A Paradox of “Adjacent” Real Points and Beyond

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Abstract. We reveal adjacent real points in the real set using a concise logical reference. This raises a paradox while the real set is believed as existing and complete. However, we prove each element in a totally ordered set has adjacent element(s); there is no densely ordered set. Furthermore, since the natural numbers can also be densely ordered under certain ordering, the set of natural numbers, which is involved with each infinite set in ZFC set theory, does not exist itself.

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Opening Words: A Paradox of “Adjacent” Real Points

It is well known that any two real points are separated by at least their midpoint. So nobody attempts to find “adjacent” real points. However, they are just there within reach of everyone.

On X-axis there is a segment, [1, 2], each point x of which is planted with a vertical segment $[0, x]$. The rightmost vertical segment is red in color and the horizontal one on X-axis is transparent whilst the others are green. When standing at the origin and looking at them (in parallel perspective), a viewer sees an otherwise totally green vertical “segment” with a red top point. There is no doubt the visible red point is the point 2 of the vertical segment [0, 2].

Next remove the upper endpoint from each of the vertical segments. Then the same viewer sees an almost same “segment”. This time the new visible red point is the upper endpoint of the vertical segment [0, 2), namely, the lower “adjacent” point of the above-mentioned point 2.

This paradox defends a crucial fact, which may have been otherwise denied or obscured.

1. Introduction

We view Georg Cantor’s theory about infinity with grave suspicion [1], and have waited for an incisive proof to finally close the book on it. When encounter the paradox of “adjacent” real points, we are not surprised at all for knowing that the set $\mathbb{R}$ is full of loopholes in the first place [1, sec. 4]. However, further reflection suggests that adjacent elements are all over a totally ordered set, and as a consequence the set $\mathbb{Q}$, which we have questioned in [1, *Thought Experiment 6.1*], is illogical. Then the natural desire that follows is to have the equally questionable set $\mathbb{N}$ [1, *Thought Experiment 5.2*] involved. We achieve the goal via rearranging the natural numbers into a dense pattern, just like

[1] This is an updated version of the original article. Some known errors have been corrected and **Appendix B** has been rewritten.

the rational numbers. The end result is far-reaching, eliminating a long list of paradoxes about infinity and continuum all at once.

Trivial Declaration. All involved multi-element sets in this paper are totally ordered (linearly ordered) sets with usual order $\leq$, each with many elements. In our discussion, point and number are merely different expressions of the same thing, and we switch between them for better intuition.

2. Adjacent Elements in a Totally Ordered Set

Because of the paradoxes exposed in [1, sec. 4], we are not astonished at the collapse of $\mathbb{R}$ as a set; instead, considering other paradoxes in [1], we would be disappointed if the crisis were limited to just the set $\mathbb{R}$.

2.1. Finding an Adjacent Element

Let the universe E be an arbitrarily given set (not necessarily complete or densely ordered) with total order $\leq$. For $a_1, a_2 \in E$ with $a_1 < a_2$, we call $\{x \in E : a_1 \leq x \leq a_2\}$ an *E-segment* and denote it by $[a_1,\, a_2]_E$; and call $\{x \in E : a_1 \leq x < a_2\}$, which is denoted by $[a_1,\, a_2)_E$, an *E-segment*, too.

Side note: *If* $E = \{1, 2, 3, \ldots, 10\}$*, then* $[2, 6]_E = \{2, 3, 4, 5, 6\}$ *and* $[2, 6)_E = \{2, 3, 4, 5\}$*, but* $\{2, 3, 5, 6\}$ *is not an* E-segment.

Since, hereafter, the universal set E is always there and equally applies to both horizontal and vertical directions, the subscript "E" is omitted if there is no ambiguity.

Theorem 2.1.1 *In a totally ordered set each element has at least one adjacent element. And each element that is neither the least nor greatest one has both an immediate predecessor and an immediate successor.*

Proof. It is sufficient to take the usual order $\leq$ as an example.

