SUPERLUMINAL LORENTZ TRANSFORMATIONS

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Abstract: In this article linear transformations of coordinates to a superluminal inertial reference frame are presented. Even if there is no need to use imaginary numbers to maintain c invariant, these functions are just intended as a mathematical curiosity not necessarily having a real physical meaning. Possible applications to our world, if any, are left to the reader.

Keywords: *Special Relativity; Extended Relativity; Superluminal Motions; Superluminal Inertial Reference Frames; Superluminal Lorentz Transformations.*

1. INTRODUCTION

Since the birth of special relativity, inertial reference frames travelling faster than light have never been considered part of physics because the Lorentz transformations are not defined when $v > c$. However, that did not prevent the growth of a huge literature attempting to extend the physical principles and the mathematical framework of Einstein's theory to the superluminal domain. In ref.[1] and ref.[2] for example, the Superluminal Lorentz Transformations in one spatial dimension are given. On the other hand, real reference structures have three spatial dimensions and so, in this case, the Superluminal Lorentz Transformations will be represented by the four following equations:

$$s_i = a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}t \qquad [1]$$

where (x,y,z,t) are the space time coordinates of the laboratory frame R, s_i $(i = 1,\cdots,4)$ are the space time coordinates (x_s,y_s,z_s,t_s) of the superluminal inertial reference frame R_s, while a_{ij} $(i,j = 1,\ldots,4)$ are constants such that the three basic kinematic conditions given in section 2 characterizing the transformations of coordinates to a superluminal inertial reference frame are satisfied. Now every time it is searched for the values of the constants a_{ij}, it is discovered that imaginary numbers are needed to satisfy the three basic kinematic conditions. In other words, this means that the problem does not have any physical solution.

At this point, if we want to find a real result the only way is to delete some condition to be satisfied. Since the main scope is to give the correct description of a physical problem, we will begin changing the mathematical framework commonly used to illustrate it. In particular, we will begin from changing equations [1]. In fact, the structure of equations [1] tries to extend into the superluminal domain the pseudo metrics defined in special relativity. Discharging this mathematical condition, we will see in section 4 that it is possible to write linear superluminal transformations of coordinates satisfying the three basic kinematic conditions given in section 2. All of that is achieved by introducing a mathematical representation of time in the faster-than-light inertial reference frame different from the one commonly used as illustrated in section 3. Pay attention to the fact that we are not changing the

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physical quantity time, but we are just changing the mathematical object representing it. In fact, to change this basic mechanical quantity one should know first what time really is.

In conclusion, before to define the inertial reference frames in section 2 the reader must keep in mind that only kinematics is considered in this paper. Other fields of physics such as dynamics and electromagnetism have not been taken into account.

2. INERTIAL REFERENCE FRAMES

First of all we begin giving the physical definition of an inertial reference frame. To do that we will take the suggestions provided in ref.[3] as framework.

In physics a reference frame is a real structure supporting an observer and the instruments needed for the experimentation. Among these structures there are some more suitable because one can easily have at his/her disposal objects at rest in them. For example, the objects of furniture are usually at rest in a room and so this latter one should be preferred to a cabin of a ship in a storm. Obviously, there are many ways to find structures where objects at rest can be easily obtained. In fact, just rotate or translate which means that the room above ours or on another side of a building can be a good choice alike. However, the experimental observation suggests that the family of suitable structures is wider. More precisely, there are structures where one can easily have at his/her disposal objects at rest but in uniform motion with respect to other proper structures. For example, the compartment of a train travelling regularly could be another suitable structure along with the rooms of a building.

Hence, we can say that an inertial reference frame is a real structure where one can easily have at his/her disposal objects at rest or in uniform motion. Anyway, another property holds in these real structures. In fact, it has been verified that in these latter ones the propagation of light is isotropic and uniform.

In conclusion, we can say that an inertial reference frame is a real structure where one can easily have at his/her disposal objects at rest or in uniform motion and the light in vacuum is seen to propagate at constant speed c in every direction.

