The Solution of the Invariant Subspace Problem.
Part I. Complex Hilbert space.

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Abstract: The incompleteness of set theory ZFC leads one to look for natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be "true". One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG. It is a non-conservative extension of ZFC and is obtaineed from other axiomatic set theories by the inclusion of Tarski's axiom which implies the existence of inaccessible cardinals [1].In this paper we look at a set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$, based on bivalent gyper infinitary logic with restricted Modus Ponens Rule [2]-[5]. In this paper we deal with set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ based on gyper infinitary logic with Restricted Modus Ponens Rule.We present a new approach to the invariant subspace problem for Hilbert spaces. Our main result will be that: if T is a bounded linear operator on an infinite-dimensional complex separable Hilbert space H, it follow that T has a non-trivial closed invariant subspace.Nonconservative extension based on set theory $\mathbf{NC}^{\#}_{\infty}$ of the model theoretical nonstandard analysis [6]is considered.

Contents

0.Introduction

1.The incompleteness of set theory ZFC leads one to look for natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”. One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG. It is a non-conservative extension of ZFC and is obtaineed from other axiomatic set theories by the inclusion of Tarski’s axiom which implies the existence of inaccessible cardinals [1].In this paper we look at a set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$, based on bivalent gyper infinitary logic with restricted Modus Ponens Rule [2]-[5].
2.In this paper we will present a new approach to the invariant subspace problem for Hilbert spaces. Our main result will be that: if T is a bounded linear operator on an infinite-dimensional complex separable Hilbert space H, it follow that T has a non-trivial closed invariant subspace.In Section 2 we give the general ideas of the approach. In Section 3 we investigate set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ based on bivalent gyper infinitary logic with
restricted Modus Ponens Rule.
3.In Section 4 we

1.The Invariant Subspace Problem.

1.1.The Invariant Subspace Problem.Positive classical results.

The problem, in a general form,is stated as follows.**The Invariant Subspace Problem**:
If T is a bounded linear operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?
The Invariant Subspace Problem (as it stands today). If T is a bounded linear

operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?
Sometime during the 1930s John von Neumann proved that compact operators have non-trivial invariant subspaces, but did not publish it. The proof was rediscovered and finally published by N. Aronszajn and K. T. Smith [1] in 1954.
Theorem 1.1.1. (von Neumann, proved in [1]). Every compact operator on H has a non-trivial invariant subspace.
In 1966 Bernstein and Robinson [2],[3] extended the result to the slightly larger class of polynomially compact operators.
Definition 1.1.1.A linear operator T on a Banach space is said to be polynomially compact if there is a non-zero polynomial $p(t) \in \mathbb{C}[t]$ such that $p(T)$ is compact.
An nonclassical aspect of Bernstein and Robinson's proof is that it used the relatively new techniques of non-standard analysis, which builds up the foundations of analysis based on a rigorous definition of infinitesimal numbers [3]. Shortly after, the proof was translated into standard analysis by Halmos [4].
The next major generalization was achieved by Arveson and Feldman [5] in 1968.
Definition 1.1.2.For a bounded linear operator T on X, the uniformly closed algebra generated by T, denoted by $\mathbf{A}(T)$, is defined to be the subspace $[\{I, T, T^2, \dots\}]$ of $\mathbf{B}(X)$. Alternatively, $\mathbf{A}(T)$ is the smallest closed subspace of $\mathbf{B}(X)$ containing T and I which is closed under function composition.
If T is a bounded operator, then $\mathbf{A}(T)$ can be thought of as the closure of the set of polynomial combinations of T, or the set of all operators which can be norm approximated by polynomial combinations of T.
Theorem 1.1.2.(Arveson and Feldman [5]). If $T : H \to H$ is a bounded quasinilpotent operator such that $\mathbf{A}(T)$ contains a non-zero compact operator, then T has a non-trivial invariant subspace.
While the techniques of von Neumann and subsequent generalizations yielded many interesting and surprising theorems during the 1950s and 60s, their effectiveness was reaching its limit by the 70s.
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1.2.Counterexamples on Banach Spaces.

In 1975 Per Enflo discovered the first example of an operator on a Banach space having only the trivial invariant subspaces. He gave an outline of the proof in 1976 However, his full solution was not submitted until 1981 and did not appear in print until 1987 [7].As Enflo's paper crawled through the publication process, C. J. Read developed a counterexample of his own and submitted it for publication [8]. The paper was of similar length and complexity to Enflo's, however it was published much earlier in 1984. He did not cite Enflo's work, except to say the following:

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2.General schematic of the Solution of the Invariant Subspace Problem.

2.1.Stage I. Embedding $l_2 \hookrightarrow l^{\#}_{\omega} \hookrightarrow V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$.

An classical sequence space that is a subspace of the set of all sequences real or complex numbers $x = (x_1, x_2, \ldots) = (x_i)_{i=1}^{\infty}$.

The set of all $\mathbb{R}$-valued (or $\mathbb{C}$-valued) countable sequences we denote by l_{ω}.

For any $k \in \mathbb{N}$ let $\mathbf{e}_k = \mathbf{e}_k[i]$ be the sequence defined by

$$\mathbf{e}_k[i] = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases} \qquad (2.1.1)$$

The space l_2 (of square-summable sequences of real or complex numbers) is the set of infinite sequences of real or complex numbers such that

$$\|x\|_2 = \sum_{i=1}^{\infty} |x_i|^2 < \infty. \qquad (2.1.2)$$

l_2 is isomorphic to all separable, infinite dimensional Hilbert spaces.

There is a canonical countable basis $\{\mathbf{e}_k\}_{k=1}^{\infty}$ such that

$\mathbf{e}_1 = (1, 0, \ldots), \mathbf{e}_2 = (0, 1, \ldots)$, etc.

Remark 2.1.1. Note that there is the canonical embeding $r \to {}^*r$ [5]:

$$\mathbb{R} \hookrightarrow_* {}^*\mathbb{R} \qquad (2.1.3)$$

and we denote emadge of this embeding by ${}^*\mathbb{R}_{\mathbf{st}} \subset {}^*\mathbb{R}$. Thus we replace (2.1.2) by

$$\|{}^*x\|_2 = \sum_{i=1}^{\infty} |{}^*x_i|^2 < \infty. \qquad (2.1.4)$$

Definition 2.1.1.The space $l_{2,\omega}$ is the set of all ${}^*\mathbb{R}_{\mathbf{st}}$-valued (or ${}^*\mathbb{C}_{\mathbf{st}}$-valued) countable sequences such that

$$\|{}^*x\|_{2,\omega} = Ext\text{-}\sum_{i=1}^{\omega} |{}^*x_i|^2 < \infty. \qquad (2.1.5)$$

Remark 2.1.2. Note that in general case $\|{}^*x\|_2 \neq \|{}^*x\|_{2,\omega}$, since in general case

$$\sum_{i=1}^{\infty} |{}^*x_i|^2 \neq Ext\text{-}\sum_{i=1}^{\omega} |{}^*x_i|^2 < \infty, \qquad (2.1.6)$$

see sect. 3.12.

The set of all ${}^*\mathbb{R}^{\#}_c$-valued (or ${}^*\mathbb{C}^{\#}_c$-valued) countable sequences we denote by $\mathcal{L}^{\#}_{\omega}$.

Definition 2.1.2.The space $l^{\#}_{2,\omega}$ is the set of all

Definition 2.1.3.

1. Let $V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$ be a hyperfinite-dimensional vector space over external non-Archimedian field ${}^*\mathbb{R}^{\#}_c$. Such vector space consists of all external and internal ${}^*\mathbb{R}^{\#}_c$-valued hyperfinite sequences (called a vector) $\mathbf{x} = \{x_i\}_{i=1}^{i=n} = \{x_i\}_{i \in n}$ of hyperreal numbers, called the coordinates or components of vector $\mathbf{x}$. The vector sum of $\mathbf{x} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{x} + \mathbf{y} = \{x_i + y_i\}_{i=1}^{i=n}. \qquad (2.1.1)$$

If $a \in \mathbb{R}^{\#}_c$ is a hyperreal number, the scalar multiple of $\mathbf{x}$ by a is

$$a \times \mathbf{x} = \{a \times x_i\}_{i=1}^{i=n}. \qquad (2.1.2)$$

there is a canonical hyperfinite basis $\{\mathbf{e}_i\}$ $1 \leq i \leq n, n \in \mathbb{N}^{\#}/\mathbb{N}$ such that
$e_1 = (1,0,\ldots), e_2 = (0,1,\ldots)$, etc.

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2.2.Stage II.Extending of the bounded operator $A : l_2 \to l_2$ up to operator $\widehat{A} : V_n \to V_n$.

Remind that bounded operators $A : l_2 \to l_2$ admit matrix representations completely analogous to the well known matrix representations of operators on finite dimensional spaces [9].

We choose any orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in l_2 and let $Ae_k = c_k \in l_2$,

$$(Ae_k, e_i) = a_{ik} \qquad (2.2.1)$$

and therefore

$$c_k = \sum_{i=1}^{\infty} a_{ik} e_i, \qquad (2.2.2)$$

where $\sum_{i=1}^{\infty} |a_{ik}|^2 < \infty, k \in \mathbb{N}$. We introduce the infinite matrix $\mathbf{A}_\omega$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \qquad (2.2.3)$$

of which the elements of the k-th column are the components of the vector into which the
operator A maps the Ath coordinate vector. If the operator A is bounded, then it is uniquely
determined by the infinite matrix (a_{ik}) or the proof of this assertion it is necessary to show how to represent (he operator in terms of the matrix and the orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Thus, we have

$$Ae_k = \sum_{i=1}^{\infty} a_{ik} e_i. \qquad (2.2.4)$$

Since the operator A is linear, it is well defined on the linear envelope of (he given basis, i.e., for all vectors each of which has only a finite number of nonzero components relative to the basis. Since $\mathbf{A}$ is continuous, the value of $\mathbf{A}f$ for an arbitrary
vector $f \in l_2$ may be found by the limit.It is not difficult to write a simple formula for the components of the vector $\mathbf{A}f$ indeed, if

$$f = \sum_{k=1}^{\infty} x_k e_k, \qquad (2.2.5)$$

then

$$\mathbf{A}f = \sum_{k=1}^{\infty} y_k e_k, \qquad (2.2.6)$$

where

$$y_k = \sum_{i=1}^{\infty} a_{ki}x_i \tag{2.2.7}$$

Definition 2.2.1. If the operator $A : l_2 \to l_2$ is defined everywhere in l_2 and if its value for any vector (1.5.5) is given by the formulas (1.5.6) and (1.5.7), then we say that the operator A admits a matrix representation relative to the orthogonal basis $\{e_k\}_{k=1}^{\infty}$.

Theorem 2.2.1.Every bounded linear operator $A : l_2 \to l_2$ defined on the entire space admits a matrix representation with respect to each orthogonal basis.

Proof.Let $f_n = \sum_{k=1}^{n} x_k e_k$, then $Af_n = \sum_{k=1}^{\infty} y_k^{(n)} e_k$, where $y_k^{(n)} = \sum_{k=1}^{n} a_{ki}x_i$. By the boundedness of the operator A, we get

$$y_k = (Af, e_k) = \lim_{n\to\infty}(Af_n, e_k) = \lim_{n\to\infty} y_k^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{n} a_{ki}x_i = \sum_{k=1}^{\infty} a_{ki}x_i. \tag{2.2.8}$$

Theorem 2.2.2. If an operator $A : l_2 \to l_2$ defined everywhere in a separable space l_2, admits a matrix representation (1.5.3) with respect to some orthogonal basis $\{e_k\}_{k=1}^{\infty}$, then it is bounded.

Proof. By hypothesis, the series $(Af, e_k) = \sum_{i=1}^{\infty} a_{ki}x_i$ converges for each vector $f = \sum_{k=1}^{\infty} x_k e_k$, where $\{e_k\}_{k=1}^{\infty}$ the orthonormal basis, mentioned in the theorem, with respect to which the operator A admits a matrix representation. Therefore, by the theorem of Landau (see [9] Section 18), one obtains

$$\sum_{i=1}^{\infty} |a_{ki}|^2 < \infty, k \in \mathbb{N}. \tag{2.2.9}$$

We introduce the sequence of vectors $c_k^* = \sum_{i=1}^{\infty} \bar{a}_{ki}e_i, k \in \mathbb{N}$ and by means of them, define the linear operator A^*. First, let $A^*e_k = c_k^*$ and then use linearity to define A^* on the linear envelope of the set of vectors e_k. Finally, extend A^* by continuity to all of l_2. It is easy to prove that for any $f, g \in l_2, (Af, g) = (f, A^*g)$ after which, to complete the proof, it remains to apply Hellinger and Toeplitz theorem, see [9] Section 26.

Remark 2.2.1.In view of the inequality (1.5.9), the expression

$$\Phi_k(f) = \sum_{i=1}^{\infty} a_{ki}x_i, k \in \mathbb{N} \tag{2.2.10}$$

defines a linear functional of $f = \sum_{k=1}^{\infty} x_k e_k$ and therefore,$P_n(f) = \sqrt{\sum_{k=1}^{n} \Phi_k^2(f)}, n \in \mathbb{N}$ defines a convex continuous functional of f. Since the sequence $\{P_n(f)\}_{n=1}^{\infty}$ is bounded for each $f \in l_2$. On the basis of the corollary of the lemma concerning convex functionals [9], the functional

$$P(f) = \sup_{n\in\mathbb{N}} P_n(f) = \lim_{n\to\infty} P_n(f) = \sqrt{\sum_{k=1}^{\infty} \Phi_k^2(f)} = \|Af\| \tag{2.2.11}$$

is continuous, i.e., there exists a constant M such that $P(f) \leq M\|f\|$, but this implies that the operator A is bounded.

Remark 2.2.2.The proof of the theorem can be formulated also in the following form: if for arbitrary numbers $x_k, k \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty \tag{2.2.12}$$

the inequality

$$\sum_{k=1}^{\infty} \left|\sum_{i=1}^{\infty} a_{ki}x_i\right|^2 < \infty \tag{2.2.13}$$

holds, then there exists a constant M such that

$$\sum_{k=1}^{\infty} \left|\sum_{i=1}^{\infty} a_{ki}x_i\right|^2 \leq M^2 \sum_{k=1}^{\infty} |x_k|^2. \tag{2.2.14}$$

Definition 2.2.2. A sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence if

$$\sum_{k=1}^{\infty}|x_k|^2 = Ext\text{-}\sum_{k=1}^{\omega}|x_k|^2. \quad (2.2.15)$$

Definition 1.5.3. Let $\{e_k\}_{k=1}^{\infty}$ be Schauder basis in l_2, i.e., the standard unit vector basis in l_2. Vector $f = \sum_{k=1}^{\infty} x_k\mathbf{e}_k \in l_2$ is admissible vector of l_2 if sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence.

Remark 1.5.3. Note that if vector $f = \sum_{k=1}^{\infty} x_k\mathbf{e}_k$ is admissible vector of l_2 then $f = \sum_{k=1}^{\infty} x_k\mathbf{e}_k = Ext\text{-}\sum_{k=1}^{\omega} x_k\mathbf{e}_k$.

Definition 1.5.4. We extend now the operator $A : l_2 \to l_2$ is given on l_2 by the Eq.(2.2.6)-Eq.(2.2.7) up to the operator $\widehat{A} : \mathcal{L}^{\#} \to \mathcal{L}^{\#}$ is given on $\mathcal{L}^{\#}$ by the ω-sum is given by the Eq.(2.2.16)-Eq.(2.2.17)

$$\widehat{A}f = Ext\text{-}\sum_{k=1}^{\omega} \widehat{y}_k\mathbf{e}_k, \quad (2.2.16)$$

where

$$\widehat{y}_k = Ext\text{-}\sum_{i=1}^{\omega} a_{ki}x_i. \quad (2.2.17)$$

Theorem 1.5.3. Assume that a sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence and let $f = \sum_{k=1}^{\infty} x_k\mathbf{e}_k \in l_2$. Let $\{\widehat{y}_k\}_{k=1}^{\infty}$ be a sequence is given by the Eq.(2.2.17). Then (i) sequence $\{\widehat{y}_k\}_{k=1}^{\infty}$ is admissible sequence and (ii) $\{\widehat{y}_k\}_{k=1}^{\infty} \in l_2$

Proof. If sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence, then by Definition 2.2.2 we get

$$\sum_{k=1}^{\infty}|x_k|^2 = Ext\text{-}\sum_{k=1}^{\omega}|x_k|^2. \quad (2.2.18)$$

From the equality (2.2.18) by Theorem 3.12.6 we obtain

$$\lim_{m\to\infty} Ext\text{-}\sum_{k=m}^{\omega}|x_k|^2 = 0. \quad (2.2.19)$$

From the equality (2.2.17) we get for all $k \in \mathbb{N}$

$$|\widehat{y}_k| \leq Ext\text{-}\sum_{i=1}^{\omega}|a_{ki}x_i| \leq \alpha_k\left(Ext\text{-}\sum_{i=1}^{\omega}|x_i|^2\right), \quad (2.2.20)$$

where $\alpha_k = Ext\text{-}\sum_{i=1}^{\omega}|a_{ki}|^2 < \infty$. We set now $x_i = 0, i \leq m$ in (2.2.20) and therefore

$$\left|\widehat{y}_{k,m}\right| = \left|Ext\text{-}\sum_{i=m}^{\omega} a_{ki}x_i\right| \leq Ext\text{-}\sum_{i=m}^{\omega}|a_{ki}x_i| \leq \alpha_k\left(Ext\text{-}\sum_{i=m}^{\omega}|x_i|^2\right), \quad (2.2.21)$$

From the inequality (2.2.21) by the equality (2.2.19) for the all $k \in \mathbb{N}$ we get

$$\lim_{m\to\infty}\left|\widehat{y}_{k,m}\right| = \lim_{m\to\infty}\left|Ext\text{-}\sum_{i=m}^{\omega} a_{ki}x_i\right| \leq \lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\infty}|x_k|^2\right) = 0 \quad (2.2.22)$$

and therefore

$$\lim_{m\to\infty}\left|\widehat{y}_{k,m}\right| = 0. \quad (2.2.23)$$

From the equality (2.2.18) by Theorem 3.12.6 we get for all $k \in \mathbb{N}$ that

$$\widehat{y}_k = Ext\text{-}\sum_{i=m}^{\omega} a_{ki}x_i = \sum_{i=m}^{\infty} a_{ki}x_i = y_k. \quad (2.2.24)$$

Thus we get for all $k \in \mathbb{N}$ that $\widehat{y}_k = y_k$.

Theorem 1.5.4. Assume that a vector $f = \sum_{k=1}^{\infty} x_k\mathbf{e}_k$ is admissible vector of l_2 then for all $n \in \mathbb{N}$ vector $\left(\widehat{A}\right)^n f$ is admissible vector of l_2.

Proof. Immediately by Definition 1.5.3 from Theorem 1.5.3.

The operator $\widehat{A} : V_n \to V_n$.

Using now the canonical imbedding $\mathbb{R} \hookrightarrow_{*} {}^{*}\mathbb{R}$(2.1.3) we imbed now the infinite matrix $\mathbf{A}_\omega = (a_{ij})_{i,j\in\mathbb{N}}$(2.2.3) into the infinite matrix $\mathbf{A}^*_\omega = ({}^*a_{ij})_{i,j\in\mathbb{N}}$ (2.2.18) such that

$$\mathbf{A}^*_\omega = \left\| \begin{array}{ccccc} {}^*a_{11} & {}^*a_{12} & \cdots & {}^*a_{1n} & \cdots \\ {}^*a_{21} & {}^*a_{22} & \cdots & {}^*a_{2n} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ {}^*a_{n1} & {}^*a_{n2} & \cdots & {}^*a_{nn} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots \end{array} \right\| \qquad (2.2.25)$$

We imbed now the infinite matrix $\mathbf{A}^*_\omega = ({}^*a_{ij})_{i,j\in\mathbb{N}}$ (2.2.3) into the hyperfinite matrix $\mathbf{A}^*_{\omega,m} = ({}^*a_{ij})_{i\leq m, j\leq m}$, where $m \in \mathbb{N}^\#\backslash\mathbb{N}$ and for $i,j \in \mathbb{N}^\#\backslash\mathbb{N}$ the conditions are satisfied (i)${}^*a_{ij} = {}^*0$ if $i \neq j$, (ii) ${}^*a_{ij} = {}^*1$ if $i = j$, see subsection (4.1.3).

Thus $\mathbf{A}^*_{\omega,m}$ is hyperfinite external matrix of the following literal form

$$\mathbf{A}^*_{\omega,m} = \left\| \begin{array}{cccccccc} {}^*a_{11} & {}^*a_{12} & \cdots & {}^*a_{1n} & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*a_{21} & {}^*a_{22} & \cdots & {}^*a_{2n} & \cdots & {}^*0 & {}^*0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*a_{n1} & {}^*a_{n2} & \cdots & {}^*a_{nn} & \cdots & {}^*0 & {}^*0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*0 & {}^*0 & \cdots & {}^*0 & \cdots & {}^*1 & {}^*0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & {}^*0 & {}^*1 & \cdots \\ {}^*0 & {}^*0 & \cdots & {}^*0 & {}^*0 & {}^*0 & {}^*0 & {}^*1 \end{array} \right\| \qquad (2.2.26)$$

The matrix $\mathbf{A}^*_{\omega,n}$ is defind an external linear operator $\widehat{A}$ on V_n by the formula

$$\widehat{A}f =$$

$$\left\| \begin{array}{cccccccc} {}^*a_{11} & {}^*a_{12} & \cdots & {}^*a_{1n} & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*a_{21} & {}^*a_{22} & \cdots & {}^*a_{2n} & \cdots & {}^*0 & {}^*0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*a_{n1} & {}^*a_{n2} & \cdots & {}^*a_{nn} & \cdots & {}^*0 & {}^*0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & {}^*0 & {}^*0 & \cdots \\ {}^*0 & {}^*0 & \cdots & {}^*0 & \cdots & {}^*1 & {}^*0 & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots & {}^*0 & {}^*1 & \cdots \\ {}^*0 & {}^*0 & \cdots & {}^*0 & {}^*0 & {}^*0 & {}^*0 & {}^*1 \end{array} \right\| \times \left\| \begin{array}{c} x_1 \\ x_2 \\ \\ x_n \\ \cdot \\ \cdot \\ x_{m-1} \\ x_m \end{array} \right\| = \left\| \begin{array}{c} y_1 \\ y_2 \\ \\ y_n \\ \cdot \\ \cdot \\ y_{m-1} \\ y_m \end{array} \right\| \qquad (2.2.27)$$

where $f = Ext\text{-}\sum_{k=1}^{m} x_k \mathbf{e}_k \in V_n$ and where the multiplication $\times$ is defined by

$$y_i = Ext\text{-}\sum_{k=1}^{m} a_{ik} x_k, \tag{2.2.28}$$

where $i \leq m,$ see subsection 4.4.

2.3.Stage III.Proof that operator $\widehat{\mathbf{A}}$ has a non-trivial infinite-dimensional invariant subspace $\widehat{l}_{2,\omega} \subsetneqq l^{\#}_{2,\omega} \subset V_m$.

Theorem 2.3.1. Suppose the operator $\mathbf{A}$ has an annihilating polynomial of the form

$$Q(\lambda) = Ext\text{-}\prod_{k=1}^{m} [Ext\text{-}(\lambda - \lambda_k)^{r_k}], \tag{2.3.1}$$

where: (i) the function $Ext\text{-}(\lambda - \lambda_k)^{r_k}$,for all $r_k \in \mathbb{N}^{\#}$ is defined by the following formula

$$Ext\text{-}(\lambda - \lambda_k)^{r_k} = Ext\text{-}\prod_{i=1}^{p} \Theta_i(\lambda, \lambda_k), \tag{2.3.2}$$

where $r_k < p \in \mathbb{N}^{\#}\backslash\mathbb{N}$, $\Theta_i(\lambda, \lambda_k) = (\lambda - \lambda_k)$ for all $1 \leq i \leq r_k$ and $\Theta_i(\lambda, \lambda_k) \equiv 1$ for all $i > r_k$, and

(ii) the function $Ext\text{-}(\lambda - \lambda_k)^{r_k}$,for $r_k = \omega$ is defined by the formula (2.3.2)

$$Ext\text{-}(\lambda - \lambda_k)^{r_k} = Ext\text{-}\prod_{i \in \mathbb{N}^{\#}} \Theta_i(\lambda, \lambda_k), \tag{2.3.3}$$

with $\Theta_i(\lambda, \lambda_k) = (\lambda - \lambda_k)$ for all $i \in \mathbb{N}$ and $\Theta_i(\lambda, \lambda_k) \equiv 1$ for all $i \in \mathbb{N}^{\#}\backslash\mathbb{N}$.In this case we denote it by

$$Ext\text{-}(\lambda - \lambda_k)^{\omega} = Ext\text{-}\prod_{i=1}^{\omega} \Theta_i(\lambda, \lambda_k). \tag{2.3.4}$$

Here $\lambda_1, \ldots, \lambda_m$ are all the (distinct) roots of $Q(\lambda)$ and r_k is the multiplicity of λ_k. For example, such a factorization is always possible (to within a numerical factor) in the field ${}^*\mathbb{C}^{\#}_c$.

Remark 2.3.1. Then the space $V_m, m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ can be represented as the direct sum

$$V_m = Ext\text{-}\bigoplus_{k=1}^{r} T_k \tag{2.3.5}$$

of r subspaces $T_1, \ldots, T_r, r \in \mathbb{N}^{\#}$ all invariant with respect to $\mathbf{A}$, where the subspace T_k is annihilated by $\mathbf{B}_k^{r_k}$, the r_k -th power of the operator $\mathbf{B}_k = \mathbf{A} - \lambda_k \mathbf{E}$,see subsection 4.17.2.

The all vectors $\mathbf{x} \in V_m$ has a form $\mathbf{x} = \{x_i\}_{i=1}^{i=n}$.Thus we can represent the vector $\mathbf{x} \in V_m$ as

the sum $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$,where

$$\mathbf{x}_1 = \left\{x_i^{(1)}\right\}_{i \in \mathbb{N}} \tag{2.3.6}$$

and

$$\mathbf{x}_2 = \left\{x_i^{(2)}\right\}_{i \in \mathbb{N}^{\#}\backslash\mathbb{N}} \tag{2.3.7}$$

with $i \leq m$.Then the space V_m can be represented as the direct sum

$$V_m = V_m^{(1)} \bigoplus V_m^{(2)}, \tag{2.3.8}$$

where the subspace $V_m^{(1)}$ contains the all vectors of the form $\mathbf{x}_1 = \left\{x_i^{(1)}\right\}_{i\in\mathbb{N}}$ and the subspace $V_m^{(2)}$ contains the all vectors of the form $\mathbf{x}_2 = \left\{x_i^{(2)}\right\}_{i\in\mathbb{N}^\#\setminus\mathbb{N}}$ with $i \leq m$.
The hyperfinite matrix $\mathbf{A}^*_{\omega,n}$ is a direct sum $\mathbf{A}^*_{\omega,m} = \mathbf{A}^*_\omega \bigoplus \mathbf{D}$, where $\mathbf{D}$ is infinite diagonal matrix $\mathbf{D} = diag({}^*0, {}^*0, \ldots, {}^*1, {}^*1, \ldots, {}^*1)$. Thus operator $\widehat{\mathbf{A}}$ has the following form $\widehat{\mathbf{A}} = \widehat{\mathbf{A}}_1 + \widehat{\mathbf{A}}_2$, where $\widehat{\mathbf{A}}_1 = \widehat{\mathbf{A}} \restriction V_m^{(1)}$ and $\widehat{\mathbf{A}}_2 = \widehat{\mathbf{A}} \restriction V_m^{(2)}$.
We will be consider now the following three possible cases.
I.There is no invariant subspace of $V_m^{(1)}$ with respect to $\widehat{\mathbf{A}}_1$. In this case we obtain $Q(\lambda) = Ext\text{-}(\lambda - \lambda_1)^\omega$, where $\lambda_1 \in \mathbb{R}_c^\#$. It follows from Theorem 1.5.4 that $\lambda_1 \in \mathbb{R}$ and infinite matrice $\mathbf{A}_\omega$ (2.2.3) is diagonal. Thus in this case operator $\mathbf{A} : l_2 \to l_2$ has a form

$$\mathbf{A} = \lambda_1 \mathbf{1}. \tag{2.3.9}$$

II.There is countable set of subspaces $\{T_k\}_{k\in\mathbb{N}}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $k \in \mathbb{N}, \dim T_k < \infty$ and where

$$V_m^{(1)} = Ext\text{-}\bigoplus\nolimits_{k\in\mathbb{N}} T_k. \tag{2.3.10}$$

Let $\mathbb{N}_1$and $\mathbb{N}_2$ be a subsets of $\mathbb{N}$ such that $\mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$ and $\mathbb{N}_1 \cap \mathbb{N}_2 = \varnothing$. Now we choose an countable set of subspaces $\{T_i\}_{i\in\mathbb{N}_1} \subsetneqq \{T_k\}_{k\in\mathbb{N}}$ such that

$$\{T_k\}_{k\in\mathbb{N}} = \{T_i\}_{i\in\mathbb{N}_1} \cup \{T_j\}_{j\in\mathbb{N}_2}, \tag{2.3.11}$$

where $\{T_j\}_{j\in\mathbb{N}_2} = \{T_k\}_{k\in\mathbb{N}} \setminus \{T_i\}_{i\in\mathbb{N}_1}$. Let $\Delta_1$and $\Delta_2$be a subspaces of $V_m^{(1)}$ such that

$$\Delta_1 = Ext\text{-}\bigoplus\nolimits_{k\in\mathbb{N}_1} T_k \tag{2.3.12}$$

and

$$\Delta_2 = Ext\text{-}\bigoplus\nolimits_{k\in\mathbb{N}_2} T_k \tag{2.3.12}$$

correspondingly and therefore

$$V_m^{(1)} = \Delta_1 \oplus \Delta_2. \tag{2.3.13}$$

III.There is only finite set of subspaces $\{T_k\}_{k\leq r}, r \in \mathbb{N}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $1 \leq k \leq r, \dim T_k = \infty$ and where

$$V_m^{(1)} = \bigoplus\nolimits_{k=1}^{r} T_k. \tag{2.3.14}$$

IV.There is countable set of subspaces $\{T_k\}_{k\in\mathbb{N}}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $k \in \mathbb{N}, \dim T_k = \infty$ and where

2.4.Stage IV.Proof that there exists an admissible vector $\Psi \in l_2 \wedge \Psi \in \widehat{l}_{2,\omega}$.

Theorem 2.3.4.There exists an admissible vector Ψ_1 in subspace Δ_1 and there exists an admissible vector Ψ_2 in subspace Δ_2.
Proof. Note that subspace Δ_1 has an countable basis $\left\{\mathbf{b}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty}$and similarly subspace Δ_2 has an countable basis $\left\{\mathbf{b}_{k_1}^{(2)}\right\}_{k_1=1}^{\infty}$and $\left\{\mathbf{b}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty} \bigcap \left\{\mathbf{b}_{k_1}^{(2)}\right\}_{k_1=1}^{\infty} = \varnothing$, where $\{k_1\}_{k_1=1}^{\infty} \cap \{k_2\}_{k_2=1}^{\infty} = \varnothing$ and $\{k_1\}_{k_1=1}^{\infty} \cup \{k_2\}_{k_2=1}^{\infty} = \mathbb{N}$.

We represent now basis vectors $\mathbf{b}_i^{(1)}, i = 1,2,\ldots$ as infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1,2,\ldots$ of the following literal form

$$\left(\mathbf{b}_1^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{11}^{(1)} \\ b_{21}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 1}^{(1)} \\ b_{k_1+11}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \left(\mathbf{b}_2^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{12}^{(1)} \\ b_{22}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 2}^{(1)} \\ b_{k_1+12}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \ldots, \left(\mathbf{b}_{k_1}^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{1k_1}^{(1)} \\ b_{2k_1}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 k_1}^{(1)} \\ b_{k_1+1k_1}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \ldots \qquad (2.4.1)$$

Remark 2.4.1.Note that the infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1,2,\ldots$are linearly independent.

Similarly we represented basis vectors $\mathbf{b}_i^{(2)}, i = 1,2,\ldots$ as infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1,2,\ldots$of the following literal form

$$\left(\mathbf{b}_1^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{11}^{(2)} \\ b_{21}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k1}^{(2)} \\ b_{k+11}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \left(\mathbf{b}_2^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{12}^{(2)} \\ b_{22}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_2 2}^{(2)} \\ b_{k_2+12}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \ldots, \left(\mathbf{b}_{k_2}^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{1k}^{(2)} \\ b_{2k}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_2 k_2}^{(2)} \\ b_{k_2+1k_2}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \ldots \qquad (2.4.2)$$

Remark 2.4.2. Note that $\left\{\left(\mathbf{b}_{k_1}^{(1)}\right)^{\top}\right\}_{k_1=1}^{\infty} \bigcap \left\{\left(\mathbf{b}_{k_2}^{(2)}\right)^{\top}\right\}_{k_2=1}^{\infty} = \varnothing$,since $\left\{\mathbf{b}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty} \bigcap \left\{\mathbf{b}_{k_2}^{(2)}\right\}_{k_2=1}^{\infty} = \varnothing$.

Using the columns (2.4.1) we formed the following infinite matrix of the following literal form

$$
\mathbf{\Omega}_{1,\omega} = \left\| \begin{matrix}
b_{11}^{(1)} & b_{12}^{(1)} & \cdots & b_{1k}^{(1)} & b_{1k_1+1}^{(1)} & \cdots \\
b_{21}^{(1)} & b_{22}^{(1)} & \cdots & b_{2k}^{(1)} & b_{2k_1}^{(1)} & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
b_{k_1 1}^{(1)} & b_{k_1 2}^{(1)} & \cdots & b_{k_1 k_1}^{(1)} & b_{k_1 k_1+1}^{(1)} & \cdots \\
b_{k_1+11}^{(1)} & b_{k_1+12}^{(1)} & \cdots & b_{k_1+1k_1}^{(1)} & b_{k_1+1k_1+1}^{(1)} & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots
\end{matrix} \right\| \qquad (2.4.3)
$$

And similarly the columns (2.4.2) we formed the following infinite matrix of the following literal form

$$
\mathbf{\Omega}_{2,\omega} = \left\| \begin{matrix}
b_{11}^{(2)} & b_{12}^{(2)} & \cdots & b_{1k}^{(2)} & b_{1k_2+1}^{(2)} & \cdots \\
b_{21}^{(2)} & b_{22}^{(2)} & \cdots & b_{2k}^{(2)} & b_{2k_2}^{(2)} & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
b_{k_2 1}^{(2)} & b_{k_2 2}^{(2)} & \cdots & b_{k_2 k_2}^{(2)} & b_{k_2 k_2+1}^{(2)} & \cdots \\
b_{k_2+11}^{(2)} & b_{k_2+12}^{(2)} & \cdots & b_{k_2+1k}^{(2)} & b_{k_2+1k_2+1}^{(2)} & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots
\end{matrix} \right\| \qquad (2.4.4)
$$

It follows directly from Theorem 4.14.2, see subsection 4.14, that

$$
\det \mathbf{\Omega}_{1,\omega} \neq 0 \qquad (2.4.5)
$$

and

$$
\det \mathbf{\Omega}_{2,\omega} \neq 0. \qquad (2.4.6)
$$

We conseder now the following infinite system of the linear equations

$$\left\| \begin{array}{cccccc} b^{(1)}_{11} & b^{(1)}_{12} & \cdots & b^{(1)}_{1k} & b^{(1)}_{1k_1+1} & \cdots \\ b^{(1)}_{21} & b^{(1)}_{22} & \cdots & b^{(1)}_{2k} & b^{(1)}_{2k_1} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ b^{(1)}_{k_1 1} & b^{(1)}_{k_1 2} & \cdots & b^{(1)}_{k_1 k_1} & b^{(1)}_{k_1 k_1+1} & \cdots \\ b^{(1)}_{k_1+11} & b^{(1)}_{k_1+12} & \cdots & b^{(1)}_{k_1+1k_1} & b^{(1)}_{k_1+1k_1+1} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \end{array} \right\| \times \left\| \begin{array}{c} y^{(1)}_1 \\ y^{(1)}_2 \\ \cdot \\ \cdot \\ \cdot \\ y^{(1)}_{k_1} \\ y^{(1)}_{k_1+1} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| = \left\| \begin{array}{c} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{k_1} \\ x_{k_1+1} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| \qquad (2.4.7)$$

and the following infinite system of the linear equations

$$\left\| \begin{array}{cccccc} b^{(2)}_{11} & b^{(2)}_{12} & \cdots & b^{(2)}_{1k} & b^{(2)}_{1k_2+1} & \cdots \\ b^{(2)}_{21} & b^{(2)}_{22} & \cdots & b^{(2)}_{2k} & b^{(2)}_{2k_2} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ b^{(2)}_{k_2 1} & b^{(2)}_{k_2 2} & \cdots & b^{(2)}_{k_2 k_2} & b^{(2)}_{k_2 k_2+1} & \cdots \\ b^{(2)}_{k_2+11} & b^{(2)}_{k_2+12} & \cdots & b^{(2)}_{k_2+1k} & b^{(2)}_{k_2+1k_2+1} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \end{array} \right\| \times \left\| \begin{array}{c} y^{(2)}_1 \\ y^{(2)}_2 \\ \cdot \\ \cdot \\ \cdot \\ y^{(2)}_{k} \\ y^{(2)}_{k_2+1} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| = \left\| \begin{array}{c} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{k_2} \\ x_{k_2+1} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| \qquad (2.4.8)$$

where a sequences $\{x_{k_1}\}_{k_1=1}^{\infty} \in l_2$ and $\{x_{k_2}\}_{k_2=1}^{\infty} \in l_2$are admissible sequences such that $\{x_{k_1}\}_{k_1=1}^{\infty} \cap \{x_{k_2}\}_{k_2=1}^{\infty} = \varnothing$, see Definition 2.2.2.
It follows directly from (2.4.5) by Theorem 4.12.2 that the system (2.4.7) has a unique solution namely $\left\{\bar{y}^{(1)}_{k_1}\right\}_{k_1=1}^{\infty}$ and similarly it follows directly from (2.4.6) by Theorem 4.12.2 that the system (2.4.8) has a unique namely $\left\{\bar{y}^{(2)}_{k_2}\right\}_{k_2=1}^{\infty}$.
Thus finally we get that there is an admissible vector $f_1 = Ext\text{-}\sum_{k=1}^{\omega} x_k \mathbf{e}_k$ such that

$$f_1 = \sum_{k_1=1}^{\infty} x_{k_1} \mathbf{e}_{k_1} = Ext\text{-}\sum_{k_1=1}^{\omega} x_{k_1} \mathbf{e}_{k_1} \in \Delta_1 \qquad (2.4.9)$$

and there is an admissible vector $f_2 = Ext\text{-}\sum_{k_2=1}^{\omega} x_k \mathbf{e}_k$ such that

$$f_2 = \sum_{k_2=1}^{\omega} x_{k_2} \mathbf{e}_{k_2} = Ext\text{-}\sum_{k_2=1}^{\omega} x_{k_2} \mathbf{e}_{k_2} \in \Delta_2. \qquad (2.4.10)$$

Note that the statements (2.4.9) and (2.4.10) finalized the proof.

2.5.Stage V.Proof the main result by a contradiction.

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3.Nonconservative extension of the canonical nonstandard analysis.

3.1.Set Theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ Based on Bivalent Gyper Infinitary Logic with Restricted Modus Ponens Rule.

Set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is formulated as a system of axioms based on bivalent hyper infinitary logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion [1]-[4],see Appendix A. The language of set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is a first-order hyper infinitary language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg(x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x,y,z,\ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x)\ \&\ \forall y(\varphi(y) \Rightarrow x = y)]$. $L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called non-classical sets; we shall use upper case letters $A,B,\ldots$ for such sets. For each non-classical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x,A)]$ is called the defining axioms for the non-classical set A.

Remark 3.1.1. Remind that in logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ is not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{RMP} \beta \qquad (3.1.1)$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{RMP} \beta \qquad (3.1.2)$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds (or logically valid) [1]-[4],see Appendix A.

Abbreviation 3.1.2. We often write for the sake of brevity instead (3.1.1) by

$$\alpha \Rightarrow_{s} \beta \qquad (3.1.3)$$

and we often write instead (3.1.2) by

$$\alpha \Rightarrow_{w} \beta. \qquad (3.1.4)$$

Remark 3.1.2. Let A be an nonclassical set.Note that in set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$the following true formula

$$\exists A \forall x[x \in A \Leftrightarrow \varphi(x,A)] \qquad (3.1.5)$$

does not always guarantee that

$$x \in A, x \in A \Rightarrow \varphi(x,A) \vdash_{RMP} \varphi(x,A) \qquad (3.1.6)$$

even if $x \in A$ holds and (or)

$$\varphi(x,A), \varphi(x,A) \Rightarrow x \in A \vdash_{RMP} x \in A; \qquad (3.1.7)$$

even $\varphi(x,A)$ holds, since for nonclassical set A for some y possible

$$y \in A,\ y \in A \Rightarrow \varphi(y,A) \nvdash_{RMP} \varphi(y,A) \qquad (3.1.8)$$

and (or)

$$\varphi(y,A),\ \varphi(y,A) \Rightarrow y \in A \nvdash_{RMP} y \in A. \tag{3.1.9}$$

Remark 3.1.3. Note that in this paper the formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u] \tag{3.1.10}$$

and more general formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \tag{3.1.11}$$

is considered as the defining axioms for the classical set a.

