general solution compared to the particular solutions. That is, the particular solutions are deduced from the given premises and need not be verified by putting back to eq.(1), but the general solution is constructed through intuitive steps of reasoning and therefore, needs to be put back to eq.(1) so as to verify that it indeed satisfies eq.(1).

RGSBOX

1. Option 1 of AEBOX and the assumption (4) together yield "$x = Ae^{kt} \ni A \neq 0, i\omega = k \neq -i\omega$" i.e. 4(a). No new symbols are necessarily required to be introduced. The solution is *deduced* from the given premises. Verification is not necessary.
2. Option 2 of AEBOX and the assumption (4) together yield "$x = Ae^{kt} \ni A \neq 0, i\omega \neq k = -i\omega$" i.e. 4(b). No new symbols are necessarily required to be introduced. The solution is *deduced* from the given premises. Verification is not necessary.
3. Option 3 of AEBOX is implemented through the following steps. We write the linear combination $A_+e^{k_+t} + A_-e^{k_-t} \ni A_+ \neq 0, A_- \neq 0, k_+ = i\omega, k_- = -i\omega$. The assumption (4) holds for each of the pieces which are linearly combined, but it does not hold for the linear combination taken as a whole. Therefore, this step is not a complete *deduction* from the given premises alone. It needs the following *intuitive* steps of reasoning to be put together: (i) introduction of new symbols accordingly (ii) forming the linear combination. It is due to this reason, the above linear combination is a proposed solution i.e. a new proposition. *Verification* is necessary to check whether the proposed solution works or not. To perform this check, we use this linear combination in place of the symbol "x" in eq.(1).It is through such verification we conclude that the proposed linear combination is a solution of eq.(1), which we call the general solution, designated by symbol x as it satisfies eq.(1). That is, we write

   $$\text{“}x = A_+e^{k_+t} + A_-e^{k_-t} \ni A_+ \neq 0, A_- \neq 0, k_+ = i\omega, k_- = -i\omega\text{”},$$

   i.e. 4(c).

This clarifies how we rely on our intuition to apply the PoS, which nevertheless is justified by the usefulness of such practice.

3 Formal reasoning: Inadequacy of the basic laws of classical logic and a decision problem

Now, we shall present what we have discussed till now, formally in terms of propositions so as to demonstrate how the law of non-contradiction and the law of excluded middle does not always hold in the process reasoning through which we write the general solution of the CHO through the application of the PoS. In order to do so, we represent the relevant statements in quotes (“”) as propositions, which can be either TRUE (T) or FALSE (F), and we use capital letters to symbolize the propositions. These propositions are the elements of our reasoning.

PROPBOX

C : “$\frac{d^2x}{dt^2} + \omega^2 x = 0$”, P : “$x(t) = Ae^{kt} \ni A \neq 0$”, D : “$(k^2 + \omega^2) = 0$”,

B : “$(k - i\omega)(k + i\omega) = 0$”, B_+ : “$k = i\omega$”, B_- : “$k = -i\omega$”,

$\neg B$: “$(k - i\omega)(k + i\omega) \neq 0$”, $\neg B_+$: “$k \neq i\omega$”, $\neg B_-$: “$k \neq -i\omega$”,

Q : “$x = A_+ e^{k_+ t} + A_- e^{k_- t} \ni k_+ = i\omega, k_- = -i\omega$”, $\neg P$: “$x(t) \neq Ae^{kt} \ni A \neq 0$”,

I_n : Intuitive steps to implement split-identity through the PoS to write the general solution.

Now, the mathematical steps which we go through to find the two roots from eq.(6), which is the factorized form of eq.(5) obtained by using assumption (4) in eq.(1), can be formally written as follows in AUXBOX below.

AUXBOX

$$C \wedge P \equiv D \quad \to \quad B.$$

The explanation of why $D \to B$ has been elaborately given in ref.[20]. So, we feel that it is unnecessary to repeat that explanation here as it is not going to play any significant role in the present discussion. Nevertheless, we provide below the truth table corresponding to the situation so as to highlight the main essence of it.