Now, after having defined an inertial reference frame qualitatively, let us begin with a quantitative approach. We consider the objects at rest in our reference frame. By the use of a suitable instrument, typically a meter, we can measure the mutual distances between them. Then, given a unit of measure for the length, we can assign to each couple of objects a real number representing their distance. The mathematical framework adequately describing this kind of relations is the Euclidean geometry. Since we are living in a three-dimensional space, the proper choice is $\mathbb{R}^3$.

At this point, we have to take into account another feature encountered when observing our world. In fact, there are physical systems that keep certain properties practically unaltered like an object at rest which is always in the same position, but there are also physical systems changing their properties periodically like a pendulum always returning in the same position with the same velocity during its motion. Hence, using these periodic systems we can build clocks. By referring to the periodic behavior of a chosen clock, given a unit of measure and an origin, it is possible to represent the indications of the clock by the physical quantity called time. This is a practical definition of time, while its real nature is a problem that has been affecting scientists and philosophers for thousands of years without reaching a final answer. Obviously, the final answer will not be found here. Much more humbly, just to give a meaning and a background to the words used here, we will suppose that in an inertial reference frame the time is something that flows perpetually and smoothly in one unreal direction (forward or backwards).

At this point, we have that the geometric representation of an inertial reference frame is given by a spatial origin, a Cartesian tern and a unit of measure for the length along with a time origin, a time axis and a unit of measure for the time.

Since we are considering simultaneously the position of an object in a spatial reference structure represented by a point in $\mathbb{R}^3$ and the time marked by a clock represented by a point on the time axis, it is natural to switch from $\mathbb{R}^3$ to $\mathbb{R}^4$ to provide a geometric representation of an inertial reference frame. Remember that this geometric representation is the best our mathematical tools allows us to do because there is a deep qualitative difference between space and time. In fact, it is nonsense affirming that the time axis has any geometric relation with the space axes, for example it is perpendicular to them, since it is like comparing temperature and mass just because they are represented by a real number.

Keeping in mind that and the previous discussion and definitions, we have that the transformation of coordinates between our inertial reference frame R (our laboratory frame) and another subluminal inertial reference frame R_c is a function from $\mathbb{R}^4$ to $\mathbb{R}^4$ such that:

1) R_c moves at constant speed $v < c$ in R;
2) uniform motions in R are turned into uniform motions in R_c or, in other words, uniform motions remain uniform motions;
3) uniform motions in R such that $w = c$ are turned into uniform motions in R_c such that $w_c = c$, namely the speed of light c is invariant.

Instead, we have that the transformation of coordinates between our inertial reference frame R and a superluminal inertial reference frame R_s is a function from $\mathbb{R}^4$ to $\mathbb{R}^{10}$ such that:

I) R_s moves at constant speed $v > c$ in R;
II) uniform motions in R are turned into uniform motions in R_s;
III) uniform motions in R such that $w = c$ are turned into uniform motions in R_s such that $w_s = c$.

In this case, the time marked by a clock is no longer represented by a point on the time axis, but it will be represented by a point in $\mathbb{R}^7$. This topic will be explained and clarified in great detail in the next section.

At this point, it is clear that the inverse transformation of coordinates from R_s to R does not have the same form of the transformation of coordinates from R to R_s because the first goes from $\mathbb{R}^{10}$ to $\mathbb{R}^4$ and the latter goes from $\mathbb{R}^4$ to $\mathbb{R}^{10}$.

This highlights the fact that the subluminal observer, us in a common room used as a laboratory, and the superluminal observer, an astronaut on a starship travelling faster than light for example, are not equivalent because they are not experiencing the same physics (readers interested in this topic can take a look at ref.[4]).

However, also the subluminal observers do not seem to be equivalent even if the Lorentz transformations have the same form when going from R to R_c and from R_c to R. In fact, considering equivalent subluminal observers we obtain that the time in R_c will flow slower than in R in the same way the time in R flows slower than in R_c providing a not usable physical result. Hence, if we want a single and well-defined number to compare with experimental measurements we have to suppose that all the subluminal observers are not equivalent (readers interested in this topic can take a look at section 5 in ref.[5]).