Remark 3.1.4. Let a be a classical set. Note that in $\mathbf{NC}^{\#}_{\infty^{\#}}$: (i) the following true formula

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \tag{3.1.12}$$

always guarantee that

$$x \in a, x \in a \Rightarrow \varphi(x,a) \vdash_{RMP} \varphi(x) \tag{3.1.13}$$

if $x \in a$ holds and

$$\varphi(x), \varphi(x) \Rightarrow x \in a \vdash_{RMP} x \in a; \tag{3.1.14}$$

if $\varphi(x)$ holds;

In order to emphasize this fact mentioned above in Remark 3.1.1-3.1.3,

we rewrite the defining axioms in general case for the nonclassical sets in the following form

$$\exists A \forall x\{[x \in A \Leftrightarrow_s \varphi(x,A)] \vee [x \in A \Leftrightarrow_w \varphi(x,A)]\} \tag{3.1.15}$$

and similarly we rewrite the defining axioms in general case for the classical sets in the following form

$$\forall x[x \in a \Leftrightarrow_s \varphi(x,a) \wedge (x \in u)]. \tag{3.1.16}$$

Abbreviation 3.1.2. We write instead (3.1.15):

$$\forall x\{[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]\} \tag{3.1.17}$$

Definition 3.1.1. (1) Let A be a nonclassical set defined by formula (3.1.17).
Assum that: (i) for some y statement $\varphi(y)$ and statement $\varphi(y) \Rightarrow y \in A$ holds and
(ii) $\varphi(y), \varphi(y) \Rightarrow y \in A \nvdash_{RMP} y \in A,\ y \in A, y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$.
Then we say that y is a <u>weak member</u> of non-classical set A and abbreviate $y \in_w A$.

Abbreviation 3.1.3. Let A be a nonclassical set defined by formula (3.1.5) or by formula (3.1.17). We abbreviate $x \in_{s,w} A$ if the following statement $x \in_s A \vee x \in_w A$ holds, i.e.

$$x \in_{s,w} A \leftrightarrow_{def} (x \in_s A \vee x \in_w A). \tag{3.1.18}$$

Definition 3.1.2. (1) Two nonclassical sets A, B are defined to be equal and we write $A = B$ if $\forall x[x \in_{s,w} A \Leftrightarrow_s x \in_{s,w} B]$. (2) A is a subset of B, and we often write $A \subset_{s,v} B$, if $\forall x[x \in_{s,w} A \Rightarrow_s x \in_{s,w} B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u \forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x)]$ and $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x,A)]$.

Remark 3.1.5. $\mathbf{CL.Set}(A)$) asserts that the set A is a classical set. For any classical set u, it follows from the defining axiom for the classical set $\{x | x \in_s u \wedge \varphi(x)\}$ that

$\mathbf{CL.Set}(\{x|x \in_s u \wedge \varphi(x)\})$.
We shall identify $\{x|x \in_s u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset_s A, u \subseteqq_s A$, etc.
Abbreviation 3.1.4. Let $\varphi(t)$ be a formula of $\mathbf{NC}^{\#}_{\infty^{\#}}$.
(i) $\forall x\varphi(x)$ and $\forall^{\mathbf{CL}} x\varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(ii) $\exists x\varphi(x)$ and $\exists^{\mathbf{CL}} x\varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(iii) $\forall X\varphi(X)$ and $\forall^{\mathbf{NCL}} X\varphi(X)$ abbreviates $\forall X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
(iv) $\exists X\varphi(X)$ and $\exists^{\mathbf{NCL}} X\varphi(X)$ abbreviates $\exists X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
Remark 3.1.6. If A is a nonclassical set, we write $\exists x \in A\ \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$ and $\forall x \in A\varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.
We define now the following sets:
1. $\{u_1,u_2,\ldots,u_n\} = \{x|x = u_1 \vee x = u_2 \vee\ldots\vee x = u_n\}$. 2. $\{A_1,A_2,\ldots,A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee\ldots\vee x = A_n\}$. 3. $\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4. $\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5. $A \cup B = \{x|x \in A \vee x \in B\}$.
5. $A \cap B = \{x|x \in A \wedge x \in B\}$. 6. $A - B = \{x|x \in A \wedge x \notin B\}$. 7. $u^+ = u \cup \{u\}$.
8. $\mathbf{P}(A) = \{x|x \subseteq A\}$. 9. $\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10. $\mathbf{V} = \{x|: x = x\}$.
11. $\varnothing = \{x|x \neq x\}$.
The system $\mathbf{NC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:
Extensionality1: $\forall u\forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A\forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCL.Set}(\mathbf{V})$
Empty Set: $\mathbf{CL.Set}(\varnothing)$
Pairing1: $\forall u\forall v\ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A\forall B\ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u\ \mathbf{CL.Set}(\cup u)$
Union2: $\forall A\ \mathbf{NCL.Set}(\cup A)$
Powerset1: $\forall u\ \mathbf{CL.Set}(\mathbf{P}(u))$
Powerset2: $\forall A\ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^+ \in a)]$
Separation1 $\forall u_1\forall u_2,\ldots\forall u_n\forall a\exists \mathbf{Cl.Set}(\{x \in_s a|\varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation2 $\forall u_1\forall u_2,\ldots\forall u_n \mathbf{NCl.Set}(\{x \in_{s,w} A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$
Comprehension1 $\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x;u_1,u_2,\ldots,u_n)]$
Comprehension 2 $\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x,A;u_1,u_2,\ldots,u_n)]$
Comprehension 3 $\forall u_1\forall u_2,\ldots\forall u_n\exists a\forall x[x \in_s a \Leftrightarrow_s (a \subset u_1) \wedge \varphi(x,a;u_1,u_2,\ldots,u_n)]$
In particular:
Comprehension 3′ $\forall u\exists a\forall x[x \in_s a \Leftrightarrow_s (a \subset u) \wedge \varphi(x,a;u)]$
Hyperinfinity: see subsection 3.1.1.
Remark 3.1.7. Note that the axiom of hyper infinity follows from the schemata Comprehension 3.
Definition 3.1.3. The ordered pair of two sets u,v is defined as usual by

$$\langle u,v\rangle = \{\{u\},\{u,v\}\}. \tag{3.1.19}$$

Definition 3.1.4. We define the Cartesian product of two nonclassical sets A and B as usual by

$$A \times_{s,w} B = \{\langle x,y\rangle|x \in_{s,w} A \wedge y \in_{s,w} B\} \tag{3.1.20}$$

Definition 3.1.5. A binary relation between two nonclassical sets A,B is a subset $R\subseteq_{s,w}A\times_{s,w}B.$ We also write $aR_{s,w}b$ for $<a,b>\in_{s,w}R.$ The doman $\mathbf{dom}(R)$ and the range $\mathbf{ran}(R)$ of R are defined by

$$\mathbf{dom}(R)=\{x|\exists y(xR_{s,w}y)\},\mathbf{ran}(R)=\{y:\exists x(xR_{s,w}y)\}. \quad (3.1.21)$$

Definition 3.1.6.A relation $F_{s,w}$ is a function, or map, written $\mathbf{Fun}(F_{s,w})$, if for each $a\in_{s,w}\mathbf{dom}(F)$ there is a unique b for which $aF_{s,w}b.$ This unique b is written $F(a)$ or $Fa.$ We write $F_{s,w}:A\to B$ for the assertion that $F_{s,w}$ s a function with $\mathbf{dom}(F_{s,w})=A$ and $\mathbf{ran}(F_{s,w})=B.$ In this case we write $a\mapsto F_{s,w}(a)$ for $F_{s,w}a.$

Definition 3.1.7.The identity map $\mathbf{1}_A$ on A is the map $A\to A$ given by $a\mapsto a.$ If $X\subseteq_{s,w}A,$ the map $x\mapsto x:X\to A$ is called the insertion map of X into $A.$

Definition 3.1.8.If $F_{s,w}:A\to B$ and $X\subseteq_{s,w}A,$ the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X\to A$ given by $x\mapsto F_{s,w}(x)$. If $Y\subseteq_{s,w}B,$ the inverse image of Y under $F_{s,w}$ is the set

$$F_{s,w}^{-1}[Y]=\{x\in_{s,w}A:F_{s,w}(x)\in_{s,w}Y\}. \quad (3.1.22)$$

Given two functions $F_{s,w}:A\to B,G_{s,w}:B\to C,$ we define the composite function $G_{s,w}\circ F_{s,w}:A\to C$ to be the function $a\mapsto G_{s,w}(F_{s,w}(a))$. If $F_{s,w}:A\to A,$ we write $F_{s,w}^2$ for $F_{s,w}\circ F_{s,w},F_{s,w}^3$ for $F_{s,w}\circ F_{s,w}\circ F_{s,w}$ etc.

Definition 3.1.9.A function $F_{s,w}:A\to B$ is said to be monic if for all $x,y\in_{s,w}A,F_{s,w}(x)=F_{s,w}(y)$ implies $x=y,$ epi if for any $b\in_{s,w}B$ there is $a\in_{s,w}A$ for which $b=F_{s,w}(a)$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that

$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w}:B\to A$ such that $F_{s,w}\circ G_{s,w}=\mathbf{1}_B$ and $G_{s,w}\circ F_{s,w}=\mathbf{1}_A.$

Definition 3.1.10.Two sets X and Y are said to be equipollent, and we write $X\approx_{s,w}Y,$ if there is a bijection between them.

Definition 3.1.11.Suppose we are given two sets I,A and an epi map $F_{s,w}:I\to A.$ Then $A=\{F_{s,w}(i)|i\in I\}$ and so, if, for each $i\in_{s,w}I,$ we write a_i for $F_{s,w}(i)$, then A can be

presented in the form of an indexed set $\{a_i:i\in_{s,w}I\}$. If A is presented as an indexed set of sets $\{X_i|i\in_{s,w}I\}$, then we write $\bigcup_{i\in I}X_i$ and $\bigcap_{i\in I}X_i$ for $\cup A$ and $\cap A,$ respectively.

Definition 3.1.12.The projection maps $\pi_1:A\times_{s,w}B\to A$ and $\pi_2:A\times_{s,w}B\to B$ are defined to be the maps $<a,b>\mapsto$ a and $<a,b>\mapsto b$ respectively.

Definition 3.1.13.For sets $A,B,$ the exponential B^A is defined to be the set of all functions from A to $B.$

3.1.1.Axiom of nonregularity and axiom of hyperinfinity

Axiom of nonregularity

Remind that a non-empty set u is called regular iff $\forall x[x\neq\varnothing\to(\exists y\in x)(x\cap y=\varnothing)].$

Let's investigate what it says: suppose there were a non-empty x such that $(\forall y\in x)(x\cap y\neq\varnothing).$ For any $z_1\in x$ we would be able to get $z_2\in z_1\cap x.$ Since $z_2\in x$ we would be able to get $z_3\in z_2\cap x.$ The process continues forever: $\ldots\in z_{n+1}\in z_n\ldots\in z_4\in z_3\in z_2\in z_1\in x.$Thus if we don't wish to rule out such an infinite regress we forced accept the following statement:

$$\exists x[x\neq\varnothing\to(\forall y\in x)(x\cap y\neq\varnothing)]. \quad (3.1.23)$$

Axiom of hyperinfinity.

Definition 3.1.14.(i) A non-empty transitive non regular set u is a well formed non regular
set iff:
(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \tag{3.1.24}$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^+, \tag{3.1.25}$$

where $a^+ = a \cup \{a\}$.
(ii) we define a function $a^{+[k]}$inductively by $a^{+[k+1]} = (a^{+[k]})^+$

Definition 3.1.15. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any
$n \in \mathbb{N}$

$$w \in u_n. \tag{3.1.26}$$

Definition 3.1.16. We say that an well formed non regular set u is infinite or hyperfinite nuber iff:
(I) For any member $w \in u$ one and only one of the following conditions are satified:
(i) $w \in \mathbb{N}$ or
(ii) $w = u_n$ for some $n \in \mathbb{N}$ or
(iii) $w \prec u$.
(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$,then by relation $(\bullet \prec \bullet)$ a set ${}_{\prec}u$ is densely ordered with no first element.
(III) $\mathbb{N} \subset u$.

Definition 3.1.17. Assume $u \in \mathbb{N}^{\#}$,then u is infinite (hypernatural) number if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Axiom of hyperinfinity

There exists unique set $\mathbb{N}^{\#}$ such that:
(i) $\mathbb{N} \subset \mathbb{N}^{\#}$
(ii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number v such that $v \prec u$
(iii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number w such that $u \prec w$
(v) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is partially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

3.2.Hypernaturals $\mathbb{N}^{\#}$. Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$

In this subsection nonstandard arithmetic $\mathbf{A}^{\#}$ related to hypernaturals $\mathbb{N}^{\#}$ is considered axiomatically.

Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$ are:

Axiom of hyperinfinity

There exists unique set $\mathbb{N}^{\#}$ such that:
(i) $\mathbb{N} \subset \mathbb{N}^{\#}$
(ii) if u is infinite (hypernatural) number then there exists infinite (hypernatural)

number v such that $v \prec u$

(iii) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$

(iv) set $\mathbb{N}^{\#}\backslash\mathbb{N}$ is partially ordered by relation $(\cdot \preceq \cdot) \triangleq (\cdot \prec \cdot) \wedge (\cdot = \cdot)$ with no first and no last element.

Axioms of infite ω-induction

(i)

$$\forall S(S \subset \mathbb{N})\left\{\left[\bigwedge_{n\in\omega}(n \in S \Rightarrow_s n^+ \in S)\right] \Rightarrow_s S = \mathbb{N}\right\}. \tag{3.2.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n\in\omega}(F(n) \Rightarrow_s F(n^+))\right] \Rightarrow_s \forall n(n \in \omega)F(n). \tag{3.2.2}$$

Definition 3.2.1.(i) Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Let $[0,\beta] \subset \mathbb{N}^{\#}$ be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and let $[0,\beta)$ be a set $[0,\beta) = [0,\beta]\backslash\{\beta\}$.

(ii) Let $\beta \in \mathbb{N}^{\#}\backslash\mathbb{N}$ and let $\beta_\infty \subset \mathbb{N}^{\#}$be a set such that

$$\forall x\{x \in \beta_\infty \Leftrightarrow \exists k(k \geq 0)[0 \leq x \leq \beta^{+[k]}]\}. \tag{3.2.3}$$

Definition 3.2.2. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is restricted on a classical set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff the following condition is satisfied

$$\forall \alpha[\alpha \in \mathbb{N}^{\#}\backslash S \Rightarrow_s \neg F(\alpha)]. \tag{3.2.4}$$

Definition 3.2.3. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is <u>strictly</u> restricted on a set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff there is no proper subset $S' \subset S$ such that a wff $F(x)$ is restricted on a set S'.

Example 3.2.1.(i)Let $\mathbf{fin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{fin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}$. Obviously wff $\mathbf{fin}(\alpha)$ is strictly restricted on a set $\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N}^{\#}\backslash\mathbb{N} \Rightarrow_s \neg\mathbf{fin}(\alpha)]$.
Let $\mathbf{hfin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{hfin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}^{\#}\backslash\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N} \Rightarrow_s \neg\mathbf{hfin}(\alpha)]$.

Definition 3.2.4. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We say that a wff $F(x)$ is unrestricted if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq \mathbb{N}^{\#}$.

Axiom of hyperfinite induction 1

$$\forall S(S \subseteq_s [0,\beta])\forall\beta(\beta \in_s \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in_s [0,\beta))\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^+ \in_s S)\right] \Rightarrow_s S = [0,\beta]\right\}. \tag{3.2.5}$$

Axiom of hyperfinite induction 1$'$

$$\forall S(S \subseteq_s [0,\beta_\infty])\forall\beta(\beta \in \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in [0,\beta_\infty])\left[\bigwedge_{0\leq\alpha<\beta_\infty}(\alpha \in S \Rightarrow \alpha^+ \in S)\right] \Rightarrow S = [0,\beta_\infty]\right\}. \tag{3.2.6}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset_s \mathbb{N}^{\#})\left\{\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^+ \in_s S)\right] \Rightarrow_s S =_s \mathbb{N}^{\#}\right\}. \tag{3.2.7}$$

Definition 3.2.5. A set $S \subset_s \mathbb{N}^{\#}$ is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha \in \mathbb{N}^{\#}} (\alpha \in_s S \Rightarrow_s \alpha^+ \in_s S). \tag{3.2.8}$$

Obviously a set $\mathbb{N}^{\#}$ is a hyper inductive. Thus axiom of hyper infinite induction 1 asserts that a set $\mathbb{N}^{\#}$ this is the smallest hyper inductive set.

Axioms of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta \in [0,\beta])\left[\bigwedge_{0 \leq \alpha < \beta} (F(\alpha) \Rightarrow_s F(\alpha^+))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta])F(\alpha). \tag{3.2.9}$$

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta_\infty]$ then

$$\left[\forall\beta(\beta \in [0,\beta_\infty])\left[\bigwedge_{0 \leq \alpha < \beta_\infty} (F(\alpha) \Rightarrow_s F(\alpha^+))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta_\infty])F(\alpha). \tag{3.2.10}$$

Axiom of hyper infinite induction 2

Let $F(x)$ be anrestricted wff of the set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0 \leq \alpha < \beta} (F(\alpha) \Rightarrow_s F(\alpha^+))\right]\right] \Rightarrow_s \forall\beta(\beta \in \mathbb{N}^{\#})F(\beta). \tag{3.2.11}$$

Rules of conclusion

MRR (Main Restricted rule of conclusion)

Let $\varphi(x)$ be a wff with one free variable and there exists $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ such that $\mathbf{A}^{\#} \vdash \varphi(n)$ then $\neg\varphi(n) \nvdash B$, i.e., if statement $\varphi(n)$ holds in $\mathbf{A}^{\#}$ we cannot obtain from $\neg\varphi(n)$ any formula B whatsoever, see Appendix A.

1.Addition operation of hypernatural numbers.

There is a function $+(m,n) \triangleq m+n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$

$m+0 = m, m+n^+ = (m+n)^+$.

This function $m+n$ satisfies all properties of addition such as:

for all $m,n,k \in \mathbb{N}^{\#}$

(i) $m+0 = m$ (ii) $m+n = n+m$ (iii) $m+(n+k) = (m+n)+k$.

2.Multiplicattion operation of gypernatural numbers.

There is a function $\times(m,n) \triangleq m \times n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$

This function $m \times n$ satisfies all properties of multiplicattion such as:

for all $m,n,k \in \mathbb{N}^{\#}$

(i) $m \times 1 = 1$ (ii) $m \times n = n \times m$ (iii) $m \times (n \times k) = (m \times n) \times k$.

4.Distributivity with respect to multiplication over addition.

$m \times (n+k) = m \times n + m \times k$.

5.Inequalities.

Definition 3.2.1. We define noe the relation $a \leq b$ such that $a \leq b \Leftrightarrow_s a \preceq b$.

(i) For all $a,b \in \mathbb{N}^{\#}, a \leq b$ if and only if there exists some $c \in \mathbb{N}^{\#}$ such that $a+c = b$.

(ii) This relation is stable under addition and multiplication: for $a,b,c \in \mathbb{N}^{\#}$, if $a \leq b$, then: (i) $a+c \leq b+c$, and (ii) $a \times c \leq b \times c$

Remark 3.2.1. Note that the functions $m+n$ and $m \times n$ as we seen later, cannot be defined inductively.

3.3.Hyper inductive definitions in general.

A function $f : \mathbb{N}^{\#} \to A$ whose domain is the set $\mathbb{N}^{\#}$ is colled an hyper infinite sequence and denoted by $\{f_n\}_{n\in\mathbb{N}^{\#}}$ or by $\{f(n)\}_{n\in\mathbb{N}^{\#}}$The set of all hyperinfinite sequences whose terms belong to A is clearly $A^{\mathbb{N}^{\#}}$; the set of all hyperfinite sequences of $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ terms in A is A^n. The set of all hyperfinite sequences with terms in A can be defined as

$$\left\{R \subset \mathbb{N}^{\#} \times A : (R \text{ is a function}) \wedge \bigvee_{n\in\mathbb{N}^{\#}} (\mathbf{Dom}(R) = n)\right\}, \qquad (3.3.1)$$

where $Dom(R)$ is domain of R. This definition implies the existence of the set of all hyper finite finite sequences with terms in A. The simplest case is the hyper inductive definition of a hyper infinite sequence $\{\varphi(n)\}_{n\in\mathbb{N}^{\#}}$ (with terms belonging to a certain set Z) satisfying the following conditions:

(a)

$$\varphi(0) = z, \varphi(n^+) = e(\varphi(n), n), \qquad (3.3.2)$$

where $z \in Z$ and e is a function mapping $Z \times \mathbb{N}^{\#}$ into Z.

More generally, we consider a mapping f of the cartesian product $Z \times \mathbb{N}^{\#} \times A$ into Z and seek a function $\varphi \in Z^{\mathbb{N}^{\#}\times A}$ satisfying the conditions :

(b)

$$\varphi(0,a) = g(a), \varphi(n^+,a) = f(\varphi(n,a),n,a), \qquad (3.3.3)$$

where $g \in Z^A$. This is a definition by hyper infinite induction with parameter a ranging over the set A. Schemes (a) and (b) correspond to induction "from n to $n^+ = n + 1$",i.e. $\varphi(n^+)$ or $\varphi(n^+,a)$ depends upon $\varphi(n)$ or $\varphi(n,a)$ respectively. More generally, $\varphi(n^+)$ may depend upon all values $\varphi(m)$ where $m \leq n$ (i.e. $m \in n^+$). In the case of induction with parameter, $\varphi(n^+,a)$ may depend upon all values $\varphi(m,a)$, where $m \leq n$; or even upon all values $\varphi(m,a)$, where $m \leq n^+$ and $b \in A$. In this way we obtain the following schemes of

definitions by hyper infinite induction:

(c) $\varphi(0) = z, \varphi(n^+) = h(\varphi|n^+, n)$,

(d) $\varphi(0,a) = g(a), \quad \varphi(n^+,a) = H(\varphi|(n^+ \times A), n, a)$.

In the scheme (c), $z \in Z$ and $h \in Z^{C\times\mathbb{N}^{\#}}$, where C is the set of hyperfinite sequences whose terms belong to Z; in the scheme (d), $g \in Z^A$ and $H \in Z^{T\times\mathbb{N}^{\#}\times A}$, where T is the set of functions whose domains are included in $\mathbb{N}^{\#} \times A$ and whose values belong to Z.

It is clear that the scheme (d) is the most general of all the schemes considered above.

By choice of functions one obtains from (d) any of the schemes (a)-(d). For example, taking the function defined by $H(c,n,a) = f(c(n,a),n,a)$ for $a \in A, n \in \mathbb{N}^{\#}, c \in Z^{\mathbb{N}^{\#}\times A}$ as H in (d), one obtain (b). We shall now show that, conversely, the scheme (d) can be obtained from (a). Let g and H be functions belonging to Z^A and $Z^{T\times\mathbb{N}^{\#}\times A}$ respectively, and let φ be a function satisfying (d). We shall show that the sequence $\Psi = \{\Psi_n\}_{n\in\mathbb{N}^{\#}}$ with $\Psi_n = \varphi|(n^+,A)$ can be defined by (a).Obviously, $\Psi_n \in T$ for every $n \in \mathbb{N}^{\#}$. The first term of the sequence Ψ is equal to $\varphi|(0^+,A)$, i.e. to the set: $z^* = \{\langle\langle 0,a\rangle, g(a)\rangle | a \in A\}$. The relation between Ψ_n, and Ψ_{n^+} is given by the formula:$\Psi_{n^+} = \Psi_n \cup \varphi|(\{n^+\} \times A)$, where the second component is

$$\{\langle\langle n^+,a\rangle, \varphi(n^+,a)\rangle | a \in A\} = \{\langle n^+,a\rangle, H(\Psi_n,n,a) | a \in A\}. \qquad (3.3.4)$$

Thus we see that the sequence Ψ can be defined by (a) if we substitute T for Z, z^* for z and let $e(c,n) = c \cup \{\langle n^+,a\rangle, H(c,n,a) | a \in A\}$ for $c \in T$.

Now we shall prove the existence and uniqueness of the function satisfying (a).

This theorem shows that we are entitled to use definitions by induction of the type (a).
According to the remark made above, this will imply the existence of functions satisfying
the formulas (b), (c), and (d). Since the uniqueness of such functions can be proved in the same manner as for (a), we shall use in the sequel definitions by induction of any of
the types (a)-(d).

Theorem 3.3.1. If Z is any set $z \in Z$ and $e \in Z^{Z\times\mathbb{N}^{\#}}$, then there exists exactly one hyper infinite sequence φ satisfying formulas (a).

Proof. Uniqueness. Suppose that $\{\varphi_1(n)\}_{n\in\mathbb{N}^{\#}}$ and $\{\varphi_2(n)\}_{n\in\mathbb{N}^{\#}}$ satisfy (a) and let

$$K = \{n|n \in \mathbb{N}^{\#} \wedge \varphi_1(n) = \varphi_2(n)\} \quad (3.3.5)$$

Then (a) implies that K is hyperinductive. Hence $\mathbb{N}^{\#} = K$ and therefore $\varphi_1(n) \equiv \varphi_2(n)$.
Existence. Let $\Phi(z,n,t)$ be the formula $e(z,n) = t$ and let $\Psi(w,z,F_n)$ be the following formula:

$$(F_n \text{ is a function}) \wedge (Dom(F) = n^+) \wedge (F(0) = z) \wedge \bigwedge_{m\in n} \Phi(F_n(m),m,F_n(m^+)). \quad (3.3.6)$$

In other words, F is a function defined on the set of numbers $\leq n \in \mathbb{N}^{\#}$ such that $F(0) = z$ and $F(m^+) = e(F(m),m)$ for all $m < n \in \mathbb{N}^{\#}$.

Assumption 3.3.1.We assume now (but without loss of generality) that predicate $\Psi(w,z,F_n)$ is unrestricted on variable $n \in \mathbb{N}^{\#}$,see Definition 3.2.4.

We prove by hyper infinite induction that there exists exactly one function F_n such that $\Psi(n,z,F_n)$.

The proof of uniqueness of this function is similar to that given in the Theorem 3.3.1.
The existence of F_n can be proved as follows: for $n = 0$ it suffices to take $\{\langle 0,z\rangle\}$ as F_n; if $n \in \mathbb{N}^{\#}$ and F_n satisfies $\Psi(n,z,F_n)$, then $F_{n^+}= F_n \cup \{\langle n^+,e(F_n(n),n)\rangle\}$
satisfies the condition $\Psi(n^+,z,F_{n^+})$.

Now, we take as φ the set of pairs $\langle n,s\rangle$ such that $n \in \mathbb{N}^{\#}, s \in Z$ and

$$\exists F[\Psi(n,z,F) \wedge (s = F(n))]. \quad (3.3.7)$$

Since F is the unique function satisfying $\Psi(n,z,F)$, it follows that φ is a function.
For $n = 0$ we have $\varphi(0) = F_0(0) = z$; if $n \in \mathbb{N}^{\#}$, then $\varphi(n^+) = F_{n^+}(n^+) = e(F_n(n),n)$ by the definition of F_n; hence we obtain $\varphi(n^+) = e(\varphi(0),n)$. Theorem 3.3.2 is thus proved.

Remark 3.3.1.Note that Assumption 3.1.1 is not necessarily,see sect. 3.6.

We frequently define not one but several functions (with the same range Z) by a simultaneous induction:

$$\varphi(0) = z, \varphi(n^+) = f(\varphi(n),\psi(n),n), \psi(0) = t, \psi(n^+) = g(\varphi(n),\psi(n),n) \quad (3.3.8)$$

where $z,t \in Z$ and $f,g \in Z^{Z\times Z\times\mathbb{N}^{\#}}$.

This kind of definition can be reduced to the previous one. It suffices to notice that the hyper infinite sequence $\vartheta_n = \langle\varphi(n),\psi(n)\rangle$ satisfies the formulas:

$$\vartheta_0 = \langle z,t\rangle, \vartheta_{n^+} = e(\vartheta_n,n), \quad (3.3.9)$$

where we set

$$e(u,n) = \langle f(K(u),L(u),n), g(K(u),F(w),n)\rangle, \quad (3.3.10)$$

and K,L denote functions such that $K(\langle x,y\rangle)$ and $L(\langle x,y\rangle) = y$ respectively. Thus the function ϑ is defined by hyper infinite induction by means of (a). We now define φ and

ψ by

$$\varphi(n) = K(\vartheta_n), \psi(n) = L(\vartheta_n). \quad (3.3.11)$$

Remark 3.3.2. We assume now that predicate $\Psi(w,z,F_n)$ is restricted on variable $n \in \mathbb{N}^{\#}$,
on a set $[0,\beta] \cup \widehat{\omega} \subset \mathbb{N}^{\#}$, see Definition 3.3.2, then there exists exactly one hyperfinite sequence φ satisfying formulas (a). Note that is a case if and only if $f,g \in Z^{Z \times Z \times [0,\beta] \cup \widehat{\omega}}$.
sequence φ satisfying formulas (a). Note that is a case if and only if $f,g \in Z^{Z \times Z \times [0,\beta] \cup \widehat{\omega}}$.

The theorem 3.3.1 on hyper inductive definitions can be generalized to the case of operations. We shall discuss only one special case. Let $\Phi(z,n,t)$ be a formula such that

$$\forall z \forall n(n \in \mathbb{N}^{\#}) \forall t_1 \forall t_2 [\Phi(z,n,t_1) \wedge \Phi(z,n,t_2) \Rightarrow t_1 = t_2]. \quad (3.3.12)$$

Theorem 3.3.2. For any set S there exists exactly one hyper infinite sequence $\varphi_n, n \in \mathbb{N}^{\#}$
such that $\varphi_0 = S$ and

$$\forall n(n \in \mathbb{N}^{\#}) \Phi(\varphi_n, n, \varphi_{n^+}). \quad (3.3.13)$$

Proof. Uniqueness can be proved as in Theorem 3.3.1 above.
To prove the existence of φ_n, let us consider the following formula $\Psi(n,S,F)$:

$$(F \text{ is a function})(D_1(F) = n^+) \wedge (F(0) = S) \wedge \forall m(m \in n) \Phi(F(m), m, F(m')), \quad (3.3.14)$$

where $D_1(F)$ is domain of F.
As in the proof of Theorem 3.3.1, it can be shown that there exists exactly one function
F_n such that $\Psi(n,S,F_n)$. To proceed further we must make certain that there exists a set containing all the elements of the form $F_n(n)$ where $n \in \mathbb{N}^{\#}$. (In the case considered in Theorem 3.3.1 this set is Z for the domain of the last variable of the formula Φ which
we used in the proof of Theorem 3.3.1 was limited to the set Z.) In the case under consideration, the existence of the required set Z follows from the axiom of replacement.
In fact, the uniqueness of F_n implies that the formula

$$\exists F_n [\Psi(n,S,F_n) \wedge (y = F_n(n))] \quad (3.3.15)$$

satisfies the assumption of axiom of replacement. Hence by means of axiom of replacement the image of $\mathbb{N}^{\#}$ obtained by this formula exists. This image is the required
set Z containing all the elements $F_n(n)$.
The remainder of the proof is analogous to that of Theorem 3.3.1.
Example 3.3.1. Let $\Phi(S,\mathbf{t})$ be the formula $\mathbf{t} = \mathbf{P}(S)$. Thus for any set S there exists exactly one hyper infinite sequence $\{\varphi_n\}_{n \in \mathbb{N}^{\#}}$ such that $\varphi_0 = S$ and $\varphi_{n^+} = \mathbf{P}(\varphi_n)$ for every
number $n \in \mathbb{N}^{\#}$

3.4.Fundamental examples of the hyper inductive definitions.

1. Let $Z = A = X^X, g(a) = I_X, f(u,n,a) = u \circ a$ in (b). Then (b) takes on the following form

$$\varphi(0,a) = I_X, \varphi(n^+,a) = \varphi(n,a) \circ a. \quad (3.4.1)$$

The function $\varphi(n,a)$ is denoted by a^n and is colled n-th iteration of the function a :

$$a^0(x) = x, a^{n^+}(x) = a^n(a(x)), x \in X, a \in X^X, n \in \mathbb{N}^\#. \quad (3.4.2)$$

2.Let $A = (\mathbb{N}^\#)^{\mathbb{N}^\#}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \quad (3.4.3)$$

The function is defined by the Eqs.(3.4.3) is denoted by

$$Ext\text{-}\sum_{i=0}^{n} a_i \quad (3.4.4)$$

3.Let $A = (\mathbb{N}^\#)^{\mathbb{N}^\#}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \quad (3.4.5)$$

The function is defined by the Eqs.(3.4.5) is denoted by

$$Ext\text{-}\prod_{i=0}^{n} a_i \quad (3.4.6)$$

Theorem 3.4.1. The following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^\#$:

(1) using distributivity

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \quad (3.4.7)$$

(2) using commutativity and associativity

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n}(a_i \pm b_i) \quad (3.4.8)$$

(3) splitting a sum, using associativity

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \quad (3.4.9)$$

(4) using commutativity and associativity, again

$$Ext\text{-}\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \quad (3.4.10)$$

(5) using distributivity

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j\right) = Ext\text{-}\sum_{i=0}^{n}\left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j\right) \quad (3.4.11)$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i\right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \quad (3.4.12)$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right)^m = Ext\text{-}\prod_{i=0}^{n} a_i^m \quad (3.4.13)$$

Proof. Imediately by hyper infinite induction principle.

Theorem 3.4.2. Suppose that $a_i \leq b_i, 1 \leq i \leq n$ then the following equalities holds for any $n \in \mathbb{N}^{\#}$:

$$Ext\text{-}\sum_{i=0}^{n} a_i \leq Ext\text{-}\sum_{i=0}^{n} b_i \qquad (3.4.14)$$

Definition 3.4.1.We define a hyperfinite number by

$$[\omega] = Ext\text{-}\sum_{i=1}^{n} a_i, \qquad (3.4.15)$$

where $a_i \equiv 1$ if $i \in \mathbb{N}$ and $a_i \equiv 0$ if $i \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Remark 3.4.1.Note that $[\omega] \in \mathbb{N}^{\#} \backslash \mathbb{N}$, since $[\omega] = \sum_{i=1}^{m} a_i + Ext\text{-}\sum_{i=m+1}^{\omega} b_i > m$ and therefore $[\omega] > m$ for all $m \in \mathbb{N}$.

Definition 3.4.2.We define a function $[n], n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ by

$$[n] = Ext\text{-}\sum_{i=1}^{n} a_i, \qquad (3.4.16)$$

where $a_i \equiv 1, 1 \leq i \leq n$.

Theorem 3.4.2. There is a function $[n] : \mathbb{N}^{\#} \to \mathbb{N}^{\#}$ such as:

(i) $[m+0] = [m], [m+n^{+}] = [(m+n)]^{+}$(ii) $[m+n] = [m]+[n]$.

This function $[n]$ satisfies a properties such as:for all $m,n,k \in \mathbb{N}^{\#}$

(i) $[m+n] = [n+m]$ (iii) $[m+(n+k)] = [(m+n)+k]$.

Theorem 3.4.3.

3.5.Nonconservative extension of the model theoretical nonstandard analysis.

In this paper we deal with set theory $\mathbf{NC}_{\infty}^{\#}$ based on gyper infinitary logic with Restricted

Modus Ponens Rule [1]-[3].

Note that analysis on a non-Archimedean field $\mathbb{R}_c^{\#}$ is essentially different in comparison with analysis on non Archimedean field ${}^{*}\mathbb{R}$ [4]-[5] known in literature as nonstandard analysis, see for example [4]-[5].

Remind that Robinson nonstandard analysis (RNA) many developed using set-theoretical objects called superstructures [5]. A superstructure $\mathbf{V}(S)$ over a set S is defined in the following way:

$$\mathbf{V}_0(S) = S, \mathbf{V}_{n+1}(S) = \mathbf{V}_n(S) \cup (P(\mathbf{V}_n(S)), \mathbf{V}(S) = \bigcup_{n \in \mathbb{N}} \mathbf{V}_n(S). \qquad (3.5.1)$$

Superstructure is a set consist of sets of infinite rank in the cumulative hierarchy and therefore do not satisfy the in…nity axiom. Making $S = \mathbb{R}$ will suffice for virtually any construction necessary in analysis.

Bounded formulas are formulas where all quantifiers occur in the form

$$\forall x(x \in y \Rightarrow \cdots), \exists x(x \in y \Rightarrow \cdots). \qquad (3.5.2)$$

A nonstandard embedding is a mapping

$$* : \mathbf{V}(X) \to \mathbf{V}(Y) \qquad (3.5.3)$$

from a superstructure $\mathbf{V}(X)$ called the standard universum, into another superstructure

$\mathbf{V}(Y)$, called nonstandard universum, satisfying the following postulates:
1. $Y = {}^*X$
2.**Transfer Principle**.For every bounded formula $\Phi(x_1,\ldots,x_n)$ and elements $a_1,\ldots,a_n \in \mathbf{V}(X)$,the property Φ is true for $a_1,\ldots,a_n$ in the standard universum if and only if it is true for ${}^*a_1,\ldots,{}^*a_n$ in the nonstandard universum:

$$\langle \mathbf{V}(X),\in\rangle \models \Phi(a_1,\ldots,a_n) \Leftrightarrow \langle \mathbf{V}(Y),\in\rangle \models \Phi({}^*a_1,\ldots,{}^*a_n). \quad (3.5.4)$$

Definition 3.5.1.[6-7].A set x is internal if and only if x is an element of ${}^*A$ for some element A of $\mathbf{V}(\mathbb{R})$. Let X be a set with $A = \{A_i\}_{i\in I}$ a family of subsets of X. Then the collection A has the infinite intersection property, if any infinite subcollection $J \subset I$ has non-empty intersection. Nonstandard universum is κ-saturated if whenever $\{A_i\}_{i\in I}$is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to κ, $\bigcap_{i\in I} A_i \neq \varnothing$.

Definition 3.5.2.[2]-[4].A set $S \subseteq {}^*\mathbb{N}$is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha\in{}^*\mathbb{N}} (\alpha \in S \Rightarrow \alpha^+ \in S), \quad (3.5.5)$$

where $\alpha^+ \triangleq \alpha+1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.As we see later there is just one hyper inductive subset of ${}^*\mathbb{N}$,namely ${}^*\mathbb{N}$itself.
In this paper we apply the following hyper inductive definitions of a sets [2]-[4]

$$\exists S \forall \beta \left[\beta \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \right], \quad (3.5.6)$$

We extend up Robinson nonstandard analysis (**RNA**) by adding the following postulate:
4.Any hyper inductive set S is internal.
Remark 3.5.1.The statement **4** is not provable in ZFC but provable in set theory $\mathbf{NC}^{\#}_{\infty}$, see [2]-[4].Thus postulates 1-4 gives an nonconservative extension of RNA and we denote such extension by **NERNA**.
Remark 3.5.2.Note that NERNA of course based on the same gyper infinitary logic with Restricted Modus Ponens Rule as set theory $\mathbf{NC}^{\#}_{\infty}$ [1]-[4].
Remind that in RNA the following induction principle holds.
Theorem 3.1.1.[6]. Assume that $S \subset {}^*\mathbb{N}$ is internal set, then

$$(1 \in S) \wedge \forall x[x \in S \Rightarrow x+1] \Rightarrow S = {}^*\mathbb{N}. \quad (3.5.7)$$

In NERNA Theorem 1.1also holds.
Remark 3.5.3.It follows from postulate 4 and Theorem 3.1.1 that any hyper inductive set S is equivalent to ${}^*\mathbb{N}$: $S \equiv {}^*\mathbb{N}$.
Remark 3.5.4. Note that the following statement is provable in $\mathbf{NC}^{\#}_{\infty}$ [2]-[4].
4′Axiom of hyper infinite induction

$$\forall S(S \subseteq {}^*\mathbb{N}) \left\{ \forall \beta(\beta \in {}^*\mathbb{N}) \left[\bigwedge_{0\leq\alpha<\beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \right] \Rightarrow S = {}^*\mathbb{N} \right\}. \quad (3.5.8)$$

Thus postulate **4 of the theory** NERNA is provable in $\mathbf{NC}^{\#}_{\infty^{\#}}$.