B_+	B_-	D	B	$D \to B$
F	F	F	F	T
T	F	T	T	T
F	T	T	T	T
T	T	F	T	T
U	U	T	F	F

The fourth row of the above truth table explains the situation where B_+ and B_- are true at the same time and this corresponds to AEBOX.3; the polynomial form does not hold corresponding to this situation and hence, D is false. The fifth row shows the analysis of the situation where D is true but B is false. Since such a situation does not arise, $D \to B$ is false and the status of B_+ and B_- remain undecided.

Now, the content of AEBOX can be formally represented as follows in FAEBOX where the symbol “$\vee$” means “logical (inclusive) disjunction (OR)”.

FAEBOX

$$\begin{array}{llll} 1.\ B & \leftarrow & B_+ \wedge \neg B_- & \\ 2.\ B & \leftarrow & \neg B_+ \wedge B_- & \therefore B \equiv (B_+ \wedge \neg B_-) \veebar (\neg B_+ \wedge B_-) \veebar (B_+ \wedge B_-) \equiv (B_+ \vee B_-). \\ 3.\ B & \leftarrow & B_+ \wedge B_- & \end{array}$$

Now, option 1 and option 2 of RGSBOX can be written in formal terms as follows in FRGSBOX12.

FRGSBOX12

- Option 1 of RGSBOX in formal terms: $(B_+ \wedge \neg B_- \wedge P) \rightarrow C$
- Option 2 of RGSBOX in formal terms: $(\neg B_+ \wedge B_- \wedge P) \rightarrow C$

Option 3 of RGSBOX can be explicated in formal terms as follows. Considering $B_+ \wedge P$ and $B_- \wedge P$ we write the pieces "$A_+ e^{k_+ t}$" and "$A_- e^{k_- t}$", respectively, where the symbols are modified in accord with (7) and (8). We write the pieces as a linear combination. This symbolic modification along with the linear combination constitute an intuitive jump that can not be formalized in terms of the other propositions. So, we have considered I_n as an independent proposition in PROPBOX. Now, the general solution of the CHO, i.e. the linear combination, declared as a proposition Q in PROPBOX can be written as follows:

$$Q \equiv (B_+ \wedge P) \wedge (B_- \wedge P) \wedge I_n \wedge D \equiv (B_+ \wedge B_- \wedge P \wedge I_n \wedge D). \tag{9}$$

We may note that D comes into the above expression because of the necessity of the verification. However,

$$Q \equiv \neg P.$$

This is because the symbol "x" can only represent the solution of eq.(1) in this present context. So, we can write

$$\neg P \equiv (B_+ \wedge B_- \wedge P \wedge I_n \wedge D). \tag{10}$$

So, option 3 of RGSBOX can be formally written below in FRGSBOX3.

FRGSBOX3

- Option 3 of RGSBOX in formal terms:

$$(B_+ \wedge B_- \wedge P \wedge I_n \wedge D) \equiv \neg P \rightarrow C.$$

In view of this we can now write down formally the whole process of reasoning that leads us to the general solution of the CHO by the application of the PoS, formally as follows in PoSBOX.

PoSBOX

$$\begin{array}{lll} C & \equiv & ((B_+ \wedge \neg B_- \wedge P) \veebar (\neg B_+ \wedge B_- \wedge P)) \veebar \neg P \\ & \equiv & (P \veebar \neg P) \qquad \text{[the law of excluded middle]} \\ & \equiv & \neg(P \wedge \neg P) \qquad \text{[the law of non-contradiction].} \end{array}$$

Thus, the truth of the CHO as a whole seems logically consistent as it is equivalent to the validity of the law of excluded middle. However, the inner inconsistency appears once we focus on the logical expression (10). This is because we can write the following:

$$P \wedge \neg P \equiv (B_+ \wedge B_- \wedge P \wedge I_n \wedge D).$$

This can be recast both as the violation of the law of excluded middle and as the violation of the law of non-contradiction as follows in VIOBOX.