3. VECTOR TIME

We consider a function from $\mathbb{R}^4$ to $\mathbb{R}^7$ which associates to every point ($\mathbf{x}$,t) of the inertial reference frame R the vector time $\mathbf{t_s}$ of the inertial reference frame R_s travelling at v > c in R.

Using a concise notation the vector time $\mathbf{t_s}$ has the following form:

$$\mathbf{t_s} = \left(\frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) - \frac{\mathbf{v}}{v}t\,,\; \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}),\; \frac{v}{c}t \right) \qquad [2]$$

where $\mathbf{v} = (v_x,v_y,v_z)$ is the superluminal speed of R_s in R, while $\mathbf{x_0}$ is the initial position at $t=0$.
More precisely the seven components of the vector time are given by the functions:

$$\begin{aligned}
t_{s1} &= \frac{v}{c^2}(x-x_0) - \frac{v_x}{v}t \\
t_{s2} &= \frac{v}{c^2}(y-y_0) - \frac{v_y}{v}t \\
t_{s3} &= \frac{v}{c^2}(z-z_0) - \frac{v_z}{v}t \\
t_{s4} &= \frac{v}{c^2}(x-x_0) \\
t_{s5} &= \frac{v}{c^2}(y-y_0) \\
t_{s6} &= \frac{v}{c^2}(z-z_0) \\
t_{s7} &= \frac{v}{c}t
\end{aligned} \qquad [3]$$

Let us notice that the initial position $\mathbf{x_0}$ depends on the motion in R taken into account. For example, given a uniform motion of constant velocity $\mathbf{w}$, we have to give the position $\mathbf{x_0}$ at $t=0$ to select the one we are interested in between the infinite straight lines in $\mathbb{R}^3$ having the same direction. So, the vector time in R_s corresponding to $t=0$ in R is $\mathbf{t_s} = \mathbf{0}$ whatever the motion $\mathbf{x}(t)$ in R we are considering because it is $\mathbf{x}(0) = \mathbf{x_0}$. Also the trivial case of a point at rest in R, $\mathbf{x}(t) = \mathbf{x_0}$ for every $t \in \mathbb{R}$, is included.

Now, since the reading of a clock provides a single result and not seven numbers, by using the vector time $\mathbf{t_s}$ defined in equation [2] we can derive the time t_s measured by the clocks of the superluminal inertial reference frame R_s. In fact, we have:

$$\begin{aligned}
t_s &= \left| \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) - \frac{\mathbf{v}}{v}t \right| - \left| \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) \right| + \frac{v}{c}t \qquad && \text{for } t \geq 0 \\
t_s &= -\left| \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) - \frac{\mathbf{v}}{v}t \right| + \left| \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) \right| + \frac{v}{c}t \qquad && \text{for } t < 0
\end{aligned} \qquad [4]$$

Let us notice that the first two terms of the sums in the above equalities are the length in $\mathbb{R}^3$ of the three-dimensional vectors obtained from the components of the vector time (t_{s1},t_{s2},t_{s3}) and (t_{s4},t_{s5},t_{s6}) respectively given in equation [3].

Furthermore, for simplicity and without loss of generality, equation [4] has been defined such that the origin of time in R_s corresponds to the origin of time in R. In other words, $t_s = 0$ for $t = 0$ whatever the motion in R we are observing. The trivial case of a point at rest in R is included as well. If this is not the case, namely the origin of time in R_s does not correspond to the origin of time in R, then you can add to the two equalities of equation [4] a suitable constant term $t_{s0} \neq 0$.

At this point, we consider a uniform motion in R such that $\mathbf{x} = \mathbf{x_0} + \mathbf{w}\,t$ where $0 \leq w < +\infty$. In this case, equation [4] becomes:

$$t_s = \left| \frac{v}{c^2}(\mathbf{x_0} + \mathbf{w}t - \mathbf{x_0}) - \frac{\mathbf{v}}{v}t \right| - \left| \frac{v}{c^2}(\mathbf{x_0} + \mathbf{w}t - \mathbf{x_0}) \right| + \frac{v}{c}t \qquad \text{for } t \geq 0$$