Without loss of fairness, suppose $0, x_1, x_2 \in E$ with $0 < x_1 < x_2$. (The presence of the constant 0 is not a must, but it helps us focus on key issues without changing the essence of the problem.) We just find an adjacent element for x_2, and the rest is evident then.

On X-axis there is an *E-segment* $[x_1,\, x_2]$, each point x of which is planted with a vertical *E-segment* $[0,\, x]$. Thus all the vertical *E-segments* together cover a vertical range $\{y \in E : 0 \leq y \leq x_2\}$.

Side note: *We are not talking about the familiar function* $f(x) = x$.

Step 1: Remove the upper endpoint from each of the vertical *E-segments*. As a consequence, the vertical range becomes $\{y \in E : 0 \leq y \leq x_2\} \setminus \{x_2\} = \{y \in E : 0 \leq y < x_2\}$, since any two of the vertical *E-segments* being comparable under proper inclusion (in terms of the ordinates of their points) determines that the vertical range of all the vertical *E-segments* is just the same as that of the tallest one. Now, the vertical *E-segment* $[0,\, x_2)$ contains all the counterparts of each of [*the other vertical ones + their lost upper endpoints respectively*] and is therefore the unique tallest.

Step 2: Then remove the tallest vertical *E-segment*, namely, $[0,\, x_2)$. The vertical range gets a new loss as the vertical *E-segment* $[0,\, x_2)$ is at least one element taller than each of the other vertical ones for the above mentioned fact. Hence the new loss is at least one element.

Finally, to make it clear that the new loss of the vertical range is only one element, we (return to the initial conditions and) access current ending status through another path.

Step I: Limit our horizontal base to $[x_1, x_2)$ on X-axis, so the vertical *E-segment* $[0, x_2]$ is dismissed; and the vertical range becomes $\{y \in E : 0 \leq y \leq x_2\} \setminus \{x_2\} = \{y \in E : 0 \leq y < x_2\}$, since in the original vertical range the element x_2 is the only one that is exclusively contributed by the vertical *E-segment* $[0, x_2]$.

Step II: Remove the upper endpoint from each of the vertical *E-segments*. Here we are at the ending status again. Let x_3 is a newly lost point from the vertical range (that means $x_3 \in E$ with $x_1 \leq x_3 < x_2$). To show that no element of E is between x_3 and x_2, we assume to the contrary there is an element $x_4 \in E$ such that $x_3 < x_4 < x_2$. As $x_1 < x_4 < x_2$ implies the former existence of the vertical *E-segment* $[0, x_4]$ and, in turn, the current existence of $[0, x_4)$, which contains x_3 (for $0 < x_3 < x_4$), we have that the final vertical range contains x_3. But this contradicts x_3 being a lost point from the range. Therefore the assumption about x_4 is false; only the lower adjacent element of x_2, if exists in E, may be a newly lost point from the vertical range.

Combining the two paths, we conclude that the latest loss for the vertical range is a missing of one point, and the point is the lower adjacent element of x_2 in E. The final vertical range is $\{y \in E : 0 \leq y < x_2\} \setminus \{\text{the lower adjacent element of } x_2 \text{ in } E\}$. □

A corollary immediately follows:

Corollary 2.1.2 *No totally ordered set is a densely ordered one.*

Aside: *It is not that we are blind to the property of a given set, but that a static set cannot afford the honor of having some properties — not every set in one's imagination can really exist. We do not object the concept of "dense", but it does not apply to a set.*

(We express the above procedure more symbolically in **Appendix A**.)