VIOBOX

- Violation of the law of non-contradiction: $\neg\neg(P \wedge \neg P) \equiv (B_+ \wedge B_- \wedge P \wedge I_n \wedge D)$.
- Violation of the law of excluded middle: $\neg(P \veebar \neg P) \equiv (B_+ \wedge B_- \wedge P \wedge I_n \wedge D)$.

Thus, the basic principles of classical logic both hold and not hold in the whole process of reasoning by which we solve the CHO and claim that the general solution is a valid solution obtained by the application of the PoS. In fact, we can not *decide* whether the laws of classical logic are true or false in the whole process of solving the CHO. So, we arrive at *a decision problem* concerning the universal validity of the basic laws of classical logic, where 'universe' means the whole process of reasoning through which we solve the CHO. Of course such dilemmas have arose as a consequence of our attempt to formalize intuition. However, without such an attempt we can not think of a way to formalize the CHO in accord with Hilbert's sixth problem[30].

We may note further that, from the above logical expression, it becomes also clear that all of B_+, B_-, D can not be true at the same time while the basic laws of classical logic are valid. This is why within the system of logic that we have constructed in terms of propositions, we have $D \rightarrow B$ and *not* $D \equiv B$, which is in tandem with what has been discussed in ref.[20].

4 Conclusion: Middle way as the foundation of human reasoning, a realization through self-inquiry

In view of our discussion regarding the process of reasoning, through which we are habituated in writing down the general solution of the classical harmonic oscillator by applying the principle of superposition, we can conclude that the same can not be explicated in terms of the basic principles of classical logic. While this should not be interpreted as some ultimate failure of classical logic, however, it certainly is a demonstration of a scenario where one can not adhere to the basic principles of classical logic through a whole process of reasoning concerning the same context i.e. finding the general solution of the classical harmonic oscillator[9]. If we consider the general solution of the classical harmonic oscillator to be a valid mathematical solution, then the process of reasoning contains the following.

1. The assumption of the form of the solution holds true (P), to begin with.
2. The assumption of the form of the solution does not hold true ($\neg P$) when we consider the principle of superposition to be a valid mathematical solution.
3. As we linearly superpose the particular solutions, albeit with the intuitive jump to modify the symbols accordingly, for each of the particular solutions the assumption of the form of the solution holds true (P). But, considering the linearly superposed expression as a whole to be a solution, the assumption regarding the form of the solution does not hold true ($\neg P$). So, we can interpret the situation as if P is true while constructing the pieces which are linearly superposed and $\neg P$ is true after we write the general solution.
4. Considering the whole process of reasoning to write down the general solution of the classical harmonic oscillator by applying the principle of superposition we can not stick to a particular decision regarding the form of the solution i.e. neither P is absolutely true, nor $\neg P$ is absolutely true.

Interestingly, the fourth among the above points, appears similar to Brouwer's view regarding admissibility of logic in mathematics, albeit now in the particular context of the PoS and the CHO, that "*it is uncertain whether the whole of logic is admissible and it is uncertain whether the problem of its admissibility is decidable.*"[34].

[9]The wordings in the introduction of ref.[3], "*the Boolean structure ...... suitable for the discourse of classical physics*", seems very inappropriate in view of the present discussion where we have dealt with possibly one of the most important equations, if not the most important equation, in classical physics.

It appears from the above process of reasoning that the first two are the opposite ends of a total process of reasoning where the middle is contained in the third and the fourth options which do not have independent formal status. It is only that, if we consider P as an isolated and complete truth, which is nevertheless the practice in classical logic, then certainly the process of reasoning looks paradoxical. However, it is the human intuition that provides the basis to implement such logic according to the context, which is sometimes the particular, sometimes the general or the universal and sometimes both universal and particular – the totality of which is conveyed by the third and the fourth statements. If we remove such associated discussions concerning the context of reasoning, the truth value of P appears to be the following.