Rules of conclusion
MRR (Main Restricted rule of conclusion)
Let $\varphi(x)$ be a wff with one free variable and there exists $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that $\mathbf{V}(Y) \models \varphi(n)$, then $\neg\varphi(n) \nvdash B$, i.e., if statement $\varphi(n)$ is satisfable in $\mathbf{V}(Y)$ we cannot obtain from $\neg\varphi(n)$ any formula B whatsoever.
Remark 3.5.5. Note that ${}^*\mathbb{N}$ is an model related to hypernaturals $\mathbb{N}^\#$ which we has introduced axiomatically in [3],[4],see section 3.3.

3.4.Nonconservative extension of the Internal Set Theory IST.

3.4.1.Internal Set Theory IST.

The axiomatics IST (Internal Set Theory) was presented in 1977 [8] and in a sense formulates within first-order language the behaviour of standard and internal sets of a nonstandard model of ZFC. This were done by adding the unary standardness predicate "st" to the language of ZFC as well as adding to the axioms of ZFC three new axiom schemes involving the predicate "st": **Idealization**, **Standardization** and **Transfer**.
Remark 3.4.1.Formulas which do not use the predicate $\mathbf{st}$ are called internal formulas (or $\in$-formulas) and formulas that use this new predicate are called external formulas (or st-$\in$-formulas).A formula φ is standard if only standard constants occur in φ.
Abbreviaion 3.4.1.We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a $\mathbf{st}$-$\in$-formula:
1.$\forall^{\mathbf{st}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow \varphi(x))$. 2.$\exists^{\mathbf{st}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.
3.$\forall^{\mathbf{fin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow \varphi(x))$. 4.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.
5.$\forall^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow \varphi(x))$.
6.$\exists^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.
The fundamental axioms of $\mathbf{IST}$:
(**I**) **Idealization**

$$\forall^{\mathbf{stfin}} F \exists y \forall x \in F[R(x,y) \Leftrightarrow \exists b \forall^{\mathbf{st}} x R(x,b)] \qquad (3.4.1)$$

for any internal relation R.
Remark 2.1.2.The idealization axiom obviously states that saying that for any fixed finite set F there is a y such that $R(x,y)$ holds for all $x \in F$ is the same as saying that there is a b such that for all fixed x the relation $R(x,b)$ holds.
(**II**) **Standardization**

$$\forall^{\mathbf{st}} A \exists^{\mathbf{st}} B \forall^{\mathbf{st}} x(x \in B \Leftrightarrow x \in A \wedge \varphi(x)) \qquad (3.4.2)$$

for every st-$\in$-formula φ with arbitrary (internal) parameters.
(**III**) **Transfer**

$$\forall^{\mathbf{st}} y_1,\ldots,y_n \forall^{\mathbf{st}} x[\varphi(x,y_1,\ldots,y_n)] \Rightarrow \forall x \varphi(x,y_1,\ldots,y_n) \qquad (3.4.3)$$

for all internal standard $\varphi(x,y_1,\ldots,y_n)$.
Remark 3.4.3. An importent consequence of (**I**) is the principle of **External Induction**, which states that for any (external or internal) formula φ,one has

$$\varphi(0) \wedge [\forall^{\mathbf{st}} n(\varphi(n) \Rightarrow \varphi(n+1))] \Rightarrow \forall^{\mathbf{st}} n\varphi(n). \qquad (3.4.4)$$

Boundedness

$$\forall x \exists^{\mathbf{st}} y (x \in y) \tag{3.4.5}$$

and since (2.5) contradicts idealization the following (bounded) form is taken instead:

(**IV**) **Bounded Idealization**

For every $\in$-formula R :

$$\forall^{\mathbf{st}} Y [\forall^{\mathbf{stfin}} F \exists y \in Y (\forall x \in F R(x,y) \Leftrightarrow \exists b (b \in Y) \forall^{\mathbf{st}} x R(x,b))]. \tag{3.4.6}$$

This gives a subsystem BST, which corresponds to the bounded sets of IST.

3.4.2.Internal Set Theory IST$^{\#}$

The axiomatics IST$^{\#}$ formulates within infinitary first-order language the behaviour of standard and internal sets of a nonstandard model of NC$^{\#}_{\infty^{\#}}$. This done by adding the unary standardness predicate "st" to the language of NC$^{\#}_{\infty^{\#}}$ as well as adding to the axioms of NC$^{\#}_{\infty^{\#}}$ three new axiom schemes involving the predicate "st": **Idealization**, **Standardization**,**Transfer** and **Axiom of internal hyper infinite induction**.

Remark 2.2.1.Formulas which do not use the predicate $\mathbf{st}$ are called internal formulas (or $\in_{sw}$ -formulas) and formulas that use this new predicate are called external formulas (or st-$\in_{sw}$ -formulas).A formula φ is standard if only standard constants occur in φ.

Abbreviaion 2.2.1.We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a $\mathbf{st}$- $\in_{sw}$ -formula:

1.$\forall^{\mathbf{st}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_{s} \varphi(x))$.

2.$\forall^{\mathbf{st}}_{sw} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_{sw} \varphi(x))$.

3.$\exists^{\mathbf{st}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.

4.$\forall^{\mathbf{fin}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \ \Rightarrow_{s} \varphi(x))$.

5.

4.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.

5.$\forall^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \ \Rightarrow \varphi(x))$.

6.$\exists^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.

The fundamental axioms of $\mathbf{IST}^{\#}$:

(**I**) **Idealization for classical sets**

$$\forall^{\mathbf{stfin}}_{s} F^{\mathbf{CL}} \exists y^{\mathbf{CL}} \forall x^{\mathbf{CL}} \in_{s} F[R^{\mathbf{CL}}(x,y) \Leftrightarrow_{s} \exists b^{\mathbf{CL}} \forall^{\mathbf{st}}_{s} x R^{\mathbf{CL}}(x,b)] \tag{2.2.1}$$

for any internal classical relation $R^{\mathbf{CL}}(x,y)$.

Remark 2.2.2.The idealization axiom obviously states that saying that for any fixed classical finite set F there is a classical y such that $R^{\mathbf{CL}}(x,y)$ holds for all classical $x \in_{s} F$ is the same as saying that there is a classical b such that for all fixed classical x the classical relation $R^{\mathbf{CL}}(x,b)$ holds.

3.5.Generalized Recursion Theorem.

Theorem 3.4.1. Let S be a set, $c \in S$ and $G : S \to S$ is any function with $\mathbf{dom}(G) = S$ and $\mathbf{range}(G) \subseteqq S$. Let $W[G] \in \mathbb{N}^{\#} \times S$ be a binary relation such that:
(a) $(1,c) \in W[G]$ and
(b) if $(x,y) \in W[G]$ then $(\mathbf{Sc}(x), G(y)) \in W[G]$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$ and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x))$.
1.The desired function $\mathcal{F}$ is a certain relation $\mathbf{W} \subseteqq \mathbb{N}^{\#} \times S$. It is to have the properties:
(ii′) $(1,c) \in \mathbf{W}$;
(iii′) if $(x,y) \in \mathbf{W}$ then $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$.
Remark 3.4.1. The latter is just another way of expressing (iii), that if

$$\mathcal{F}(x) = y \tag{3.4.1}$$

then

$$\mathcal{F}(\mathbf{Sc}(x)) = G(y). \tag{3.4.2}$$

Remark 3.4.2.Note that any relation $\mathbf{W}$ mentioned above is hyper inductiverelation since the hyper inductivity conditions (ii′)-(iii′) are satisfied.
However there are many hyper inductive relations which satisfy the conditions (ii′)-(iii′); on such is $\mathbb{N}^{\#} \times S$. What distinguishes the desired function from all these other relations is that we want (a,b) to be on it only as required by (ii′) and (iii′). In other words, it is to be the smallest relation satisfying (ii′)-(iii′). This can be expressed precisely as follows:
(1) Let $\mathbf{M}$ be a set of the relations $\mathbf{W}$ satisfying the conditions (ii′) and (iii′); then we define

$$\mathcal{F} = \bigcap_{\mathbf{W} \in \mathbf{M}} \mathbf{W}.$$

Hence
(2) whenever $\mathbf{W} \in \mathbf{M}$ then $\mathcal{F} \subseteq \mathbf{W}$.
We shall now show that we can derived from (1) that $\mathcal{F}$ is also one relation in $\mathbf{M}$.
(3) $(1,c) \in \mathcal{F}$.
This follows immediately from the definition of $\bigcap_{\mathbf{W} \in \mathbf{M}}$ and the fact that $(1,c) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$.
(4) If $(x,y) \in \mathcal{F}$ then $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$.
For if $(x,y) \in \mathcal{F}$ then $(x,y) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$; hence by (iii′) $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$ so that $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$ by (1).
We must now verify that $\mathcal{F}$ ís actually a function, i,e., we wish to show that for any $x, z_1, z_2 \in \mathbb{N}^{\#}$, if $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$, then $z_1 = z_2$.
We shall prove this by hyper infinite induction on x. Let
(5) $A = \{x | x \in \mathbb{N}^{\#}$ and for all $z_1, z_2 \in \mathbb{N}^{\#}$, if $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$ then $z_1 = z_2\}$.
We shall show $A = \mathbb{N}^{\#}$ by applying hyper infinite induction. First we have
(6) $1 \in A$.
To prove (6), it suffices to show that for any z, if $(1,z) \in \mathcal{F}$ then $z = c$.

We prove this by contradiction; in other words, suppose to tbe contrary that there is some z with $(1,z) \in \mathcal{F}$ but $z \neq c$. Consider the relation $W = \mathcal{F} \backslash \{(1,z)\}$. Since $(1,c) \in \mathcal{F}$ and $(1,c) \neq (1,z)$, it follows that $(1,c) \in W$. Moreover, whenever $(u,y) \in W$ then $(u,y) \in \mathcal{F}$ and hence $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$ but $\mathbf{Sc}(u) \neq 1$, so $(\mathbf{Sc}(u), G(y)) \neq (1,z)$, and hence $(\mathbf{Sc}(u), G(y)) \in W$. Thus W satisfies both conditions (ii$'$) and (iii$'$); in other words, $\mathbf{W} \in \mathbf{M}$. But then it follows from (2) that $\mathcal{F} \subseteq \mathbf{W}$ however this is elearly false sinee $(1,z) \in \mathcal{F}$ and $(1,z) \notin \mathbf{W}$. Tbus our hypothesis has led us to a contradiction, and henee (6) is proved. Next we show that

(7) for any $x \in \mathbb{N}^{\#}$ if $x \in A$ then $\mathbf{Sc}(x) \in A$.

Suppose that $x \in A$, so that whenever $(x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$ then $z_1 = z_2$. We must show that whenever $(\mathbf{Sc}(x), w_1) \in \mathcal{F}$and $(\mathbf{Sc}(x), w_2) \in \mathcal{F}$ then $w_1 = w_2$. To prove this, it suffices to show that

(8) whenever $(\mathbf{Sc}(x), w) \in \mathcal{F}$ then there exists a z with $w = G(z)$ and $(x,z) \in \mathcal{F}$.

For if (8) ia true, we would have for the given w_1, w_2 some $z_1 = z_2$ with $w_1 = G(z_1)$, $w_2 = G(z_2)$, $(x,z_1) \in \mathcal{F}$ and $(x,z_2) \in \mathcal{F}$. Then, since $x \in A, z_1 = z_2$ and henee $G(z_1) = G(z_2)$, that is, $w_1 = w_2$.

Now to prove (8) suppose, to the contrary, that it is not true; in other words, suppose that we have some w with $(\mathbf{Sc}(x), w) \in \mathcal{F}$ but such that for all z which $(x,z) \in \mathcal{F}$ we have $w \neq G(z)$. Consider the relation $\mathbf{W} = \mathcal{F} \backslash \{(\mathbf{Sc}(x), w)\}$. We shall show that $\mathbf{W} \in \mathbf{M}$. First of all $(1,c) \in \mathcal{F}$and $(1,c) \neq (\mathbf{Sc}(x), w)$; hence $(1,c) \in \mathbf{W}$. Suppose tbat $(u,y) \in \mathbf{W}$; then $(u,y) \in \mathcal{F}$ and $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$. Clearly if $u \neq x$ then $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w)$, so that in this case $(Sc(u), G(y)) \in \mathbf{W}$. On the other hand, if $u = x$ and $(\mathbf{Sc}(u), G(y)) = (\mathbf{Sc}(x), w)$, then $w = G(y)$, where $(x,y) \in \mathcal{F}$, contrary to the choice of w henee $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w))$, so again $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Thus whenever $(u,y) \in \mathbf{W}$, also $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Now that we have shown $\mathbf{W} \in \mathbf{M}$ we see by (2) that $\mathcal{F} \subseteq \mathbf{W}$ but this is false since $(\mathbf{Sc}(x), w) \in \mathcal{F}$ and $(\mathbf{Sc}(x), w) \notin \mathbf{W}$. Thus our hypothesis that (8) is incorrect has led to a contradiction, and now (8) is proved. Sinee (7) follows from (8), we have by hyper infinite induction from (6) that $A = \mathbb{N}^{\#}$. Hence

(9) $\mathcal{F}$ is a function.

We have still to prove that $\mathcal{F}$ satisfies,condition (i); we must show that for each $x \in \mathbb{N}^{\#}$ there· is a y with $(x,y) \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathbb{N}^{\#} \times S$, it will then follow that $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$ and $\mathbf{range}(\mathcal{F}) \subseteq S$. Let $B = \mathbf{dom}(\mathcal{F})$, that is,

(10) $B = \{x | x \in \mathbb{N}^{\#}$ and for some $y, (x,y) \in \mathcal{F}\}$.

We prove now by hyper infinite induction that $B = \mathbb{N}^{\#}$. First, $1 \in B$, sínce $(1,c) \in \mathcal{F}$ by (3). Next, if $x \in B$, pick some y with $(x,y) \in \mathcal{F}$; then by (4), $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$, and henee $\mathbf{Sc}(x) \in B$.

Thus concludes the first part of the proof, that there is at least one function $\mathcal{F}$ satisfying conditions (i)-(iii).

Part 2. We prove that there cannot be more than one such function.

Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ both satisfy the conditions (i)-(iii) we wish to show $\mathcal{F}_1 = \mathcal{F}_2$, i.e., that for all $x \in \mathbb{N}^{\#}, \mathcal{F}_1(x) = \mathcal{F}_2(x)$. Thus is proved by hyper infinite induction on X. By (ii), $\mathcal{F}_1(1) = c$ and $\mathcal{F}_2(1) = c$, so $\mathcal{F}_1(1) = \mathcal{F}_2(1)$. Suppose that $\mathcal{F}_1(x) = \mathcal{F}_2(x)$; then $\mathcal{F}_1(\mathbf{Sc}(x)) = G(\mathcal{F}_1(x))$ and $\mathcal{F}_2(\mathbf{Sc}(x)) = G(\mathcal{F}_2(x))$, so $\mathcal{F}_1(\mathbf{Sc}(x)) = \mathcal{F}_2(\mathbf{Sc}(x))$.

Theorem 3.4.2. Let S be a set, $c \in S$ and $G : S \times \mathbb{N}^{\#} \to S$ is a binary funetion with

$\mathbf{dom}(G) = S \times \mathbb{N}^{\#}$ and $\mathbf{range}(G) \subseteqq S$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x), x)$.
We omit the proof of the Theorem 3.4.2 since it can be given by simple modification of the proof to Theorem 3.4.1.

3.5.General associatíve and commutative laws.

Definition 3.5.1. Suppose that S is a set on which a binary operation $+$ is defined and under which S is closed. Let $\{x_k\}_{k \in \mathbb{N}^{\#}}$ be an hyper infinite sequence of terms of S. For every $n \in \mathbb{N}^{\#}$ we denote by $Ext\text{-}\sum_{k=1}^{n} x_k$ the element of S uniquely determined by the following conditions:
(i) $Ext\text{-}\sum_{k=1}^{1} x_k = x_1$; (ii) $Ext\text{-}\sum_{k=1}^{n+1} x_k = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1}$ for all $n \in \mathbb{N}^{\#}$.

Remark 3.5.1.This definition is justified on the following grounds. The sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$is a given external function H with domain $\mathbb{N}^{\#}, H(x_k) = x_k$ for every k. We seek a function F with domain $\mathbb{N}^{\#}$ whose value $F(n)$ is to be $Ext\text{-}\sum_{k=1}^{n} x_k$. Then the conditions (i), (ii) above correspond to the following conditions on F :
(i′) $F(1) = H(1)$; (ii′) $F(n+1) = F(n) + H(n+1)$, for all $n \in \mathbb{N}^{\#}$.
Let (1) $c = H(1)$; (2) $G(n,z) = z + H(n+1)$.
Thus the conditions (i′) and (ii′) are equivalent to
(i″) $F(1) = c$;
(ii″) $F(n+1) = G(n, F(n))$ for all $n \in \mathbb{N}^{\#}$.
Given the function H, the element c of S and the function G are well-defined by (1)-(2). Then by Theorem 3.4.1 we see that there is a unique function F satisfying (1)-(2) with $\mathbf{dom}(F) = \mathbb{N}^{\#}$ and $\mathbf{range}(F) \subseteq S$. Thus (i′)-(ii′) is just another form of recursive definition.
(Hence it should be expected that various properties of $Ext\text{-}\sum_{k=1}^{n} x_k$ will have to be verified
by hyper infinite induction on $n \in \mathbb{N}^{\#}$.)
Definition 3.5.2. Suppose that S is a set on which a binary operation $\times$ is defined and under which S is dosed. Let $\{x_k\}_{k \in \mathbb{N}^{\#}}$ be an hyper infinite sequence of terms of S. For every $n \in \mathbb{N}^{\#}$ we denote by $Ext\text{-}\prod_{k=1}^{n} x_k$ the element of S uniquely determined by the following conditions:
(i) $Ext\text{-}\prod_{k=1}^{n} x_k = x_1$; (ii) $Ext\text{-}\prod_{k=1}^{n+1} x_k = \left(Ext\text{-}\prod_{k=1}^{n} x_k\right) \times x_{n+1}$ for all $n \in \mathbb{N}^{\#}$.

Theorem 3.5.1.(1) Suppose that S is a set closed under a binary operation $+$ and that $+$ is associative on S, i.e.,for all $x,y,z \in S, x + (y+z) = (x+y) + z$. Let $\{x_k\}_{k \in \mathbb{N}^{\#}}$ be any hyper infinite sequence of terms in S. Then for any $n, m \in \mathbb{N}^{\#}$. we have

$$Ext\text{-}\sum_{k=1}^{n+m} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{m} x_{n+k}\right). \qquad (3.5.1)$$

(2) Suppose that S is a set closed under a binary operation $\times$ and that $\times$ is associative on S, i.e.,for all $x,y,z \in S, x \times (y \times z) = (x \times y) \times z$. Let $\{x_k\}_{k \in \mathbb{N}^\#}$ be any hyper infinite sequence of terms in S. Then for any $n,m \in \mathbb{N}^\#$. we have

$$Ext\text{-}\prod_{k=1}^{n+m} x_k = \left(Ext\text{-}\prod_{k=1}^{n} x_k\right) \times Ext\text{-}\prod_{k=1}^{m} x_{n+k}. \qquad (3.5.2)$$

Proof. We prove (3.5.1); the proof of (2) is completely similar. Let n be fixed; we proceed by hyper infinite induction on m. For $m = 1$ from Eq.(3.5.1) we get

$$Ext\text{-}\sum_{k=1}^{n+1} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{1} x_{n+k}\right). \qquad (3.5.3)$$

By Definition 3.5.1(i) we obtain

$$Ext\text{-}\sum_{k=1}^{1} x_{n+k} = x_{n+1}. \qquad (3.5.4)$$

Suppose Eq.(3.5.1) is true for $m \in \mathbb{N}^\#$. We show that is true for $m+1$, i.e.,that

$$Ext\text{-}\sum_{k=1}^{n+(m+1)} x_k = \left(Ext\text{-}\sum_{k=1}^{n} x_k\right) + \left(Ext\text{-}\sum_{k=1}^{m+1} x_{n+k}\right). \qquad (3.5.1)$$

By associativity $+$ on $\mathbb{N}^\#$ we get

$$Ext\text{-}\sum_{k=1}^{n+(m+1)} x_k = Ext\text{-}\sum_{k=1}^{(n+m)+1} x_k. \qquad (3.5.6)$$

From Eq.(3.5.6) by Definition 3.5.1(ii) we obtain

$$Ext\text{-}\sum_{k=1}^{(n+m)+1} x_k = Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{(n+m)+1} = Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{n+(m+1)}. \qquad (3.5.7)$$

From Eq.(3.5.7) by induction hypothesis we obtain

$$Ext\text{-}\sum_{k=1}^{n+m} x_k + x_{n+(m+1)} = \left(Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m} x_k\right) + x_{n+(m+1)}. \qquad (3.5.8)$$

From Eq.(3.5.8) by associativity $+$ on S we get

$$\left(Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m} x_k\right) + x_{n+(m+1)} = Ext\text{-}\sum_{k=1}^{n} x_k + \left(Ext\text{-}\sum_{k=n}^{m} x_k + x_{n+(m+1)}\right). \qquad (3.5.9)$$

From Eq.(3.5.9) by Definition 3.5.1(ii) we obtain

$$Ext\text{-}\sum_{k=1}^{n} x_k + \left(Ext\text{-}\sum_{k=n}^{m} x_k + x_{n+(m+1)}\right) = Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=n}^{m+1} x_k. \qquad (3.5.10)$$

This equality completes the inductive step and hence the proof of the theorem.

Definition 3.5.3. Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^\#$.

Then $Ext\text{-}\sum_{k=m}^{n} x_k$ and $Ext\text{-}\prod_{k=m}^{n} x_k$ are defined for any $n,m \in \mathbb{N}^\#$ by the recursions

(i) $Ext\text{-}\sum_{k=m}^{n} x_k = 0$ and $Ext\text{-}\prod_{k=m}^{n} x_k = 1$ if $n < m$;

(ii) $Ext\text{-}\sum_{k=m}^{n} x_k = \left(Ext\text{-}\sum_{k=m}^{n-1} x_k\right) + x_n$ and

(iii) $Ext\text{-}\prod_{k=m}^{n} x_k = x_n \times \left(Ext\text{-}\prod_{k=m}^{n-1} x_k\right)$ if $m < n$.

The condition (ii) of the above definitíon is justified by recursive definition, see Appendix B.

Definition 4. Let $\langle x_1,\ldots,x_j,\ldots\rangle, j \in \mathbb{N}$ be a countable sequence of elements of $\mathbb{R}_c^\#$. Then ω-sum $Ext\text{-}\sum_{j=m}^{\omega} x_k$ and ω-product $Ext\text{-}\prod_{j=m}^{\omega} x_k$ are defined for any $m \in \mathbb{N}$ by

(iv) $Ext\text{-}\sum_{j=m}^{\omega} x_j \triangleq Ext\text{-}\sum_{j=m}^{n} y_j$, where $\langle y_1,\ldots,y_j,\ldots,y_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 0$ for all $j \in \mathbb{N}^\#\backslash\mathbb{N}$;

(v) $Ext\text{-}\prod_{j=m}^{\omega} x_j \triangleq Ext\text{-}\prod_{j=m}^{\omega} y_j$, where $\langle y_1,\ldots,y_j,\ldots,y_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 1$ for all $j \in \mathbb{N}^\#\backslash\mathbb{N}$.

Theorem 3.5.2.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^\#$. Then we have

$$Ext\text{-}\sum_{k=m}^{n} x_k = Ext\text{-}\sum_{k=m}^{n-m+q} x_{k+m-q} \tag{3.5.11}$$

and

$$z \times \left(Ext\text{-}\sum_{k=m}^{n} x_k\right) = Ext\text{-}\sum_{k=m}^{n} z \times x_k, \tag{3.5.12}$$

$z \in \mathbb{R}_c^\#$.

Proof.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^\#\backslash\mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^\#$. Consider now any hyperfinite nonnegative integers

$n_1, n_2, .., n_i, \ldots, n_{\mathbf{t}}, n_i \in \mathbb{N}^\#\backslash\mathbb{N}, 1 \leq i \leq \mathbf{t}$,

and set

$$n = n_1 + n_2 + \ldots + n_{\mathbf{t}}. \tag{3.5.13}$$

Given $x_1,\ldots,x_n$, we can group these as:

$$x_1,\ldots,x_{n_1};\ x_{n_1+1},\ldots,x_{n_1+n_2};\ x_{n_1+n_2+1},\ldots,x_{n_1+n_2+n_3};\ldots x_{n_1+n_2+\ldots n_i+1},\ldots,x_{n_1+n_2+\ldots n_{i+1}};\ldots \tag{3.5.14}$$

Here, if $n_i = 0$, the corresponding subsequence is regarded as being empty.

Theorem 3.5.3. Let $\langle x_1,\ldots,x_k,\ldots\rangle$ be an hyper infinite sequence of elements of $\mathbb{R}_c^\#$. Let $\langle n_1,\ldots,n_{\mathbf{t}}\rangle$ be a sequence of nonnegalive integers. For each $i = 1,\ldots,\mathbf{t} \in \mathbb{N}^\#$, let $m_i = \sum_{j=1}^{i-1} n_j$ and let $n = m_t + n_t$. Then

$$Ext\text{-}\sum_{k=1}^{n} x_k = \sum_{i=1}^{\mathbf{t}}\left(Ext\text{-}\sum_{k=1}^{n_i} x_{m_i+k}\right) \tag{3.5.15}$$

and

$$Ext\text{-}\prod_{k=1}^{n} x_k = \prod_{i=1}^{\mathbf{t}}\left(Ext\text{-}\prod_{k=1}^{n_i} x_{m_i+k}\right). \tag{3.5.16}$$

Proof. By hyper infinite induction.

Definition 3.5.5. A function F is said to be a permutation of a set S if it is one-to-one and $\mathbf{dom}(F) = \mathbf{range}(F) = S$.

Definition 3.5.6. Let $[1,n]$ a set $\{k|k \in \mathbb{N}^{\#} \wedge (1 \leq k \leq n)\}$

Theorem 3.5.4.Let $\langle x_1,\ldots,x_n\rangle, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be an hyperfinite external sequence of elements

of $\mathbb{R}^{\#}_c$. Then for any $n \in \mathbb{N}^{\#}$ and any permutalion $\mathbf{F}$ of $[1,n]$ following holds

$$Ext\text{-}\sum_{k=1}^{n} x_k = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}. \quad (3.5.17)$$

The same holds if we replace $Ext\text{-}\sum$ by $Ext\text{-}\prod$.

Proof. The proof is by hyper infinite induction on $n \in \mathbb{N}^{\#}$. For $n = 1$ it is trivial.
Suppose that it is true for n. Let $\mathbf{G}$ be a permutation of $[1,n+1]$.Then $G(m) = n+1$ for a unique $m,$ such that $1 \leq m \leq n+1.$ Then by Eq.(3.5.15)

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} \quad (3.5.18)$$

and by Eq.(3.5.18)

$$Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \quad (3.5.19)$$

Thus by Eq.(3.5.11) we obtain

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \quad (3.5.20)$$

To reduce this to the inductive hypothesis, we wish to rewrite the external sum of the first

two terms as $Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}$ for suitable $\mathbf{F}$. Define $\mathbf{F}$ by

$$\mathbf{F}(k) = \begin{cases} \mathbf{G}(k) & \text{if} \quad 1 \leq k < m \\ \\ \mathbf{G}(k+1) & \text{if} \quad m \leq k \leq n \end{cases} \quad (3.5.21)$$

Since all valucs of $\mathbf{G}(k)$ for $k \neq m,$ we have for all $k \leq n$

$$1 \leq \mathbf{F}(k) \leq n \quad (3.5.22)$$

Now we claim that

$$\mathbf{F} \text{ is a permutation of } [1,n]. \quad (3.5.23)$$

By (3.5.21) and (3.5.22) we need only check that $\mathbf{F}$ is one-to one. Suppose that $\mathbf{F}(k_1) = \mathbf{F}(k_2)$.

If both k_1,k_2 are $< m$ or both are $\geq m,$ it lollows from (3.5.21) and the fact that $\mathbf{G}$ is a permutation that $k_1 = k_2$.If, say, $k_1 < m \leq k_2$, we have $\mathbf{G}(k_1) = \mathbf{G}(k_2+1),$ hence $k_1 = k_2+1,$ which contradicts our assumption. Thus neither this case· nor, by symmetry, the case $k_2 < m \leq k_1$ can occur. We have from (3.5.20) and (3.5.21) that

$$Ext\text{-}\sum_{k=1}^{m+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{F}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} \quad (3.5.24)$$

by (3.5.23) and inductive hypothesis

$$Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1} = Ext\text{-}\sum_{k=1}^{n+1} x_k \quad (3.5.25)$$

This equality completes the inductive step and hence the proof of the theorem.

3.6.External non-Archimedian field ${}^*\mathbb{R}_c^\#$ by Cauchy completion of the internal non-Archimedean field ${}^*\mathbb{R}$.

Definition 3.6.1. A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a : \mathbb{N}^\# \to {}^*\mathbb{R}$ from hypernatural numbers $\mathbb{N}^\#$ into the hyperreal numbers ${}^*\mathbb{R}$.
We usually denote such a function by $n \mapsto a_n$, or by $a : n \to a_n$, so the terms in the sequence are written $\{a_1, a_2, a_3, \ldots, a_n \ldots\}$. To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^\#}$, or $\{a_n\}_{n\in\mathbb{N}^\#}$, or for the sake of brevity simply $\{a_n\}$.
Definition 3.6.2. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^\# \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon$. We denote this symbolically by $a_n \to_\# 0$.
We can also, at this point, define what it means for a hyper infinite ${}^*\mathbb{R}$-valued sequence #-tends to any given number $q \in {}^*\mathbb{R}$: $\{a_n\}$ #-tends to q if the hyper infinite sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0$.
Definition 3.6.3. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence. We call $\{a_n\}$ a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N, |a_n - a_m| < \varepsilon$.
Theorem 3.6.1.If $\{a_n\}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence (that is, $a_n \to_\# q$ for some hyperreal number $q \in {}^*\mathbb{R}$), then $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.
Proof. We know that $a_n \to_\# q$. Here is a ubiquitous trick: instead of using ε in the definition Definition 3.6.3, start with an arbitrary infinite small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^\# \backslash \mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N$. Then if $m, n > N$, we have $|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\{a_n\}_{n\in\mathbb{N}^\#}$ is a Cauchy sequence.
Theorem 3.6.2. If $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is bounded or hyper bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.
Proof.Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^\#$ such that $|a_m - a_n| < 1$whenever $m, n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$. This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}$. Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then
(as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$. Hence, $M \in {}^*\mathbb{R}$ is a bound for the sequence $\{a_n\}$.
Definition 3.6.4. Let S be a set. A relation $x \sim y$ among pairs of elements of S is said to be an equivalence relation if the following three properties hold:
Reflexivity: for any $s \in S, s \sim s$.

Symmetry: for any $s, t \in S,$ if $s \sim t$ then $t \sim s.$
Transitivity: for any $s, t, r \in S,$ if $s \sim t$ and $t \sim r,$ then $s \sim r.$
Theorem 3.6.3. Let S be a set, with an equivalence relation $(\bullet \sim \bullet)$ on pairs of elements. For $s \in S,$ denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s, t \in S,$ either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.
The hyperreal numbers ${}^*\mathbb{R}_c^\#$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences. Let $\mathcal{F}_{{}^*\mathbb{R}}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Definition 3.6.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{{}^*\mathbb{R}}$. Say they are #-equivalent if $a_n - b_n \to_\# 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n - b_n\}$ tends to 0.
Theorem 3.6.4.Definition 3.6.5 yields an equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Proof. We need to show that this relation is reflexive, symmetric, and transitive.
Reflexive: $a_n - a_n = 0,$ and the hyper infinite sequence all of whose terms are 0 clearly #-converges to $0.$ So $\{a_n\}$ is related to $\{a_n\}$.
Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_\# 0.$
But $b_n - a_n = -(a_n - b_n),$ and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 10.2, it follows that $b_n - a_n \to_\# 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.
Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 10.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_\# 0$ and $b_n - c_n \to_\# 0.$ To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^\#$ such that for all $n > N, |a_n - b_n| < \varepsilon/2;$ also, there exists an M such that for all $n > M$, $|b_n - c_n| < \varepsilon/2.$ Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$
So, choosing L equal to the max of $N, M,$ we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon.$ This means that $a_n - c_n \to_\# 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.
Definition 3.6.6. The external hyperreal numbers ${}^*\mathbb{R}_c^\#$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per Definition 3.6.5. That is, each such equivalence class is an external hyperreal number.
Definition 3.6.7. Given any hyperreal number $q \in {}^*\mathbb{R}$, define a hyperreal number $q^\#$ to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence

$$q^\# = ({}^*q, {}^*q, {}^*q, {}^*q, \ldots)$$

consisting entirely of ${}^*q, q \in \mathbb{R}.$ So we view ${}^*\mathbb{R}$ as being inside ${}^*\mathbb{R}_c^\#$ by thinking of each hyperreal number q as its associated equivalence class $q^\#$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.
Definition 3.6.8. Let $s, t \in {}^*\mathbb{R}_c^\#,$ so there are Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences $\{a_n\}, \{b_n\}$ of hyperreal numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.
(a) Define $s + t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.
(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.
Theorem 3.6.5.The operations $+, \times$ in Definition 10.8 (a),(b) are well-defined.
Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that $a_n - c_n \to_\# 0$ and $b_n - d_n \to_\# 0.$ Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n).$
Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to $0,$ and

so $\mathbf{cl}[\{a_n + b_n\}] = \mathbf{cl}[\{c_n + d_n\}]$.

Multiplication is a little trickier; this is where we will use Theorem 3.6.3. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $\mathbf{cl}[(a_n)] = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_{\#} 0.$ Well, we add and subtract one of the other cross terms, say

$b_n \times c_n$: $a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$

$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n)$.

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 3.6.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^{\#}$. Taking some number K which is bigger than both, we have

$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|)$.

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_{\#} 0$.

Theorem 3.6.6. Given any hyperreal number $s \in {}^*\mathbb{R}^{\#}_c$, $s \neq 0$, there is a hyperreal number $t \in {}^*\mathbb{R}^{\#}_c$ such that $s \times t = 1$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0,0,0,0,\ldots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1,1,1,1,\ldots\}]$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$.

That is:

Lemma 3.6.1. If $\{a_n\}$ is a Cauchy sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N, a_n \neq 0$.

Definition 3.6.9. Let $s \in {}^*\mathbb{R}^{\#}_c$. Say that s is positive if $s \neq 0$, and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperreal numbers such that for some $N \in \mathbb{N}^{\#}, a_n > 0$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 3.6.7. Let $s, t \in {}^*\mathbb{R}^{\#}_c$ be hyperreal numbers such that $s > t$, and let $r \in {}^*\mathbb{R}^{\#}_c$. Then $s + r > t + r$.

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}]$, and $r = \mathbf{cl}[\{c_n\}]$. Since $s > t$ i.e., $s - t > 0$, we know that there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N$, $a_n - b_n > 0$. So $a_n > b_n$ for $n > N$. Now, adding c_n to both sides of this inequality (as we know we can do for hyperreal numbers ${}^*\mathbb{R}$), we have $a_n + c_n > b_n + c_n$ for $n > N$, or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N$. Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to 0, by the assumption that $s - t > 0$. Thus, by Definition 10.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r$.

Theorem 3.6.8. Let $s,t \in {}^*\mathbb{R}_c^\#$ $s,t > 0$ be hyperreal numbers.Then there is $m \in \mathbb{N}^\#$ such that $m \times s > t.$

Proof. Let $s,t > 0$ be hyperreal numbers. We need to find a natural number m so that $m \times s > t.$ First, recall that, by m in this context, we mean $\mathbf{cl}[\{m,m,m,m,\ldots\}].$ So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$, what we need to show is that there exists m with $\mathbf{cl}[\{m,m,m,m,\ldots\}] \times \mathbf{cl}[\{a_1,a_2,a_3,a_4,\ldots\}] = \mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4,\ldots\}] >$ $> \mathbf{cl}[\{b_1,b_2,b_3,b_4,\ldots\}].$

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}]$, or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 3.6.9, just to say that there is $N \in \mathbb{N}^\#$ such that $m \times a_n - b_n > 0$ for all $n > N,$ while $m \times a_n - b_n \nrightarrow_\# 0.$ To be precise, the first statement is:

There exist $m,N \in \mathbb{N}^\#$ so that $m \times a_n > b_n$ for all $n > N.$

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and $N,$ there exists an $n > N$ so that $m \times a_n \leq b_n.$

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 3.6.2 it is hyperbounded - there is a hyperreal number $M \in {}^*\mathbb{R}$ such that $b_n \leq M$ for all $n \in \mathbb{N}^\#.$ Now, by the properties for the hyperreal numbers ${}^*\mathbb{R},$ given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0,$ there is an $m \in \mathbb{N}^\#$ such that $M/m < \varepsilon/2.$ Fix such an $m.$ Then if $m \times a_n \leq b_n,$ we have $a_n \leq b_n/m \leq M/m < \varepsilon/2.$

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for $n,k > N, |a_n - a_k| < \varepsilon/2.$

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n,$ which means that $a_n < \varepsilon/2.$ But then for every $k > N,$ we have that $a_k - a_n < \varepsilon/2$, so $a_k < a_n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ Hence, $a_k < \varepsilon$ for all $k > N.$ This proves that $a_k \to_\# 0,$ which by Definition 3.6.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0.$

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^\# \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow 0.$

Actually, it is possible that $m \times a_n - b_n \to 0$ (for example if $\{a_n\} = \{1,1,1,\ldots\}$ and $\{b_n\} = \{m,m,m,\ldots\}$). But that's okay: then we can simply choose a larger $m.$ That is: let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^\# \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow 0,$ then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \to 0,$ then take instead the integer $m + 1.$ Since $s = \mathbf{cl}[\{a_n\}] > 0,$ we have a n > 0 for all infinite large n, so $(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m + 1$ works just as well as m did in this regard; and since $m \times a_n - b_n \to 0,$ we have $(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow 0$).

It will be handy to have one more Theorem about how the hyperreals ${}^*\mathbb{R}$ and hyperreals ${}^*\mathbb{R}_c^\#$ compare before we proceed. This theorem is known as the density of ${}^*\mathbb{R}$ in ${}^*\mathbb{R}_c^\#$, and it follows almost immediately from the construction of the ${}^*\mathbb{R}_c^\#$ from ${}^*\mathbb{R}$.

Theorem 3.6.9. Given any hyperreal number $r \in {}^*\mathbb{R}_c^\#,$ and any hyperreal number $\varepsilon > 0,\ \varepsilon \approx 0,$ there is a hyperreal number $q \in {}^*\mathbb{R}$ such that $|r - q| < \varepsilon.$

Proof. The hyperreal number r is represented by a Cauchy ${}^*\mathbb{R}$-valued sequence $\{a_n\}.$ Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0,$ there is $N \in \mathbb{N}^\#$ so that for all $m,n > N,$

$|a_n - a_m| < \varepsilon.$ Picking some fixed $l > N,$ we can take the hyperreal number q given by

$q = \mathbf{cl}[\{a_l, a_l, a_l, \ldots\}]$. Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n \in \mathbb{N}^\#}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n \in \mathbb{N}^\#}]$.

Now, since $l > N$, we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon$, which means by Definition 3.6.9 that $r - q < \varepsilon$ and $q - r < \varepsilon$; hence, $|r - q| < \varepsilon$.

Definition 3.6.10.Let $S \subsetneqq {}^*\mathbb{R}^\#_c$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in {}^*\mathbb{R}^\#_c$ is called an upper bound for S if $x \geq s$ for all $s \in S$.

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of S.

Remark 3.6.1.The order $\leq$ given by Definition 3.6.9 obviously is $\leq$-incomplete.

Definition 10.11. Let $S \subsetneqq {}^*\mathbb{R}^\#_c$ be a nonempty subset of ${}^*\mathbb{R}^\#_c$. We we will say that:

(1) S is $\leq$-admissible above if the following conditions are satisfied:

(i) S bounded or hyperbounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 3.6.10. (i) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}^\#_c$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset {}^*\mathbb{R}^\#_c$ has the greatest lower bound property.

Proof. Let $S \subset {}^*\mathbb{R}^\#_c$ be a nonempty subset, and let M be an upper bound for S. We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S$. Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $l_1, l_2, l_3, \ldots, l_n, \ldots$

(i) Set $u_0 = M$ and $l_0 = s$.

(ii) Suppose that we have already defined u_n and l_n. Consider the number $m_n = (u_n + l_n)/2$, the average between u_n and l_n.

(1) If m_n is an upper bound for S, define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.

(2) If m_n is not an upper bound for S, define $u_{n+1} = u_n$ and $l_{n+1} = l_n$.

Remark 10.1.Since $s < M$, it is easy to prove by hyper infinite induction that

(i) $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^\#$and $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^\#$, (ii) u_n is an upper bound for S for all $n \in \mathbb{N}^\#$ and l_n is never an upper bound for S for any $n \in \mathbb{N}^\#$,(iii) $u_n - l_n = 2^{-n}(M - s)$.

This gives us the following lemma.