$$t_s = -\left| \frac{v}{c^2}(\mathbf{x_0} + \mathbf{w}t - \mathbf{x_0}) - \frac{\mathbf{v}}{v}t \right| + \left| \frac{v}{c^2}(\mathbf{x_0} + \mathbf{w}t - \mathbf{x_0}) \right| + \frac{v}{c}t \qquad \text{for } t < 0$$

or, being $|\mathbf{w}| = w$,

$$t_s = \left| \frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v} \right| |t| - \frac{v}{c^2} w |t| + \frac{v}{c}t \qquad \text{for } t \geq 0$$

$$t_s = -\left| \frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v} \right| |t| + \frac{v}{c^2} w |t| + \frac{v}{c}t \qquad \text{for } t < 0$$

Since $|t| = t$ for $t \geq 0$ and $|t| = -t$ for $t < 0$, we finally obtain for $t \in \mathbb{R}$:

$$t_s = \left(\left| \frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v} \right| - \frac{v}{c^2} w + \frac{v}{c} \right) t \qquad [5]$$

Now, considering that it is $|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$ for any two vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^3$, it follows:

$$\left| \frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v} \right| \geq \left| \frac{v}{c^2}\mathbf{w} \right| - \left| \frac{\mathbf{v}}{v} \right| = \frac{v}{c^2} w - 1 > \frac{v}{c^2} w - \frac{v}{c}$$

Hence, for every $\mathbf{w} \in \mathbb{R}^3$ in equation [5] it will be $dt_s \backslash dt > 0$ meaning that the time goes forward in R_s as well. For example, we consider a clock travelling at superluminal velocity $\mathbf{v}$ in the reference frame R or, correspondingly, at rest in the reference frame R_s. Using equation [5] we have:

$$t_s = \left(\left| \mathbf{v} \left(\frac{v}{c^2} - \frac{1}{v} \right) \right| - \frac{v^2}{c^2} + \frac{v}{c} \right) t = \left(\left| \frac{v^2}{c^2} - 1 \right| - \frac{v^2}{c^2} + \frac{v}{c} \right) t = \left(\frac{v}{c} - 1 \right) t \qquad [6]$$

where we can also see that the time in R_s flows more quickly that in R when $v > 2c$, at the same rate when $v = 2c$ and more slowly when $c < v < 2c$. On the other hand, if one multiplies each component of the vector time $\mathbf{t_s}$ given in equation [3] for $k > 0$, it follows that the time in R_s flows more quickly that in R when $v > c\,(k + 1)\,k^{-1}$, at the same rate when $v = c\,(k + 1)\,k^{-1}$ and more slowly when the superluminal speed is in the range $c < v < c\,(k + 1)\,k^{-1}$.

If the signs in the two equalities of equation [4] are reversed, (–,+,–) instead of (+,–,+) and (+,–,–) instead of (–,+,+), it follows that for every $\mathbf{w} \in \mathbb{R}^3$ in equation [5] it will be $dt_s \backslash dt < 0$ and so the time will go backwards in R_s.

Likely someone has already noticed that equation [2] is linear in the variables ($\mathbf{x}$,t), while equation [4] is not. Since we rely on equation [4] to define the time t_s measured in the superluminal reference frame R_s, one can think that the vector time $\mathbf{t_s}$ defined in equation [2] is just a trick to introduce a fake linearity. But this is not the case as it can be understood by the very elementary example given in Appendix A.

Hence, in conclusion, given a motion in R we have to establish the vector time along the corresponding trajectory in R_s to know, by using equation [4], the corresponding readings provided by the clocks of the superluminal inertial reference frame.

4. SUPERLUMINAL TRANSFORMATIONS OF COORDINATES

We consider the following transformations of coordinates from our laboratory reference frame R to a reference frame R_s:

$$\mathbf{x_s} = \mathbf{x} - \mathbf{v}\,t \qquad [7]$$

$$\mathbf{t_s} = \left(\frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}) - \frac{\mathbf{v}}{v}t,\ \frac{v}{c^2}(\mathbf{x}-\mathbf{x_0}),\ \frac{v}{c}t \right) \qquad [8]$$

where $\mathbf{v}$ is such that $v > c$, while $\mathbf{t_s}$ is the vector time in R_s. These functions will be called SLT (Superluminal Lorentz Transformations).