1. P is true.
2. P is false.
3. P is both true and false.
4. P is neither true nor false.

This is how the middle way[41, 42, 43, 44, 45], often viewed as Chatuskoti[39, 40], is formally interpreted in terms of classical logic where the proposition P is presumed to be *either completely true or completely false* which provides the appearance to the third and the fourth possibilities as paradoxical and hence, impossibilities. Considering classical logic to be adequate in describing truth, one is led to the non-acceptance of such possibility, i.e. the middle way, based on the belief that there is no role of intuition alongside classical logic. However, if one accepts the necessity of a discussion of the associated context, which is here the intricate process of reasoning that leads to the construction of the general solution of the classical harmonic oscillator, then the role of intuition becomes vivid. As it stands, the propositions are mere conventions to analyze the situation and do not hold the complete truth irrespective of the context of the use; it is the context that provides the essence of the propositions and the reasoning becomes refined. Consequently, as far as the present work is concerned, the essence of the middle way can be understood as the basis of human reasoning as far as the principle of superposition is concerned in the context of the classical harmonic oscillator. Such an analysis has not appeared in the literature of Western science till date because of the lack of self-inquiry i.e. a refinement of one's own reasoning by asking oneself, at each and every step of reasoning, that how truthful "I" can be with my own reasoning. In such context, we find it tempting to mention the following quote by Potter, on page no. 2 of ref.[46]: "*Indian logic is never conceived as 'formal' in the Western sense, but as an account of sane processes of reasoning it has few equals in the West for attention to detail.*".

We may note that views based on Eastern philosophy have sometimes been associated with only quantum mechanics in particular, and not basic physics in general, by physicists like Schroedinger[16] in the past and more recently by some prominent researchers of the modern era like Finkelstein[17], Zeilinger[17], Rovelli[18] and many others[19]. However, our investigation only showcases that the single most crucial element of Eastern philosophy that has not been discussed by such authors is self-inquiry[10], even though they have drawn analogies with many other aspects of the same in course of the development of quantum physics, in particular, only.

In light of our discussion, we think that a bit of self-awareness concerning our own steps of reasoning to do basic physics can provide us better insight and understanding of science as a whole rather than focusing on and getting bamboozled by the paradoxical outcomes of quantum mechanics only. As words of caution we may note that our work should not be confused as any attempt of drawing mere analogies between quantum mechanics and classical mechanics because of the principle of superposition being involved in linear differential equations appearing in both the subjects. Any attempt to judge this work in such a fashion will be an act of complete misunderstanding of this work, and hence of the showcased demonstration of self-inquiry, by the concerned reader.

Acknowledgment: A. M. has been supported by the Department of Science and Technology of India through the INSPIRE Faculty Fellowship, Grant no.- IFA18-PH208.

[10] Such misunderstanding of the most crucial element of Eastern philosophy has often led to the judgment of the same by Western minded thinkers as mysticism. For example, Schroedinger himself used the word "mystic" in ref.[16] while discussing Eastern philosophy. The overall attitude of such Western minded thinkers can be seen to be manifested to some extent in ref.[11] and the relevant references therein. As a note of clarification, we should emphasize that by "Western minded thinkers" we mean thinkers both from geographical East and West who were and are engaged in doing science as it stands today.

Declaration: The authors have equal contributions.

A Classical negation, choices of ignorance and the equivalence between $(P \vee \neg P)$ and $(P \veebar \neg P)$