Lemma 3.6.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy ${}^*\mathbb{R}$-valued sequences of hyperreal numbers.

Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^\#$. Since $\{l_n\}$ is non-decreasing and $u_n - l_n = 2^{-n}(M - s)$, it follows directly that $\{l_n\}$ is Cauchy.

For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^\#$, and so $-u_n \leq -s_0$.

Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.

The following Lemma shows that $\{u_n\}$ does #-tend to a hyperreal number $u \in {}^*\mathbb{R}^\#_c$.

Lemma 3.6.3. There is a hyperreal number $u \in {}^*\mathbb{R}^\#_c$ such that $u_n \to_\# u$.

Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 10.9, there is a hyperreal number $q_n \in {}^*\mathbb{R}, n \in \mathbb{N}^\#$ such that $|u_n - q_n| < 1/n$. Consider the sequence

$\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperreal numbers. We will show this sequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0$. By the Theorem 3.6.8, we can choose $N \in \mathbb{N}^{\#}$ so that $1/N < \varepsilon/3$. We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^{\#}$ such that for $n, m > M$, $|u_n - u_m| < \varepsilon/3$. Then, so long as $n, m > max\{N, M\}$, we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of internal hyperreal numbers, and so it represents the external hyperreal number $u = \mathbf{cl}[\{q_n\}]$. We must show that $u_n - u \to_{\#} 0$, but this is practically built into the definition of u. To be precise, letting q_n^* be the hyperreal number
$\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}]$, we see immediately that $q_n^* - u \to_{\#} 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_{\#} u$ and $u_n - q_n^* \to_{\#} 0$, then $u_n \to_{\#} u$. So $\{u_n\}$, a non-increasing sequence of upper bounds for S, tends to a hyperreal
number u. As you've guessed, u is the least upper bound of our set S. To prove this, we
need one more lemma.

Lemma 3.6.4. $l_n \to_{\#} u$.

Proof. First, note in the first case above, we have that
$u_{n+1} - l_{n+1} = m_n - l_n = \dfrac{u_n + l_n}{2} - l_n = \dfrac{u_n - l_n}{2}$.
In the second case, we also have
$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \dfrac{u_n + l_n}{2} = \dfrac{u_n - l_n}{2}$.
Now, this means that $u_1 - l_1 = \frac{1}{2}(M - s)$, and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(L - s)$, and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(M - s)$. Since $M > s$ so $M - s > 0$, and since $2^{-n} < 1/n$, by the Theorem 3.6.8, we have for any $\varepsilon > 0$ that $2^{-n}(M - s) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^{\#}$. Thus, $u_n - l_n = 2^{-n}(M - s) < \varepsilon$ as well, and so $u_n - l_n \to_{\#} 0$. Again, it is easily verify that, since $u_n \to_{\#} u$, we have $l_n \to_{\#} u$ as well.

Remark 3.6.2. Note that assumption in Theorem 10.10 that S is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$ is necessarily, othervice Theorem 10.10 is not holds.

Theorem 3.6.11.(Generalized Nested Intervals Theorem)
Let $\{I_n\}_{n \in \mathbb{N}^{\#}} = \{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}, [a_n, b_n] \subset \mathbb{R}_c^{\#}$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:
(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$,
(ii) $b_n - a_n \to_{\#} 0$ as $n \to \infty^{\#}$.
Then $\bigcap_{n=1}^{\infty^{\#}} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^{\#}$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.

Proof. Note that: (a) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is hyperbouded above by b_1 and
(b) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$.
By Theorem **3.6**.10 there exists $\sup A$. Let $\xi = \sup A$.
Since I_n are nested, for any positive hyperintegers m and n we have
$a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$, so that $\xi \leq b_n$ for each $n \in \mathbb{N}^{\#}$. Since we obviously have $a_n \leq \xi$ for each $n \in \mathbb{N}^{\#}$, we have $a_n \leq \xi \leq b_n$ for all $n \in \mathbb{N}^{\#}$, which implies $\xi \in \bigcap_{n=1}^{\infty^{\#}} I_n$. Finally, if

$\xi, \eta \in \bigcap_{n=1}^{\infty^{\#}} I_n$, with $\xi \le \eta$, then we get $0 \le \eta - \xi \le b_n - a_n$, for all $n \in \mathbb{N}^{\#}$, so that $0 \le \eta - \xi \le \inf_{n \in \mathbb{N}^{\#}} |b_n - a_n| = 0$.

Theorem 3.6.12.(**Generalized Squeeze Theorem**)
Let $\{a_n\}, \{c_n\}$ be two hyper infinite sequences #-converging to L, and $\{b_n\}$ a hyper infinite sequence. If $\forall n \ge K, K \in \mathbb{N}^{\#}$ we have $a_n \le b_n \le c_n$, then $\{b_n\}$ also #-converges to L.

Proof. Choose an $\varepsilon > 0, \varepsilon \approx 0$. By definition of the #-limit,there is an $N_1 \in \mathbb{N}^{\#}$ such that for all $n > N_1$ we have $|a_n - L| < \varepsilon$, in other words $L - \varepsilon < a_n < L + \varepsilon$.Similarly, there is an $N_2 \in \mathbb{N}^{\#}$ such that for all $n > N_2$ we have $L - \varepsilon < c_n < L + \varepsilon$. Denote $N = \max(N_1, N_2, K)$. Then for $n > N, L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$, in other words $|b_n - L| < \varepsilon$. Since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary, by definition of the #-limit this says that #-$\lim_{n \to \infty^{\#}} b_n = L$.

Theorem 3.6.13.(**Corollary of the Generalized Squeeze Theorem**).
If #-$\lim_{n \to \infty^{\#}} |a_n| = 0$ then #-$\lim_{n \to \infty^{\#}} a_n = 0$.

Proof.We know that $-|a_n| \le a_n \le |a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that #-$\lim_{n \to \infty^{\#}} |a_n| = 0$.This also implies that #-$\lim_{n \to \infty^{\#}} (-|a_n|) = 0$. So by the Generalized Squeeze Theorem, #-$\lim_{n \to \infty^{\#}} a_n = 0$.

Theorem 3.6.14. (**Generalized Bolzano-Weierstrass Theorem**)
Every hyperbounded hyper infinite ${}^*\mathbb{R}_c^{\#}$-valued sequence has a #-convergent hyper infinite subsequence.

Proof. Let $\{w_n\}_{n \in \mathbb{N}^{\#}}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1, b_1]$ such that $a_1 \le w_n \le b_1$ for all $n \in \mathbb{N}^{\#}$.
Either $\left[a_1, \frac{a_1 + b_1}{2}\right]$ or $\left[\frac{a_1 + b_1}{2}, b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyper infinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[a_1, \frac{a_1 + b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[\frac{a_1 + b_1}{2}, b_1\right]$. If $\left[a_1, \frac{a_1 + b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2, b_2] = \left[a_1, \frac{a_1 + b_1}{2}\right]$. Otherwise, let $[a_2, b_2] = \left[\frac{a_1 + b_1}{2}, b_1\right]$.
Either $\left[a_2, \frac{a_2 + b_2}{2}\right]$ or $\left[\frac{a_2 + b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$. If $\left[a_2, \frac{a_2 + b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3, b_3] = \left[a_2, \frac{a_2 + b_2}{2}\right]$. Otherwise, let $[a_3, b_3] = \left[\frac{a_2 + b_2}{2}, b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}$ such that:
(i) for each $n \in \mathbb{N}^{\#}, [a_n, b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$,
(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n, b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n \in \mathbb{N}^{\#}}$ which will #-converge to w.
Since $[a_1, b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that w_{k_1} is in $[a_1, b_1]$. Since $[a_2, b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1$, such that w_{k_2} is in $[a_2, b_2]$. Since $[a_3, b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2$, such that w_{k_3} is in $[a_3, b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n \in \mathbb{N}^{\#}}$ such that $w_{k_n} \in [a_n, b_n]$ for each $n \in \mathbb{N}^{\#}$. The sequence $\{w_{k_n}\}_{n \in \mathbb{N}^{\#}}$ is a subsequence of $\{w_n\}_{n \in \mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since $a_n \to_{\#} w$, and

$a_n \leq w_n \leq b_n$ for each $n \in \mathbb{N}^\#$, the squeeze theorem implies that $w_{k_n} \to_\# w$.
Definition 3.6.12. Let $\{a_n\}$ be a hyperreal sequence i.e.,$a_n \in {}^*\mathbb{R}_c^\#, n \in \mathbb{N}^\#$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N \in \mathbb{N}^\# \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that,for all $n > N$, $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_\# 0$. We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in {}^*\mathbb{R}_c^\#$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0$.
Definition 3.6.13. Let $\{a_n\}, n \in \mathbb{N}^\#$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N, |a_n - a_m| < \varepsilon$.
Theorem 3.6.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_\# b$ for some hyperreal number $b \in \mathbb{R}_c^\#$), then $\{a_n\}$ is a Cauchy hyperreal sequence.
Theorem 3.6.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is hyper bounded; that is, there is some $M \in \mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.
Theorem 3.6.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in ${}^*\mathbb{R}_c^\#$ i.e., there exists $b \in {}^*\mathbb{R}_c^\#$ such that $a_n \to_\# b$.
Proof.By Definition 3.6.13 given $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \tag{3.6.1}$$

From (3.6.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \tag{3.6.2}$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_\# b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^\#$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in {}^*\mathbb{R}_c^\#$.
We can choose $k \in \mathbb{N}^\#$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \tag{3.6.3}$$

We choose now in (3.6.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \tag{3.6.4}$$

From (3.6.3) and (3.6.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \tag{3.6.5}$$

Thus $a_n \to_\# b$ as well.
Remark 3.6.3.Note that there exist canonical natural embedings

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^\#. \tag{3.6.6}$$

3.10.The Extended Hyperreal Number System ${}^*\hat{\mathbb{R}}_c^\#$

Definition 3.10.1.(a) A set $S \subset \mathbb{N}^\#$ is hyperfinite if $card(S) = card(\{x|0 \leq x \leq n\})$, $n \in \mathbb{N}^\# \backslash \mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^\#$ is hyper infinite if $card(S) = card(\mathbb{N}^\#)$.
Notation 3.10.1. If F is an arbitrary collection of subsets of ${}^*\mathbb{R}_c^\#$, then $\cup\{S|S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S|S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyperfinitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^\#$ are also written as

$\bigcup_{k=0}^{n} S_k$ and $\bigcap_{k=0}^{n} S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^{\#}$ of sets are written as $\bigcup_{k=0}^{\infty^{\#}} S$ or $\bigcup_{n \in \mathbb{N}^{\#}} S$ and $\bigcap_{k=0}^{\infty^{\#}} S$ or $\bigcap_{n \in \mathbb{N}^{\#}} S$ correspondingly.

A nonempty set S of hyperreal numbers ${}^{*}\mathbb{R}_c^{\#}$ is unbounded above if it has no hyperfinite
upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^{\#}$ (which we also write more simply as $\infty^{\#}$) and $-\infty^{\#}$,and to define the order relationships between them and any hyperreal number $x \in {}^{*}\mathbb{R}_c^{\#}$ by $-\infty^{\#} < x < \infty^{\#}$.

We call $-\infty^{\#}$ and $\infty^{\#}$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^{\#}$ to indicate that S is unbounded above, and $\inf S = -\infty^{\#}$ to indicate that S is unbounded below.

#-Open and #-Closed Sets on ${}^{*}\hat{\mathbb{R}}_c^{\#}$.

Definition 3.10.15.If a and b are in the extended hyperreals and $a < b$, then the open interval (a,b) is defined by $(a,b) \triangleq \{x|a < x < b\}$. :

The open intervals $(a,+\infty^{\#})$ and $(-\infty^{\#},b)$ are semi-hyperinfinite if a and b are finite or hyperfinite, and $(-\infty^{\#},\infty^{\#})$ is the entire hyperreal line.

If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] \triangleq \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#},a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$,where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$,respectively.

Definition 10.16.If $x_0 \in \mathbb{R}_c^{\#}$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the open interval
$(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset {}^{*}\mathbb{R}_c^{\#}$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S.

The set of #-interior points of S is the #-interior of S, denoted by #-$Int(S)$.

(i) If every point of S is an #-interior point (that is, $S = \#\text{-}Int(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = {}^{*}\mathbb{R}_c^{\#} \backslash S$ is #-open.

Example 10.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and $\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$

Remark 10.4.The entire hyperline ${}^{*}\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#},\infty^{\#})$ is #-open, and therefore $\varnothing$ is #-closed.

However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, ${}^{*}\hat{\mathbb{R}}_c^{\#}$ is #-closed. Thus, ${}^{*}\hat{\mathbb{R}}_c^{\#}$ and $\varnothing$ are both #-open and #-closed.

Remark 10.5.They are not the only subsets of ${}^{*}\hat{\mathbb{R}}_c^{\#}$ with this property.

Definition 10.17.A deleted #-neighborhood of a point x_0 is a set that contains every point
of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x|0 < |x - x_0| < \varepsilon\}$, where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 3.10.18.(a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G|G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0. Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H|H \in F\}$. Then $T^c = \cup\{H^c|H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 3.10.2. If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] = \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#}a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x|a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^{\#} < a < b < \infty^{\#}$, as is $(a,b] = \{x|a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a,\infty^{\#}) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$, where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$, respectively.

Definition 10.18. Let S be a subset of $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#},\infty^{\#})$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted by #-∂S. The #-closure of S, denoted by #-$\bar{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 10.19. A set S is #-closed if and only if no point of S^c is a #-limit point of S.

Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained in S^c. Therefore, S^c is #-open and S is #-closed.

Corollary 10.1. A set S is #-closed if and only if it contains all its #-limit points.

If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.

Proposition 10.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S. .

#-Open Coverings

Definition 10.19. A collection H of #-open sets of $\mathbb{R}_c^{\#}$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F|F \in H\}$.

Definition 10.20. A set $S \subset \mathbb{R}_c^{\#}$ is called #-compact (or hyper compact) if each of its #-open covers has a hyperfinite subcover. .

Theorem 10.20. (**Generalized Heine–Borel Theorem**) If H is an #-open covering of a

#-closed and hyper bounded subset S of the hyperreal line ${}^{*}\mathbb{R}_c^{\#}$ (or of the ${}^{*}\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$) then S has an #-open covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.

Proof. If a set S in ${}^{*}\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.

Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^{\#}}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^{\#}}$, otherwise $C_{\infty^{\#}}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1. Likewise, the sides of T_1 can be bisected, yielding 2^n sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^{\#}}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes: $T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_k \supset \ldots, k \in \mathbb{N}^{\#}$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^{\#}$. Let us define a hyper infinite sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy, so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k \in \mathbb{N}^{\#}}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^{\#}$. Since $C_{\infty^{\#}}$ covers T_0, then it has some member $U \in C_{\infty^{\#}}$ such that $L \in U$. Since U is open, there is an n-ball $B(L) \subseteq U$. For large enough k, one has $T_k \subseteq B(L) \subseteq U$, but then the hyper infinite number of members of $C_{\infty^{\#}}$ needed to cover T_k can be replaced by just one: U, a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set T_0, then S is also #-compact.

As an application of the Generalized Heine–Borel theorem, we give a short proof of the

Generalized Bolzano–Weierstrass Theorem.

Theorem 3.10.21.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset {}^{*}\mathbb{R}_c^{\#}$ has at least one #-limit point.

Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is #-closed (Theorem 9.) and every point $x \in S$ has an #-open neighborhood N_x that contains no point of S other than x.The collection $H = \{N_x | x \in S\}$ is an #-open covering for S. Since S is also hyper bounded, Theorem 3.10.20 implies that S can be covered by finite or a hyper finite collection of sets from H, say $N_{x_1}, \ldots, N_{x_n}, n \in \mathbb{N}^{\#}$. Since these sets contain only $x_1, \ldots, x_n$ from S, it follows that $S = \{x_k\}_{1 \leq k \leq n}, n \in \mathbb{N}^{\#}$.

3.11.External hyperfinite sum of the ${}^{*}\mathbb{R}_c^{\#}$- valued hyperfinite sequences.Main properties.

Theorem 3.11.1. Let $\{a_i\}_{i=1}^{n}$and $\{b_i\}_{i=1}^{n}$be ${}^{*}\mathbb{R}_c^{\#}$- valued hyperfinite sequences.The following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^{\#} \backslash \mathbb{N}$:

(1) using distributivity

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i \right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \tag{3.11.1}$$

(2) using commutativity and associativity

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n}(a_i \pm b_i) \qquad (3.11.2)$$

(3) splitting a sum, using associativity

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \qquad (3.11.3)$$

(4) using commutativity and associativity, again

$$Ext\text{-}\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \qquad (3.11.4)$$

(5) using distributivity

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j\right) = Ext\text{-}\sum_{i=0}^{n}\left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j\right) \qquad (3.11.5)$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i\right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \qquad (3.11.6)$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right)^m = Ext\text{-}\prod_{i=0}^{n} a_i^m \qquad (3.11.7)$$

Proof. Imediately by hyper infinite induction principle.

Theorem 3.11.2. Let $\{a_i\}_{i=1}^{n}$and $\{b_i\}_{i=1}^{n}$be ${}^*\mathbb{R}_c^{\#}$- valued hyperfinite sequences. Suppose that $a_i \leq b_i, 1 \leq i \leq n$ then the following equalities holds for any $n \in \mathbb{N}^{\#}$:

$$Ext\text{-}\sum_{i=0}^{n} a_i \leq Ext\text{-}\sum_{i=0}^{n} b_i \qquad (3.11.8)$$

Theorem 3.11.3. Let $\{a_i\}_{i=1}^{n}$and $\{b_i\}_{i=1}^{n}$be ${}^*\mathbb{R}_c^{\#}$- valued hyperfinite sequences.Then the following equalities holds for any $n \in \mathbb{N}^{\#}$:

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i \times b_i\right)^2 \leq \left(Ext\text{-}\sum_{i=0}^{n} a_i^2\right)\left(Ext\text{-}\sum_{i=0}^{n} b_i^2\right). \qquad (3.11.9)$$

3.12.External countable sum $Ext\text{-}\sum_{n\in\mathbb{N}} a_n$ from external hyperfinite sum.

Definition 3.12.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be ${}^*\mathbb{R}$-valued countable sequence. Let $\{a_n\}_{n=1}^{m}$ be any ${}^*\mathbb{R}$-valued hyperfinite sequence with $m \in {}^*\mathbb{N}\backslash\mathbb{N}$ and such that $a_n = 0$ if $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Then we define external sum of the countable sequence $\{a_n\}_{n\in\mathbb{N}}$ (or ω-sum) as the following hyperfinite sum

$$Ext\text{-}\sum_{n=1}^{m} a_n \in {}^*\mathbb{R} \qquad (3.12.1)$$

and denote such sum by the symbol

$$Ext\text{-}\sum_{n\in\mathbb{N}} a_n \qquad (3.12.2)$$

or by the symbol

$$Ext\text{-}\sum_{n=k}^{\omega} a_n. \tag{3.12.3}$$

Remark 3.12.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{R}$-valued countable sequence. Note that: (i) for canonical
summation we always apply standard notation

$$\sum_{n=k}^{\infty} a_n. \tag{3.12.4}$$

(ii) the countable summ (ω-sum) (3.12.3) in contrast with (3.12.4) abviously always exists even if a series (3.12.3) diverges absolutely i.e., $\sum_{n=k}^{\infty}|a_n| = \infty$.

Definition 3.12.2.[5].(i) Let U be a free ultrafilters on $\mathbb{N}$ and introduce an equivalence relation on sequences in $\mathbb{R}^{\mathbb{N}}$ as $f_1 \sim_U f_2$ iff $\{i \in \mathbb{N} | f_1(i) = f_2(i)\} \in U$.
(ii) $\mathbb{R}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{R}$, the hyperreals; in symbols, ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\sim_U$ and similarly $\mathbb{N}^{\mathbb{N}}$ divided out by the equivalence relation $\sim_U$ gives us the nonstandard extension ${}^*\mathbb{N}$, the hyperintegers; in symbols,${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\sim_U$.

Abbreviation 3.12.1.If $f \in \mathbb{R}^{\mathbb{N}}$, we denote its image in ${}^*\mathbb{R}$ by

$$\mathbf{cl}(f), \tag{3.12.5}$$

i.e., $\mathbf{cl}(f) = \{g \in \mathbb{R}^{\mathbb{N}} | g \sim_U f\}$.

Assumption 3.12.1.We assume now that there is an embedding ${}^*\mathbb{N} \hookrightarrow \mathbb{N}^{\#}$.

Remark 3.12.2.For any real number $r \in \mathbb{R}$ let $\mathbf{r}$ denote the constant function $\mathbf{r} : \mathbb{N} \to \mathbb{R}$ with value r, i.e.,$\mathbf{r}(n) = r$, for all $n \in \mathbb{N}$. We then have a natural embedding $* : \mathbb{R} \to {}^*\mathbb{R}$ by setting ${}^*r = \mathbf{cl}(\mathbf{r})$ for all $r \in \mathbb{R}$.

Example 3.12.1. Let ${}^*\mathbf{1}(n) : \mathbb{N} \to {}^*\mathbb{R}$ be the constant ${}^*\mathbb{R}$-valued function with value ^{*1}, i.e., ${}^*\mathbf{1}(n) = {}^*1$, for all $n \in \mathbb{N}$ and ${}^*\mathbf{1}(n) = {}^*0$, for all $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$.
The ω-sum $Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n) \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.3.1.
Let ${}^*\mathbf{1}^{\#}(n) : \mathbb{N} \to {}^*\mathbb{R}$ be the constant ${}^*\mathbb{R}$-valued function with value ^{*1}, i.e., ${}^*\mathbf{1}^{\#}(n) = {}^*1$, for all $n \in \mathbb{N}^{\#}$. The hyperfinite sum

$$Ext\text{-}\sum_{n=1}^{\nu} {}^*\mathbf{1}^{\#}(n) \in {}^*\mathbb{R}\backslash\mathbb{R}, \nu \in \mathbb{N}^{\#}\backslash\mathbb{N} \tag{3.12.6}$$

exists for all $\nu \in \mathbb{N}^{\#}\backslash\mathbb{N}$ by Theorem 3.3.1.
We denote the value of ω-sum $Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n)$ by $\widehat{\omega}$. Note that

$$\widehat{\omega} \neq \mathbf{cl}(1,2,\ldots,n,\ldots) = \widetilde{\omega}, \tag{3.12.7}$$

since $\widehat{\omega} = Ext\text{-}\sum_{n\in\mathbb{N}} {}^*\mathbf{1}(n) = Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}(n) < Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}^{\#}(n) = \widetilde{\omega}$. Note that the inequality $Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}(n) < Ext\text{-}\sum_{n=1}^{\widetilde{\omega}} {}^*\mathbf{1}^{\#}(n)$ holds by Theorem 3.11.2.

Example 3.12.2. The ω-sum $Ext\text{-}\sum_{n=1}^{\omega} \frac{1}{n} \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.3.1, however $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Theorem 3.12.1. Let $Ext\text{-}\sum_{n=k}^{\omega} a_n = A$ and $Ext\text{-}\sum_{n=k}^{\omega} b_n = B,$ where $A, B, C \in {}^*\mathbb{R}$. Then

(1)

$$Ext\text{-}\sum_{n=k}^{\omega} C \times a_n = C \times \left(Ext\text{-}\sum_{n=k}^{\omega} a_n \right) \tag{3.12.8}$$

(2)

$$Ext\text{-}\sum_{n=k}^{\omega} (a_n \pm b_n) = A \pm B. \tag{3.12.9}$$

(3)

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{\omega} a_i \tag{3.12.10}$$

Proof. It follows directly from Theorem 3.11.1 by Definition 3.12.1.
Example 3.11.2. Consider the ω-sum

$$S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega} r^n, -1 < r < 1. \tag{3.12.11}$$

The ω-sum $Ext\text{-}\sum_{n=0}^{\omega} {}^*r^{-n} \in {}^*\mathbb{R} \backslash \mathbb{R}$ exists by Theorem 3.3.1.It follows from (3.12.11)

$$S_\omega(r) = 1 + Ext\text{-}\sum_{n=1}^{\omega} r^n = 1 + r\left(Ext\text{-}\sum_{n=0}^{\omega} r^n \right) = 1 + rS_\omega(r) \tag{3.12.12}$$

Thus

$$S_\omega(r) \equiv \frac{1}{1-r}. \tag{3.12.13}$$

Remark 3.12.3. Note that for $|r| < 1$

$$S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega} r^n = S_\infty(r) = \sum_{n=0}^{\infty} r^n \tag{3.12.14}$$

since as we know for $|r| < 1$

$$S_\infty(r) = \lim_{n\to\infty} \sum_{n=0}^{n} r^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{3.12.15}$$

Definition 3.12.2.[5]. An element $x \in {}^*\mathbb{R}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q}, r > 0$.
Abbreviation 3.12.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ if x is finite.
Remark 3.12.4.[5]. Let $x \in \mathbb{Q}^\#$ be finite. Let D_1, be the set of $r \in \mathbb{Q}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{Q}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}_d$. A simple argument shows that $|x - r_0|$ is infinitesimal,i.e., $|x - r_0| \approx 0$.
Definition 3.12.3.[5].This unique r_0 is called the standard part of x and is denoted by

$$^\circ x. \tag{3.12.16}$$

Definition 3.12.4. An element $x \in {}^*\mathbb{R}_{\mathbf{fin}}$ is called standard if

$$x = {}^\circ x. \tag{3.12.17}$$

Abbreviation 3.12.2.For $x \in {}^*\mathbb{R}$ we abbreviate $x \in {}^*\mathbb{R}_{\mathbf{st}}$ if x is standard.
Theorem 3.12.4.[5]. If $x \in \mathbb{R}$, then ${}^\circ x = x$; if $x, y \in {}^*\mathbb{R}_{\mathbf{fin}}$ are both finite, then

$$^{\circ}(x+y) = {}^{\circ}(x) + {}^{\circ}(y), {}^{\circ}(x-y) = {}^{\circ}(x) - {}^{\circ}(y). \quad (3.12.18)$$

Definition 3.12.5.Let $\{a_i\}_{i=0}^{\infty}$ be countable ${}^*\mathbb{R}_{\mathbf{fin}}$-valued sequence. We say that a sequence $\{a_i\}_{i=0}^{\infty}$ converges to the standard limit $a \in {}^*\mathbb{R}_{\mathbf{fin}}$ and abbreviate $a = \text{st-}\lim_{i\to\infty} a_i$ if for every $\epsilon > 0, \epsilon \not\approx 0$ there is an integer $N \in \mathbb{N}$ such that $|a_i - a| < \epsilon$ if $i \geq N$.

Theorem 3.12.5. Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be a hyperfinite ${}^*\mathbb{R}_{\mathbf{fin}}$-valued sequence such that: (i) ${}^{\circ}a_i = a_i$ for any $i \leq n$ and (ii) for any $m \leq n$: $Ext\text{-}\sum_{i=0}^{m}|a_i| < \mu \in {}^*\mathbb{R}_{\mathbf{fin}}$, then

$$^{\circ}\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} a_i. \quad (3.12.19)$$

Proof. From Eq.(3.12.18) by the condition (ii) and hyper infinite induction we get

$$^{\circ}\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} {}^{\circ}a_i. \quad (3.12.20)$$

From Eq.(3.12.20) by the condition (i) we obtain Eq.(3.12.19).

Theorem 3.12.6. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^{\circ}a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that: (i) $\sum_{i=0}^{\infty}|a_i| < \infty$ and therefore there exists $\text{st-}\lim_{m\to\infty} \sum_{i=0}^{m} a_i = \sum_{i=0}^{\infty} a_i$; (ii) $Ext\text{-}\sum_{i=0}^{\omega}|a_i| < \infty$ and (iii) $\text{st-}\lim_{k\to\infty} \sum_{i=k}^{\omega}|a_i| = 0$. Then

$$^{\circ}\left(Ext\text{-}\sum_{i=0}^{\omega} a_i\right) \equiv Ext\text{-}\sum_{i=0}^{\omega} a_i. \quad (3.12.21)$$

and

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i. \quad (3.12.22)$$

From (3.12.22) it follows directly

$$\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\omega} a_i\right) = 0 \quad (3.12.22')$$

Proof. The Eq.(3.12.21) follows directly from Eq.(3.12.19) and Definition 3.12.1.
From the Eq.(3.12.10) we get

$$Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i = Ext\text{-}\sum_{i=k}^{\omega} a_i. \quad (3.12.23)$$

From the Eq.(3.12.23)

$$\left|Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i\right| = \left|Ext\text{-}\sum_{i=k}^{\omega} a_i\right| \leq \sum_{i=k}^{\omega}|a_i|. \quad (3.12.24)$$

From the Eq.(3.12.24) by condition (ii) we get

$$\text{st-}\lim_{k\to\infty}\left|Ext\text{-}\sum_{i=0}^{\omega} a_i - \sum_{i=0}^{k} a_i\right| \leq \text{st-}\lim_{k\to\infty} \sum_{i=k}^{\omega}|a_i| = 0. \quad (3.12.25)$$

It follows from the Eq.(3.12.25)

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \text{st-}\lim_{k\to\infty} \sum_{i=0}^{k} a_i = \sum_{i=0}^{\infty} a_i \quad (3.12.26)$$

and therefore the equality (3.12.22) also holds. Assum that the equality (3.12.22) holds. Then from (3.12.22) one obtains for any $m \in \mathbb{N}$

$$Ext\text{-}\sum_{i=m}^{\omega} a_i = \sum_{i=m}^{\infty} a_i \qquad (3.12.26')$$

and therefore

$$\text{st-}\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\omega} a_i\right) = \text{st-}\lim_{m\to\infty}\sum_{i=m}^{\infty} a_i = 0.$$

Example 3.12.2. Let $\rho : \mathbb{N} \to {}^*\mathbb{R}$ be the ${}^*\mathbb{R}$-valued function such that $\rho(n) = {}^*r^n$, $|r| < 1$, for all $n \in \mathbb{N}$ and $\rho(n) = {}^*0$, for all $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. The ω-sum $S_\omega(r) = Ext\text{-}\sum_{n=0}^{\omega}{}^*r^n \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 3.3.1 and by Theorem 3.12.6 we obtain $S_\omega(r) = S_\infty(r) = \sum_{n=0}^{\infty} r^n = (1-r)^{-1}$ the same result as obtained above by direct calculation (3.12.14), see Remark 3.12.3.

Remark 3.12.4. Note that in general case the conditions (i) $\sum_{i=0}^{\infty}|a_i| < \infty$ and (ii) $Ext\text{-}\sum_{i=0}^{\omega}|a_i| < \infty$ are not imply the condition (iii), but without condition (iii) the equality (3.12.22) obviously is not holds.

Theorem 3.12.7. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^\circ a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that:(i) $a_i > 0$ for $i \geq m$ and (ii) st-$\overline{\lim}_{i\to\infty}\dfrac{a_{n+1}}{a_n} < {}^*1$. Then st-$\lim_{k\to\infty}\sum_{i=k}^{\omega}|a_i| = 0$ and therefore

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i\ . \qquad (3.12.27)$$

Proof. Note that if st-$\overline{\lim}_{i\to\infty}(a_{n+1}/a_n) < {}^*1$, there is a number $r \in {}^*\mathbb{R}_{\mathbf{st}}$ such that $0 < r < {}^*1$ and $a_{n+1}/a_n \leq r$ for $n \geq N$. Thus we obtain $a_{N+1} \leq ra_N, a_{N+2} \leq ra_{N+1} \leq r^2a_N$, $\ldots,\ a_{N+k} \leq r^k a_N, \ldots$ and therefore

$$Ext\text{-}\sum_{i=N+k}^{\omega} a_i \leq Ext\text{-}\sum_{i=k}^{\omega} r^i a_N = r^k a_N\left(Ext\text{-}\sum_{i=0}^{\omega} r^i\right) = \frac{r^k a_N}{1-r}. \qquad (3.12.28)$$

It follows from (3.12.22) st-$\lim_{k\to\infty}\left(Ext\text{-}\sum_{i=N+k}^{\omega} a_i\right)$ =st-$\lim_{k\to\infty}\dfrac{r^k a_N}{1-r} = 0$ and by Theorem 3.12.6 the equality (3.12.27) holds.

Theorem 3.12.8. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e., ${}^\circ a_i = a_i \in {}^*\mathbb{R}_{\mathbf{st}}$ for any $i \in \mathbb{N}$. Assume that:(i) $a_i > 0$ for $i \geq m$ and (ii) st-$\overline{\lim}_{i\to\infty}\left(a_i^{1/i}\right) < {}^*1$. Then st-$\lim_{k\to\infty}\sum_{i=k}^{\omega}|a_i| = 0$ and therefore

$$Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i\ . \qquad (3.12.29)$$

Theorem 3.12.9. Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be ${}^*\mathbb{R}$- valued hyperfinite sequences such that $Ext\text{-}\sum_{i=1}^{n} a_i^2 = A \in {}^*\mathbb{R}_{\mathbf{fin}}$ and $Ext\text{-}\sum_{i=1}^{n} b_i^2 = B \in {}^*\mathbb{R}_{\mathbf{fin}}$. Then the following inequality holds

$$\left(Ext\text{-}\sum_{i=1}^{n}a_ib_i\right)^2 \leq \left(Ext\text{-}\sum_{i=1}^{n}a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{n}b_i^2\right). \tag{3.12.30}$$

Proof. The inequality can be proved using only elementary algebra in this case. Consider

the following quadratic polynomial in $x \in {}^*\mathbb{R}$

$$0 \leq Ext\text{-}\sum_{i=1}^{n}(a_ix+b_i)^2 = \left(Ext\text{-}\sum_{i=1}^{n}a_i^2\right)x^2 + 2\left(Ext\text{-}\sum_{i=1}^{n}a_ib_i\right) + Ext\text{-}\sum_{i=1}^{n}b_i^2 \tag{3.12.31}$$

Since this polynomial is nonnegative, it has at most one real root for x, hence its discriminant is less than or equal to zero. That is,

$$\left(Ext\text{-}\sum_{i=1}^{n}a_ib_i\right)^2 - \left(Ext\text{-}\sum_{i=1}^{n}a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{n}b_i^2\right) \leq 0. \tag{3.12.32}$$

which yields (3.12.30).

Theorem 3.12.10. Let $\{a_i\}_{i=1}^{\infty}$and $\{b_i\}_{i=1}^{\infty}$be ${}^*\mathbb{R}$- valued countable sequences such that $Ext\text{-}\sum_{i=1}^{\omega}a_i^2 = A \in {}^*\mathbb{R}_{\mathbf{fin}}$and $Ext\text{-}\sum_{i=1}^{\omega}b_i^2 = B \in {}^*\mathbb{R}_{\mathbf{fin}}$.Then the following inequality holds

$$\left(Ext\text{-}\sum_{i=1}^{\omega}a_ib_i\right)^2 \leq \left(Ext\text{-}\sum_{i=1}^{\omega}a_i^2\right)\left(Ext\text{-}\sum_{i=1}^{\omega}b_i^2\right). \tag{3.12.33}$$

Proof.It follows from Theorem 3.12.9 by Definition 3.12.1.

4.External hyperfinite matrices and determinants

4.1. Definitions and notations

A rectangular external hyperfinite array of ordered elements wich is hyperreal numbers from external field $\mathbb{R}_c^{\#}$ or field $\mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$, is known as hyperfinite $\mathbb{R}_c^{\#}$-valued (or $\mathbb{C}_c^{\#}$-valued) matrix.

The literal form of a hyperfinite external matrix in general is written symbolically as

$$\left\|\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right\| \tag{4.1.1}$$

where $a_{ij} \in {}^*\mathbb{R}_c^{\#}; 1 \leq i \leq m, 1 \leq j \leq n; m \in \mathbb{N}^{\#}\backslash\mathbb{N}$.

We use boldface type to represent a matrix, and we enclose the array itself in square brackets. The horizontal lines are called rows and the vertical lines are called columns. Each element is associated with its location in the matrix. Thus the element a_{ij} is defined as the element located in the i-th row and the j-th column. Using this notation, we may also use the notation $[a_{ij}]_{m\times n}$ to identify a matrix of order $m \times n$, i.e. a matrix having m rows (the number of rows is given first) and n columns. Some frequently used

matrices have special names. A matrix of one column but any number of rows is known

as a column matrix or a column vector. Frequently, for such a matrix, only a single

subscript is used for the elements of the array. Another type of matrix which is given a special name is one which contains only a single row. This is called a row matrix, or a row vector. A matrix which has the same number of rows and columns, i.e. $m = n$, is a square matrix of order $(n \times n)$ or just of order $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. The main or prin-ciple diagonal of a square matrix consists of the elements $a_{11}, a_{22}, \ldots, a_{nn}$. A square matrix in which all
elements except those of the principal diagonal are zero is known as a diagonal matrix.
If, in addition, all elements of a diagonal matrix are unity, the matrix is known as a unit or identity matrix, denotet by $\mathbf{U}$ or $\mathbf{1}$. If all elements of a matrix are zero, $a_{ij} = 0$, the matrix is called a zero matrix, $\mathbf{0}$. A subclass of a square matrix which is frequently encountered in circuit analysis is a symmetric matrix. The elements of such a matrix satisfy the equality $a_{ij} = a_{ji}$ for all values of i and j, or in other words, this matrix is symmetrical about the main diagonal.
Let $\mathbf{A}_{\omega} = [a_{ij}]$ be a countable matrix, where $a_{ij} \in {}^{*}\mathbb{R}_{c}^{\#}; i,j \in \mathbb{N}$. The literal form of a countable matrix in general is written symbolically as

$$\mathbf{A}_{\omega} = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \\ \bullet & \bullet & \bullet & \bullet & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots \\ \bullet & \bullet & \cdots & \bullet & \bullet \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \cdots \\ \bullet & \bullet & \cdots & \bullet & \cdots \end{matrix} \right\| \tag{4.1.2}$$

Remark 4.1. Note there is canonical embeding $\mathbf{A}_{\omega} \hookrightarrow \mathbf{A}_{\omega,n}$, where $\mathbf{A}_{\omega,n}$ is hyperfinite external matrix of the following literal form

$$\mathbf{A}_{\omega,n} = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots & 0 & 0 & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots & 0 & 0 & \cdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & 0 & 0 & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & 0 & 0 & \cdots \\ \bullet & \bullet & \cdots & \bullet & \cdots & 0 & 0 & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \cdots & 0 & 0 & \cdots \\ \bullet & \bullet & \cdots & \bullet & \cdots & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\| \tag{4.1.3}$$

where $a_{nn} = 1$ for all $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ and $a_{mn} = 0$ for all $m \neq n$, $m,n \in \mathbb{N}^{\#}\backslash\mathbb{N}$.

1.2. Matrix equality

Two matrixes are equal if and only if (1) they are of the same order, and (2) each element of one matrix is equal to its associated (placed in the row of the same number and the colunm of the same number) element in the other matrix. Thus, for two matrices,

$\mathbf{A}$ and $\mathbf{B}$, of the same order and with elements a_{ij} and b_{ij} respectively, if $\mathbf{A} = \mathbf{B}$, then all

the elements have to be equal, i.e. $a_{ij} = b_{ij}$ for all values of i and j.

4.2.Addition and subtraction of external hyperfinite matrices.

If two external hyperfinite matrices $\mathbf{A}$ and $\mathbf{B}$ are of the same order, i.e. have the same hyperfinite number of rows and the same hyperfinite number of columns, we may determine their sum by adding the corresponding elements. Thus if the elements of $\mathbf{A}$ are a_{ij} and those of $\mathbf{B}$ are b_{ij}, then the elements of the resulting matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ are

$$c_{ij} = a_{ij} + b_{ij} \tag{4.2.1}$$

Clearly $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for hyperfinite matrices.Subtraction is similarly defined, i.e. $\mathbf{C} = \mathbf{A} - \mathbf{B}$ are

$$c_{ij} = a_{ij} - b_{ij}. \tag{4.2.2}$$

4.3. Multiplication by a scalar

The multiplication of external hyperfinite matrix $\mathbf{A}$ by a scalar $k \in \mathbb{R}_c^{\#}$ or $k \in \mathbb{C}_c^{\#}$ means that every element of the matrix $\mathbf{A}$ is multiplied by the scalar. Thus, if k is a scalar and $\mathbf{A}$ is external hyperfinite matrix with elements a_{ij}, the elements of the matrix $k\mathbf{A}$ are ka_{ij} :

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} \tag{4.3.1}$$

4.4.Multiplication of the external hyperfinite matrices.

For the case where $\mathbf{A}$ is an n-th-order square matrix and $\mathbf{Y}$ and $\mathbf{X}$ are column matrices with n rows, the elements of the resulting matrix $\mathbf{Y} = \mathbf{AX}$ is defined by the relation

$$y_i = Ext\text{-}\sum_{k=1}^{n} a_{ik}x_k, \tag{4.4.1}$$

where $1 \leq i \leq n$.