2.2. The Same Story for $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$

The above reasoning is based on the total ordering on a set (that is, independent of other special characters of the set), thereby applying to the sets of $\mathbb{R}$ and $\mathbb{Q}$. The set $\mathbb{R}$, which we have questioned in [1, sec. 4] for the suspicious completeness property, is the first to be trapped this time. Certainly there are many other reals between any two adjacent real elements found in the given set E, but those in between are not in E (no matter what one might expect E to be). The set $\mathbb{Q}$ has been caught out by a paradox rising from imposing dense ordering on a set [1, *Thought Experiment 6.1*] (we include a visualized version of that thought experiment in **Appendix B**), so it is no wonder that the set $\mathbb{Q}$ is trapped here again. Though there are many other rationals between any two adjacent rational elements found in the given set E, those others are not in E. That explains the related paradox of [1, *Thought Experiment 6.1*].

Now it is natural numbers' turn as they can also be densely ordered (like rationals) under certain ordering, and we present an illustration below.

Put a decimal point and a 0 after each natural number, then reverse each expanded natural number (just like reversing a string) to make a pure decimal. Let each of the decimal fractions represent its original natural number, e.g., 0.01325 represents 52310. That defines a bijection between natural numbers and positive terminating pure decimals (in base-10 number system, for instance). When such a terminating decimal is used for representing a natural number, we call it an *N-decimal.*

Then we can make use the experience gained from dealing with normal decimals and notice that the usual ordering $\leq$ of the *N-decimals* (when treated as normal decimals) is a total ordering, and between any two distinct *N-decimals* there is always another one. It is easy to throw the *N-decimals* into trouble by simply emulating [1, *Thought Experiment*

6.1], which is roughly introduced in **Appendix B**. Again, the ubiquitous adjacent elements offer a compelling explanation.

Let the universal set E be the supposed set $\{0.0$ and all *N-decimals*$\}$, which represents the supposed set $\{0$ and all natural numbers$\}$, so as to apply the reasoning in **Subsection 2.1** mechanically. The adjacent elements emerge in due course, thereby denying the existence of the supposed set $\{0.0$ and all *N-decimals*$\}$ and in turn the so-called set $\mathbb{N}$ behind it.

Aside: *Why do we eagerly disturb an otherwise "peaceful" scene? Because we have noticed something more than a suspicion that the set of the natural numbers violates the law of contradiction [1,* Thought Experiment 5.2*] [1,* p. 15, par. 1*].*

And as a consequence no "infinite set" is tenable in ZFC set theory, since there the "set of the natural numbers" is the simplest "infinite set" and every "infinite set" contains a subset equivalent with $\mathbb{N}$. Therefore we conclude:

Theorem 2.2.1 *The so-called set* $\mathbb{N}$ *does not exist.*

Corollary 2.2.2 *There is no such thing as superset of a so-called countably infinite set.*

Aside: *Such being the case, where does "the set of the natural numbers" come from? So far as we know, it appears to come from nowhere but has roots in some people's personal belief. Later, in axiomatic set theory, it is introduced by* the Axiom of Infinity*, which states that* there exists an infinite set*, or in other words* there exists an inductive set.

3. Conclusion

At this point, things become clearer than ever before. A lot of work relating to infinity and continuum, especially the part following Cantor's ideas, needs to be rethought completely.

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Appendix

A. Expressing the Procedure in Symbols More

(For more details of the reasoning please refer to **Subsection 2.1**.)

Let the universe E be an arbitrary given set with total order $\leq$. We aim to isolate an adjacent element of a given element.

Without loss of generality, suppose $x_0, x_1, x_2 \in E$ with $x_0 < x_1 < x_2$, and try to find an adjacent element of x_2. (As for the variable x_0, we use the constant 0 instead in **Subsection 2.1** for a simplified image.)

There is a set of *E-segments* $A = \{[x_0, x] : x \in E \text{ and } x_1 \leq x \leq x_2\}$, and all the *E-segments* together cover a range $\bigcup A = \{y \in E : x_0 \leq y \leq x_2\}$.

Step 1: Remove the upper endpoint of each element of A, the stage result is $A_1 = \{[x_0, x) : x \in E \text{ and } x_1 \leq x \leq x_2\}$ and the set A_1 covers the range $\bigcup A_1 = \{y \in E : x_0 \leq y < x_2\}$; for the range, the missing element is point x_2.