Now we verify that the transformations of coordinates SLT, equation [7] and equation [8], satisfy the properties I÷III given in section 2 which define the transformations of coordinates to a superluminal inertial reference frame.

First, $\forall\, \mathbf{x_s} \in R_s$ we obtain from relation [7] the equation of motion $\mathbf{x} = \mathbf{x_s} + \mathbf{v}\,t$ in R. This means that the reference frame R_s translates rigidly in R at constant speed $\mathbf{v}$ satisfying in this way property I. In particular, the two structures overlap for $t=0$.

Then, we consider a uniform motion in R such that $\mathbf{x} = \mathbf{x_0} + \mathbf{w}\,t$ where $0 \le w < +\infty$. Substituting into the transformations SLT this motion will be seen in R_s as:

$$\mathbf{w_s} = \frac{d\mathbf{x_s}}{dt_s} = \frac{d\mathbf{x_s}}{dt}\frac{dt}{dt_s} = \frac{\mathbf{w}-\mathbf{v}}{\left|\frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v}\right| - \frac{v}{c^2}w + \frac{v}{c}} \qquad [9]$$

where dt/dt_s is the inverse of the derivative with respect to t of equation [5]. Because $\mathbf{v}$, $\mathbf{w}$ and c are constants, it follows that uniform motions in R are transformed into uniform motions in R_s satisfying in this way property II.

Finally, if the uniform motion is such that $w = c$, equation [9] becomes:

$$\mathbf{w_s} = \frac{\mathbf{w}-\mathbf{v}}{\left|\frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v}\right| - \frac{v}{c^2}c + \frac{v}{c}} = \frac{\mathbf{w}-\mathbf{v}}{\left|\frac{v}{c^2}\mathbf{w} - \frac{\mathbf{v}}{v}\right|} \qquad [10]$$

At this point, let us verify that $w_s = c$ as well satisfying in this way property III. In fact, we have:

$$|\mathbf{w_s}|^2 = \frac{(w_x - v_x)^2 + (w_y - v_y)^2 + (w_z - v_z)^2}{\left(\frac{v}{c^2} w_x - \frac{v_x}{v}\right)^2 + \left(\frac{v}{c^2} w_y - \frac{v_y}{v}\right)^2 + \left(\frac{v}{c^2} w_z - \frac{v_z}{v}\right)^2} =$$

$$= \frac{w_x^2 - 2 w_x v_x + v_x^2 + w_y^2 - 2 w_y v_y + v_y^2 + w_z^2 - 2 w_z v_z + v_z^2}{\frac{v^2}{c^4} w_x^2 - 2\frac{w_x v_x}{c^2} + \frac{v_x^2}{v^2} + \frac{v^2}{c^4} w_y^2 - 2\frac{w_y v_y}{c^2} + \frac{v_y^2}{v^2} + \frac{v^2}{c^4} w_z^2 - 2\frac{w_z v_z}{c^2} + \frac{v_z^2}{v^2}} =$$

$$= \frac{c^2 + v^2 - 2(w_x v_x + w_y v_y + w_z v_z)}{\frac{v^2}{c^4} c^2 + \frac{v^2}{v^2} - \frac{2}{c^2}(w_x v_x + w_y v_y + w_z v_z)} = \frac{c^2 + v^2 - 2(w_x v_x + w_y v_y + w_z v_z)}{\frac{v^2}{c^2} + 1 - \frac{2}{c^2}(w_x v_x + w_y v_y + w_z v_z)} =$$

$$= \frac{c^2 + v^2 - 2(w_x v_x + w_y v_y + w_z v_z)}{\frac{1}{c^2}\left[v^2 + c^2 - 2(w_x v_x + w_y v_y + w_z v_z)\right]} = c^2$$

In conclusion, we have shown that the functions SLT are the transformations of coordinates to a superluminal inertial reference frame. Moreover, these functions are a family of transformations of coordinates because multiplying both $\mathbf{x_s}$ and $\mathbf{t_s}$ for the same constant $k > 0$ the fundamental properties continue to be satisfied.