The multiplication of two external hyperfinite matrices $\mathbf{A}$ and $\mathbf{B}$ is defined only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is of order $(m \times n)$ and B is of order $(n \times p)$ (such a pair of matrices is said to be conform able for multiplication), then the product $\mathbf{A} \cdot \mathbf{B}$ is a matrix $\mathbf{C}$ of order $(m \times p)$

$$\mathbf{A}_{(m\times n)} \cdot \mathbf{B}_{(m\times p)} = \mathbf{C}_{(m\times p)} \tag{4.4.2}$$

The elements of $\mathbf{C}$ are found from the elements of $\mathbf{A}$ and $\mathbf{B}$ by multiplying the i-th row elements of $\mathbf{A}$ and the corresponding j-th column elements of $\mathbf{B}$ and summing these products to give c_{ij}

$$c_{ij} = Ext\text{-}\sum_{k=1}^{n} a_{ik}b_{kj}, \tag{4.4.3}$$

where $1 \leq i \leq m, 1 \leq j \leq p$.

4.5.The Determinant of the external hyperfinite matrices.

Suppose we are given a square hyperfinite matrix $\mathbf{A}$, i.e., an array of n^2 hyper real numbers

$$\mathbf{A} = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{matrix} \right\| \tag{4.5.1}$$

where $a_{ij} \in {}^*\mathbb{R}_c^\#; 1 \leq i \leq n, 1 \leq j \leq n, n \in \mathbb{N}^\# \backslash \mathbb{N}$. The number of rows and columns of the matrix (4.5.1) is called its order. The numbers a_{ij} are called the elements of the matrix. The first index indicates the row and the second index the column in which a_{ij} appears. The elements $a_{ii}, 1 \leq i \leq n$ form the principal diagonal of the matrix.
Consider any product of n elements which appear in different rows and different columns of the matrix (4.5.1), i.e., a product containing jast one element from each row and each column. Such external product can be written in the form

$$Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m} = Ext\text{-}(a_{\alpha_1 1} \times a_{\alpha_2 2} \times \ldots \times a_{\alpha_n n}). \tag{4.5.2}$$

Actually, for the first factor we can always choose the element appearing in the first column of the matrix (4.5.1); then, if we denote by oq the number of the row in which the element appears, the indices of the element will be $\alpha_1, 1$. Similarly, for the second factor we can choose the element appearing in the second column; then its indices will be $\alpha_2, 2$, where α_2 is the number of the row in which the element appears, and so on. Thus, the indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the numbers of the rows in which the factors of the product (4.5.2) appear, when we agree to write the column indices in increasing order.
Definition 4.5.1. A function F is said to be a permutation of a set S if it is one-to-one and $\mathbf{dom}(F) = \mathbf{range}(F) = S$.
Definition 4.5.2. Let $[1, n]$ a set $\{k | k \in \mathbb{N}^\# \wedge (1 \leq k \leq n)\}$.
Since, by hypothesis, the elements $a_{\alpha_1 1}, a_{\alpha_2 2}, \ldots, a_{\alpha_n n}$ appear in different rows of the matrix (4.5.1), one from each row, then the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all different and represent some permutation of the set $[1, n]$. By an inversion in the sequence $\{\alpha_i\}_{i=1}^{n}$, we mean an arrangement of two indices such that the larger index comes before the smaller index. The total number of inversions will be denoted by

$$\pi(\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{4.5.3}$$

If the number of inversions in the sequence $\{\alpha_i\}_{i=1}^{n}$ is even, we put a plus sign before the product (4.5.2); if the number is odd, we put a minus sign before the product.
In other words, we agree to write in front of each product of the form (4.5.2) the sign determined by the expression

$$(-1)^{\pi(\alpha_1, \alpha_2, \ldots, \alpha_n)}. \tag{4.5.4}$$

The total number of products of the form (4.5.2) which can be formed from the elements of a given matrix of order n is equal to the total number of permutations of the set $[1,n]$. As is well known, this number is equal to $n!$.

Definition 4.5.3. By the determinant $\mathbf{D}$ of the matrix (4.5.1) is meant the external sum of the $n!$ products of the form (4.5.2), each preceded by the sign determined by the rule just given, i.e.,

$$\mathbf{D} = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}\left(Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m}\right) = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}(Ext\text{-}(a_{\alpha_1 1} \times a_{\alpha_2 2} \times\ldots\times a_{\alpha_n n})). \quad (4.5.5)$$

Henceforth, the products of the form (4.5.2) will be called the terms of the determinant $\mathbf{D}$. The elements a_{ij} of the matrix (4.5.1) will be called the elements of $\mathbf{D}$ and the order of (4.5.1) will be called the order of $\mathbf{D}$. We denote the determinant $\mathbf{D}$ corresponding to the matrix (4.5.1) by one of the following symbols:

$$\mathbf{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{vmatrix} = \det\|a_{ij}\| = |\mathbf{A}|. \quad (4.5.6)$$

4.6.Determinant and Cofactors.

General procedure for evaluating the determinants of any order is by expanding determinant in terms of a row or column, which is called Laplaces' expansion. If such an expansion is made along the i-th row of an array, it has the following form

$$|\mathbf{A}| = Ext\text{-}\sum a_{ik}A_{ik}, \quad (4.6.1)$$

where all a_{ik} are the elements of $\mathbf{A}$ and all A_{ik} are cofactors. These cofactors are formed by deleting the i-th row and k-th column of the array (so that the remaining elements form a determinant, called minor, $\mathbf{M}$, which is of order one less than $|\mathbf{A}|$) and prefixing the result by the multiplier $(-1)^{i+k}$, which predetermines the sign of the minor.

4.7.The transposition of the external hyperfinite matrix

Let $\mathbf{A}^t$ be a hyperfinite matrix

$$\mathbf{A}^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \quad (4.7.1)$$

is obtained from a hyperfinite matrix (4.7.2) by interchanging rows an columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (4.7.2)$$

The determinant $|\mathbf{A}^t|$ obtained from the determinant $|\mathbf{A}|$ by interchanging rows and columns with the same indices is said to be the transpose of the determinant $|\mathbf{A}|$. We now show that the transpose of a determinant has the same value as the original determinant. In fact, the determinants $|\mathbf{A}|$ and $|\mathbf{A}^t|$ obviously consist of the same terms; therefore it is enough for us to show that identical terms in the determinants $|\mathbf{A}|$ and $|\mathbf{A}^t|$ have identical signs. Transposition of the matrix of a determinant is clearly the result of rotating it (in space) through 180° about the principal diagonal an, $a_{22}, \ldots, a_{nn}$. As a result of this rotation, every segment with negative slope (e.g., making an angle $\alpha < 90^\circ$ with the rows of the matrix) again becomes a segment with negative slope (i.e., making the angle 90° - a with the rows of the matrix). Therefore the number of segments with negative slope joining the elements of a given term does not change after transposition.

Consequently the sign of the term does not change either. Thus the signs of all the terms are preserved, which means that the value of the determinant remains unchanged.

The property just proved establishes the equivalence of the rows and columns of a determinant. Therefore further properties of determinants will be stated and proved only for columns.

4.8.The antisymmetry property.

By the property of being antisymmetric with respect to columns, we mean the fact that a determinant changes sign when two of its columns are interchanged. We consider first the case where two adjacent columns are interchanged, for example columns j and $j+1$. The determinant which is obtained after these columns are interchanged obviously still consists of the same terms as the original determinant. Consider any of the terms of the original determinant. Such a term contains an element of the j-th column and an element of the $(j+1)$-th column. If the segment joining these two elements originally had negative slope, then after the interchange of columns, its slope becomes positive, and conversely. As for the other segments joining pairs of elements of the term in question, each of these segments does not change the character of its slope after the column interchange. Consequently the number of segments with negative slope joining the elements of the given term changes by one when the two columns are interchanged; therefore each term of the determinant, and hence the determinant itself, changes sign when the columns are interchanged.

Suppose now that two nonadjacent columns are interchanged, e.g., column j and column k with $j < k$, where there are hyper finitely many $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ other columns between. This interchange can be accomplished inductively by successive interchanges of adjacent columns as follows:
First column j is interchanged with column $j+1$, then with columns $j+2, j+3, \ldots, k$. Then the column $k-1$ so obtained (which was formerly column k) is interchanged with columns $k-2, k-3, \ldots, j$. In all, $m+1+m = 2m+1$ interchanges of adjacent columns are required, each of which, according to what has just been proved, changes the sign of the determinant. Therefore, at the end of the process, the determinant will have a sign opposite to its original sign (since for any hyperinteger $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$, the number $2m+1$ is odd).
Remark 4.8.1. Note that the process mention above is well defined by hyperfinite induction axiom [2]-[4].
Corollary 4.8.1. A hyperfinite determinant with two identical columns vanishes.
Proof. Interchanging the columns does not change the determinant $\mathbf{D}$. On the other hand, as just proved, the determinant must change its sign. Thus $\mathbf{D} = -\mathbf{D}$, which implies that $\mathbf{D} = 0$.

4.9. The linear properties of determinant

This property can be formulated as follows:
Theorem 4.9.1. If all the elements of the j-th column of a determinant $\mathbf{D}$ are linear combinations of two columns of numbers, i.e., if

$$a_{ij} = b_i + c_i, 1 \leq i \leq n, \tag{4.9.1}$$

where $\lambda, \mu \in \mathbb{R}_c^{\#}$ or $\lambda, \mu \in \mathbb{C}_c^{\#}$ are fixed numbers, then D is equal to a linear combination of two determinants

$$\mathbf{D} = \lambda \mathbf{D}_1 + \mu \mathbf{D}_2 \tag{4.9.2}$$

Here both determinants $\mathbf{D}_1$ and $\mathbf{D}_2$ have the same columns as the determinant $\mathbf{D}$ except for the j-th column', the j-th column of $\mathbf{D}_1$ consists of the numbers b_i, while the j-th column of $\mathbf{D}_2$ consists of the numbers c_i.
Proof. Every term of the determinant D can be represented in the form

$$\begin{gathered}\mathbf{D} = \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i} \right) a_{\alpha_j j} \left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i} \right) = \\ \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i} \right) (\lambda b_{\alpha_j j} + \mu c_{\alpha_j j}) \left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i} \right) = \\ \lambda \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i} \right) b_{\alpha_j j} \left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i} \right) + \mu \left(Ext\text{-}\prod_{i=1}^{j-1} a_{\alpha_i i} \right) c_{\alpha_j j} \left(Ext\text{-}\prod_{i=j+1}^{n} a_{\alpha_i i} \right).\end{gathered} \tag{4.9.3}$$

Adding up all the first terms (with the signs which the corresponding terms have in the original determinant), we clearly obtain the determinant $\mathbf{D}_1$, multiplied by the number λ.
Similarly, adding up all the second terms, we obtain the determinant $\mathbf{D}_2$, multiplied by the number μ.

Remark 4.9.1. It is convenient to write this formula in a somewhat different form. Let $\mathbf{D}$ be an arbitrary fixed determinant. Denote by $\mathbf{D}_j(p_i)$ the determinant which is obtained by replacing the elements of the j-th column of $\mathbf{D}$ by the numbers $p_i, 1 \leq i \leq n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Then Eq.(4.9.2) takes the form

$$\mathbf{D}_j(\lambda b_i + \mu c_i) = \lambda \mathbf{D}_j(b_i) + \mu \mathbf{D}_j(c_i) \quad (4.9.4)$$

The linear property of determinants can be extended to the case where every element of the j-th column is a linear combination not of two terms but of any other number of terms, i.e.

$$a_{ij} = Ext\text{-}\sum_{k=1}^{r} \lambda_k b_i^k. \quad (4.9.5)$$

In this case

$$\mathbf{D}_j(a_{ij}) = \mathbf{D}_j\left(Ext\text{-}\sum_{k=1}^{r} \lambda_k b_i^k\right) = Ext\text{-}\sum_{k=1}^{r} \lambda_k \mathbf{D}_j(b_i^k). \quad (4.9.6)$$

Corollary 4.9.1. Any common factor of a column of a determinant can be factored out of the determinant.

Proof. If $a_{ij} = \lambda b_i,$ then by (4.9.6) we have

$\mathbf{D}_j(a_{ij}) = \mathbf{D}_j(\lambda b_i) = \lambda \mathbf{D}_j(b_i).$

Corollary 4.9.2. If a column of a determinant consists entirely of zeros, then the determinant vanishes.

Proof. Since 0 is a common factor of the elements of one of the columns, we can factor it out of the determinant, obtaining $D_j(0) = D_j(0 \cdot 1) = 0 \cdot D_j(1)$.

4.10.Addition of an arbitrary multiple of one column to another column.

Theorem 4.10.1. The value of a determinant is not changed by adding the elements of one column multiplied by an arbitrary number to the corresponding elements of another column.

Proof. Suppose we add the /cth column multiplied by the number λ to the j-th column $(\kappa \neq j)$. The j-th column of the resulting determinant consists of elements of the form $a_{ij} + \lambda a_{ik}, 1 \leq i \leq n.$ By (4.9.2) we have $\mathbf{D}_j(a_{ij} + \lambda a_{ik}) = \mathbf{D}_j(a_{ij}) + \lambda \mathbf{D}_j(a_{ik})$.

The j-th column of the second determinant consists of the elements aik, and hence is identical with the /cth column. It follows from Corollary 3.8.1 that $\mathbf{D}_j(a_{ik}) = 0,$ so that $\mathbf{D}_j(a_{ij} + \lambda a_{ik}) = \mathbf{D}_j(a_{ij})$.

Remark 4.10.1.Theorem 4.10.1 can be formulated in the following more general form: The value of a determinant is not changed by adding to the elements of its j-th column first the corresponding elements of the k-th column multiplied by λ, next the elements of the l-th column multiplied by μ, etc., and finally the elements of the p-th column multiplied by τ $(\kappa \neq j, l \neq j, \ldots, p \neq j)$.

Remark 4.10.2.Because of the invariance of determinants under transposition, all the properties of determinants proved above for columns remain valid for rows as well.

4.11.Cofactors and minors

Consider any column, the j-th say, of the determinant $\mathbf{D}$. Let a$_{ij}$ be any element of this column. Add up all the terms containing the element a_{ij} appearing in the right-hand side of equation (4.5.5),i.e.,

$$\mathbf{D} = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}\left(Ext\text{-}\prod_{m=1}^{n} a_{\alpha_m m}\right) = Ext\text{-}\sum(-1)^{\pi(\alpha_1,\alpha_2,\ldots,\alpha_n)}(Ext\text{-}(a_{\alpha_1 1}\times a_{\alpha_2 2}\times\ldots\times a_{\alpha_n n})). \quad (4.11.1)$$

and then factor out the element ais. The quantity which remains, denoted by A_{ij}, is called the cofactor of the element a_{ij} of the determinant $\mathbf{D}$. Since every term of the determinant $\mathbf{D}$ contains an element from the j-th column, (4.11.1) can be written in the form

$$Ext\text{-}\sum_{i=1}^{n} a_{ik}A_{ij} = Ext\text{-}\left(a_{1k}A_{1j} + a_{2k}A_{2j} + \ldots + a_{nk}A_{nj}\right) \quad (4.11.2)$$

called the expansion of the determinant $\mathbf{D}$ with respect to the (elements of the) j-th column. Naturally, we can write a similar formula for any row of the determinant $\mathbf{D}$. For example, for the ith row we have the formula

$$Ext\text{-}\sum_{j=1}^{n} a_{ij}A_{ij} = Ext\text{-}\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}\right). \quad (4.11.3)$$

Thus one obtains.

Theorem 4.11.1. The sum of all the products of the elements of any column (or row) of the determinant $\mathbf{D}$ with the corresponding cofactors is equal to the determinant $\mathbf{D}$ itself.

Remark 4.10.1.Equations (4.11.2) and (4.11.3) can be used to calculate determinants, but first we must know how to calculate cofactors.

Remark 4.10.2.Next we note a consequence of (4.11.2) and (4.11.3) which will be useful later. Equation (4.11.2) is an identity in the quantities $a_{1j}, a_{2j}, \ldots, a_{nj}$. Therefore it remains valid if we replace a_{ij} $(1 \leq i \leq n)$ by any other quantities. The quantities $A_{1j}, A_{2j}, \ldots, A_{nj}$ remain unchanged when such a replacement is made, since they do not depend on the elements ais. Suppose that in the right and left-hand sides of the equality (4.11.2) we replace the elements $a_{1j}, a_{2j}, \ldots, a_{nj}$ by the corresponding elements of any other column, say the k-th. Then the determinant in the left-hand side of (4.11.2) will have two identical columns and will therefore vanish, according to Corollary 4.8.1. Thus one obtains the relation

$$Ext\text{-}\sum_{i=1}^{n} a_{ik}A_{ij} = Ext\text{-}\left(a_{1k}A_{1j} + a_{2k}A_{2j} + \ldots + a_{nk}A_{nj}\right) = 0 \quad (4.11.4)$$

for $k \neq j$. Similarly from Eq.(4.11.3) one obtains the relation

$$Ext\text{-}\sum_{j=1}^{n} a_{lj}A_{ij} = Ext\text{-}(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{ln}A_{in}) = 0 \quad (4.11.5)$$

for $l \neq i$. Thus one obtains the following.

Theorem 4.11.2. The sum of all the products of the elements of a column (or row) of the determinant $\mathbf{D}$ with the cofactors of the corresponding elements of another column (or row) is equal to zero.

Remark 4.10.3.If we delete a row and a column from a matrix of hyperfinite order n, then, of course, the remaining elements form a hyperfinite matrix of order $n-1$. The determinant of this matrix is called a minor of the original n-th-order matrix (and also a minor of its determinant $\mathbf{D}$).

If we delete the j-th row and the j-th column of $\mathbf{D}$, then the minor so obtained is denoted by $\mathbf{M}_{ij}$ or $\mathbf{M}_{ij}(\mathbf{D})$.

We now show that the relation

$$A_{ij} = (-1)^{i+j}\mathbf{M}_{ij} \qquad (4.11.6)$$

holds, so that the calculation of cofactors reduces to the calculation of the corresponding minors. First we prove (4.11.6) for the case $i = 1, j = 1$. We add up all the terms in the right-hand side of (4.11.1) which contain the element a_{11}, and consider one of these terms. It is clear that the product of all the elements of this term except a_{11} gives a term c of the minor $\mathbf{M}_{11}$. Since in the matrix of the determinant $\mathbf{D}$, there are no segments of negative slope joining the element an with the other elements of the term selected, the sign ascribed to the term $a_{11}c$ of the determinant $\mathbf{D}$ is the same as the sign ascribed to the term c in the minor $\mathbf{M}_{11}$. Moreover, by suitably choosing a term of the determinant $\mathbf{D}$ containing a_{11} and then deleting a_{11}, we can obtain any term of the minor $\mathbf{M}_{11}$. Thus the algebraic hyperfinite external sum of all the terms of the determinant $\mathbf{D}$ containing a_{11}, with a_{11} deleted, equals the product $\mathbf{M}_{11}$.

But according to results obtained above, this sum is equal to the product A_{11}. Therefore, $A_{11} = \mathbf{M}_{11}$ as required.

Now we prove (4.11.6) by hyper infinite induction for arbitrary i and j, making essential use of the fact that the formula is valid for $i = j = 1$. Consider the element $a_{ii} = a$. appearing in the i-th row and the j-th column of the determinant $\mathbf{D}$. By successively interchanging adjacent rows and columns, we can move the element a over to the upper left-hand corner of the matrix; to do this, we need $i-1+j-1 = i+j-2$ hyper interchanges. As a result, we obtain the determinant $\mathbf{D}_1$ with the same terms as those of the original determinant $\mathbf{D}$ multiplied by $(-1)^{i+j-2} = (-1)^{i+j}$.

The minor $\mathbf{M}_{11}(\mathbf{D}_1)$ of the determinant $\mathbf{D}_1$ is clearly identical with the minor $\mathbf{M}_{ij}(\mathbf{D})$ of the determinant $\mathbf{D}$. By what has been proved already, the sum of the terms of the determinant $\mathbf{D}_1$ which contain the element a, with a deleted, is equal to $\mathbf{M}_{11}(\mathbf{D}_1)$. Therefore the sum of the terms of the original determinant $\mathbf{D}$ which contain the element $a_{ij} = a$, with a deleted, is equal to

$$(-1)^{i+j}\mathbf{M}_{11}(\mathbf{D}_1) = (-1)^{i+j}\mathbf{M}_{ij}(\mathbf{D}). \qquad (4.11.7)$$

According to results obtained above, this sum is equal to A_{ij}. Consequently $A_{ij} = (-1)^{i+j}\mathbf{M}_{ij}$, which completes the proof of (4.11.6).

Theorem 4.11.3. Formulas (4.11.2) and (4.11.3) can now be written in the following form

$$\mathbf{D} = Ext\text{-}\sum_{k=1}^{n} Ext\text{-}(-1)^{k+j}a_{kj}\mathbf{M}_{kj} = Ext\text{-}\left((-1)^{1+j}a_{1j}\mathbf{M}_{1j} + (-1)^{2+j}a_{2j}\mathbf{M}_{2j} + \ldots + (-1)^{n+j}a_{nj}\mathbf{M}_{nj}\right) \qquad (4.11.8)$$

and

$$\mathbf{D} = Ext\text{-}\sum_{k=1}^{n}(-1)^{i+k}a_{ik}\mathbf{M}_{ik} = Ext\text{-}\left((-1)^{i+1}a_{i1}\mathbf{M}_{i1} + (-1)^{i+2}a_{i2}\mathbf{M}_{i2} + \ldots + (-1)^{i+n}a_{in}\mathbf{M}_{in}\right). \qquad (4.11.9)$$

Example 4.10.1.An hyperfinite n-th-order determinant

$$\mathbf{D}_n = \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \qquad (4.11.10)$$

is called triangular. Expanding $\mathbf{D}_n$ with respect to the first row, we find that $\mathbf{D}_n$ equals the product of the element a_{11} with the triangular determinant

$$\mathbf{D}_{n-1} = \begin{vmatrix} a_{22} & 0 & 0 & \cdots & 0 \\ a_{32} & a_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & 0 \\ a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix} \qquad (4.11.11)$$

of the order $n-1$. Again expanding $\mathbf{D}_{n-1}$ with respect to the first row, we find that

$$\mathbf{D}_{n-1} = a_{22}\mathbf{D}_{n-2}, \qquad (4.11.12)$$

where $\mathbf{D}_{n-2}$ is triangular determinant of the order $n-2$. By hyper infinite induction finally we obtain

$$\mathbf{D}_n = Ext\text{-}\prod_{i=1}^{n} a_{ii}. \qquad (4.11.13)$$

4.12. Generalized Cramer's Rule for hyperfinite system.

We are now can to solve external hyperfinite systems of linear equations.
First we consider hyperfinite system of the special form

$$\begin{aligned} Ext\text{-}\sum_{i=1}^{n} a_{1i}x_i &= b_1, \\ Ext\text{-}\sum_{i=1}^{n} a_{2i}x_i &= b_2, \\ &\cdots\cdots\cdots\cdots\cdots \\ Ext\text{-}\sum_{i=1}^{n} a_{ni}x_i &= b_n. \end{aligned} \qquad (4.12.1)$$

i.e., a system which has the same number of unknowns and equations. The coefficients a_{ij} $(i,j = 1,2,\ldots,n)$ form the coefficient matrix of the system; we assume that the determinant of this matrix is different from zero. We now show that such a system is always compatible and determinate, and we obtain a formula which gives the unique solution of the system.
We begin by assuming that $c_1, c_2, \ldots, c_n$ is a solution of (4.12.1),so that

$$Ext\text{-}\sum_{i=1}^{n} a_{1i}c_i = b_1,$$
$$Ext\text{-}\sum_{i=1}^{n} a_{2i}c_i = b_2, \qquad (4.12.2)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$Ext\text{-}\sum_{i=1}^{n} a_{ni}c_i = b_n.$$

We multiply the first of the equations (4.12.2) by the cofactor A_{11} of the element a_{11} in the coefficient matrix, then we multiply the second equation by A_{21}, the third by A_{31}, and so on, and finally the last equation by A_{n1}. Then we add all the equations so obtained. The result is

$$\begin{aligned} &Ext\text{-}(a_{11}A_{11} + a_{21}A_{21} +\ldots+a_{n1}A_{n1})c_1 + \\ &+Ext\text{-}(a_{12}A_{11} + a_{22}A_{21} +\ldots+a_{n2}A_{n1})c_2 +\ldots+ \\ &+Ext\text{-}(a_{1n}A_{11} + a_{2n}A_{21} +\ldots+a_{nn}A_{n1})c_n = b_1A_{11} + b_2A_{21} +\ldots+b_nA_{n1}. \end{aligned} \qquad (4.12.3)$$

By Theorem 4.11.1, the coefficient of c_1 in (4.12.3) equals the determinant $\mathbf{D}$ itself. By Theorem 4.11.2, the coefficients of all the other $c_j (j \neq 1)$ vanish. The expression in the right-hand side of (4.12.3) is the expansion of the determinant

$$\mathbf{D}_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad (4.12.4)$$

with respect to its first column. Therefore (19) can now be written in the form $\mathbf{D}c_1 = \mathbf{D}_1$, so that

$$c_1 = \frac{\mathbf{D}_1}{\mathbf{D}}. \qquad (3.12.5)$$

In a completely analogous way, we can obtain the expression

$$c_j = \frac{\mathbf{D}_j}{\mathbf{D}}, \qquad (4.12.6)$$

$1 \leq j \leq n$, where

$$\mathbf{D}_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix} \qquad (4.12.7)$$

is the determinant obtained from the determinant $\mathbf{D}$ by replacing its j-th column by the numbers $b_1 b_2, \ldots, b_n$. Thus we obtain the following result.

Theorem 4.12.1. If a solution of the system (4.12.1) exists, then (4.12.6) expresses the solution in terms of the coefficients of the system and the numbers in the right-hand side of (4.12.1). In particular, we find that if a solution of the system (4.12.3) exists, it is unique.

Remark 4.12.1.We must still show that a solution of the system (4.12.1) always exists. Consider the quantities $c_j = \mathbf{D}_j/\mathbf{D}, 1 \leq j \leq n$ and substitute them into the system (4.12.1) in place of the unknowns $x_1 x_2, \ldots, x_n.$ Then this reduces all the equations of the system (4.12.1) to identities. In fact, for the i-th equation we obtain

$$\begin{gathered} Ext\text{-}(a_{i1}c_1 + a_{i2}c_2 + \ldots + a_{in}c_n) = a_{i1}\frac{\mathbf{D}_1}{\mathbf{D}} + a_{i2}\frac{\mathbf{D}_2}{\mathbf{D}} + \ldots + a_{in}\frac{\mathbf{D}_n}{\mathbf{D}} = \\ \mathbf{D}^{-1}[a_{i1}(Ext\text{-}(b_1A_{11} + b_2A_{21} + \ldots + b_nA_{n1})) + \\ + a_{i2}(Ext\text{-}(b_1A_{12} + b_2A_{22} + \ldots + b_nA_{n2})) + \ldots + \\ + a_{in}(Ext\text{-}(b_1A_{1n} + b_2A_{2n} + \ldots + b_nA_{nn}))] = \\ = \mathbf{D}^{-1}[b_1(Ext\text{-}(a_{i1}A_{11} + a_{i2}A_{12} + \ldots + a_{in}A_{1n})) + \ldots + \\ + b_2(Ext\text{-}(a_{i1}A_{21} + a_{i2}A_{22} + \ldots + a_{in}A_{2n})) + \ldots + \\ + b_n(Ext\text{-}(a_{i1}A_{n1} + a_{i2}A_{n2} + \ldots + a_{in}A_{nn}))]. \end{gathered} \tag{4.12.8}$$

By Theorems 4.11.1 and 4.11.2, only one of the coefficients of the quantities $b_1, b_2, \ldots, b_n$ is different from zero, namely the coefficient of b$_i$, which is equal to the determinant D itself. Consequently, the above expression reduces to

$$\mathbf{D}^{-1} b_i \mathbf{D} = b_i, \tag{4.12.9}$$

i.e., is identical with the right-hand side of the i-th equation of the system.

Thus the quantities c_j $(1 \leq j \leq n)$ actually constitute a solution of the system (4.12.1), and we have found the following prescription (Generalized Cramer's rule) for obtaining solutions of hyperfinite system (4.12.1).

Theorem 4.12.2. If the determinant of the system (4.12.1) is different from zero, then (4.12.1) has a unique solution, namely, for the value of the unknown x_j $(1 \leq j \leq n)$ we take the fraction whose denominator is the determinant $\mathbf{D}$ of (4.12.1) and whose numerator is the determinant obtained by replacing the j-th column of $\mathbf{D}$ by the column consisting of the constant terms of (4.12.1),i.e., the numbers in the right-hand sides of the system.

Remark 4.12.2. One sometimes encounters systems of linear equations whose constant terms are not numbers but vectors, e.g., in analytic geometry or in mechanics.

Cramer's rule and its proof remain valid in this case as well; one must only bear in mind that the values of the unknowns $x_1, x_2, \ldots, x_n$ will then be vectors rather than numbers.

4.13.Minors of arbitrary hyperfinite order. Generalized Laplace's Theorem.

Theorem 4.11.3 on the expansion of a determinant with respect to a row or a column is a special case of a more general theorem on the expansion of a determinant with respect to a whole set of rows or columns. Before formulating this general theorem (Generalized Laplace's theorem), we introduce some new notation.

Suppose that in a square external matrix of hyperfinite order $n \in \mathbb{N}^{\#}/\mathbb{N}$ we specify any $k \leq n$ different rows and the same number of different columns. The elements appearing at the intersections of these rows and columns form a square matrix of hyperfinite order $k.$ The determinant of this matrix is called a minor of order k of the original matrix of order n (also a minor of order k of the determinant $\mathbf{D}$); it is denoted by

$$\mathbf{M} = \mathbf{M}^{i_1, i_2, \dots, i_k}_{j_1, j_2, \dots, j_k}, \qquad (4.13.1)$$

where $i_1, i_2, \dots, i_k$, are the numbers of the deleted rows, and $j_1, j_2, \dots, j_k$ are the numbers of the deleted columns.

If in the original matrix we delete the rows and columns which make up the minor $\mathbf{M}$, then the remaining elements again form a square matrix, this time of order $n - k$. The determinant of this matrix is called the complementary minor of the minor M, and is denoted by the symbol

$$\overline{\mathbf{M}} = \overline{\mathbf{M}}^{i_1, i_2, \dots, i_k}_{j_1, j_2, \dots, j_k}. \qquad (4.13.2)$$

In particular, if the original minor is of order 1, i.e., is just some element a_{ij} of the determinant $\mathbf{D}$, then the complementary minor is the same as the minor $\mathbf{M}_{ij}$ discussed in Sec. .Consider now the minor

$$\mathbf{M}_1 = \mathbf{M}^{1,2,\dots,k}_{1,2,\dots,k} \qquad (4.13.3)$$

formed from the first k rows and the first k columns of the determinant $\mathbf{D}$; its complementary minor is

$$\mathbf{M}_2 = \overline{\mathbf{M}}^{1,2,\dots,k}_{11,2,\dots,k}. \qquad (4.13.4)$$

In the right-hand side of equation (4.5.5), put group together all the terms of the determinant whose first k elements belong to the minor $\mathbf{M}_1$ (and thus whose remaining $n - k$ elements belong to the minor $\mathbf{M}_2$). Let one of these terms be denoted by c; we now wish to determine the sign which must be ascribed to c. The first k elements of c belong to a term c_1, of the minor $\mathbf{M}_1$. If we denote by N_1 the number of segments of negative slope corresponding to these elements, then the sign which must be put in front of the term c_1 in the minor $\mathbf{M}_1$ is $(-1)^{N_1}$.The remaining $n - k$ elements of c belong to a term c_2 of the minor $\mathbf{M}_2$; the sign which must be put in front of this term in the minor $\mathbf{M}_2$ is $(-1)^{N_2}$, where N_2 is the number of segments of negative slope corresponding to the $n - k$ elements of c_2. Since in the matrix of the determinant $\mathbf{D}$ there is not a single segment with negative slope joining an element of the minor $\mathbf{M}_1$ with an element of the minor $\mathbf{M}_2$, the total number of segments of negative slope joining elements of the term c equals the sum $N_1 + N_2$. Therefore the sign which must be put in front of the term c is given by the expression $(-1)^{N_1+N_2}$, and hence is equal to the product of the signs of the terms c_1 and c_2 in the minors $\mathbf{M}_1$ and $\mathbf{M}_2$. Moreover, we note that the product of any term of the minor $\mathbf{M}_1$ and any term of the minor $\mathbf{M}$, gives us one of the terms of the determinant $\mathbf{D}$ that have been grouped together. It follows that the sum of all the terms that we have grouped together from the expression for the determinant $\mathbf{D}$ given by (4.5.5) is equal to the product of the minors $\mathbf{M}_1$ and $\mathbf{M}_2$.Next we solve the analogous problem for an arbitrary minor

$$\mathbf{M}_1 = \mathbf{M}^{i_1, i_2, \dots, i_k}_{j_1, j_2, \dots, j_k} \qquad (4.13.5)$$

with complementary minor $\mathbf{M}_2$. By successively interchanging adjacent rows and columns, we can move the minor $\mathbf{M}_1$ over to the upper left-hand comer of the determinant $\mathbf{D}$; to do so, we need a total of $Ext\text{-}\sum_{\alpha=1}^{k}(i_\alpha - \alpha) + Ext\text{-}\sum_{\alpha=1}^{k}(j_\alpha - \alpha)$ interchanges. As a result, we obtain a determinant $\mathbf{D}_1$ with the same terms as in the original determinant but multiplied by $(-1)^{i+j}$, where $i = Ext\text{-}\sum_{\alpha=1}^{k}(i_\alpha - \alpha)$, $j = Ext\text{-}\sum_{\alpha=1}^{k}(j_\alpha - \alpha)$ by what has just been proved, the sum of all the terms in the determinant $\mathbf{D}_1$ whose first k elemeflts appear in the minor $\mathbf{M}_1$ is equal to the product $\mathbf{M}_1\mathbf{M}_2$. It follows from this that the sum of the corresponding terms of the determinant $\mathbf{D}$ is equal to the product $(-1)^{i+j}\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_1A_2$, where the quantity $A_2 = (-1)^{i+j}\mathbf{M}_2$ is called the cofactor of the minor $\mathbf{M}_1$ in the determinant $\mathbf{D}$. Sometimes one uses the notation $A_2 = \overline{A}_{j_1,j_2,\ldots,j_k}^{i_1,i_2,\ldots,i_k}$, where the indices indicate the numbers of the deleted rows and columns.Finally, let the rows of the determinant $\mathbf{D}$ with indices $i_1, i_2, \ldots, i_k$ be fixed; some elements from these rows appear in every term of $\mathbf{D}$. We group together all the terms of $\mathbf{D}$ such that the elements from the fixed rows $i_1, i_2, \ldots, i_k$ belong to the columns with indices $j_1, j_2, \ldots, j_k$. Then, by what has just been proved, the sum of all these terms equals the product of the minor with the corresponding cofactor. In this way, all the terms of $\mathbf{D}$ can be divided into groups, each of which is characterized by specifying k columns. The sum of the terms in each group is equal to the product of the corresponding minor and its cofactor. Therefore the entire determinant can be represented as the sum

$$\mathbf{D} = Ext\text{-}\sum \mathbf{M}_{j_1,j_2,\ldots,j_k}^{i_1,i_2,\ldots,i_k}\overline{A}_{j_1,j_2,\ldots,j_k}^{i_1,i_2,\ldots,i_k}, \qquad (4.13.6)$$

where the indices $i_1, i_2, \ldots, i_k$ (the indices selected above) are fixed, and the sum is over all possible values of the column indices $j_1, j_2, \ldots, j_k$ $(1 < j_1 < j_2 < \ldots < j_k < n)$. The expansion of $\mathbf{D}$ given by (4.13.6) is called Laplace's theorem. Clearly, Laplace's theorem constitutes a generalization of the formula for expanding a determinant with respect to one of its rows. There is an analogous formula for expanding the determinant

$\mathbf{D}$ with respect to a fixed set of columns.

4.14.Linear dependence between hyperfinite columns.

Suppose we are given m columns of hyperreal numbers with n numbers in each:

$$A_1 = \left\| \begin{matrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{matrix} \right\|, A_2 = \left\| \begin{matrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{matrix} \right\|, \ldots, A_m = \left\| \begin{matrix} a_{1m} \\ a_{2m} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{matrix} \right\|. \qquad (4.14.1)$$

We multiply every element of the first column by some number λ_1, every element of the second column by λ_2, etc., and finally every element of the last (mth) column by λ_m; we then add corresponding elements of the columns.

As a result, we get a new column of numbers, whose elements we denote by $c_1, c_2, \ldots, c_n$. We can represent all these operations schematically as follows:

$$Ext\text{-}\left(\lambda_1\left\|\begin{matrix}a_{11}\\a_{21}\\\bullet\\\bullet\\\bullet\\a_{n1}\end{matrix}\right\|+\lambda_2\left\|\begin{matrix}a_{12}\\a_{22}\\\bullet\\\bullet\\\bullet\\a_{n2}\end{matrix}\right\|+\ldots+\lambda_m\left\|\begin{matrix}a_{1m}\\a_{2m}\\\bullet\\\bullet\\\bullet\\a_{nm}\end{matrix}\right\|\right)=\left\|\begin{matrix}c_1\\c_2\\\bullet\\\bullet\\\bullet\\c_n\end{matrix}\right\|, \qquad (4.14.2)$$

or more briefly as

$$Ext\text{-}\sum_{i=1}^{m}\lambda_i A_i = C, \qquad (4.14.3)$$

where C denotes the column whose elements are $c_1, c_2, \ldots, c_n \in {}^*\mathbb{R}_c^\#$. The column C is called a linear combination of the columns $A_1, A_2, \ldots, A_m$, and the hyperreal numbers $\lambda_1, \lambda_2, \ldots, \lambda_m \in {}^*\mathbb{R}_c^\#$ are called the coefficients of the linear combination. As special cases of the linear combination C, we have the sum of the columns if $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 1$ and the product of a column by a number if $m = 1$.

Suppose now that our columns are not chosen independently, but rather make up a determinant $\mathbf{D}$ of order $n \in \mathbb{N}^\#/\mathbb{N}$. Then we have the following

Theorem 4.14.1. If one of the columns of the determinant $\mathbf{D}$ is a linear combination of the other columns, then $\mathbf{D} = 0$.

Proof. Suppose, for example, that the q-th column of the determinant $\mathbf{D}$ is a linear combination of the j-th, k-th, . . . , p-th columns of $\mathbf{D}$, with coefficients $\lambda_j, \lambda_k, \ldots, \lambda_p$, respectively. Then by subtracting from the q-th column first the j-th column multiplied by λ_j, then the k-th column multiplied by λ_k, etc., and finally the p-th column multiplied by λ_p, we do not change the value of the determinant $\mathbf{D}$.

However, as a result, the q-th column consists of zeros only, from which it follows that $\mathbf{D} = 0$.

Remark 4.14.1.It is remarkable that the converse is also true, i.e., if a given determinant $\mathbf{D}$ is equal to zero, then (at least) one of its columns is a linear combination of the other columns. The proof of this theorem requires some preliminary considerations, to which we now turn.

Again suppose we have $m \in \mathbb{N}^\#\backslash\mathbb{N}$ columns of numbers with $n \in \mathbb{N}^\#\backslash\mathbb{N}$ elements in each.

We can write them in the form of a matrix

$$\mathbf{A} = \left\|\begin{matrix}a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \bullet & \bullet & \cdots & \bullet\\ a_{n1} & a_{n2} & \cdots & a_{nm}\end{matrix}\right\| \qquad (4.14.4)$$

with n rows and m columns. If k columns and k rows of this matrix are held fixed, then the elements appearing at the intersections of these columns and rows form a square matrix of order κ, whose determinant is a minor of order κ of the original matrix $\mathbf{A}$; this determinant may either be vanishing or nonvanishing. If, as we shall always assume, not all of the a_{ik} are zero, then we can always find an integer r

which has the following two properties:

1.The matrix **A** has a minor of order r which does not vanish;

2. Every minor of the matrix **A** of order $r+1$ and higher (if such actually exist) vanishes.

Definition 4.14.1.The number r which has these properties is called the rank of the matrix A. If all the a_{ik} vanish, then the rank of the matrix **A** is considered to be zero ($r=0$). Henceforth we shall assume that $r>0$. The minor of order r which is different from zero is called the basis minor of the matrix **A**. (Of course, **A** can have several basis minors, but they all have the same order r.) The columns which contain the basis minor are called the basis columns.

Concerning the basis columns, we have the following important

Theorem 4.14.2. (**Basis minor theorem**). Any column of the matrix A is a linear combination of its basis columns.

