A New Representation for Dirac δ-function

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A polynomial power series is constructed for the one-sided step function using a modified Taylor series, whose derivative results in a new representation for Dirac δ-function.

It's well-known that the Kronecker delta arises whenever an inner-product is performed between any two orthogonal vectors belonging to a real/complex vector space spanned by a countably finite/infinite number of dimensions and the Dirac delta function replaces the Kronecker delta if the dimensionality of vector space is uncountable - labeled by a continuous real-parameter. The Kronecker delta and the Dirac delta function are respectively given as below:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \ \forall \ i \neq j \end{cases} \tag{1}$$

where, $i, j = 1, 2, 3, \cdots$ and

$$\delta(x - x_0) = \begin{cases} \infty \ \text{ if} x = x_0 \\ 0 \ \text{ if} x \neq x_0 \end{cases} \tag{2}$$

where, $x \in \mathbf{R}$ = set of real numbers such that,

$$\int_{-\infty}^{+\infty} dx \ \delta(x - x_0) = 1. \tag{3}$$

For all properties and various existing representations of delta function, see Refs. [1–4]. From the Heaviside unit step function,

$$H(x) = \begin{cases} 1 \ \text{ if} x > 0 \\ 0 \ \text{ if} x < 0 \end{cases} \tag{4}$$

the delta function can be obtained as a derivative:

$$\delta(x) = \frac{dH(x)}{dx}. \tag{5}$$

The one-sided step function, say $R^+(r)$, can be defined as,

$$R^+(r) = \begin{cases} 1 \text{ if } r > 0 \\ 0 \text{ if } r = 0, \end{cases} \tag{6}$$

and still, the delta function - on the positive real line - can be obtained as a derivative:

$$\delta(r) = \frac{dR^+(r)}{dr}. \tag{7}$$

In the present paper, a polynomial power series is constructed for $R^+(r)$, from which the $\delta(r)$ follows as a derivative as shown below:

Let $f(x)$ be an analytical function defined on the real line with $|f(0)| < \infty$. Therefore,

$$\begin{aligned} f(x) - f(0) &= \frac{1}{1!}\int_0^x dt\ f'(t) \\ &= \frac{x}{1!}f'(x) - \frac{1}{2!}\int_0^x dt\ f''(t) \\ &= \frac{x}{1!}f'(x) - \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}\int_0^x dt\ f'''(t) \\ &= -\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n!}f^{(n)}(x) \end{aligned} \tag{8}$$

and hence, a modified Taylor series can be obtained as,

$$f(0) = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}f^{(n)}(x), \tag{9}$$

where, $f^{(0)}(x) \equiv f(x)$.

Let $f(x) = x^r\ \forall\ \{r|r \in \mathbf{R}^+\}$, then,

$$\begin{aligned} f(0) &= x^r\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{j=1}^{n}(r+1-j)\right) \\ &\equiv x^r\eta(r), \end{aligned} \tag{10}$$

where,

$$\eta(r) := 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{j=1}^{n}(r+1-j). \tag{11}$$

$f(0) = 0\ \forall\{r \in \mathbf{R}^+|r \neq 0\} \implies \eta(r) = 0\ \forall\{r \in \mathbf{R}^+|r \neq 0\}$, because, x^r is not identically zero on entire $\mathbf{R}$ and $f(0) = 1$ if $r = 0 \implies \eta(0) = 1$ if $r = 0$.

Let

$$R^+(r) := 1 - \eta(r) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\prod_{j=1}^{n}(r+1-j), \tag{12}$$

then,

$$\delta(r) = \frac{dR^+(r)}{dr} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \prod_{j=1}^{n} (r+1-j) \sum_{k=1}^{n} (r+1-k)^{-1}. \tag{13}$$

The usefulness of the above polynomial power series representation for delta function given in Eq. (13) is not clear at the present moment, but still it's presented here as a mathematical possibility.

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