A crucial step in our reasoning is the explanation that we gave to assert the relation $\neg P \equiv Q$. Certainly, if x does not have the exponential form, it does not necessarily mean that it has a form that is a linear combination of two exponential terms – it can be of any arbitrary functional form. However, if we really want to mean "solution of the CHO", by the symbol "x", then there is no choice but to consider only those functional forms which satisfy the equation for the CHO. Thus, the arbitrariness of the truth associated with $\neg P$ is fixed by the context that necessitates us to investigate the content of the proposition P itself, and consequently the meaning of the symbol "x" itself. Therefore, the understandings of the meaning of the symbol "x" and the content of the proposition P are necessary for truthful reasoning. This is quite opposite to the founding motive of doing mathematical reasoning or mathematical logic that becomes evident from the following words of Hilbert and Ackermann, on page no. 5 of ref.[31]: "*.... the truth or falsehood of a sentential combination depends solely upon the truth or falsehood of the sentences entering into the combination, and not upon their content.*" So, it is simple to observe that mathematical reasoning can lead to contradictions when the meaning or the "content" of the propositions are not taken care of. And when any such contradiction arises, inner investigation (or "inner inquiry") concerning the content of the symbols and sentences can be of essence to resolve the contradiction through a refinement (or "subtilization") of the middle, that is otherwise excluded by choice, to state the 'laws' or conventions of classical logic.

In view of this, it is interesting to have the following observation regarding the role of classical negation through the following truth table below which reveals how the equivalence between $(P \vee \neg P)$ and $(P \veebar \neg P)$ is related to how classical negation is defined through the truth table, the reason being founded on the law of non-contradiction considered as a tautology. In the following truth table P and $\neg P$ are treated independently to show all the possibilities so that the choice can be demonstrated.

P	$\neg P$	$\neg P \vee P$	$\neg P \veebar P$	$\neg(P \wedge \neg P)$
T	F	T	T	T
F	T	T	T	T
T	T	T	F	F
F	F	F	F	T

From the above truth table we can find that $\neg P \vee P$ and $\neg P \veebar P$ are not logically equivalent i.e. the law of excluded middle should be written as $\neg P \veebar P$. Now, we consider the law of non-contradiction as a tautology i.e. we shall ignore the rows which yield the falsity of the law. We show this by striking out the third row to obtain the following truth table.

P	$\neg P$	$\neg P \vee P$	$\neg P \veebar P$	$\neg(P \wedge \neg P)$
T	F	T	T	T
F	T	T	T	T
~~T~~	~~T~~	~~T~~	~~F~~	~~F~~
F	F	F	F	T

Consequently, now we can see that $\neg P \vee P$ and and $\neg P \veebar P$ are equivalent in terms of the truth values in the above truth table. However, there is the fourth row for which both are false, which correspond to the undecidability of P because both P and $\neg P$ are false i.e. neither P nor $\neg P$. This undecidability can be removed by claiming that the law of excluded middle is also a tautology i.e. $\neg P \vee P \equiv \neg P \veebar P$ is always true. We do this by striking out the fourth row to obtain the following truth table.

P	$\neg P$	$\neg P \vee P$	$\neg P \veebar P$	$\neg(P \wedge \neg P)$
T	F	T	T	T
F	T	T	T	T
~~T~~	~~T~~	~~T~~	~~F~~	~~F~~
~~F~~	~~F~~	~~F~~	~~F~~	~~T~~

Thus, when it comes to P and $\neg P$ the equivalence of the logical connectives "$\vee$" and "$\veebar$", considered in the middle of the two, are made equivalent by these two choices of ignorance i.e. the two strike outs in the truth table. Considering the validity of the following $A \veebar B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$, for two arbitrary propositions A and B, we can now see in the following way that the relation "$\neg\neg P \equiv P$" needs to be necessarily valid.

$$\begin{aligned} \neg P \veebar P &\equiv ((\neg P) \wedge (\neg P)) \vee ((\neg\neg P) \wedge P) \\ &\equiv \neg P \vee (P \wedge P) \quad [\text{if and only if } \neg\neg P \equiv P] \\ &\equiv \neg P \vee P \end{aligned}$$

This explains how the choices of ignorance to exclude the middle and the undecidability is related to the classical negation. In view of this we may observe that the law of excluded middle does not apply to itself in the process of reasoning i.e. the proposition "a proposition is either true or false" is both true and false in one and the same process of reasoning through which we write the general solution of the CHO. However, we have performed this whole process of reasoning based on the assumption that "a proposition is either true or false" and hence, we can not hold fast to the above conclusion. In a nutshell, the Middle Way is self-destructive and leads to an infinite maze of reasoning – an ornament that is empty of its essence – which can be realized through a process of self-inquiry/self-referencing.