Proof. To be explicit, we assume that the basis minor of the matrix is located in the first r rows and first r columns of A. Let s be any integer from 1 to m, let κ be any integer from 1 to n, and consider the determinant

$$\mathbf{D}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2s} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{rs} \\ a_{k1} & a_{k2} & \cdots & a_{kr} & a_{ks} \end{vmatrix} \qquad (4.14.5)$$

of order $r+1$. If $k\leq n$, the determinant **D** is obviously zero, since it then has two identical rows. Similarly, $\mathbf{D}=0$ for $s<r$. If $k>r$ and $s>r$, then the determinant **D** is also equal to zero, since it is then a minor of order $r+1$ of a matrix of rank r. Consequently $\mathbf{D}=0$ for any values of k and s. We now expand **D** with respect to its last row, obtaining the relation

$$Ext\text{-}(a_{k1}A_{k1}+a_{k2}A_{k2}+a_{kr}A_{kr})+a_{ks}A_{ks}=0, \qquad (4.14.6)$$

where the numbers $A_{k1},A_{k2},\ldots,A_{kr},A_{ks}$ denote the cofactors of the elements $a_{k1},a_{k2},\ldots,a_{kr},a_{ks}$ appearing in the last row of **D**. These cofactors do not depend on the number k, since they are formed by using elements a_{ij} with $i<r$. Therefore we can introduce the notation

$$A_{k1}=c_1,A_{k2}=c_2,\ldots,A_{kr}=c_r,A_{ks}=c_s. \qquad (4.14.7)$$

Substituting the values $\kappa=1,2,\ldots,n$ in turn into (4.14.6), we obtain hyperfinite system of equations

$$Ext\text{-}\sum_{j=1}^{r} c_j a_{1j} + c_s a_{1s} = Ext\text{-}(c_1 a_{11} + c_2 a_{12} + \ldots + c_r a_{1r}) + c_s a_{1s} = 0,$$

$$Ext\text{-}\sum_{j=1}^{r} c_j a_{2j} + c_s a_{2s} = Ext\text{-}(c_1 a_{21} + c_2 a_{22} + \ldots + c_r a_{2r}) + c_s a_{2s} = 0, \quad (4.14.8)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Ext\text{-}\sum_{i=1}^{r} c_j a_{nj} + c_s a_{ns} = Ext\text{-}(c_1 a_{n1} + c_2 a_{n2} + \ldots + c_r a_{nr}) + c_s a_{ns} = 0.$$

The number $c_s = A_{ks}$ is different from zero, since A_{ks} is a basis minor of the matrix A. Dividing each of the equations (4.14.8) by cs, transposing all the terms except the last to the right-hand side, and denoting $-c_j/c_s$ by λ_j $(1 \leq j \leq r)$, we obtain

$$Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{1j} = Ext\text{-}(\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_r a_{1r}) = a_{1s},$$

$$Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{2j} + c_s a_{2s} = Ext\text{-}(\lambda_1 a_{21} + \lambda_2 a_{22} + \ldots + \lambda_r a_{2r}) = a_{2s}, \quad (4.14.9)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Ext\text{-}\sum_{j=1}^{r} \lambda_j a_{nj} + \lambda_s a_{ns} = Ext\text{-}(\lambda_1 a_{n1} + \lambda_2 a_{n2} + \ldots + \lambda_r a_{nr}) = a_{ns}.$$

These equations show that the s-th column of the matrix A is a linear combination of the first r columns of the matrix (with coefficients $\lambda_1, \lambda_2, \ldots, \lambda_r$). The proof of the theorem is now complete, since s can be any number such that $1 \leq s \leq m$.

Theorem 4.14.2. If the determinant $\mathbf{D}$ vanishes, then it has at least one column which is a linear combination of the other columns.

Proof. Consider the matrix of the determinant $\mathbf{D}$. Since $\mathbf{D} = 0$, the basis minor of this matrix is of order $r < n$. Therefore, after specifying the r basis columns, we can still find at least one column which is not one of the basis columns. By the basis minor theorem, this column is a linear combination of the basis columns. Thus we have found a column of the determinant $\mathbf{D}$ which is a linear combination of the other columns. Note that we can include all the remaining columns of the determinant $\mathbf{D}$ in this linear combination by assigning them zero coefficients.

Remark 4.14.2.The results obtained above can be formulated in a somewhat more symmetric way. If the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ of a linear combination of m columns $A_1, A_2, \ldots, A_m$ are equal to zero, then obviously the linear combination is just the zero column, i.e., the column consisting entirely of zeros. But it may also be possible to obtain the zero column from the given columns by using coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ which are not all equal to zero. In this case, the given columns $A_1, A_2, \ldots, A_m$ are called linearly dependent.

A more detailed statement of the definition of linear dependence is the following: The hyperfinite columns

$$A_1 = \left\| \begin{matrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{matrix} \right\|, A_2 = \left\| \begin{matrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{matrix} \right\|, \ldots, A_m = \left\| \begin{matrix} a_{1m} \\ a_{2m} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{matrix} \right\|. \quad (4.14.10)$$

are called linearly dependent if there exist numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all equal to zero, such that the system of equation

$$\begin{aligned} &Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{1j} = Ext\text{-}(\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots + \lambda_m a_{1m}) = 0, \\ &Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{2j} = Ext\text{-}(\lambda_1 a_{21} + \lambda_2 a_{22} + \ldots + \lambda_m a_{2m}) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &Ext\text{-}\sum_{j=1}^{m} \lambda_j a_{nj} = Ext\text{-}(\lambda_1 a_{n1} + \lambda_2 a_{n2} + \ldots + \lambda_r a_{nr}) = 0. \end{aligned} \quad (4.14.11)$$

is satisfied, or equivalently such that

$$Ext\text{-}\sum_{i=1}^{m} \lambda_i A_i = \mathbf{0}, \quad (4.14.12)$$

where the symbol $\mathbf{0}$ on the right-hand side denotes the zero column. If one of the columns $A_1, A_2, \ldots, A_m$, (e.g., the last column) is a linear combination of the others, i.e.,

$$A_m = Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i. \quad (4.14.13)$$

then the columns $A_1, A_2, \ldots, A_m$ are linearly dependent. In fact, (4.14.13) is equivalent to the relation

$$A_m - Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i = 0 \quad (4.14.14)$$

Consequently, there exists a linear combination of the columns $A_1, A_2, \ldots, A_m$, whose coefficients are not equal to zero (e.g., with the last coefficient equal to -1 whose sum is the zero column; this just means that the columns $A_1, A_2, \ldots, A_m$ are linearly dependent.

Conversely, if the columns $A_1, A_2, \ldots, A_m$ are linearly dependent, then {at least) one of the columns is a linear combination of the other columns. In fact, suppose that in the relation

$$\lambda_m A_m + Ext\text{-}\sum_{i=1}^{m-1} \lambda_i A_i = 0 \quad (4.14.15)$$

expressing the linear dependence of the columns $A_1, A_2, \ldots, A_m$, the coefficient λ_m, say, is nonzero. Then (4.14.15) is equivalent to the relation

$$A_m = -\left(Ext\text{-}\sum_{i=1}^{m-1} \frac{\lambda_i}{\lambda_m} A_i \right). \quad (4.14.16)$$

Remark 4.14.3.Theorems 4.14.1 and 4.14.2 show that the determinant $\mathbf{D}$ vanishes if and only if one of its columns is a linear combination of the other columns. Using the results obtained above, we have the following.

Theorem 4.14.3. The determinant $\mathbf{D}$ vanishes if and only if there is linear dependence between its columns.

Remark 4.14.4.Since the value of a determinant does not change when it is transposed and since transposition changes columns to rows, we can change columns to rows in all the statements made above. In particular, the determinant $\mathbf{D}$ vanishes if and only if there is linear dependence between its rows.

4.15.External hyperfinite dimensional linear spaces. Subspaces,direct summ and factor spaces. Basic results and definitions.

A vector space over a field ${}^*\mathbb{R}_c^\#$ is a set V together with two operations that satisfy the eight axioms listed below.

The first operation, called vector addition or simply addition $+ : V \times V \to V,$ takes any two vectors $\mathbf{x}$ and $\mathbf{y}$ and assigns to them a third vector which is commonly written as $\mathbf{x}+\mathbf{y},$ and called the sum of these two vectors.

The second operation, called scalar multiplication $\times : F \times V \to V,$ takes any scalar a and any vector $\mathbf{v}$ and gives another vector $a \times \mathbf{x}$.

Axioms:

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$;

(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$;

(3) There exists $\mathbf{0} \in V$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in V$;

(4) For every $\mathbf{x} \in V$ there exists $\mathbf{y} \in V$ such that $\mathbf{x}+\mathbf{y} = \mathbf{0}$;

(5) $1 \times \mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in V$;

(6) $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ for every $\mathbf{x} \in V$ and every $\alpha,\beta,\gamma \in {}^*\mathbb{R}_c^\#$;

(7) $(\alpha+\beta)\times\mathbf{x} = \alpha\times\mathbf{x} + \beta\times\mathbf{x}$ for every $\mathbf{x} \in V$ and every $\alpha,\beta \in {}^*\mathbb{R}_c^\#$;

(8) $\alpha\times(\mathbf{x}+\mathbf{y}) = \alpha\times\mathbf{x}+\alpha\times\mathbf{y}$ for every $\mathbf{x},\mathbf{y} \in V$ and every $\alpha \in {}^*\mathbb{R}_c^\#$.

Axioms (1)-(8) have a number of implications:

Theorem 4.15.1.The zero vector $\mathbf{0}$ in a linear space V is unique.

Proof. The existence of at least one zero vector is asserted in axiom (3). Suppose there are two zero vectors $\mathbf{0}_1$ and $\mathbf{0}_2$ in the space V. Setting $\mathbf{x} = \mathbf{0}_1, \mathbf{0} = \mathbf{0}_2$ in axiom (3), we obtain $\mathbf{0}_1+\mathbf{0}_2 = \mathbf{0}_1$. Setting $\mathbf{x} = \mathbf{0}_2, \mathbf{0} = \mathbf{0}_1$ in axiom (3),we obtain $\mathbf{0}_1+\mathbf{0}_2 = \mathbf{0}_2$. Comparing the first of these relations with the second and using axiom (1), we find that $\mathbf{0}_1 = \mathbf{0}_2$.

Theorem 4.15.2.Every element in a linear space has a unique negative.

Proof. The existence of at least one negative element is asserted in axiom (4). Suppose an element $\mathbf{x} \in V$ has two negatives y_1 and y_2. Adding y_2 to both sides of the

equation $\mathbf{x}+\mathbf{y}_1=\mathbf{0}$ and using axioms (1)-(3), we get
$\mathbf{y}_2+(\mathbf{x}+\mathbf{y}_1) = (\mathbf{y}_2+\mathbf{x})+\mathbf{y}_1=\mathbf{0}+\mathbf{y}_1=\mathbf{y}_1, \mathbf{y}_2+(\mathbf{x}+\mathbf{y}_1) = \mathbf{y}_2+\mathbf{0}=\mathbf{y}_2$, whence $\mathbf{y}_1=\mathbf{y}_2$.
Theorem 4.15.3.The relation $0\times\mathbf{x}=\mathbf{0}$ holds for every $\mathbf{x}\in V$.
Theorem 4.15.4.For any $\mathbf{x}\in V$ the element $\mathbf{y}=(-1)\times\mathbf{x}$ is a negative of $\mathbf{x}$.
Definition 4.15.1.Let $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k, k\in\mathbb{N}^\#$ be vectors of the linear space V over a field ${}^*\mathbb{R}_c^\#$, and let $\alpha_1,\alpha_2,\ldots,\alpha_k$ be numbers from ${}^*\mathbb{R}_c^\#$. Then the vector

$$\mathbf{y} = Ext\text{-}\sum_{i=1}^{k}\alpha_i\mathbf{x}_i \qquad (4.15.1)$$

is called a linear combination of the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$, and the numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ are called the coefficients of the linear combination.If $\alpha_i=0, 1\leq i\leq k,$ then $\mathbf{y}=0$ by **Theorem 4.15.5**. However, there may exist a linear combination of the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ which equals the zero vector, even though its coefficients are not all zero. In this case, the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are called linearly dependent. In other words, the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are said to be linearly dependent if there exist numbers $\alpha_1,\alpha_2,\ldots,\alpha_k,$ not all equal to zero, such that

$$Ext\text{-}\sum_{i=1}^{k}\alpha_i\mathbf{x}_i = 0. \qquad (4.15.2)$$

If (4.15.2) holds if and only if $\alpha_i\equiv 0, 1\leq i\leq k,$ the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are said to be linearly dependent over ${}^*\mathbb{R}_c^\#$
Next we note two simple properties of systems of vectors, both involving the notion of linear dependence.
Theorem 4.15.6. If some of the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are linearly dependent, then the whole system $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ is also linearly dependent.
Proof. Without loss of generality, we can assume that the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_j, j<k$ are linearly dependent. Thus there is a relation

$$Ext\text{-}\sum_{i=1}^{j}\alpha_i\mathbf{x}_i = 0,$$

where at least one of the constants $\alpha_1,\alpha_2,\ldots,\alpha_j$ is different from zero.
By Theorem 4.15.3 and axiom (3), we have

$$Ext\text{-}\sum_{i=1}^{j}\alpha_i\mathbf{x}_i + Ext\text{-}\sum_{i=j+1}^{k}0\times\mathbf{x}_i = 0.$$

But then the vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are also linearly dependent, since at least one of the constants $\alpha_1,\alpha_2,\ldots,\alpha_j,0,\ldots,0$ is different from zero. |
Theorem 4.15.7. The vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k$ are linearly dependent if and only if one of the vectors can be expressed as a linear combination of the others.
Proof A similar statement has already been encountered; in fact, it was proved for columns of hyperreal numbers in Sec.4.14. Inspecting the proof given there, we see that it is based only on the possibility of performing on columns the operations of addition and multiplication by hyperreal numbers. Hence the proof can be carried through for the elements of any linear space, i.e., this theorem is valid for any linear space.
Definition 4.15.2. A hyperfinite system of linearly independent vectors $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n,$

$n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ in a linear space V over a field ${}^{*}\mathbb{R}_{c}^{\#}$ is called a basis for V if, given any $\mathbf{x} \in V$, there exists an expansion

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \zeta_i \mathbf{e}_i, \tag{4.15.3}$$

where $\zeta_i \in {}^{*}\mathbb{R}_{c}^{\#}, 1 \leq i \leq n$.
It is easy to see that under these conditions the coefficients in the expansion (4.15.3) are uniquely determined. In fact, if we can write two expansions

$$\begin{gathered} \mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \zeta_i \mathbf{e}_i, \\ \mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \eta_i \mathbf{e}_i, \end{gathered} \tag{4.15.4}$$

for a vector $\mathbf{x}$, then, subtracting them term by term, we obtain the relation

$$Ext\text{-}\sum_{i=1}^{n} (\zeta_i - \eta_i)\mathbf{e}_i = 0 \tag{4.15.5}$$

from which, by the assumption that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent, we obtain that

$$\zeta_i = \eta_i, , 1 \leq i \leq n. \tag{4.15.6}$$

Definition 4.15.3.The uniquely defined numbers $\zeta_i \in {}^{*}\mathbb{R}_{c}^{\#}, 1 \leq i \leq n,$ are called the components of the vector $\mathbf{x}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.
Example 4.15.2 An example of a basis in the space $V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$ is the hyperfinite system of vectors $e_1 = (1, 0, \ldots), e_2 = (0, 1, \ldots), \ldots, e_n = (0, 0, \ldots 1)$. Indeed it is obvious that the relation

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \zeta_i e_i \tag{4.15.7}$$

holds for every vector

$$\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n). \tag{4.15.8}$$

This fact, together with the linear independence of the vectors $e_i, 1 \leq i \leq n$ already proved, shows that these vectors form a basis in the space V_n. In particular, we see that the hyperreal numbers $\xi_i, 1 \leq i \leq n$ are just the components of the vector $\mathbf{x}$ with respect to the basis $e_i, 1 \leq i \leq n$.
Theorem 4.15.8.When two vectors of a linear space V_n are added, their components (with respect to any basis) are added. When a vector is multiplied by a number $\lambda \in {}^{*}\mathbb{R}_{c}^{\#}$, all its components are multiplied by λ.
Proof. Let

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \zeta_i \mathbf{e}_i, \mathbf{y} = Ext\text{-}\sum_{i=1}^{n} \eta_i \mathbf{e}_i. \tag{4.15.9}$$

Then

$$\mathbf{x} + \mathbf{y} = Ext\text{-}\sum_{i=1}^{n} (\zeta_i + \eta_i)\mathbf{e}_i, \lambda\mathbf{x} = Ext\text{-}\sum_{i=1}^{n} \lambda\zeta_i \mathbf{e}_i \tag{4.15.10}$$

by the axioms.

Definition 4.15.3.If in a linear space V we can find $n \in \mathbb{N}^{\#}$ linearly independent vectors while every $n+1$ vectors of the space are linearly dependent, then the number
$n \in \mathbb{N}^{\#}/\mathbb{N}$
is called the dimension of the space V and the space V itself is called n-dimensional and denoted V_n. A linear space in which we can find an hyperfinite number of linearly independent vectors also is called hyperfinite-dimensional.

Theorem 4.15.8.In a space V of dimension $n \in \mathbb{N}^{\#}$ there exists a basis consisting of n vectors. Moreover, any set of n linearly independent vectors of the space V is a basis for the space.

Proof. Let $\mathbf{e}_i, 1 \leq i \leq n$ be a hyperfinite system of n linearly independent vectors of the given n-dimensional space V. If $\mathbf{x}$ is any vector of the space, then the set of $n+1$ vectors $\mathbf{x}, \mathbf{e}_i, 1 \leq i \leq n$ is linearly dependent, i.e., there exists a relation of the form

$$\alpha_0 \mathbf{x} + Ext\text{-}\sum_{i=1}^{n} \alpha_i \mathbf{e}_i = 0, \tag{4.15.11}$$

where at least one of the coefficients $\alpha_0, \alpha_i, 1 \leq i \leq n$ is different from zero. Clearly α_0 is different from zero, since otherwise the vectors $\mathbf{e}_i, 1 \leq i \leq n$ would be linearly dependent, contrary to hypothesis. Thus, in the usual way, i.e., by dividing (4.15.11) by α_0 and transposing all the other terms to the other side, we find that $\mathbf{x}$ can be expressed as a linear combination of the vectors $\mathbf{e}_i, 1 \leq i \leq n$. Since $\mathbf{x}$ is an arbitrary vector of the space V, we have shown that the vectors $\mathbf{e}_i, 1 \leq i \leq n$ form a basis for the space.

The preceding theorem has the following converse.

Theorem 4.15.9.If there is a basis in the space V, then the dimension of V equals the number of basis vectors.

Proof. Let the vectors $\mathbf{e}_i, 1 \leq i \leq n$ be a basis for V. By the definition of a basis, the vectors $\mathbf{e}_i, 1 \leq i \leq n$ are linearly independent; thus we already have n linearly independent vectors. We now show that any $n+1$ vectors of the space V are linearly dependent. Suppose we are given $n+1$ vectors of the space V :

$$\begin{aligned} x_1 &= Ext\text{-}\sum_{i=1}^{n} \xi_i^{(1)} \mathbf{e}_i, \\ x_2 &= Ext\text{-}\sum_{i=1}^{n} \xi_i^{(2)} \mathbf{e}_i, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n+1} &= Ext\text{-}\sum_{i=1}^{n} \xi_i^{(n+1)} \mathbf{e}_i \end{aligned} \tag{4.15.12}$$

Writing the components of each of these vectors as a column of numbers, we form the
matrix

$$A = \left\| \begin{matrix} \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(n+1)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \cdots & \xi_2^{(n+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \xi_n^{(1)} & \xi_n^{(2)} & & \xi_n^{(n+1)} \end{matrix} \right\| \qquad (4.15.13)$$

with n rows and $n+1$ columns. The basis minor of the matrix A is of order $r < n$. If $r = 0$, the linear dependence is obvious. Let $r > 0$. After specifying the r basis columns,we can still find at least one column which is not one of the basis columns. But then,according to the basis minor theorem, this column is a linear combination of the basis columns. Thus the corresponding vector of the space V is a linear combination of some other vectors among the given $x_1, x_2, \ldots, x_{n+1}$. But in this case, according to Theorem 4.15.6, the vectors $x_1, x_2, \ldots, x_{n+1}$ are linearly dependent.

Definition 4.15.4. A Complex linear space $V = V[{}^*\mathbb{C}_c^\#]$ that is a linear space over field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$

Note that $V[{}^*\mathbb{C}_c^\#]$ is obviously a real space as well, since the domain of the external complex numbers${}^*\mathbb{C}_c^\#$contains the domain of hyperreal numbers ${}^*\mathbb{R}_c^\#$. However, the dimension $\dim_{{}^*\mathbb{C}_c^\#}(V)$ of V as a complex space does not coincide with dimension $\dim_{{}^*\mathbb{R}_c^\#}(V)$ of $V[{}^*\mathbb{C}_c^\#]$ as a real space. In fact, if the vectors $\mathbf{e}_i, 1 \leq i \leq n$ are linearly independent in V regarded as a complex space, then the vectors $\mathbf{e}_i, i\mathbf{e}_i, 1 \leq i \leq n$, are linearly independent in V regarded as a real space. Hence the dimension of V regarded
as a real space is twice as large as that of V regarded as a complex space.

4.15.1.Subspaces

Suppose that a set L of elements of a linear space V over field ${}^*\mathbb{R}_c^\#$has the following properties:

(a) If $\mathbf{x} \in L, y \in L$, then $\mathbf{x} + \mathbf{y} \in L$;

(b) If $x \in L$ and $\lambda \in {}^*\mathbb{R}_c^\#$ then $\lambda\mathbf{x} \in L$.

Thus L is a set of elements with linear operations defined on them.

We now show that this set is also a linear space. To do so, we must verify that the set L with the operations (a) and (b) satisfies the **Axioms** (1), (2) and (5)-(8) are satisfied, since they hold quite generally for all elements of the space V. It remains to verify axioms (3) and (4). Let $\mathbf{x}$ be any element of L. Then, by hypothesis, $\lambda\mathbf{x} \in L$ for every $\lambda \in {}^*\mathbb{R}_c^\#$. First we choose $\lambda = 0$. Then, since $0 \times x = 0$, the zero vector belongs to the set L, i.e., axiom (3) is satisfied. Next we choose $\lambda = -1$. Then, by Theorem 4.15.4, $(-1) \times x$ is the negative of the element x. Thus, if an element x belongs to the set L, so does the negative of x. This means that axiom (4) is also satisfied, so that L is a linear space, as asserted.

Definition 4.15.5.Every set $L \subset V$ with properties (a) and (b) is called a linear subspace (or simply a subspace) of the space V.

Definition 4.15.6.Let L_1 and L_2 be two subspaces of the same linear space V. Then the set of all vectors $\mathbf{x} \in V$ belonging to both L_1 and L_2 forms a subspace called the intersection of the subspaces L_1 and L_2. The set of all vectors of the form $\mathbf{y} + \mathbf{z}$ where $\mathbf{y} \in L_1, \mathbf{z} \in L_2$ forms a subspace, denoted by $L_1 + L_2$ and called the sum of the subspaces L_1 and L_2.

We now consider some properties of subspaces which are related to the definitions above. First of all, we note that every linear relation which connects the vectors $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$ in a subspace L is also valid in the whole space V, and conversely. In particular, the fact that the vectors $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z} \in L$ are linearly dependent holds true simultaneously in the subspace L and in the space V. For example, if every set of $n+1$ vectors is linearly dependent in the space V, then this fact is true a fortiori in the subspace L. It follows that the dimension of any subspace L of an n-dimensional space V does not exceed the number n, According to Theorem 4.15.9, in any subspace $L \subset V$ there exists a basis with the same number of vectors as the dimension of L. Of course, if a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is chosen in V, then in the general case we cannot choose the basis vectors of the subspace L from the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, because none of these vectors may belong to L. However, it can be asserted that if a basis $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ is chosen in the subspace L {which, to be explicit, is assumed to have dimension $l < n$), then additional vectors $\mathbf{f}_{l+1}, \ldots, \mathbf{f}_n$ can always be chosen in the whole space V such that the system $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l, \ldots, \mathbf{f}_n$ is a basis for all of V.

To prove this, we argue as follows: In the space V there are vectors which cannot be expressed as linear combinations of $\mathbf{f}_1, \ldots, \mathbf{f}_l$. Indeed, if there were no such vectors, then the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l$, which are linearly independent by hypothesis, would constitute a basis for the space V, and then by Theorem 4.15.9 the dimension of V would be l rather than n. Let $\mathbf{f}_{l+1}$ be any of the vectors that cannot be expressed as a linear combination of $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l$. Then the System $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l, \mathbf{f}_{l+1}$ is linearly independent. In fact, suppose there were a relation of the form

$$Ext\text{-}\sum_{i=1}^{l+1} \alpha_i \mathbf{f}_i = 0. \qquad (4.15.14)$$

Then if $\alpha_{l+1} \neq 0$, the vecto $\mathbf{f}_{l+1}$ could be expressed as a linear combination of $\mathbf{f}_1, \ldots, \mathbf{f}_l$ while if $\alpha_{l+1} = 0$, the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l$ would be linearly dependent. But both these results contradict the construction. If now every vector of the space V can be expressed as a linear combination of $\mathbf{f}_1, \ldots, \mathbf{f}_l, \mathbf{f}_{l+1}$, then the system $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l, \mathbf{f}_{l+1}$ forms a basis for V with $l+1 = n$, which concludes our construction. If $l+1 < n$, then there is a vector $\mathbf{f}_{l+2}$ which cannot be expressed as a linear combination $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l, \mathbf{f}_{l+1}$, and hence we can continue the construction. Eventually, after $n-l$ steps, we obtain a basis for the space V.

Definition 4.15.7. We say that the vectors $g_1, \ldots, g_k$ are linearly independent over the subspace $L \subset V$ if the relation

$$Ext\text{-}\sum_{i=1}^{k} \alpha_i \mathbf{g}_i \in L \qquad (4.15.15)$$

implies $\alpha_i = 0, 1 \leq i \leq k$.

If L is the subspace consisting of the zero vector alone, then linear independence over L means ordinary linear independence. Linear dependence of the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_k$

over the subspace L means that there exists a linear combination $Ext\text{-}\sum_{i=1}^{k}\alpha_i\mathbf{g}_i$ belonging to L, where at least one of the coefficients $\alpha_i, 1 \leq i \leq k$ is nonzero.

Definition 4.15.8. The largest possible number of vectors of the space V which are linearly independent over the subspace $L \subset V$ is called the dimension of V over L. If the vectors $\mathbf{g}_1,\ldots,\mathbf{g}_k$ are linearly independent over the space $L \subset V$ and if the vectors $f_1\ldots,f_l$ are linearly independent in the subspace L, then the vectors $\mathbf{g}_1,\ldots,\mathbf{g}_k,\ \mathbf{f}_1\ldots,\mathbf{f}_l$ are linearly independent in the whole space V. In fact, if there were a relation of the form

$$Ext\text{-}\sum_{i=1}^{l}\alpha_i\mathbf{f}_i + Ext\text{-}\sum_{i=1}^{k}\beta_i\mathbf{g}_i = 0, \tag{4.15.16}$$

or equivalently

$$Ext\text{-}\sum_{i=1}^{k}\beta_i\mathbf{g}_i = -\left(Ext\text{-}\sum_{i=1}^{l}\alpha_i\mathbf{f}_i\right) \in L, \tag{4.15.17}$$

then $\beta_i = 0, 1 \leq i \leq k,$ by the assumed linear independence of the vectors $\mathbf{g}_1,\ldots,\mathbf{g}_k$ over L. It follows that $\alpha_i = 0, 1 \leq i \leq k,$ by the linear independence of the vectors $\mathbf{f}_1\ldots,\mathbf{f}_l$.

Remark 4.15.1.The vectors $\mathbf{f}_{l+1}\ldots,\mathbf{f}_n$ constructed above are linearly independent over the subspace L. In fact, if there were a relation of the form

$$Ext\text{-}\sum_{i=l+1}^{n}\alpha_i\mathbf{f}_i = Ext\text{-}\sum_{i=1}^{l}\alpha_i\mathbf{f}_i \tag{4.15.17$'$}$$

with at least one of the numbers $\alpha_{l+1},\ldots,\alpha_n$ not equal to zero, then the vectors $\mathbf{f}_1,\ldots,\mathbf{f}_n$ would be linearly dependent, contrary to the construction. Hence the dimension of the space V over L is no less than $n-l$. On the other hand, this dimension cannot be greater than $n-l$, since if $n-l+1$ vectors $\mathbf{h}_1\ldots,\mathbf{h}_{n-l+1}$ say, were linearly independent over L, then the vectors $\mathbf{h}_1\ldots,\mathbf{h}_{n-l+1},\mathbf{f}_1,\ldots,\mathbf{f}_l$ of which there are more than n, would be linearly independent in V. Therefore the dimension of V over L is precisely $n-l$.

4.15.2.The hyperfinite direct sum

Definition 4.15.9.We say that a linear space L is the hyperfinite direct sum of given subspaces $L_1,\ldots,L_m \subset L, m \in {}^{\#}\mathbb{N}\backslash\mathbb{N}$ if: (a) For every $\mathbf{x} \in L$ there exists an expansion

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{m}x_i, \tag{4.15.18}$$

where $x_1 \in L_1,..,x_m \in L_m$;
(b) This expansion is unique, i.e., if

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{m}x_i = Ext\text{-}\sum_{i=1}^{m}y_i, \tag{4.15.19}$$

where $x_j \in L_j, y_j \in L_j, 1 \leq j \leq m,$ then $z_i = 0, 1 \leq j \leq m.$
However, the validity of condition (b) is a consequence of the following simpler condition: (b$'$) If

$$Ext\text{-}\sum_{i=1}^{m} z_i = 0 \qquad (4.15.20)$$

where $z_i \in L_i, 1 \leq i \leq m,$ then $z_i = 0, 1 \leq i \leq m.$

In fact, given two expansions $\mathbf{x} = Ext\text{-}\sum_{i=1}^{m} x_i$, $\mathbf{x} = Ext\text{-}\sum_{i=1}^{m} y_i$ suppose (b′) holds. Then subtracting the second expansion from the first, we get $\mathbf{0} = Ext\text{-}\sum_{i=1}^{m}(x_i - y_i),$ and hence $x_1 = y_1, \ldots, x_m = y_m,$ because of (b′). Conversely, (b′) follows from (b) if we set $x = \mathbf{0}, x_1 = \ldots = x_m = \mathbf{0}.$ It follows from condition (b) that every pair of subspaces $L_1, \ldots, L_m$ has only the element $\mathbf{0}$ in common. In fact, if $z \in L_j$ and $z \in L_k,$ then using (b) and comparing the two expansions $z = z + \mathbf{0}, z \in L_j, \mathbf{0} \in L_k$ and $z = \mathbf{0} + z, \mathbf{0} \in L_j,$ $z \in L_k,$ we find that $z = \mathbf{0}.$ Thus an n-dimensional space V_n is the hyperfinite direct sum of the n one-dimensional subspaces determined by any n linearly independent vectors. Moreover, the space V_n can be represented in various ways as a direct sum of subspaces not all of dimension 1.

Remark 4.15.2. Let L be a fixed subspace of an n-dimensional space $V_n.$ Then there always exists a Subspace $M \subset V_n$ such that the whole space V_n is the direct sum of L and $M.$ To prove this, we use the vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, \ldots, \mathbf{f}_n$ constructed above, which are linearly independent over the subspace $L.$ Let M be the subspace consisting of all linear combinations of the vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, \ldots, \mathbf{f}_n$. Then M satisfies the stipulated requirement. In fact, since the vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ form a basis in $V_n,$ every vector $x \in L$ has an expansion of the form

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{f}_i + Ext\text{-}\sum_{i=l+1}^{n} \alpha_i \mathbf{f}_i = \mathbf{y} + \mathbf{z}, \qquad (4.15.21)$$

where

$$\mathbf{y} = Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{f}_i \in L, \mathbf{z} = Ext\text{-}\sum_{i=l+1}^{n} \alpha_i \mathbf{f}_i \in M. \qquad (4.15.22)$$

Moreover $x = \mathbf{0}$ implies $\alpha_i = 0, 1 \leq i \leq n,$ since the vectors $\mathbf{f}_i, 1 \leq i \leq n$ are linearly independent. Therefore conditions (a)-(b′) are satisfied, so that V_n is the direct sum of L and $M.$

Remark 4.15.3. If the dimension of the space L_k equals $r_k,\ 1 \leq k \leq m$ and if r_k linearly independent vectors $\mathbf{f}_{k_1}, \mathbf{f}_2, \ldots, \mathbf{f}_{k_{r_k}}$ are selected in each space $L_k,$ then every vector $\mathbf{x}$ of the sum $L = Ext\text{-}\sum_{i=1}^{k} L_i$ can be expressed as a linear combination of these vectors. Hence the dimension of the sum of the spaces $L_1, \ldots, L_k$ does not exceed the sum of the dimensions of the separate spaces. If the hyperfinite sum $Ext\text{-}\sum_{i=1}^{k} L_i$ is direct, then the vectors $\mathbf{f}_{1_1}, \ldots, \mathbf{f}_{1_{r_1}} \ldots, \mathbf{f}_{k_1}, \ldots, \mathbf{f}_{m_1}, \ldots f_{m_{r_m}},$ are all linearly independent, so that in this case the dimension of the sum is precisely the hyperfinite sum of the dimensions.

Remark 4.15.4. In the general case, the dimension of the sum is related to the dimensions of the summands in a more complicated way. Here we consider only the problem of determining the dimension of the sum of two hyperfinite-dimensional subspaces P and Q of the space V, of dimensions p and $q,$ respectively. Let L be the intersection of the subspaces P and Q, and let L have dimension l. First we choose a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_l$ in L. Then, using the argument mentioned above, we augment the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_l$ by the vectors $\mathbf{f}_{l+1}, \mathbf{f}_{l+2}, \ldots, \mathbf{f}_p$ to make a basis for the whole subspace P

and by the vectors $\mathbf{g}_{I+1}, \mathbf{g}_{I+2}, \ldots, \mathbf{g}_q$ to make a basis for the whole subspace Q. By definition, every vector in the sum $P + Q$ is the sum of a vector from P and a vector from $Q,$ and hence can be expressed as a linear combination of the vectors

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_l, \mathbf{f}_{l+1}, \mathbf{f}_{l+2}, \ldots, \mathbf{f}_p, \mathbf{g}_{I+1}, \mathbf{g}_{I+2}, \ldots, \mathbf{g}_q. \qquad (4.15.23)$$

We now show that these vectors form a basis for the subspace $P + Q$. To show this, it remains to verify their linear independence. Assume that there exists a linear relation of

the form

$$Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{e}_i + Ext\text{-}\sum_{i=l+1}^{p} \beta_i \mathbf{f}_i + Ext\text{-}\sum_{i=l+1}^{q} \gamma_i \mathbf{g}_i, \qquad (4.15.24)$$

where at least one of the coefficients $\alpha_1, \ldots, \gamma_q$ is different from zero. We can then assert that at least one of the numbers $\gamma_{l+1}, \ldots \gamma_q,$ is different from zero, since otherwise the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_l, \mathbf{f}_{l+1}, \mathbf{f}_{l+2}, \ldots, \mathbf{f}_p$ would be linearly dependent, which is impossible in view of the fact that they form a basis for the subspace $P.$ Consequently the vector

$$\mathbf{x} = Ext\text{-}\sum_{i=l+1}^{q} \gamma_i \mathbf{g}_i \neq 0 \qquad (4.15.25)$$

for otherwise the vectors $\mathbf{g}_{I+1}, \mathbf{g}_{I+2}, \ldots, \mathbf{g}_q$ would be linearly dependent. But it follows from (4.15.24) that

$$-x = Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{e}_i + Ext\text{-}\sum_{i=l+1}^{p} \beta_i \mathbf{f}_i \qquad (4.15.26)$$

while (4.15.25) shows that $\mathbf{x} \in Q.$ Thus $\mathbf{x}$ belongs to both P and $Q,$ and hence belongs to the subspace $L.$ But then

$$\mathbf{x} = Ext\text{-}\sum_{i=l+1}^{q} \gamma_i \mathbf{g}_i = Ext\text{-}\sum_{i=1}^{l} \lambda_i \mathbf{e}_i \qquad (4.15.27)$$

and since the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_l, \mathbf{g}_{I+1}, \mathbf{g}_{I+2}, \ldots, \mathbf{g}_q$ are linearly independent, we have $\gamma_{l+1}, \ldots, \gamma_q = 0.$ This contradiction shows that the vectors (4.15.23) are actually linearly independent, and hence form a basis for the subspace $P + Q.$ It follows from Theorem 4.15.9 that the dimension of $P + Q$ equals the number of basis vectors (4.15.23). But this number equals $p + q - l.$

Theorem 4.15.10. The dimension of the sum of two subspaces is equal to the sum of their dimensions minus the dimension of their intersection.

Corollary 4.15.1. Let $V_p,$ and $V_q,$ be two subspaces of dimensions p and q

respectively,
of an n-dimensional space $V_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$, and suppose $p+q > n$. Then the intersection $V_p \cap V_q$ is of dimension no less than $p+q-n$.

4.15.3.Factor spaces

Definition 4.15.10.(a) Given a subspace $\mathbf{L}$ of a linear space V, an element $y \in \mathbf{V}$ is said to be comparable with an element $y \in \mathbf{V}$ (or comparable relative to $\mathbf{L}$) if $x - y \in \mathbf{L}$. Obviously, if x is comparable with y, then y is comparable with x, so that the relation of comparability is symmetric. Every element $x \in \mathbf{V}$ is comparable with itself. Moreover, if x is comparable with y and y is comparable with z, then x is comparable with z, since $x - z = (x-y)+(y-z) \in \mathbf{L}$.
(b) The set of ail elements $y \in \mathbf{V}$ comparable with a given element $x \in V$ is called a class, and is denoted by $[x]$. As just shown, a class $[x]$ contains the element y itself, and every pair of elements $y \in [x], z \in [x]$ are comparable with each other. Moreover, if $u \neq [x]$, then u is not comparable with any element of $[x]$. Therefore two classes either
have no elements in common or else coincide completely. The subspace $\mathbf{L}$ itself is a class. This class is denoted by $[0]$, since it contains the zero element of the space $\mathbf{V}$.
(c) The whole space V can be partitioned into a set of nonintersecting classes $[x],[y],\ldots$
This set of classes will be denoted by $\mathbf{V}/\mathbf{L}$.
We now introduce linear operations in $\mathbf{V}/\mathbf{L}$ as follows: Given two classes $[x],[y]$ and two elements $\alpha,\beta$of the field ${}^{*}\mathbb{R}_c^{\#}$, we wish to define the class $\alpha[x]+\beta[y]$. To do this, we choose arbitrary elements $x_1 \in [x]$, $y_1 \in [y]$ and find the class $[z]$ containing the element $z = \alpha x_1 + \beta y_1$. This class is then denoted by $\alpha[x] + \beta[y]$. Clearly, $\alpha[x] + \beta[y]$ is uniquely defined. In fact, suppose we choose another element x_1 of the class $[x]$ and another ement y_1 of the class $[y]$. Then $(\alpha x_1 + \beta y_1) - (\alpha x + \beta y) = \alpha(x_1 - x) + \beta(y_1 - y)$ belongs to the space $\mathbf{L}$, since $x_1 - x$ and $y_1 - y$ both belong to L. It follows that $\alpha x_1 + \beta y_1$
belongs to the same class as $\alpha x + \beta y$.
In particular, the above prescription defines addition of two classes $[x]$ and $[y]$, as well as multiplication of a class by a number $\alpha \in {}^{*}\mathbb{R}_c^{\#}$. We now show that these operations obey the axioms of a linear space, mentioned above. In fact, the validity of these axioms
for classes follows at once from their validity for elements of the space V. Moreover, the
zero element of the space $\mathbf{V}/\mathbf{L}$ is the class $[0]$ (consisting of all elements of the subspace
$\mathbf{L}$), while the inverse of the class $[x]$ is the class consisting of all inverses of elements of
the class $[x]$. Thus all axioms are satisfied for the set of classes $\mathbf{V}/\mathbf{L}$. The resulting linear
space $\mathbf{V}/\mathbf{L}$ is called the factor space of the space $\mathbf{V}$ with respect to the subspace $\mathbf{L}$.
Theorem 4.15.10. Let $\mathbf{V} = \mathbf{V}_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be an n-dimensional linear space over the field
${}^{*}\mathbb{R}_c^{\#}$, and let $\mathbf{L} = \mathbf{L}_l \subset \mathbf{V}$ be an l-dimensional subspace of $\mathbf{V}$. Then the factor space $\mathbf{V}/\mathbf{L}$

is of dimension $n - l$.