At the end of this section, we consider the simplest case defined by $\mathbf{v} = (v,0,0)$ and $\mathbf{w} = (\pm c,0,0)$. When $\mathbf{w} = (c,0,0)$ from equation [10] we get:

$$w_{xs} = \frac{c - v}{\sqrt{\left(\frac{v}{c^2} c - \frac{v}{v}\right)^2}} = \frac{c - v}{\sqrt{\left(\frac{v}{c} - 1\right)^2}} = \frac{c - v}{v - c} c = -c$$

and $w_{ys} = w_{zs} = 0$. This result is in agreement with the fact that the inertial reference frame R_s travels faster than light and so the superluminal observer will see the light standing back and losing ground. Instead when $\mathbf{w} = (-c,0,0)$ from equation [10] we have:

$$w_{xs} = \frac{-c - v}{\sqrt{\left(-\frac{v}{c^2} c - \frac{v}{v}\right)^2}} = \frac{-1(c + v)}{\sqrt{(-1)^2 \left(\frac{v}{c} + 1\right)^2}} = \frac{-1(c + v)}{v + c} c = -c$$

and again $w_{ys} = w_{zs} = 0$. Finally, if the time goes backwards in R_s as already discussed in section 3, then equation [10] will be multiplied by -1 giving for $\mathbf{w} = (\pm c,0,0)$ the result $\mathbf{w_s} = (c,0,0)$ in agreement with the fact that the processes should be seen reversed in the superluminal inertial reference frame.

5. CONCLUSIONS

As shown in section 4, the transformations of coordinates SLT have been built to agree with the observed kinematic properties typical of real structures defined as inertial reference frames (in the hypothesis that an inertial reference frame can really travels faster than light). However, this does not mean that the transformations SLT are correctly describing reality. In fact, not everything coming out from mathematics is physics. If it were not so, it would be like to invent a new word and then pretend that the corresponding object shows up out of nowhere by magic.

For this reason, the functions SLT should be regarded as just being a mathematical curiosity. Applications to the real world, if any, are left to the reader.

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APPENDIX A
TRAVELLING STARSHIPS

We suppose to have a starship, the starship VV Cephei, going from the space base 0 located in the origin O of the reference frame R to the space base 2 located in the point $\mathbf{x_2}$. The starship leaves the space base 0 at $t_0 = 0$ and reaches its final destination at time t_2 moving along a straight path at constant speed $\mathbf{w_2} = \mathbf{x_2} \backslash t_2$.

Now we suppose to have another starship, the starship Vega, leaving the space base 0 at $t_0 = 0$ and reaching the space base 2 at the same time t_2 but along a different path from the one followed by the starship VV Cephei. More precisely, first the starship Vega reaches the space base 1 located in the point $\mathbf{x_1}$ at time $t_1 < t_2$ moving along a straight path at constant speed $\mathbf{w_1} = \mathbf{x_1} \backslash t_1$. Then the starship Vega leaves the space base 1 at time t_1 and reaches the space base 2 at time t_2 moving along a straight path at constant speed $\mathbf{w_a} = (\mathbf{x_2} - \mathbf{x_1}) \backslash (t_2 - t_1)$.

At this point, the superluminal observer in R_S will have to see the two starships arriving to the space base 2 at the same time t_{s2}. Instead using directly equation [4] given in section 3, bypassing in this way the vector time, in general the previous condition is not verified giving rise to an unphysical situation. To show that we consider the data in Table 1.