References

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955); (English translation by R. T. Beyer from the original German version published in 1932). 2

[2] G. Birkhoff, J. von Neumann, *The Logic of Quantum Mechanics*, Annals of Mathematics, 37(4), 823–843 (1936), https://doi.org/10.2307/1968621. 2

[3] *Quantum Logic in Historical and Philosophical Perspective in Internet Encyclopedia of Philosophy*, https://iep.utm.edu/qu-logic/. 2, 9

[4] T. Maudlin, *The labyrinth of quantum logic*, https://arxiv.org/abs/1802.01834. 2

[5] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics (Collected Papers on Quantum Philosophy)*, 2nd Edition, Cambridge University Press (2004). 2

[6] D. Dieks, *Von Neumann's Impossibility Proof: Mathematics in the Service of Rhetorics*, Studies in History and Philosophy of Modern Physics 60 (2017) 136-148, https://arxiv.org/abs/1801.09305v1. 2

[7] W. Heisenberg, *The physical principles of quantum theory*, Dover Publication (1930). 2

[8] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press (1930). 2

[9] L. Brillouin, *Wave Propagation in Periodic Structures*, Second Edition, Dover Publications, Inc. (1953). 2

[10] S. Deffner and S. Campbell, *Quantum speed limits: from Heisenberg's uncertainty principle to optimal quantum control*, Journal of Physics A: Mathematical and Theoretical. 50, 453001 (2017), https://iopscience.iop.org/article/10.1088/1751-8121/aa86c6. 2

[11] J. M. Marin, *'Mysticism' in quantum mechanics: the forgotten controversy*, European Journal of Physics 30, 807 (2009). 10

[12] G. Boole, *An Investigation of the Laws of Thought*, Cambridge: Macmillan and Co. (1854). 3

[13] G. Boole, *A Treatise on Differential Equations*, Nabu Press (2010). 2, 3, 4

[14] E. Schroedinger (translated by J. D. Trimmer), *The present situation in quantum mechanics: a translation of Schroedinger's "Cat Paradox" paper*, http://hermes.ffn.ub.es/luisnavarro/nuevo_maletin/Schrodinger_1935_cat.pdf. 2

[15] A. Zeilinger A, *Experiment and the foundations of quantum physics*, Rev. Mod. Phys. 71 (2): S288 - S297 (1999); https://doi.org/10.1103/RevModPhys.71.S288. 2

[16] E. Schroedinger, *What is life? The Physical Aspect of the Living Cell*, Cambridge University Press (1944). 2, 10

[17] B. A. Wallace, *Buddhism and Science*, Columbia University Press (2003). 2, 10

[18] C. Rovelli (translated by E. Segre and S. Carnell), Helgoland: Making Sense of the Quantum Revolution, Penguin Random House (2021). 10

[19] Tuszynski, Jack A. (Ed.), *The Emerging Physics of Consciousness*, Frontiers Collection, Springer-Verlag Berlin Heidelberg (2006). 2, 10

[20] A. Majhi, R. Radhakrishnan, *Problem of identity and quadratic equation*, https://vixra.org/abs/2110.0121. 2, 5, 6, 7, 9

[21] Rieffel, E. G., and Polak, W. H. (2011). *Quantum computing: A gentle introduction.* MIT Press. 2

[22] Chiara et al, *Quantum Computation and Logic*, Springer International Publishing (2018).http://10.1007/978-3-030-04471-8. 2

[23] E. Kreyszig, Advanced Engineering Mathematics, Tenth Edition, John Wiley & Sons, Inc. 4

[24] K. F. Riley, M. P. Hobson, S. J. Bence, Mathematical Methods for Physics and Engineering, Third Edition, Cambridge University Press (2006). 4