Proof. Choose any basis $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l \in \mathbf{L}$, and augment it, as mentioned above, by vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, \ldots, \mathbf{f}_n$ to make a basis for the whole space V. Then the classes $[\mathbf{f}_{l+1}], [\mathbf{f}_2], \ldots, [\mathbf{f}_n]$ form a basis in the space $\mathbf{V}/\mathbf{L}$. To see this, we note that given any $x \in \mathbf{V}$, there is a representation

$$x = Ext\text{-}\sum_{k=1}^{n} \alpha_k \mathbf{f}_k,$$

and hence a representation

$$[x] = Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k]$$

for the class $[x]$. Moreover, the classes $[\mathbf{f}_{l+1}], [\mathbf{f}_2], \ldots, [\mathbf{f}_n]$ are linearly independent. In fact, if $Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k] = [0] \in \mathbf{V}/\mathbf{L}$ for any $\alpha_k, 1 \leq k \leq n$ in ${}^*\mathbb{R}_c^\#$, then, in particular, there would be a relation $Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k] \in L$. But $\mathbf{f}_{l+1}, \mathbf{f}_{l+2}, \ldots, \mathbf{f}_n$ are linearly independent over L, and hence $\alpha_i = 0, l+1 \leq i \leq n$ as required. Thus the $n - l$ classes $[x_{l+1}], \ldots, [x_n]$ form a basis in V/L. It follows from Theorem 4.15.9 that $\mathbf{V}/\mathbf{L}$ is of dimension $n - l$.

4.15.4. Linear Manifolds

An important way of constructing subspaces is to form the linear manifold spanned by a given hyperfinite system of vectors.

Definition 4.15.11. Let $x_i, 1 \leq i \leq k, k \in \mathbb{N}^\# \backslash \mathbb{N}$ be a system of vectors of a linear space V.

Then the linear manifold spanned by $x_i, 1 \leq i \leq k$ is the set of all linear combinations

$$Ext\text{-}\sum_{i=1}^{k} \alpha_i x_i \qquad (4.15.28)$$

with coefficients $\alpha_i, 1 \leq i \leq k$ in the field ${}^*\mathbb{R}_c^\#$.

It is easily verified that this set has properties (a) and (b) of Sec. 4.15.1. Therefore the linear manifold spanned by a system $x_i, 1 \leq i \leq k$ is a subspace of the space V.

Obviously, every subspace containing the vectors $x_i, 1 \leq i \leq k$ also contains all their linear combinations (4.15.28). Consequently, the linear manifold spanned by the vectors $x_i, 1 \leq i \leq k$. is the smallest subspace containing these vectors. The linear manifold spanned by the vectors $x_i, 1 \leq i \leq k$ is denoted by $\mathbf{L}\left(\{x_i\}_{i=1}^{k}\right)$.

Examples

(i) The linear manifold spanned by the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ of a space V_n is obviously the whole space $\mathbf{V}_n$.

(ii) The linear manifold spanned by the system of functions $1, t, t^2 \ldots, t^n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ consists of the set of all external hyper polynomials in the variable t with coefficients in the field ${}^*\mathbb{R}_c^\#$ of degree no higher than n.

(iii) The linear manifold spanned by the system of functions $1, t, t^2 \ldots, t^n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ consists of the set of all external hyper polynomials in the variable t with coefficients in the field ${}^*\mathbb{C}_c^\#$ of degree no higher than n.

Lemma 4.15.1. If the vectors $\{x_i'\}_{i=1}^k$ belong to the linear manifold spanned by the vectors $\{x_i\}_{i=1}^k$, then the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ contains the whole linear manifold $\mathbf{L}\left(\{x_i'\}_{i=1}^k\right)$.

Proof. Since the vectors $\{x_i'\}_{i=1}^k$ belong to the subspace $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ then all their linear combinations, whose totality constitutes the linear manifold $\mathbf{L}\left(\{x_i'\}_{i=1}^k\right)$, also belong to the subspace of the $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$.

Lemma 4.15.2. Every vector of the system $\{x_i\}_{i=1}^k$ which is linearly dependent on the other vectors of the system can be eliminated without changing the linear manifold spanned by $\{x_i\}_{i=1}^k$.

Proof. If the vector x_1, say, is linearly dependent on the vectors $\{x_i\}_{i=2}^k$ this means that $x \in \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$. It follows from Lemma 4.15.1 that $\mathbf{L}\left(\{x_i\}_{i=1}^k\right) \subset \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$. On the other hand, obviously $\mathbf{L}\left(\{x_i\}_{i=2}^k\right) \subset \mathbf{L}\left(\{x_i\}_{i=1}^k\right)$. Together these two relations imply $\mathbf{L}\left(\{x_i\}_{i=1}^k\right) = \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$.

We now will consider the problem of constructing a basis for a linear manifold and determining the dimension of a linear manifold. In solving this problem, we will assume that the number of vectors $\{x_i\}_{i=1}^k$ spanning the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ is hyperfinite, although some of our conclusions do not actually require this assumption.

Suppose that among the vectors $\{x_i\}_{i=1}^k$ spanning the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ we can find $r \in \mathbb{N}^\#$ linearly independent vectors $\{\widetilde{x}_i\}_{i=1}^r$, say, such that every vector of the system $\{x_i\}_{i=1}^k$ is a linear combination of $\{\widetilde{x}_i\}_{i=1}^r$. Then the vectors $\{\widetilde{x}_i\}_{i=1}^r$ form a basis for the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$. Indeed, by the very definition of a linear manifold, every vector z can be expressed as a linear combination of a hyperfinite number of vectors of the system $\{x_i\}_{i=1}^k$. But, by hypothesis, each of these vectors can be expressed as a linear combination of $\{\widetilde{x}_i\}_{i=1}^r$. Thus eventually the vector z can also be expressed as a linear combination of the vectors $\{\widetilde{x}_i\}_{i=1}^r$. This, together with the assumption that the vectors $\{\widetilde{x}_i\}_{i=1}^r$ are linearly independent, shows that $\{\widetilde{x}_i\}_{i=1}^r$ indeed form a basis, as asserted. According to Theorem 4.15.9, the dimension of the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ is equal to the number r. Since there can be no more than r linearly independent vectors in an r-dimensional space, we get the following:

(a) If the number of vectors $\{x_i\}_{i=1}^k$ spanning $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ is larger than the number r, then the vectors $\{x_i\}_{i=1}^k$ are linearly dependent. If the number of these vectors equals r, then the vectors are linearly independent.

(b) Every set of $r+1$ vectors from the system $\{x_i\}_{i=1}^k$ is linearly dependent.

(c) The dimension of the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ can be defined as the maximum number of linearly independent vectors in the system $\{x_i\}_{i=1}^k$.

4.16.Algebra of external hyperfinite Polynomials.

Definition 4.16.1. A linear space V over field ${}^*\mathbb{R}_c^\#$ or over field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$ is called an algebra over ${}^*\mathbb{R}_c^\#$ (or over ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$) if there is defined on the elements $x,y,\ldots$ of V an operation of multiplication, denoted by $xy,$ which satisfies the following conditions:

(1) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every $x,y \in V$(or $\in {}^*\mathbb{C}_c^\#$) and every $\alpha \in {}^*\mathbb{R}_c^\#({}^*\mathbb{C}_c^\#)$;

(2) $(xy)z = x(yz)$ for every $x,y,z \in V$ (the associative law);

(3) $(x+y)z = xz + yz$ for every $x,y,z \in V$ (the distributive law).

In general, multiplication may not be commutative, i.e., we may have $xy \neq yx$.

Definition 4.16.2.If multiplication is commutative, i.e., if
(4) $xy = yx$ for every $x,y \in V,$ then the algebra V is said to be commutative.

Definition 4.16.3.An element $e \in V$ is called a left unit if $ex = x$ for every $x \in V$, a right unit if $xe = x$ for every $x \in V$, and a two-sided unit or simply a unit if $ex = xe = x$ for every $x \in V$.

Definition 4.16.4.An element $x \in V$ is called a left inverse of the element $y \in V$ if xy is the unit of the algebra V; in this case, y is called a right inverse of $x.$ If an element z has
both a left and a right inverse, then the two inverses are unique and in fact coincide. The element z is then said to be invertible, and its inverse is denoted by z^{-1}.
The product zu of an invertible element z and an invertible element u is an invertible element with inverse $u^{-1}z^{-1}.$ If the element u is invertible, then the equation $ux = v$ has the solution $x = u^{-1}v.$ This solution is unique, being obtained by multiplying the equation $ux = v$ on the left by u^{-1}. In the commutative case, we write $x = v/u$ or $x = v : u,$ calling the element x the quotient of the elements v and u.

Definition 4.16.5.An algebra V over field ${}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$) is said to have hyperfinite dimension n if V has dimension $n \in \mathbb{N}^\#\backslash\mathbb{N}$ regarded as a linear space. We will denote such algebra by V_n.

Example 4.16.1.An example of a nontrivial commutative algebra over a field ${}^*\mathbb{R}_c^\#({}^*\mathbb{C}_c^\#)$ is given by the set $\Pi^\#$ of all hyperfinite polynomials

$$P(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k, \qquad (4.16.1)$$

$m \in \mathbb{N}^\#\backslash\mathbb{N},$with coefficients in ${}^*\mathbb{R}_c^\#({}^*\mathbb{C}_c^\#)$, equipped with the usual operations of addition and multiplication. This "polynomial algebra" has a unit, namely the polynomial $e(\lambda)$ with $a_0 = 1$ and all other coefficients equal to 0.

Example 4.16.2.The linear Space $M_n({}^*\mathbb{R}_c^\#)$ of all matrices of order $n \in \mathbb{N}^\#\backslash\mathbb{N}$ with elements in ${}^*\mathbb{R}_c^\#$, with the usual definition of matrix multiplication, is an example of a hyperfinite dimensional noncommutative algebra of dimension n^2.

Example 4.16.3.A more general example of a hyperfinite dimensional noncommutative algebra $L_n({}^*\mathbb{R}_c^\#)$ with a unit is the linear space of all linear operators acting in a linear space $V_n, n \in \mathbb{N}^\#\backslash\mathbb{N}$ with the usual definition of operator multiplication.

Definition 4.16.6.A subspace $L \subset V_n$ is called a subalgebra of the algebra V_n if $x \in L,$ $y \in L,$implies $xy \in L.$ A subspace $L \subset V_n$ is called a right ideal in V_n if $x \in L, y \in K$ implies $xy \in L$ and a left ideal in V_n if $x \in L, y \in K$ implies $yx \in L.$ An ideal which is both a left and a right ideal is called a two-sided ideal. In a commutative algebra there is no distinction between left, right and two-sided ideals. There are two obvious two-sided ideals in every algebra V_n, i.e., the algebra V_n itself and the ideal $\{0\}$ consisting of the zero element alone. All other one-sided and two-sided ideals are called proper ideals. Every ideal is a subalgebra, but the converse is in general false. Thus the set of all polynomials $P(\lambda)$ satisfying the condition $P(0) = P(1)$ is a subalgebra of the algebra Π which is not an ideal, while the set of all polynomials $P(\lambda)$ satisfying the condition $P(0) = 0$ is a proper ideal of the algebra Π.

Definition 4.16.7.Let $L \subset V_n$ be a subspace of the algebra V_n, and consider the factor space V_n/L, i.e., the linear space consisting of the classes X of elements $x \in V_n$ which are comparable relative to L. If L is a two-sided ideal in V_n, then, besides linear operations, we can introduce an operation of multiplication for the classes $X \in V_n/L$. In fact, given two classes X and Y, choose arbitrary elements $x \in X, y \in Y$ and interpret XY as the class containing the product xy. This uniquely defines XY, since if $x' \in X, y \in Y$, then $xy' - xy = x'(x-y) + (x-x)y$,
and hence $xy - xy$ belongs to L together with $y' - y$ and $x' - x$. Moreover, since conditions (1)-(3) of Definition 4.16.1 hold in V_n, the analogous conditions hold for the classes $X \in V_n/L$. Therefore the factor space V_n/L equipped with the above operation of multiplication, is also an algebra, called the factor algebra of the algebra V_n with respect to the two-sided ideal L. If the algebra V_n is commutative, then obviously so is the factor algebra V_n/L.

Definition 4.16.7.Let V_n' and V_n'' be two algebras over a field ${}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$). Then a morphism ω of the space V_n' into the space V_n'' is called a morphism of the algebra V_n' into the algebra V_n'' if besides satisfying the two conditions:

(i) $\omega(x' + y') = \omega(x') + \omega(y')$ for every $x', y' \in V_n'$,

(ii) $\omega(\alpha x') = \alpha\omega(x')$ for every $x' \in V_n'$ and every $\alpha \in {}^*\mathbb{R}_c^\#(\alpha \in {}^*\mathbb{C}_c^\#)$,

(iii) $\omega(x'y') = \omega(x')\omega(y')$ for every $x', y' \in V_n'$.

Remark 4.16.1.Let ω be a morphism of an algebra V_n' into an algebra V_n''. Then the set If of all vectors $x' \in V_n'$ such that $\omega(x') = 0$, which is obviously a subspace of V_n', is a two-sided ideal of the algebra V_n'. In fact, if $x' \in L'$, $y' \in V_n'$, then $\omega(x'y') = \omega(x')\omega(y') = \mathbf{0}$,
so that $x'y' \in L'$, and similarly $y'x' \in L'$, i.e., L' is a two-sided ideal of V_n', as asserted. As in Remark 4.16.2.let Ω be the monomorphism of the space V_n'/L' into the space V_n'' which assigns to each class $X' \in V_n'/L'$ the (unique) element $\omega(x'), x' \in X'$. Then is a monomorphism of the algebra V_n'/L' into the algebra V_n''. In fact, choosing $x' \in X', y' \in Y'$, we have $x'y' \in X'Y'$ and

$$\Omega(\mathbf{X}'\mathbf{Y}') = \omega(x'y') = \omega(x')\omega(y') = \Omega(\mathbf{X}')\Omega(\mathbf{Y}').$$

If the morphism ω is an epimorphism of the algebra V' into the algebra V'', then the morphism Ω is an isomorphism of the algebra V'/L' onto the algebra V''.

Let $\mathbf{A}$ be a linear operator acting in a space V over a field ${}^*\mathbb{C}_c^\#$. Since addition and multiplication by constants in ${}^*\mathbb{C}_c^\#$ are defined for linear operators acting in V, with every

polynomial $P(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k$ we can associate the operator

$$P(\mathbf{A}) = Ext\text{-}\sum_{k=0}^{m} a_k\mathbf{A}^k \qquad (4.16.2)$$

acting in the same space V as $\mathbf{A}$ itself. Then the rule associating $P(\lambda)$ with $P(\mathbf{A})$ has the properties: (1) if

$$P(\lambda) = P_1(\lambda) + P_2(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k + Ext\text{-}\sum_{k=0}^{m} b_k\lambda^k = Ext\text{-}\sum_{k=0}^{m}(a_k + b_k)\lambda^k, \quad (4.16.3)$$

then

$$P(\mathbf{A}) = Ext\text{-}\sum_{k=0}^{m}(a_k + b_k)\mathbf{A}^k = Ext\text{-}\sum_{k=0}^{m} a_k\mathbf{A}^k + Ext\text{-}\sum_{k=0}^{m} b_k\mathbf{A}^k = P_1(\mathbf{A}) + P_2(\mathbf{A}). \quad (4.16.4)$$

Similarly (2) if

$$Q(\lambda) = P_1(\lambda)P_2(\lambda) = \left(Ext\text{-}\sum_{i=0}^{m} a_i\lambda^i\right)\left(Ext\text{-}\sum_{k=0}^{m} b_k\lambda^k\right) = Ext\text{-}\sum_{i=0}^{m}\left(Ext\text{-}\sum_{k=0}^{m} a_i b_k\lambda^{i+k}\right), \quad (4.16.5)$$

then

$$Q(\mathbf{A}) = Ext\text{-}\sum_{i=0}^{m}\left(Ext\text{-}\sum_{k=0}^{m} a_i b_k\mathbf{A}^{i+k}\right) = P_1(\mathbf{A})P_2(\mathbf{A}) \quad (4.16.6)$$

by the distributive law for operators .

Note that the operators $P_1(A)$ and $P_2(A)$ always commute with each other, regardless of the choice of the polynomials $P_1(X)$ and $P_2(X)$.

The resulting morphism of the algebra $\Pi^\#$ of polynomials into the algebra $L_n({}^*\mathbb{R}_c^\#)$ of linear operators acting in V_n (Example 4.16.3) is in general not an epimorphism, if only because operators of the form $P(\mathbf{A})$ commute with each other, while the whole algebra $L_n({}^*\mathbb{R}_c^\#)$ is noncommutative.

There exists an isomorphism between the algebra $L_n({}^*\mathbb{R}_c^\#)$ of all linear operators acting
in the n-dimensional space V_n and the algebra $M_n({}^*\mathbb{R}_c^\#)$ of all matrices of order n with elements from the field ${}^*\mathbb{R}_c^\#$.

This isomorphism is established by fixing a basis $e_1\ldots,e_n$ in the space V_n and assigning
for every operator $\mathbf{A} \in M_n({}^*\mathbb{R}_c^\#)$ its matrix in this basis. Both algebras $L_n({}^*\mathbb{R}_c^\#)$ and $M_n({}^*\mathbb{R}_c^\#)$ have the same hyperfinite dimension n^2.

The set of all hyperfinite polynomials of the form $P(\lambda)Q_0(\lambda)$, where $Q_0(\lambda)$ is a fixed polynomial and $P(\lambda)$ an arbitrary polynomial, is obviously an ideal in the commutative algebra $\Pi^\#$ of all polynomials $P(\lambda)$ with coefficients in a field ${}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$) (Example 4.16.1).

Conversely, we now show that every ideal $\mathbf{I} \neq \{0\}$ of the algebra $\Pi^\#$ is of this structure,
i.e., is obtained from some polynomial $Q_0(\lambda)$ by multiplication by an arbitrary polynomial
hyperfinite $P(\lambda)$. To this end, we find the nonzero polynomial of lowest degree, say q, in
the ideal $\mathbf{I}$, and denote it by $Q_0(\lambda)$. We then assert that every polynomial $Q(\lambda)$ in $\mathbf{I}$ is of the form $P(\lambda)Q_0(\lambda)$, where $P(\lambda) \in \Pi^\#$. In fact, as is familiar from elementary algebra,

$$Q(\lambda) = P(\lambda)Q_0(\lambda) + R(\lambda), \quad (4.16.7)$$

where $R(\lambda)$ is the quotient obtained by dividing $Q(\lambda)$ by $Q_0(\lambda)$ and $P(\lambda)$ is the remainder,
of degree less than the divisor $Q_0(\lambda)$, i.e., less than the number q. But the polynomials $Q(\lambda)$ and $Q_0(\lambda)$ belong to the ideal $\mathbf{I}$, and hence, as is apparent from (4.16.7), so does the remainder $P(\lambda)$. Since the degree of $P(\lambda)$ is less than q and since $Q_0(\lambda)$ has the lowest degree, namely q, of all nonzero polynomials in $\mathbf{I}$, it follows that $P(\lambda) = 0$, and the requireed assertion is proved.The polynomial $Q_0(\lambda)$ is said to generate the ideal $\mathbf{I}$.

Remark 4.16.1.The polynomial $Q_0(\lambda)$ is uniquely determined by the ideal $\mathbf{I}$ to within a numerical factor. In fact, if the polynomial $Q_1(\lambda)$ has the same property as the polynomial $Q_0(\lambda)$, then, as just shown, $Q_1(\lambda) = P_1(\lambda)Q_0(\lambda), Q_n(\lambda) = P_0(\lambda)Q_1(\lambda)$.
It follows that the degrees of the polynomials $Q_1(\lambda)$ and $Q_0(\lambda)$ coincide and that $P_1(\lambda)$ and $P_0(\lambda)$ do not contain λ and hence are numbers, as asserted.

Remark 4.16.2.Given polynomials $Q_1(\lambda),\dots,Q_m(\lambda)$ not all equal to zero and with no common divisors of degree $> I$, we now show that there exist polynomials $P_1^0(\lambda),\dots,P_m^0(\lambda)$ such that

$$Ext\text{-}\sum_{i=0}^{m} P_i^0(\lambda)Q_i(\lambda) \equiv 1. \quad (4.16.8)$$

In fact, let $\mathbf{I}$ be the set of all polynomials of the form

$$Ext\text{-}\sum_{i=0}^{m} P_i^0(\lambda)Q_i(\lambda) \quad (4.16.9)$$

with arbitrary $P_1(\lambda),\dots,P_m(\lambda)$ in $\Pi^{\#}$. Then $\mathbf{I}$ is obviously an ideal in $\Pi^{\#}$.
In particular

$$Q_1(\lambda) = S_1(\lambda)G_0(\lambda),\dots,Q_m(\lambda) = S_m(\lambda)G_0(\lambda), \quad (4.16.10)$$

where $S_1(\lambda),\dots,S_m(\lambda)$ are certain polynomials, from which it follows that $Q_0(\lambda)$ is a common divisor of the polynomials $Q_1(\lambda),\dots,\ Q_m(\lambda)$. But, by hypothesis, the degree of $Q_0(\lambda)$ is zero, and hence $Q_0(\lambda)$ is a constant a_0, where $a_0 \neq 0$ since otherwise $\mathbf{I} = \{0\}$. Multiplying (4.16.9) by a_0^{-1} and writing $P_k^0(\lambda) = \widetilde{P}_k^0(\lambda)a_0^{-1}$, we get (4.16.8), as required.

4.17.Canonical Form of the Matrix of an Arbitrary Operator

Let $\mathbf{A}$ denote an arbitrary linear operator acting in an n-dimensional space $V_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ Since the operations of addition and multiplication are defined for such operators , with every hyperfinite external polynomial

$$P(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k \quad (4.17.1)$$

we can associate an operator

$$P(\mathbf{A}) = Ext\text{-}\sum_{k=0}^{m} a_k\mathbf{A}^k \quad (4.17.2)$$

acting in the same space V_n, where addition and multiplication of polynomials corresponds to addition and multiplication of the associated operators in the sense of Sec. 4.16. In fact, if

$$P(\lambda) = P_1(\lambda) + P_2(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k + Ext\text{-}\sum_{k=0}^{m} b_k\lambda^k = Ext\text{-}\sum_{k=0}^{m}(a_k + b_k)\lambda^k, \quad (4.17.3)$$

then

$$P(\mathbf{A}) = Ext\text{-}\sum_{k=0}^{m}(a_k + b_k)\mathbf{A}^k = Ext\text{-}\sum_{k=0}^{m} a_k\mathbf{A}^k + Ext\text{-}\sum_{k=0}^{m} b_k\mathbf{A}^k = P_1(\mathbf{A}) + P_2(\mathbf{A}). \quad (4.17.4)$$

Similarly, if

$$Q(\lambda) = P_1(\lambda)P_2(\lambda) = \left(Ext\text{-}\sum_{i=0}^{m} a_i\lambda^i\right)\left(Ext\text{-}\sum_{k=0}^{m} b_k\lambda^k\right) = Ext\text{-}\sum_{i=0}^{m}\left(Ext\text{-}\sum_{k=0}^{m} a_i b_k\lambda^{i+k}\right), \quad (4.17.5)$$

then

$$Q(\mathbf{A}) = Ext\text{-}\sum_{i=0}^{m}\left(Ext\text{-}\sum_{k=0}^{m} a_i b_k\mathbf{A}^{i+k}\right) = P_1(\mathbf{A})P_2(\mathbf{A}) \quad (4.17.6)$$

by the distributive law for operator multiplication. In particular, the operators $P_1(\mathbf{A})$ and $P_2(\mathbf{A})$ always commute.

Thus the mapping $\omega(P(\lambda)) = P(\mathbf{A})$ is an epimorphism of the algebra $\Pi^{\#}$ of all hyperfinite polynomials with coefficients in the field ${}^*\mathbb{R}_c^{\#}$ (${}^*\mathbb{C}_c^{\#}$) into the algebra $\Pi_{\mathbf{A}}^{\#}$ of all linear operators of the form $P(\mathbf{A})$ acting in the space V_n. By Sec. 4.16, the algebra $\Pi_{\mathbf{A}}^{\#}$ is isomorphic to the factor algebra $\Pi_{\mathbf{A}}^{\#}/\mathbf{I}_A$, where $\mathbf{I}_A$ is the ideal consisting of all polynomials $P(\lambda)$ such that $\omega(P(\lambda)) = P(\mathbf{A}) = 0$.

We now analyze the structure of this ideal.

As noted in Example 4.16.3, the set of all linear operators acting in a space $V_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is an algebra of hyperfinite dimension n^2 over the field ${}^*\mathbb{R}_c^{\#}$ (${}^*\mathbb{C}_c^{\#}$). Hence, given any operator $\mathbf{A}$, it follows that the first $n^2 + 1$ terms of the hyperfinite sequence $\mathbf{A}_0 = \mathbf{E}, \mathbf{A}, \mathbf{A}^2, , , , ., \mathbf{A}^m, \ldots$ must be linearly dependent. Suppose that

$$Ext\text{-}\sum_{k=0}^{m} a_k\mathbf{A}^k = 0, \quad (4.17.7)$$

where $m \leq n^2$. Then, by the correspondence between polynomials and operators mentioned above the hyperfinite polynomial

$$Q(\lambda) = Ext\text{-}\sum_{k=0}^{m} a_k\lambda^k \quad (4.17.7)$$

must correspond to the zero operator. Every polynomial $Q(\lambda)$ for which the operator $Q(\mathbf{A})$ is the zero operator is called an annihilating polynomial of the operator $\mathbf{A}$, Thus we have just shown that every operator $\mathbf{A}$ has an annihilating polynomial of degree $\leq n^2$. The set of all annihilating polynomials of the operator $\mathbf{A}$ is an ideal in the algebra $\Pi^{\#}$. By Sec. 4.16 there is a polynomial $Q_0(\lambda)$ uniquely determined to within a numerical factor such that all annihilating polynomials are of the form $P(\lambda)Q_0(\lambda)$ where $P(\lambda)$ is an arbitrary polynomial in $\Pi^{\#}$, In particular, $Q_0(\lambda)$ is the annihilating polynomial of lowest degree among all annihilating polynomials of the operator $\mathbf{A}$. Hence $Q_0(\lambda)$ is called the minimal annihilating polynomial of the operator $\mathbf{A}$.

Theorem 4.17.1. Let $Q(\lambda)$ be an annihilating polynomial of the operator $\mathbf{A}$, and suppose that $Q(\lambda) = Q_1(\lambda)Q_2(\lambda)$,where the factors $Q_1(\lambda)$ and $Q_2(\lambda)$ are relatively prime. Then the space V_n can be represented as the direct sum $V_n = \mathbf{T}_1 \oplus \mathbf{T}_2$ of two subspaces $\mathbf{T}_1$ and $\mathbf{T}_2$ both invariant with respect to the operator $\mathbf{A}$, where $Q_1(\mathbf{A})x_2 = 0, Q_2(\mathbf{A})x_1 = 0$ for arbitrary $x_1 \in \mathbf{T}_1, x_2 \in \mathbf{T}_2$, so that $Q_1(\lambda)$ and $Q_2(\lambda)$ are annihilating polynomials for the operator $\mathbf{A}$ acting in the subspaces $\mathbf{T}_2$ and $\mathbf{T}_1$, respectively.

Proof. By Sec. 4.16 there exist polynomials $P_1(\lambda)$ and $P_2(\lambda)$ such that

$$P_1(\lambda)Q_1(\lambda) + P_2(\lambda)Q_2(\lambda) \equiv 1, \qquad (4.17.8)$$

and hence

$$P_1(\mathbf{A})Q_1(\mathbf{A}) + P_2(\mathbf{A})Q_2(\mathbf{A}) \equiv \mathbf{E}. \qquad (4.17.9)$$

Let $\mathbf{T}_k, k = 1,2$ denote the range of the operator $Q_k(\mathbf{A})$, i.e., the set of all vectors of the form $Q_k(\mathbf{A})x, x \in V_n$. Then obviously $y = Q_k(\mathbf{A})x \in \mathbf{T}_k$ implies $\mathbf{A}y = Q_k(\mathbf{A})\mathbf{A}x \in \mathbf{T}_k$, so that the subspace $\mathbf{T}_k$ is invariant with respect to the operator $\mathbf{A}$,

Given any $x_1 \in \mathbf{T}_1$, there is a vector $y \in V_n$ such that
$Q_2(\mathbf{A})x_1 = Q_1(\mathbf{A})Q_2(\mathbf{A})z = Q(\mathbf{A})z = \mathbf{0}$, and similarly, given any $x_2 \in \mathbf{T}_2$, there is a vector $z \in V_n$ such that $Q_1(\mathbf{A})x_2 = Q_1(\mathbf{A})Q_2(\mathbf{A})z = Q(\mathbf{A})z = \mathbf{0}$.

Moreover, given any $x \in V_n$, we have
$x = Q_1(\mathbf{A})P_1(\mathbf{A})x + Q_2(\mathbf{A})P_2(\mathbf{A})x = x_1 + x_2$, where
$x_k = Q_k(\mathbf{A})P_k(\mathbf{A})x \in \mathbf{T}_k,\ k = 1,2.$

It follows that V_n is the sum of the subspaces $\mathbf{T}_1$ and $\mathbf{T}_2$. If $x_0 \in \mathbf{T}_1 \cap \mathbf{T}_2$, then $Q_1(\mathbf{A})x_0 = Q_2(\mathbf{A})x_0 = \mathbf{0}$, and hence $x_0 = P_1(\mathbf{A})Q_1(\mathbf{A})x_0 + P_2(\mathbf{A})Q_2(\mathbf{A})x_0 = 0$.
Therefore $\mathbf{T}_1 \cap \mathbf{T}_2 = \{0\}$, and the sum $V_n = \mathbf{T}_1 \oplus \mathbf{T}_2$ is direct.

Remark 4.17.1. By construction, the operator $Q_1(\mathbf{A})$ annihilates the subspace $\mathbf{T}_2$, while the operator $Q_2(\mathbf{A})$ annihilates the subspace $\mathbf{T}_1$, We now show that every vector x annihilated by the operator $Q_1(\mathbf{A})$ belongs to $\mathbf{T}_2$, while every vector x annihilated by the operator $Q_2(\mathbf{A})$ belongs to $\mathbf{T}_1$. In fact, suppose $Q_1(\mathbf{A})x = 0$. We have $x = x_1 + x_2$, where $x_1 \in \mathbf{T}_1, x_2 \in \mathbf{T}_2$, and hence $Q_1(\mathbf{A})x_1 = Q_1(\mathbf{A})x - Q_1(\mathbf{A})x_2 = 0$ since $Q_1(\mathbf{A})x_2 = 0$. But $Q_2(\mathbf{A})x_1 = 0$ as well, since $x_1 \in \mathbf{T}_1$. It follows that
$x_1 = P_1(\mathbf{A})Q_1(\mathbf{A})x_1 + P_2(\mathbf{A})Q_2(\mathbf{A})x_1 = \mathbf{0}, x = x_2 \in \mathbf{T}_2$.
Similarly, $Q_2(\mathbf{A})x = 0$ implies $x \in \mathbf{T}_1$, and our assertion is proved.

Remark 4.17.2.Representing the polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$ themselves as products of further prime factors, we can decompose the space V_n into smaller subspaces invariant with respect to the operator $\mathbf{A}$ and annihilated by the appropriate factors of $Q_1(\lambda)$ and $Q_2(\lambda)$. Suppose the annihilating polynomial $Q(\lambda)$ has a factorization of the form

$$Q(\lambda) = Ext\text{-}\prod_{k=1}^{m}(\lambda - \lambda_k)^{r_k}, \qquad (4.17.10)$$

where $\lambda_1, \ldots, \lambda_m$ are all the (distinct) roots of $Q(\lambda)$ and r_k is the multiplicity of λ_k. For example, such a factorization is always possible (to within a numerical factor) in the field ${}^*\mathbb{C}_c^\#$.

Theorem 4.17.2. Suppose the operator A has an annihilating polynomial of the form

(4.17.10). Then the space $V_n, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ can be represented as the direct sum

$$V_n = \bigoplus_{k=1}^{m} T_k \qquad (4.17.11)$$

of m subspaces $T_1, \ldots, T_m$, all invariant with respect to $\mathbf{A}$, where the subspace Tfc is annihilated by Bfc*, the rkth power of the operator $\mathbf{B}_k = \mathbf{A} - \lambda_k \mathbf{E}$.

Proof. Apply Theorem **4.17.1** repeatedly to the factorization (4.17.10) of $Q(\lambda)$ into m relatively prime factors of the form $(\lambda - \lambda_j)^{r_j}$.

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4.17. Spectra and external hyperfinite Polynomials.

Definition 4.17.1.By a spectrum, denoted by $\mathbf{Spec}$, we mean any hyperfinite set of numbers $\lambda_1, \ldots, \lambda_n \in {}^*\mathbb{R}_c^{\#}$ or $\lambda_1, \ldots, \lambda_n \in {}^*\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + i\,{}^*\mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}/\mathbb{N}$ where it is assumed that each point λ_k is assigned a “multiplicity,” i.e., a positive integer $r_k, 1 \leq k \leq m, m \in \mathbb{N}^{\#}$ denoted by

$$\mathbf{Spec} = \{\lambda_1^{r_1}, \ldots, \lambda_m^{r_m}\}. \qquad (4.16.1)$$

Moreover, we assume that each point λ_k is assigned a set of r_k numbers from the field ${}^*\mathbb{R}_c^{\#}$ (or ${}^*\mathbb{C}_c^{\#}$), denoted by

$$(4.16.2)$$

Such a set of numbers will be called a jet /, defined on $\mathbf{Spec}$.

We now introduce the following algebraic operations in Jr(S), the set of all jets on a given

spectrum S:

a. Addition of jets. By the sum f + g of two jets f = {/°>(Xfc)} and g — {g0>(Xfc)} we mean the jet defined by the set of numbers

(fc = I,. . . , m\j = 0, I,. . . , rk — I).

b. Multiplication of a jet by a number. By the product of of a jet f = {f{i)(Xfc)) and a number a e K we mean the jet defined by the set of numbers

= *f iiKW-

These two operations obviously convert the set Jr(S) into a linear space, whose zero element is the jet 0 whose “components” are all zero.

c. Multiplication of jets. By the product fg of two jets f = {/(Я(XA.)} and g — {#ш(^й:)} we mean the jet defined byt

(fg)(\) =/(XfcJg(Xfc),

(Jg)XW =f(Wg’(W +/’(XfcJg(Xfc),

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5.Invariant subspaces in external hyperfinite dimensional linear space over external non-Archimedean field ${}^*\mathbb{R}_c^\#$.

5.1.Basic results and definitions.

Definition 5.1.1. Let $V_n, n \in \mathbb{N}^\#/\mathbb{N}$ be a hyperfinite-dimensional vector space over external non-Archimedean field ${}^*\mathbb{R}_c^\#$ [4].Such vector space consists of all external and internal ${}^*\mathbb{R}_c^\#$-valued hyperfinite sequences (called a vector) $\mathbf{x} = \{x_i\}_{i=1}^{i=n} = \{x_i\}_{i\in n}$ of hyperreal numbers, called the coordinates or components of vector $\mathbf{x}$. The vector sum of $\mathbf{x} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{x}+\mathbf{y} = \{x_i + y_i\}_{i=1}^{i=n}. \quad (5.1.1)$$

If $a \in \mathbb{R}_c^\#$ is a hyperreal number, the scalar multiple of $\mathbf{x}$ by a is

$$a \times \mathbf{x} = \{a \times x_i\}_{i=1}^{i=n}. \quad (5.1.2)$$

there is a canonical hyperfinite basis $\{e_i\}\ 1 \leq i \leq n, n \in \mathbb{N}^\#/\mathbb{N}$ such that $e_1 = (1,0,\ldots), e_2 = (0,1,\ldots)$, etc.

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5.2.Invariant subspaces in external hyperfinite dimensional linear space.

Definition 5.2.1. Let V be a nonzero F-vector space. Let $T \in \mathbf{End}(V)$ be a linear endomorphism of V. A T-invariant subspace of V is a subspace $W \subset V$ such that $T(W) \subset W$.

Remark 5.2.1.Actually though we will just say “invariant subspace”: throughout this section we work with only one endomorphism at a time, so the dependence on T in the terminology and notation will usually be suppressed.

Definition 5.2.2.We call invariant subspaces $\{0\}$ and V trivial subspaces.

Definition 5.2.3.We say V is simple if it has no nontrivial invariant subspaces. We say V is semisimple if it is a direct sum of simple invariant subspaces. We say V is diagonalizable if there is a basis $\{e_i\},\ i \in \mathbf{n} \in \mathbb{N}^\#\backslash\mathbb{N}$ such that for all $i \in I, Te_i \in \langle e_i \rangle$ equivalently, V is a direct sum of one-dimensional invariant subspaces: i.e. $V = Ext\text{-}\bigoplus_{i\in\mathbf{n}}\langle e_i \rangle$. Thus diagonalizable implies semisimple.

Theorem 4.2.1. The following are equivalent:

(i) V is semisimple.

(ii) If $W \subset V$ is an invariant subspace, it has an invariant complement: i.e., there is an invariant subspace W' such that $V = W \oplus W'$.

(iii) V is spanned by its simple invariant subspaces.