Superluminal Inertial Reference Frame R_S	$\mathbf{v_x \backslash c}$	2.2	*Starship VV Cephei from space base 0 to space base 2*	$\mathbf{w_{2x} \backslash c}$	0.2	*Starship Vega from space base 0 to space base 1*	$\mathbf{w_{1x} \backslash c}$	0.5	*Starship Vega from space base 1 to space base 2*	$\mathbf{w_{ax} \backslash c}$	−1.3
	$\mathbf{v_y \backslash c}$	1.5		$\mathbf{w_{2y} \backslash c}$	1.1		$\mathbf{w_{1y} \backslash c}$	0.7		$\mathbf{w_{ay} \backslash c}$	3.1
	$\mathbf{v_z \backslash c}$	1.4		$\mathbf{w_{2z} \backslash c}$	0.8		$\mathbf{w_{1z} \backslash c}$	0.3		$\mathbf{w_{az} \backslash c}$	3.3
	$\mathbf{v \backslash c}$	3.0		$\mathbf{w_2 \backslash c}$	1.4		$\mathbf{w_1 \backslash c}$	0.9		$\mathbf{w_a \backslash c}$	4.7
				$\mathbf{t_0}$	0		$\mathbf{t_0}$	0		$\mathbf{t_0 = t_1}$	5000
				$\mathbf{t_2}$	6000		$\mathbf{t_1}$	5000		$\mathbf{t_2}$	6000

Table 1 – *The velocities in the table are in units of c, while the times are given in seconds.*

Putting this data into equation [4] with $\mathbf{x_0} = \mathbf{0}$, or equivalently into equation [5] in this case, we get that the superluminal observer will see the starship VV Cephei reaching the space base 2 at:

$$t_{2S} = \left(\sqrt{\left(\frac{v}{c^2} w_{2x} - \frac{v_x}{v}\right)^2 + \left(\frac{v}{c^2} w_{2y} - \frac{v_y}{v}\right)^2 + \left(\frac{v}{c^2} w_{2z} - \frac{v_z}{v}\right)^2} - \frac{v}{c^2} \sqrt{w_{2x}^2 + w_{2y}^2 + w_{2z}^2} + \frac{v}{c} \right) t_2 =$$

$$= \left(\sqrt{\left(3.0 \times 0.2 - \frac{2.2}{3.0}\right)^2 + \left(3.0 \times 1.1 - \frac{1.5}{3.0}\right)^2 + \left(3.0 \times 0.8 - \frac{1.4}{3.0}\right)^2} - 3.0 \times \sqrt{0.2^2 + 1.1^2 + 0.8^2} + 3.0 \right) \times$$

$$\times 6000 \text{ s} = 13745 \text{ s} \cong 229 \text{ min}$$

For the other path instead, using again equation [4] (or equivalently equation [5]) with $\mathbf{x_0} = \mathbf{0}$, first we obtain that the superluminal observer will see the starship Vega reaching the space base 1 at:

$$t_{1S} = \left(\sqrt{\left(\frac{v}{c^2} w_{1x} - \frac{v_x}{v}\right)^2 + \left(\frac{v}{c^2} w_{1y} - \frac{v_y}{v}\right)^2 + \left(\frac{v}{c^2} w_{1z} - \frac{v_z}{v}\right)^2} - \frac{v}{c^2} \sqrt{w_{1x}^2 + w_{1y}^2 + w_{1z}^2} + \frac{v}{c} \right) t_1 =$$

$$= \left(\sqrt{\left(3.0\times0.5-\frac{2.2}{3.0}\right)^2 + \left(3.0\times0.7-\frac{1.5}{3.0}\right)^2 + \left(3.0\times0.3-\frac{1.4}{3.0}\right)^2} - 3.0\times\sqrt{0.5^2+0.7^2+0.3^2} + 3.0 \right) \times$$

$$\times\, 5000 \text{ s} = 10519 \text{ s} \cong 175 \text{ min}$$

Then, using equation [4] (or equivalently equation [5]) with $\mathbf{x_0} = \mathbf{x_1}$, we have that the starship Vega will spend the time in its journey from the space base 1 to the space base 2 equal to:

$$t_{as} = \left(\sqrt{\left(\frac{v}{c^2}w_{ax} - \frac{v_x}{v}\right)^2 + \left(\frac{v}{c^2}w_{ay} - \frac{v_y}{v}\right)^2 + \left(\frac{v}{c^2}w_{az} - \frac{v_z}{v}\right)^2} - \frac{v}{c^2}\sqrt{w_{ax}^2 + w_{ay}^2 + w_{az}^2} + \frac{v}{c} \right) t_a =$$