Step 2: Remove the element $[x_0, x_2)$ from A_1, the final result is $A_2 = A_1 \setminus \{[x_0, x_2)\} = \{[x_0, x) : x \in E \text{ and } x_1 \leq x \leq x_2\} \setminus \{[x_0, x) : x = x_2\} = \{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$ and the set A_2 covers the range $\bigcup A_2 = \bigcup\{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$; for the range, there is definitely a new loss.

Now, return to the initial conditions and revisit the same final scene through another passage.

Step I: Remove the element $[x_0, x_2]$ from A, the stage result is $A_3 = A \setminus \{[x_0, x_2]\} = \{[x_0, x] : x \in E \text{ and } x_1 \leq x \leq x_2\} \setminus \{[x_0, x] : x = x_2\} = \{[x_0, x] : x \in E \text{ and } x_1 \leq x < x_2\}$ and the set A_3 covers the range $\bigcup A_3 = \{y \in E : x_0 \leq y < x_2\}$; for the range, the missing element is point x_2. Notice that $\bigcup A_3 = \bigcup A_1$.

Step II: Remove the upper endpoint of each element of A_3, the final result is $A_4 = \{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$ and the set A_4 covers the range $\bigcup A_4 = \bigcup\{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$; by logic only the lower adjacent element of x_2, if exists in E, may be a newly lost point from the range. (Obviously, $A_4 = A_2$.)

The two passages together show the existence of the lower adjacent point of x_2 in E.

We denote z and its neighbors in E by $\langle z\rangle_E^{-2}$, $\langle z\rangle_E^{-1}$, z, $\langle z\rangle_E^{+1}$, $\langle z\rangle_E^{+2}$, and so on in both directions. Usually we omit the subscript "E" if the context makes it unnecessary. We have $\{\langle x_2\rangle^{-1}\} = \bigcup A_1 \setminus \bigcup A_2$, or $\{\langle x_2\rangle^{-1}\} = \bigcup A_3 \setminus \bigcup A_4$, that is, $\{\langle x_2\rangle^{-1}\} = \{y \in E : x_0 \leq y < x_2\} \setminus \bigcup\{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$. And $\langle x_2\rangle^{-1}$ is the upper endpoint of the *E-segment* $[x_0, x_2)$.

Side note: *It is worth noting that* $\bigcup\{[x_0, x) : x \in E \text{ and } x_1 \leq x < x_2\}$ *might be mistaken as the equivalent of* $\bigcup\{[x_0, x) : x \in E \text{ and } x_1 \leq x \leq x_2\}$ *or* $\bigcup\{[x_0, x] : x \in E \text{ and } x_1 \leq x < x_2\}$. *(Nevertheless, the latter two equal each other.)*

Along the same line of thought, $\langle x_2\rangle^{-2}$ and others are available; and the case in the other direction is entirely analogous.

B. The Partition Paradox of a "Densely Ordered Set"

Below is a visualized version of [1, *Thought Experiment 6.1*].

Containing or not?

Consider all rational numbers between 0 and 1, excluding 0 but including 1, in the form of reduced fraction. Divide them into two groups, the blue group for those have an even number as numerator or denominator, and the red group for all others (in particular, 1 is in this group). Notice that in between any two of these rationals there are both a blue one (that belongs to the blue group) and a red one (that is in the red group).

From a number line take the continuous piece that just contains all the rationals under discussion. Cut the piece at all red points while keeping each red point as the right endpoint of its own fragment. Of course, each fragment contains exactly one red point. Our question is: Is there any fragment containing a blue point?

References

[1] Zhang Ke (2021), *The Fog Covering Cantor's Paradise: Some Paradoxes on Infinity and Continuum*, figshare. Preprint. https://doi.org/10.6084/m9.figshare.16727125.v1
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