Proof.(i) ⇒ (ii): Suppose $V = \bigoplus_{i\in I} S_i$, with each S_i a simple invariant. For each

$J \subset I$, put $V_J = \bigoplus_{i \in I} S_i$. Now let W be an invariant subspace of V. There is a maximal subset $J \subset I$ such that $W \cap V_J = 0$. For $i \notin J$ we have $(V_J \oplus S_i) \cap W \neq 0$, so choose $0 \neq x = y + z, x \in W, y \in V_J, z \in S_i$. Then $z = x - y \in (Vj + W) \cap S_i$,and if $z = 0$, then $x = y \in W \cap V_j = 0$, contradiction.
So $(V_J \oplus W) \cap S_i \neq 0$. Since S_i is simple, this forces $S_i \subset V_J \oplus W$. It follows that $V = V_J \oplus W$.
(ii) $\Rightarrow$ (i): The hypothesis on V passes to all invariant subspaces of V. We claim that every nonzero invariant subspace $C \subset V$ contains a simple invariant subspace.
Proof of claim: Choose $0 \neq c \in C$, and let D be an invariant subspace of C that is maximal with respect to not containing c. By the observation of the previous subsection, we may write $C = D \oplus E$. Then E is simple. Indeed, suppose not and let $0 \subsetneqq F \subsetneqq E$. Then $E = F \oplus G$ so $C = D \oplus F \oplus G$. If both $D \oplus F$ and $D \oplus G$ contained c, then $c \in (D \oplus F) \cap (D \oplus G) = D$, contradiction. So either $D \oplus F$ or $D \oplus G$ is a strictly larger invariant subspace of C than D which does not contain c, contradiction. So E is simple, establishing our claim. Now let $W \subset V$ be maximal with respect to being a direct sum of simple invariant subspaces, and write $V = W \oplus C$. If $C \neq 0$, then by the claim C contains a nonzero simple submodule, contradicting the maximality of W. Thus $C = 0$ and V is a direct sum of simple invariant subspaces.
(i) $\Rightarrow$ (iii) is immediate.
(iii) $\Rightarrow$ (i): There is an invariant subspace W of V that is maximal with respect to being a direct sum of simple invariant subspaces. We must show $W = V$. If not, since V is assumed to be generated by its simple invariant subspaces, there exists a simple invariant subspace $S \subset V$ that is not contained in W. Since S is simple we have $S \cap W = 0$ and thus $W + S = W \oplus S$ is a strictly larger direct sum of simple invariant subspaces than W : contradiction.
Proposition 4.2.1. Both the kernel and image of T are invariant subspaces.
Proof.If $v \in \mathbf{Ker}(T)$ then $Tv = 0$, and then $T(Tv) = T\mathbf{0} = \mathbf{0}$, so $Tv \in \mathbf{Ker}(T)$. As for the image $T(V)$ we have $T(T(V)) \subset T(V)$.
Remark 4.2.2.As a modest generalization of the invariance of $T(V)$, we observe that if $W \subset V$ is an invariant subspace, then $T(W) \subset W$ so $T(T(W)) \subset T(W)$ and thus $T(W)$ is also an invariant subspace. It follows that $T^2(W) = T(T(W))$ is an invariant subspace, and so forth: we get a descending sequence of invariant subspaces:

$$W \supset T(W) \supset T^2(W) \supset \ldots \supset T^n(W) \supset \ldots$$

If W is hyperfinite-dimensional, this sequence eventually stabilizes on $m \in \mathbb{N}^\# / \mathbb{N}$.
In general it need not.
Similarly $\mathbf{Ker}(T^k)$ is an invariant subspace for all $k \in \mathbb{N}^\#$, as is easily checked. This yields an increasing sequence of invariant subspaces $0 \subset \mathbf{Ker}(T) \subset \mathbf{Ker}(T^2) \subset \ldots$
A toothier generalization is the following.
Proposition 4.2.4. Let $f(T) \in F[t]$. Then $\mathbf{Ker}(f(T))$ and $\mathbf{Image}(f(T))$ are invariant subspaces of V.
Proof. Suppose $v, w \in \mathbf{Ker}(f(T))$. Then for all $\alpha \in F$ we have $f(T)(\alpha v + w) = \alpha f(T)v + f(T)w = 0$,so $\mathbf{Ker}(f(T))$ is a subspace. If $v \in \mathbf{Ker}(f(T))$, then $f(T)(Tv) = T(f(T)v) = T\mathbf{0} = \mathbf{0}$,so $Tv \in \mathbf{Ker}(f(T))$.
Lemma 4.2.1. Let $W_1, W_2 \subset V$ be invariant subspaces. Then $W_1 + W_2$ and $W_1 \cap W_2$ are invariant subspaces.
Definition 4.2.4.Let $v \in V$. **The orbit of $\mathbf{T}$ on $\mathbf{V}$** is the set $\{T^k v\}_{k=0}^{\infty^\#}$;

the **linear orbit** $[v]$ of v is the subspace spanned by the orbit of T on v.
Definition 4.2.5.For $v \in V$, there is a natural linear map given by evaluation at v :
1.$E_v : \mathbf{End}[V] \to V, A \mapsto Av$.
For $T \in \mathbf{End}[V]$, there is a natural linear map given by evaluation at T :
2.$E_T : F[t] \to \mathbf{End}[V], p(t) \mapsto p(T)$.
Consider the composition of these maps:
3.$E = E_{T,v} := E_v \circ E_T : F[t] \to V, p \mapsto p(T)v$.
Lemma 4.2.2. (a) The image $E(F[t])$ of the map E is $[v]$, the linear orbit of v.
(b) The kernel $\mathbf{K}$ of E is an ideal of $F[t]$.
Proof. (a) This follows immediately upon unwinding the definitions.
(b) Since E is an F-linear map, its kernel is an F-subspace of $F[t]$. However E is not a ring homomorphism (e.g. because V is not a ring!) so we do need to check that K is an ideal. But no problem: suppose $p \in K$ and $q \in F[t]$. Then $E(qp) = (q(T)p(T))v = q(T)(p(T)v) = q(T)\mathbf{0} = \mathbf{0}$.
Recall that every ideal $\mathbf{I}$ of $F[t]$ is principal: this is clear if $\mathbf{I} = (\mathbf{0})$; otherwise $\mathbf{I}$ contains a monic polynomial a(t) of least degree. Let $b(t) \in \mathbf{I}$. By polynomial division, there are $q(t), r(t) \in F[t]$ with $\deg r < \deg a$ such that $b(t) = q(t)a(t) + r(t)$. But $r(t) = b(t) - q(t)a(t) \in \mathbf{I}$. If $r(t) \neq 0$, then multiplying by the inverse of its leading coefficient, we would get a monic polynmial in $\mathbf{I}$ of degree smaller than that of $a(t)$, contradicting the definition of $a(t)$. So $r(t) = 0$ and $\mathbf{I} = (a(t))$.
Consider the ideal $\mathbf{K}$ of $F[t]$ defined in Lemma 4.2.2. There is a clear dichotomy:
Case 1: $\mathbf{K} = \mathbf{0}$. In this case $E : F[t] \cong [v]$, so every invariant subspace of V containing v is hyper infinite-dimensional. We put $P_v(t) = 0$ (the zero polynomial, which generates $\mathbf{K}$). In this case we say that the vector v is **transcendental**.
Case 2: $\mathbf{K} = (P_v(t))$ for some nonzero monic $P_v(t)$. Then $E : F[t]/K \overset{\sim}{\to} [v]$, so $\deg(P_v) = \dim[v]$ are both hyperfinite. We say that the vector v is **algebraic**.
In either case we call P_v the **local minimal polynomial of $\mathbf{T}$ at $\mathbf{v}$**.
Lemma 4.2.3.Let $v \in V$ be a transcendental vector.
(a) For every monic polynomial $f(t) \in F[t], f(T)[v]$ is an invariant subspace of V.
(b) For distinct monic polynomials f_1, f_2 , the invariant subspaces $f_1(T)[v], f_2(T)[v]$ are distinct. Thus [v] has hyper infinitely many invariant subspaces and is not simple.
(c) Every nonzero invariant subspace of $[v]$ is of the form $f(T)[v]$ for some monic polynomial $f(t) \in F[t]$.
Proof.(a) Apply Proposition 4.2.4 with $[v]$ in place of V.
(b) We claim $f_1(T)v \in f_2(T)[v]$ if and only if $f_2(t)|f_1(t)$; if so, $f_1(T)[v] = f_2(T)[v]$ implies $f_1|f_2$ and $f_2|f_1$ so $f_1 = f_2$. If $f_2|f_1$, write $f_1 = gf_2$ and then $f_1(T)v = g(T)f_2(T)v \in f_2(T)[v]$. Conversely, if $f_1(T)v \in f_2(T)[v]$, then there is a polynomial $g(t) \in F[t]$ such that $f_1(T)v = f_2(T)g(T)v$, so $(f_1(T) - f_2(T)g(T))v = 0$, and thus the local minimal polynomial of v divides $f_1(t) - f_2(t)g(t)$. But since v is transcendental, its local minimal polynomial is zero and thus $0 = f_1 - f_2 g$ and thus $f_2|f_1$. The second sentence of part b) follows immediately.
(c) Let $W \subset [v]$ be a nonzero invariant subspace. It therefore contains a nonzero vector, which may be written as $f(T)v$ for a monic polynomial $f \in F[t]$. Among all nonzero vectors choose one which may be written in this way with f(t) of least degree: we claim $W = f(T)[v]$. Indeed, consider any nonzero $w = g(T)v \in W$. By polynomial division there are $q(t), r(t) \in F[t]$ with $\deg(r) < \deg(f)$ such that

$g(t) = q(t)f(t) + r(t)$, and thus $w = q(T)f(T)v + r(T)v$. Then $r(T)v = w - q(T)f(T)v \in W$; since $\deg(r) < \deg(q)$ we get a contradiction unless $r = 0$, in which case $w = q(T)f(T)v = f(T)(q(T)v) \in f(T)[v]$.

Lemma 4.2.4.Consider the linear map $F[t] \to \mathbf{End}[V]$ given by $p(t) \mapsto P(T)$.
(a) Its kernel M is an ideal of $F[t]$, and thus of the form $(P(t))$ where $P(t)$ is either monic or zero. It is called the **minimal polynomial** of T on V.
(b) For all $v \in V$, $P_v(t)|P(t)$.
(c) $P(t)$ is the least common multiple of $\{P_v(t)\}_{v\in V}$.
(d) Suppose that $V = [v]$ for some $v \in V$, then $P(t) = P_v(t)$.

Definition 4.2.6.We say that V is **locally algebraic** if each vector $v \in V$ is algebraic, i.e., that for all $v \in V$, the local minimal polynomial $P_v(t)$ is nonzero. We say that V is algebraic if the minimal polynomial P is nonzero.

Proposition 4.2.4. (a) If V is hyperfinite-dimensional, it is algebraic: $P_V \neq 0$.
(b) If V is algebraic, it is locally algebraic.

Proof. (a) Let P be the minimal polynomial of V. By Lemma 4.2.3 we have an injection $F[t]/(P) \hookrightarrow \mathbf{End}[V]$. Since V is hyperfinite-dimensional, so is $\mathbf{End}[V]$, hence so is $F[t]/(P)$, which implies $P \neq 0$.
(b) This is immediate from the fact that $P_v|P$ for all $v \in V$.

5.3.Invariant subspaces in external hyperfinite dimensional linear space over external non-Archimedean field ${}^*\mathbb{R}^{\#}_c$.

Remark 5.3.1.From now on we assume that V is nonzero and hyperfinite-dimensional over field ${}^*\mathbb{R}^{\#}_c$.

Proposition 5.3.1. The degree of the minimal polynomial is at most $\dim(V) \in \mathbb{N}^{\#}\backslash\mathbb{N}$.

Proof. We go by <u>hyper infinite induction</u> on $\dim(V)$, the case $\dim(\mathbf{V}) = 1$ being handled, for instance, by the bound $\dim(P) \leq (\dim(V))^2$ .Now let $\dim(V) = \mathbf{d} \in \mathbb{N}^{\#}\backslash\mathbb{N}$ and suppose the result holds in smaller dimension. Choose a nonzero $v \in V$,and let $P_v$be the local minimal polynomial, so deg $P_v > 0$. Let $W = \mathbf{Ker}(P_v(T))$, so that W is a nonzero invariant subspace. If $W = V$ then $P_v = P$ and we're done. Otherwise we consider the induced action of T on the quotient space V/W. Let P_W and $P_{V/W}$ be the minimal polynomials of T on W and V/W. By <u>hyper infinite induction</u>, $\deg P_W \leq \dim(W)$ and $\deg(P_{V/W}) \leq \dim(V/W)$, so $\deg(P_W P_{V/W}) = \deg(PW) + \deg(P_{V/W}) \leq \dim(W) + \dim(V/W) = \dim(V)$. Finally, we claim that $P_W(T)P_{V/W}(T)V = 0$. Indeed, for all $v \in V, P_{V/W}(T)v \in W$ so $P_W(T)P_{V/W}(T)v = 0$.

Definition 5.3.1.For an invariant subspace $W \subset V$, we let P_W be the minimal polynomial of $P|W$.

Proposition 5.3.1. Let $W \subset V$ be an invariant subspace.Then
a) $P_W = lcm\{P_w\}_{w\in W}$.
b) For all $v \in V$, $P_{\langle v\rangle} = P_v$.

Proposition 5.3.2. Let W_1, W_2 be invariant subspaces of V; put $W = W_1 + W_2$. Then $P_W = lcm\,\{P_{W_1}, P_{W_2}\}$.

Proof. Put $P = P_{W_1+W_2}, P_1 = P_{W_1} = Ext\text{-}\prod_{i=1}^r p_i^{a_i}$, $P_2 = P_{W_2} = Ext\text{-}\prod_{i=1}^r p_i^{b_i}$ with $a_i, b_i \in \mathbb{N}^{\#}$, and then $P_3 = lcm(P_1,P_2) = Ext\text{-}\prod_{i=1}^r p_i^{\max(a_i,b_i)}$.
We may write $P_3 = f_1 P_1 = f_2 P_2$. Then every vector $w \in W_1 + W_2$ is of the form $w = w_1 + w_2$ for $w_1 \in W_1, w_2 \in W_2$ and $P_3(T)w = P_3(T)w_1 + P_3(T)w_2 = f_1(T)P_1(T)w_1 +$

$+f_2(T)P_2(T)w_2 = 0,$ so $P|P_3$. To show that $P_3|P$, since $P = lcm\{P_v\}_{v\in W_1+W_2}$ and $lcm\left(Ext\text{-}\prod_{i=1}^r p_i^{c_i}\right) = Ext\text{-}\prod_{i=1}^r p_i^{c_i}$, it is enough to find for each $1 \le i \le r$ a vector $v_i \in W_1 + W_2$ such that $p_i^{\max(a_i,b_i)}|P_{v_i}$. But since $p_i^{a_i}|P_1$, there is $w_{i,1} \in W_1$ with $p_i^{a_i}|w_{i,1}$ and $w_{i,2} \in W_2$ with $p_i^{b_i}|w_{i,2}$. One of these vectors does the job.
For any polynomial $f \in F[t]$, put $V_f := \{v \in V|f(T)v = 0\}$.
Proposition 5.3.3. Let W be an invariant subspace of V, and let $f|P_V$. Then $W_f = W \cap V_f$.
Proof. Although this is a distinctly useful result, its proof is absolutely trivial: $W_f = \{v \in W|f(T)v = 0\} = W \cap \{v \in V|f(T)v = \mathbf{0}\}$.
Note that $V_0 = V$. Henceforth we restrict attention to nonzero polynomials.
Proposition 5.3.4. Let $f \in F[t]$.
(a) V_f is an invariant subspace of V.
(b) V_f is the set of vectors v such that $P_v|f$.
(c) If $f|g$, then $V_f \subset V_g$.
(d) For $\alpha \in F^\times$, we have $V_{\alpha f} = V_f$.
(e) We have $V_f = V_{gcd(P,f)}$, where P is the minimal polynomial.
Proof. (a) It is immediate to check that V_f is linear subspace of V. Further, if $f(T)v = 0$, then $f(T)(Tv) = T(f(T)v) = T\mathbf{0} = \mathbf{0}$.
(b) This follows from the fact that P v is the generator of the ideal of all polynomials g with $g(T)v = \mathbf{0}$.
(c) If $f|g$ then $g = h(t)f(t)$, so if $f(T)v = \mathbf{0}$ then $g(T)v = (h(T)f(T))v = h(T)(f(T)v) = h(T)\mathbf{0} = \mathbf{0}$.
(d) For any $v \in V, f(T)v = \mathbf{0} \Leftrightarrow \alpha f(T)v = \mathbf{0}$.
(e) Since $gcd(P,f)|f$, $V_{gcd(P,f)} \subset V_f$. Conversely, let $v \in V_f$. Then $P_v|P$ and $P_v|f$, so $P_v|gcd(P,f)$, so $gcd(P,f)(T)(v) = \mathbf{0}$ and $v \in V_{gcd(P,f)}$.
Remark 5.3.2. In view of Proposition 4.3.4(e) there are only finitely many distinct spaces V_f, since there are only hyperfinitely many monic polynomials dividing P.
Definition 5.3.2. If there is a vector $v \in V$ with $P = P_v$, we say that the minimal polynomial P is **locally attained**. Since it was immediate from the definition that $\deg(P_v) \le \dim(V)$, if the minimal polynomial is locally attained then we get another, better, proof that $\deg(P) \le \dim(V)$.
Proposition 5.3.5. For $n \ge 2, n \in \mathbb{N}^\#$, let $f_1,\ldots,f_n \in F[t]$ be pairwise coprime. Then the subspaces $V_{f_1},\ldots,V_{f_n}$ are independent and we have $\bigoplus_{i=1}^n V_{f_i} = V_{f_1\cdots f_n}$.
Proof. We go by hyper infinite induction on n.
Base Case ($n = 2$): let $v \in V_{f_1} \cap V_{f_2}$. Since f_1 and f_2 are coprime, there are a(t), b(t) $\in$ F[t] such that $af_1 + bf_2 = 1$, and then
$v = 1v = (a(T)f_1(T) + b(T)f_2(T))v = a(T)(f_1(T)v) + b(T)(f_2(T)v)) = \mathbf{0}$,
which shows that $W := V_{f_1} + V_{f_2} = V_{f_1} \oplus V_{f_2}$. It is easy to see that $W \subset V_{f_1f_2}$: every $w \in W$ is a sum of a vector w_1 killed by $f_1(T)$ and a vector w_2 killed by $f_2(T)$, so $f_1(T)f_2(T)w = 0$. Conversely, let $v \in V$ be such that $f_1(T)f_2(T)v = 0$. As above, we have $v = a(T)f_1(T)v + b(T)f_2(T)v$. Then $a(T)f_1(T)v \in Vf_2$ and $b(T)f_2(T) \in Vf_1$, so $v \in Vf_1 \oplus Vf_2$.
Induction Step: Suppose $n \ge 3$ and that the result holds for any $(n-1) \in \mathbb{N}^\#$ pairwise coprime polynomials. Put $W = V_{f_1} + \ldots + V_{f_{n-1}}$. By hyper infinite induction,

$$W = \bigoplus_{i=1}^{n-1} f_i = V_{f_1 \cdots f_{n-1}}.$$

The polynomials $f_1 \cdots f_{n-1}$ and f_n are coprime, so applying the base case we get

$$W + V_{f_n} = \bigoplus_{i=1}^{n-1} f_i \oplus V_{f_n} = \bigoplus_{i=1}^{n} f_i = V_{f_1 \cdots f_n}.$$

Lemma 5.3.1. Let $v \in V$. For any monic polynomial $f | P_v$, we have $P_{f(T)v} = \dfrac{P_v}{f}$.

Proof. Write $P_v = fg$. Since $P_v f(T) f(T) v = P_v(T) v = 0$, we have $P_{f(T)} v | P_v / f$. If there were a proper divisor h of g such that $h(T)(f(T)v) = 0$, then $hf(T)v = 0$. That is, hf kills v but has smaller degree than $gf = P_v$, contradiction.

Theorem 5.3.1. (**Local Attainment Theorem**). Every monic divisor f of the minimal polynomial is a local minimal polynomial $f = P_v$ for some $v \in V$.

Proof. Step 1: Let $P = p^{a_1} \cdots p^{a_r}$. Since P is the lcm of the local minimal polynomials, there is $w_i \in V$ such that the exponent of p_i in P_{w_i} is a_i. Let $v_i = P/p_i^{a_i}(T) w_i$. By Lemma 5.3.1, $P_{v_i} = p_i^{a_i}$.

Step 2: Put $v = v_1 + \ldots + v_r$. We claim that $P_v = P$. Indeed, since $p_1^{a_1} \cdots p_r^{a_r}$ are pairwise coprime, the spaces $V_{p^{a_1}}, \ldots, V_{p^{a_r}}$ are independent invariant subspaces. It follows that for all $f \in F[t]$, the vectors $f(T)v_1, \ldots, f(T)v_r$ are lin early independent. In particular, if $\mathbf{0} = f(T)v = f(T)v_1 + \ldots + f(T)v_r$, then $f(T)v_1 = \ldots = f(T)v_r = \mathbf{0}$. This last condition occurs iff $p_i^{a_i} | f$ for all i, and again by coprimality this gives $P = p_1^{a_1} \cdots p_r^{a_r} | f$.

Step 3: Now suppose that we have monic polynomials f, g with $fg = P$. By Step 2, there is $v \in V$ with $P_v = P$. By Lemma 5.3.1, $P_{g(T)v} = P_g = f$.

Let $W \subset V$ be an invariant subspace. Then T induces a linear endomorphism on the quotient space V/W given by $T(v + W) = T(v) + W$. Let's check that this is well-defined, i.e., that if $v' + W = v + W$, then $T(v) + W = T(v') + W$. There is $w \in W$ such that $v' = v + w$, so $T(v') + W = T(v + w) + W = T(v) + T(w) + W = T(v) + W$, since $T(W) \subset W$. We call V /W an **invariant quotient**.

Proposition 5.3.6. Let $W \subset V$ be an invariant subspace.

(a) For $v \in V$, let $\bar{v}$ be its image in V/W. Then $P_{\bar{v}} | P_v$.

(b) For every $\bar{v} \in V/W$, there is $v' \in V$ such that $P_{\bar{v}} = Pv'$.

(c) $P_{V/W} | P_V$.

Proof. (a) Since $P(T)v = \mathbf{0}$, also $P(T)v \in W$; the latter means $P(T)\bar{v} = \mathbf{0}$.

(b) Let v be any lift of $\bar{v}$ to V . By part (a) we may write $P_{v(t)} = f(t) P_{\bar{v}(t)}$ for some hyperfinite polynomial f. By Lemma 5.3.1, $P_{f(T)v} = P_{\bar{v}}$.

(c) Since $P_V(T)$ kills every vector of V, it sends every vector of V into W. (One could also use the characterizations of the global minimal polynomial as the lcm of the local minimal polynomials together with part (b).)

5.4. Eigenvectors, ${}^*\mathbb{R}_c^{\#}$-valued eigenvalues and corresponding eigenspaces.

Definition 5.4.1. A nonzero vector $v \in V$ is an **eigenvector** for T if $Tv = \lambda v$ for some $\lambda \in {}^*\mathbb{R}_c^{\#}$, and we say that λ is the corresponding **eigenvalue**. A scalar $\lambda \in {}^*\mathbb{R}_c^{\#}$ is an **eigenvalue** of T if there is some nonzero vector $v \in V$ such that $Tv = v$.

Remark 5.4.1. In fact this is a special case of a concept studied in the last section. Namely, for $v \in V$, we have $Tv = \lambda v$ iff $(T - \lambda)v = \mathbf{0}$ iff $v \in V_{t-\lambda}$. Thus v is an eigenvector iff the local minimal polynomial P_v is linear.

Proposition 5.4.1. The following are equivalent:

(i) 0 is an eigenvalue.

(ii) T is not invertible.

Proposition 5.4.2. Let $P(t)$ be the minimal polynomial for T on V.

(a) For $\lambda \in {}^*\mathbb{R}_c^\#$, the following are equivalent:

(i) λ is an eigenvalue of T.

(ii) $P(\lambda) = 0$.

(b) It follows that T has at most $\dim(V)$ eigenvalues.

Proof. (a) By Proposition 1.13e), we have $V_{t-\lambda} = V_{gcd(t-\lambda,P)}$. It follows that if $P(\lambda) \neq 0$ then $t-\lambda \nmid P(t)$ and thus $V_{t-\lambda} = V_1 = \{0\}$, so λ is not an eigenvalue. If λ is an eigenvalue, there is $v \in V$ with $P_v = t-\lambda$. Since $P_v = t-\lambda|P$, so $P(\lambda) = 0$. (b) By Proposition 1.9, P 6= 0, and by Proposition 1.11, deg P $\leq$ dim V , so P has at most dim V roots.

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Appendix A. Bivalent Hyper Infinitary first-order logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.Generalized Deduction Theorem.

Hyper infinitary language $L^{\#}_{\infty^{\#}}$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \aleph_0^{\#} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernaturals indexed hyper infinite sequence $\{A_\delta\}_{\delta\in\mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\aleph_0^{\#}$ itself.

The syntax of bivalent hyper infinitary first-order logics ${}^{2}L^{\#}_{\infty^{\#}}$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\aleph_0^{\#} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \aleph_0^{\#}$ many variables, and we suppose there is a supply of $\kappa < \aleph_0^{\#}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules.

If $\phi, \psi, \{\phi_\alpha : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $L^{\#}_{\infty^{\#}}$, the following are also formulas:

(i) $\bigwedge_{\alpha<\gamma} \phi_\alpha, \bigwedge_{\alpha\leq\gamma} \phi_\alpha,$

(ii) $\bigvee_{\alpha<\gamma} \phi_\alpha, \bigvee_{\alpha\leq\gamma} \phi_\alpha,$

(iii) $\phi \to \psi, \phi \wedge \psi, \phi \vee \psi, \neg\phi$
(iv) $\forall_{\alpha<\gamma} x_\alpha \phi$ (also written $\forall \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),
(v) $\exists_{\alpha<\gamma} x_\alpha \phi$ (also written $\exists \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),
(vi) the statement $\bigwedge_{\alpha<\gamma} \phi_\alpha$ holds if and only if for any α such that $\alpha < \gamma$ the statement holds ϕ_α,
(vii) the statement $\bigvee_{\alpha<\gamma} \phi_\alpha$ holds if and only if there exist α such that $\alpha < \gamma$ the statement holds ϕ_α.

Definition 1.[7]. A valuation of a syntactic system is a function that as signs $\top$ (true) to some of its sentences, and/or $\bot$ (false) to some of its sentences.Precisely, a valuation maps a nonempty subset of the set of sentences into the set $\{\top, \bot\}$.
We call a valuation bivalent iff it maps all the sentences into $\{\top, \bot\}$.

Definition 2.[7]. L is a bivalent propositional language iff its admissible valuations are the functions v such that for all sentences A, B of L,
(a) $v(A) \in \{\top, \bot\}$
(b) $v(\neg A) = \top$ iff $v(A) = \bot$
(c) $v(A \wedge B) = \top$ iff $v(A) = v(B) = \top$.
(d) by definition of the implication $A \Rightarrow B$ the following truth table holds

	$v(A)$	$v(B)$	$v(A \Rightarrow B)$
(1)	$\top$	$\top$	$\top$
(2)	$\top$	$\bot$	$\bot$
(3)	$\bot$	$\top$	$\top$
(4)	$\bot$	$\bot$	$\top$

Truth table 1.

Remark 1.Note that in the case (4) on a truth table 1
In this case we call implication $A \Rightarrow B$ a weak implication and abbreviate

$$A \Rightarrow_w B \tag{1}$$

We call a statement (1) as a weak statement and often abbreviate $v(A \Rightarrow B) = \top_w \neq \top$ instead (1).

Remark 2.

Definition 3.[7-8]. A is a valid (logically valid) sentence (in symbols, $\Vdash A$) in L iff every admissible valuation of L satisfies A.
The axioms of hyper infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. **Logical axiom**

A 1. $A \to [B \to A]$
A 2. $[A \to [B \to C] \to [[A \to B] \to [A \to C]]]$
A 3. $[\neg B \to \neg A] \to [A \to B]$
A 4. $[\bigwedge_{i<\alpha}[A \to A_i]] \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^{\#}$
A 5. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$
A 6. $[\forall \mathbf{x}[A \to B] \to [A \to \forall \mathbf{x} B]]$
provided no variable in $\mathbf{x}$ occurs free in A;
A 7. $\forall \mathbf{x} A(\mathbf{x}) \to S_f(A)$,
where $S_f(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language; in particular:

A 7$'$. $\forall x_i[A(x_i)] \Rightarrow A(\mathbf{t})$ is a wff of ${}^2L^{\#}_{\infty^{\#}}$ and $\mathbf{t}$ is a term of ${}^2L^{\#}_{\infty^{\#}}$ that is free for x_i in $A(x_i)$. Note here that $\mathbf{t}$ may be identical with x_i; so that all wffs $\forall x_i A \Rightarrow A$ are axioms by virtue of axiom (7),see [8].
A 8.Gen (Generalization).
$\forall x_i B$ follows from B.
II.Restricted rules of conclusion.
Let $\mathcal{F}_{\mathbf{wff}}$ be a set of the all closed wffs of $L^{\#}_{\infty^{\#}}$.
R1.RMP (Restricted Modus Ponens).
There exist subsets $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.
From A and $A \Rightarrow B$, we conclude B iff $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$.
If $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$ we also abbraviate by $A, A \Rightarrow B \vdash_{RMP} B$.
R2.RMT (Restricted Modus Tollens)
There exist subsets $\Delta_1', \Delta_2' \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.
$P \Rightarrow Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \Rightarrow Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subset \mathcal{F}_{\mathbf{wff}}$.
R3.MRR (Main Restricted rule of conclusion)
There exists subset $\Delta_3 \subset \mathcal{F}_{\mathbf{wff}}$ such that if $A \in \Delta_3$, then $\neg A \nvdash B$, i.e.,
if $A \in \Delta_3$ we cannot obtain from $\neg A$ any formula B whatsoever.
Remark 2.Note that RMP and RMT easily prevent any paradoxes of naive Cantor set theory (NC), see [1],[9].
III.Additional derived rule of conclusion.
Particularization rule (RPR)
Remind that canonical unrestricted particularization rule (UPR) reads
UPR: If $\mathbf{t}$ is free for x in $B(x)$, then $\forall x[B(x)] \vdash B(\mathbf{t})$,see [8].
Proof.From $\forall x[B(x)]$ and the instance $\forall x[B(x)] \Rightarrow B(\mathbf{t})$ of axiom (A7), we obtain $B(\mathbf{t})$ by unrestricted modus ponens rule.Since x is free for x in $B(x)$, a special case of unrestricted particularization rule is:$\forall x B \vdash B$.
Definition 4.Any formal theory L with a hyper infinitary lenguage $L^{\#}_{\infty^{\#}}$ is defined when the following conditions are satisfied:
1. A hyper infinite set of symbols is given as the symbols of L. A finite or hyperfinite sequence of symbols of L is called an expression of L.
2. There is a subset of the set of expressions of L called the set of well formed formulas (wffs) of L. There is usually an effective procedure to determine whether a given expression is a wff.
3. There is a set of wfs called the set of axioms of L. Most often, one can effectively decide whether a given wff is an axiom; in such a case, L is called an axiomatic theory.
4. There is a finite set $R_1, \ldots, R_n$, of relations among wffs, called rules of conclusion. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wff B, one can effectively decide whether the given j wffs are in the relation R_i to B, and, if so, B is said to follow from or to be a direct consequence of the given wffs by virtue of R_j.
Definition 5.A proof in L is a finite or hyperfinite sequence $B_1, \ldots, B_k, k \in \mathbb{N}^{\#}$ of wffs such that for each i,either B_i is an axiom of L or B_i is a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of L.

Definition 6. A theorem of L is a wff B of Y such that B is the last wff of some proof in $L.$ Such a proof is called a proof of B in $L.$
Definition 7. A wff E is said to be a consequence in L of a set of Γ of wffs if and only if there is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that E is B_k and, for each $i,$ either B_i is an axiom or B_i is in Γ, or B_i is a direct consequence by some rule of inference of some of the preceding wffs in the sequence. Such a sequence is colled a proof (or deduction) E from Γ. The members of Γ are called the hypotheses or premisses of the proof.
We use $\Gamma \vdash E$ as an abbreviation for E as a consequence of Γ.
In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_L E,$ adding the subscript L to indicate the theory in question.
If Γ is a finite or hyperfinite set $\{H_i\}_{1 \leq i \leq m}, m \in \mathbb{N}^{\#}$ we write $H_1,\ldots,H_m \vdash E$ instead of $\{H_i\}_{1 \leq i \leq m} \vdash E.$
Lemma 1.[8]. $\vdash B \Rightarrow B$ for all wffs $B.$
Theorem 1.(Generalized Deduction Theorem1). If Γ is a set of wffs and B and E are wffs, and $\Gamma, B \vdash E,$ then $\Gamma \vdash B \Rightarrow_s E.$ In pticular, if $B \vdash E$ then $\vdash B \Rightarrow E.$
Proof. Let $E_1,\ldots,E_n, n \in \mathbb{N}^{\#}$ be a proof of E form $\Gamma \cup \{B\}$, where E_n is $E.$
Let us prove, by hyperfinite induction on $j,$ that $\Gamma \vdash B \Rightarrow_s E_j$ for $1 \leq j \leq n.$
First of all, E_1 must be either in Γ or an axiom of L or B itself.
By axiom schema A1, $E_1 \Rightarrow_s (B \Rightarrow_s E_1)$ is an axiom. Hence, in the first two cases, by MP, $\Gamma \vdash B \Rightarrow_s E_1$ For the third case, when E_1 is B, we have $\vdash B \Rightarrow_s E_1$ by Lemma 1, and, therefore, $\Gamma \vdash B \Rightarrow_s E_1.$ This takes care of the case $j = 1.$
Assume now that: $\vdash B \Rightarrow_s E_k$ for all $k < j, j \in \mathbb{N}^{\#}.$ Either E_j is an axiom, or E_j is in Γ, or E_j is $B,$ or E_j follows by modus ponens from some E_l and E_m where $l < j,$ $m < j,$ and E_m has the form $E_l \Rightarrow_s E_j.$ In the first three cases, $\Gamma \vdash B \Rightarrow_s E_j$ as in the case $j = 1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s E_l$ and $\Gamma \vdash B \Rightarrow_s (E_l \Rightarrow_s E_j)$ But, by axiom schema (A2),
$\vdash B \Rightarrow_s (E_l \Rightarrow_s E_j) \Rightarrow_s ((B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j))$
Hence, by MP, $\Gamma \vdash (B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j)$ and, again by MP, $\Gamma \vdash B \Rightarrow_s E_j.$
Thus, the proof by hyperfinite induction is complete.
The case $j = n \in \mathbb{N}^{\#}$ is the desired result. Notice that, given a deduction of E from Γ and $B,$ the proof just given enables us to construct a deduction of $B \Rightarrow_s E$ from Γ. Also note that axiom schema A3 was not used in proving the generalized deduction theorem.
Remark 3.For the remainder of the chapter, unless something is said to the contrary, we shall omit the subscript L in $\vdash_L$. In addition, we shall use $\Gamma, B \vdash E$ to stand for $\Gamma \cup \{B\} \vdash E$. In general, we let $\Gamma, B_1,\ldots,B_n \vdash E$ stand for $\Gamma \cup \{B_i\}_{1 \leq i \leq n} \vdash E.$
Remark 4.We shall use the terminology proof, theorem, consequence, axiomatic, etc. and notation $\Gamma \vdash E$ introduced above.
Proposition 1. Every wff B of K that is an instance of a tautology is a theorem of $K,$ and it may be proved using only axioms A1-A3 and MP.
Proposition 2.If E does not depend upon B in a deduction showing that $\Gamma, B \vdash E$, then $\Gamma \vdash E.$
Proof.Let $D_1,\ldots,D_n$ be a deduction of E from Γ and $B,$ in which E does not depend upon $B.$ In this deduction, D_n is $E.$ As an inductive hypothesis, let

us assume that the proposition is true for all deductions of length less than $n \in \mathbb{N}^{\#}$
If E belongs to Γ or is an axiom, then $\Gamma \vdash E.$ If E is a direct consequence of one or two preceding wffs by Gen or MP, then, since E does not depend upon B, neither do these preceding wfs. By the inductive hypothesis, these preceding wfs are deducible from Γ alone. Consequently, so is E .

Theorem 2.(Generalized Deduction Theorem 2).Assume that, in some deduction showing that $\Gamma, B \vdash E,$ no application of Gen to a wff that depends upon B has as its quantified variable a free variable of $B.$ Then $\Gamma \vdash B \Rightarrow_s E.$

Proof.Let $D_1, \ldots, D_n$ be a deduction of E from Γ and B satisfying the assumption of this theorem. In this deduction, D_n is $E.$ Let us show by hyperfinite induction that $\Gamma \vdash B \Rightarrow_s D_i$ for each $i \leq n \in \mathbb{N}^{\#}$. If D_i is an axiom or belongs to Γ, then $\Gamma \vdash B \Rightarrow_s D_i$, since $D_i \Rightarrow_s (B \Rightarrow_s D_i)$ is an axiom. If D_i is B, then $\Gamma \vdash B \Rightarrow_s D_i,$ since, by Proposition 1, $\vdash B \Rightarrow_s B.\cdot$ If there exist j and k less than i such that D_k is $\vdash D_j \Rightarrow_s D_i,$ then, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$ and $\Gamma \vdash B \Rightarrow_s (D_j \Rightarrow_s D_i)$. Now, by axiom A2,
$\vdash B \Rightarrow_s (D_j \Rightarrow_s D_i) \Rightarrow_s ((B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s D_i))$. Hence, by MP twice, $\Gamma \vdash B \Rightarrow_s D_i$. Finally, suppose that there is some $j < i$ such that D_i is $\forall x_k D_j$. By the inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$, and, by the hypothesis of the theorem, either D_j does not depend upon B or x_k is not a free variable of B. If D_j does not depend upon $B,$ then, by Proposition 2, $\Gamma \vdash D_j$ and, consequently, by Gen, $\Gamma \vdash \forall x_k D_j$. Thus, $\Gamma \vdash D_i$. Now, by axiom A1, $\vdash D_i \Rightarrow_s (B \Rightarrow_s D_i)$. So, $\Gamma \vdash B \Rightarrow_s D_i$ by MP. If, on the other hand, x_k is not a free variable of $B,$ then, by axiom A5, $\vdash \forall x_k (B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s \forall x_k D_j)$ Since $\Gamma \vdash B \Rightarrow_s D_j$, we have, by Gen, $\Gamma \vdash \forall x_k (B \Rightarrow_s D_j)$, and so, by MP, $\Gamma \vdash B \Rightarrow_s \forall x_k D_j$ that is, $\Gamma \vdash B \Rightarrow_s D_i$. This completes the induction, and our proposition is just the special case $i = n.$

.

Appendix B.Generalized fundamental theorem of algebra.

Remind that a hyperfinite polynomial in a single indeterminate x can always be written symbollicaly in the form

$$p(x) = Ext\text{-}(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0), \quad (1)$$

where $a_0, \ldots, a_n$ are constants and x is the indeterminate.The word indeterminate means
that x represents no particular value, although any value may be substituted for it. The mapping that associates the result of this substitution to the substituted value is a function, called a polynomial function of the hyperfinite degree $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Definition 1. The external polynomial function of the hyperfinite degree $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is given by

$$p(x) = Ext\text{-}\sum_{k=0}^{n} a_k x^k. \tag{2}$$

B.1.Topological proof.

For topological proof by contradiction, suppose that the polynomial $p(z)$ has no roots, and consequently is never equal to 0. Think of the polynomial as a map from the complex plane $\mathbb{C}_c^{\#}$ into the complex plane $\mathbb{C}_c^{\#}$. It maps any circle $|z| = R$ into a closed loop, a curve $P(R)$. We will consider what happens to the winding number of $P(R)$ at the extremes when R is very large and when $R = 0$. When R is a sufficiently infinite large number, then the leading term z^n of $p(z)$ dominates all other terms combined; in other words,

$$|z^n| > |Ext\text{-}(a_{n-1}z^{n-1} + \cdots + a_0)|. \tag{1.1}$$

When z traverses the circle $R \times (Ext\text{-}\exp(i\theta))$ ($0 \leq \theta \leq 2\pi_{\#}$) , then $z^n = R^n \times (Ext\text{-}\exp(in\theta))$ winds n times counter-clockwise ($0 \leq \theta \leq 2\pi_{\#} n$) around the origin $(0,0)$, and $P(R)$ likewise. At the other extreme, with $|z| = 0$, the curve $P(0)$is merely the single point $p(0)$, which must be nonzero because $p(z)$ is never zero. Thus $p(0)$ must be distinct from the origin $(0,0)$, which denotes 0 in the complex plane $\mathbb{C}_c^{\#}$. The winding number of $P(0)$ around the origin $(0,0)$ is thus 0. Now changing R continuously will deform the loop continuously. At some R the winding number must change. But that can only happen if the curve $P(R)$ includes the origin $(0,0)$ for some R. But then for some z on that circle $|z| = R$ we have $p(z) = 0$, contradicting our original assumption. Therefore, $p(z)$ has at least one zero.

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B.2.Complex #-analytic proofs

We assume by contradiction that $a = p(z_0) \neq 0$, then, expanding $p(z)$ in powers of $z - z_0$ we can write

$$p(z) = Ext\text{-}(a + c_k(z - z_0)^k + c_{k+1}(z - z_0)^{k+1} + \cdots + c_n(z - z_0)^n). \tag{2.1}$$

Here, the c_j are simply the coefficients of the polynomial $z \to p(z + z_0)$, and we let k be the index of the first coefficient following the constant term that is non-zero. But now we see that for z sufficiently close to z_0 this has behavior asymptotically similar to the simpler polynomial $q(z) = a + c_k(z - z_0)^k$ in the sense that (as is easy to check) the function $\left|\dfrac{p(z) - q(z)}{(z - z_0)^{k+1}}\right|$ is bounded by some positive constant $M \in \mathbb{R}_c^{\#}$ in some neighborhood of z_0. Therefore, if we define $\theta_0 = (\arg(a) + \pi_{\#} - \arg(c_k))$ and let $z = z_0 + r \times (Ext\text{-}\exp(i\theta_0))$, then for any sufficiently small positive number $r \in \mathbb{R}_c^{\#}$, since the bound M mentioned above holds and using the triangle inequality we see that

$$|p(z)| \leq |q(z)| + r^{k+1}\left|\frac{p(z)-q(z)}{r^{k+1}}\right| \leq$$
$$|a+(-1)c_k r^k|(Ext\text{-}\exp[i(arg(a)-arg(c_k))])| + Mr^{k+1} = \tag{2.2}$$
$$= |a| - |c_k| r^k + Mr^{k+1}.$$

When r is sufficiently close to 0 this upper bound for $|p(z)|$ is strictly smaller than $|a|$, in contradiction to the definition of z_0. (Geometrically, we have found an explicit direction θ_0 such that if one approaches z_0 from that direction one can obtain values $p(z)$ smaller in absolute value than $|p(z_0)|$.)

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B.3.Proof by generalized Liouville's theorem

Another analytic proof can be obtained along this line of thought observing that, since $|p(z)| > |p(0)|$ outside $D,$ the minimum of $|p(z)|$ on the whole complex plane is achieved at z_0. If $|p(z_0)| > 0,$ then $1/p(z)$ is a bounded #-holomorphic function in the entire complex plane since, for each complex number $z, |1/p(z)| \leq |1/p(z_0)|$. Applying generalized Liouville's theorem [4], which states that a bounded entire function must be constant, this would imply that $1/p(z)$ is constant and therefore that $p(z)$ is constant. This gives a contradiction, and hence $p(z_0) = 0$.

B.4.Proof by the argument principle.

Yet another analytic proof uses the argument principle. Let R be a positive hyperreal number large enough so that every root of $p(z)$ has absolute value smaller than R, such a number must exist because every non-constant polynomial function of degree $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ has at most n zeros. For each $r > R,$ consider the number

$$\frac{1}{2\pi_{\#}i}\int_{c(r)}\frac{p'(z)}{p(z)}d^{\#}z, \tag{2.3}$$

where $c(r)$ is the circle centered at 0 with radius r oriented counterclockwise; then the argument principle says that this number is the number N of zeros of $p(z)$ in the open ball centered at 0 with radius $r,$ which, since $r > R,$ is the total number of zeros of $p(z)$. On the other hand, the integral of n/z along $c(r)$ divided by $2\pi_{\#}i$ is equal to $n.$ But the difference between the two numbers is

$$\frac{1}{2\pi_{\#}i}\int_{c(r)}\left(\frac{p'(z)}{p(z)} - \frac{n}{z}\right)d^{\#}z = \frac{1}{2\pi_{\#}i}\int_{c(r)}\frac{zp'(z)-np(z)}{zp(z)}d^{\#}z. \tag{2.4}$$

The numerator of the rational expression being integrated has degree at most $n-1$ and the degree of the denominator is $n+1.$ Therefore, the number above tends to 0 as $r \to +\infty^{\#}$. But the number is also equal to $N-n$ and so $N = n.$