[25] R. P. Feynman , Leighton and Sands, The Feynman Lectures on Physics, Volume I (mainly mechanics, radiation, and heat), https://www.feynmanlectures.caltech.edu/I_21.html. 2, 4

[26] R. Hooke, *De Potentia Restitutiva, or of Spring. Explaining the Power of Springing Bodies*, London, 1678. 2

[27] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. 47, 777 (1935), http://www.drchinese.com/David/EPR.pdf. 2

[28] J. S. Bell, *On the Einstein Rosen Podolsky Paradox*, Physics Vol. 1, No. 3, pp. 195 - 200 (1964) https://journals.aps.org/ppf/pdf/10.1103/PhysicsPhysiqueFizika.1.195. 2

[29] A. Einstein, *Physics and Reality*, Journal of the Franklin Institute, Volume 221, Issue 3, Pages 349-382 (1936), https://www.sciencedirect.com/science/article/abs/pii/S0016003236910475. 3

[30] D. Hilbert, *Mathematical Problems*, Bull. Amer. Math. Soc. 8(10): 437-479 (1902); https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-8/issue-10/Mathematical-problems/bams/1183417035.full 3, 9

[31] D. Hilbert, W. Ackermann, *Principles of Mathematical Logic*, Chelsea Publishing Company (1950). 11

[32] L. E. J. Brouwer, *Collected Works, Vol.1 - Philosophy and Foundations of Mathematics, Edited by A. Heyting*, North-Holland Publishing Company (1975). 14

[33] L. E. J. Brouwer, *Consciousness, philosophy, and mathematics*, Proceedings of the Tenth International Congress of Philosophy (Amsterdam, August 11–18, 1948), North-Holland Publishing Company, Amsterdam1949, pp. 1235–1249. 3

[34] L. E. J. Brouwer, *The Unreliability of the Logical Principles*, pp 107-111, in ref.[32]. 3, 9

[35] S. C. Kleene, *Introduction to Metamathematics*, North Holland (1971). 4

[36] D. E. Hesseling, *Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the1920s*, Springer Basel AG (2003). 4

[37] N. Gisin, *Mathematical languages shape our understanding of time in physics*, Nat. Phys. 16, 114–116 (2020), http://doi.org/10.1038/s41567-019-0748-5. 4

[38] N. Gisin, Synthese 199, 13345-13371 (2021); https://philpapers.org/archive/GISIIP-2.pdf. 4

[39] P. T. Raju, *Principle of Four-Cornered Negation in Indian Philosophy*, The Review of Metaphysics, Vol. 7, No. 4 (Jun., 1954), pp. 694-713. 10

[40] G. Priest, *The Logic of Chatuskoti*, Comparative Philosophy Volume 1, No. 2 (2010): 24-54, https://scholarworks.sjsu.edu/cgi/viewcontent.cgi?article=1032&context=comparativephilosophy. 10

[41] G. C. Nayak, *Madhyamika Sunyata: A Reappraisal*, Indian Council of Philosophical Research (2001). 3, 10

[42] J. Garfield, *The Fundamental Wisdom of the Middle Way: Nagarjuna's Mulamadhyamakakarika*, Oxford University Press, USA (1995). 3, 10

[43] M. Siderits, S. Katsura, *Nagarjuna's Middle way – the Mulamadhyamakakarika (Classics of Indian Buddhism)*, Wisdom Publications (2013). 3, 10

[44] F. Stcherbatsky, *Buddhist Logic* (two volumes), Dover Publications (1962). 3, 10

[45] Bhikkhu Bodhi, *The Great Discourse on Causation: The Mahānidāna Sutta and Its Commentaries*, Buddhist Publication Society (1995). 3, 10

[46] K. Potter (editor) - *Encyclopedia of Indian Philosophies, Volume II – Indian Metaphysics and Epistemology, The Tradition of Nyaya-Vaisesika up to Gangesa*, Motilal Banarsidass (1995). 10