$$= \left(\sqrt{\left(-3.0\times1.3-\frac{2.2}{3.0}\right)^2 + \left(3.0\times3.1-\frac{1.5}{3.0}\right)^2 + \left(3.0\times3.3-\frac{1.4}{3.0}\right)^2} - 3.0\times\sqrt{(-1.3)^2+3.1^2+3.3^2} + 3.0 \right) \times$$

$$\times\, 1000 \text{ s} = 2585 \text{ s} \cong 43 \text{ min}$$

where $t_a = (t_2 - t_1)$ because in this case the origin of time is t_1. In fact, from a practical point of view, the observer in R must count the number of periodic changes of the physical properties defining the clocks to determine the time spent by the starship Vega in its journey from the space base 1 to the space base 2. Since this number does not depend on the previous number of periodic changes occurred during the journey of the starship Vega from the space base 0 to the space base 1, in other words this number does not depend on t_1, it is natural to take t_1 as the origin of time to know how long the second fly of the starship Vega lasts.

Said this, it follows that the superluminal observer will see the starship Vega reaching the space base 2 at time:

$$t_{2s} = t_{1s} + t_{as} = (10519 + 2585) \text{ s} = 13104 \text{ s} \cong 218 \text{ min}$$

In conclusion, using equation [4] directly we have that the superluminal observer will see the starship VV Cephei in the space base 2 eleven minutes later than the starship Vega instead of seeing them arriving at same time as it happens in the reference frame R.

Now, we suppose that the flowing of time in R_s is no longer described by the evolution of a point along an axis (the time axis) but it is described by the evolution of a point in $\mathbb{R}^7$. Hence, we have to use equation [2] first to set the time in R_s and then equation [4] to know the reading provided by the clocks in the superluminal inertial reference frame. In this way the preceding unphysical situation is solved. In fact, the vector time for the starship VV Cephei when reaching the space base 2 from space base 0 is given by ($\mathbf{x_0} = \mathbf{0}$):

$$\mathbf{t_{2s}} = \left(\frac{v}{c^2}\mathbf{x_2} - \frac{\mathbf{v}}{v}t_2,\ \frac{v}{c^2}\mathbf{x_2},\ \frac{v}{c}t_2 \right)$$

On the other hand, in a similar way to the one-dimensional time, the vector time corresponding to the arrival of the starship Vega in the space base 2 is given by the sum of the vector time corresponding to the travel from space base 0 to space base 1 and the vector time corresponding to the travel from space base 1 to space base 2. More precisely, it is:

$$\mathbf{t_{2s}} = \mathbf{t_{1s}} + \mathbf{t_{as}} = \left(\frac{v}{c^2}\mathbf{x_1} - \frac{\mathbf{v}}{v}t_1 , \frac{v}{c^2}\mathbf{x_1} , \frac{v}{c}t_1 \right) + \left(\frac{v}{c^2}(\mathbf{x_2} - \mathbf{x_1}) - \frac{\mathbf{v}}{v}(t_2 - t_1), \frac{v}{c^2}(\mathbf{x_2} - \mathbf{x_1}), \frac{v}{c}(t_2 - t_1) \right) =$$

$$= \left(\frac{v}{c^2}(\mathbf{x_1} + \mathbf{x_2} - \mathbf{x_1}) - \frac{\mathbf{v}}{v}(t_1 + t_2 - t_1), \frac{v}{c^2}(\mathbf{x_1} + \mathbf{x_2} - \mathbf{x_1}), \frac{v}{c}(t_1 + t_2 - t_1) \right) =$$

$$= \left(\frac{v}{c^2}\mathbf{x_2} - \frac{\mathbf{v}}{v}t_2 , \frac{v}{c^2}\mathbf{x_2} , \frac{v}{c}t_2 \right)$$

where $\mathbf{x_0} = \mathbf{0}$ ($t_0 = 0$) for the vector time $\mathbf{t_{1s}}$, while $\mathbf{x_0} = \mathbf{x_1}$ and $t_0 = t_1$ for the vector time $\mathbf{t_{as}}$.
At this point, since the two vector times are equal, using equation [4] subsequently it follows that the superluminal clocks will provide the same reading when the two starships reach the space base 2 as it should be.