The gaps between primes

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Abstract

It is proved that

· For any positive integer d, there are infinitely many prime gaps of size $2d$.

· Every even number greater than 2 is the sum of two prime numbers.

Our method from the analysis of distribution density of pseudo primes in specific set is to transform them into upper bound problem of the maximum gaps between overlapping pseudo primes, then the two are essentially the same problem.

Keywords: Polignac's conjecture; Twin prime conjecture; Goldbach's conjecture; Sieve; Prime gaps.

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1. Introduction

In number theory, Polignac's conjecture was made by Alphonse de Polignac in 1849 and states:

For any positive even number n, there are infinitely many prime gaps of size n. In other words: There are infinitely many cases of two consecutive prime numbers with difference n. The case $n = 2$, it is the twin prime conjecture.

Although the conjecture has not yet been proven or disproven for any given value of n, in 2013 an important breakthrough was made by Zhang Yitang who proved that there are infinitely many prime gaps of size n for some value of $n < 70,000,000$. Later that year, James Maynard announced a related breakthrough which proved that there are infinitely many prime gaps of some size less than or equal to 600. As of April 14, 2014, one year after Zhang's announcement, according to the Polymath project wiki, n has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that n has been reduced to 12 and 6, respectively [6].

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It was proposed by the German mathematician Christian Goldbach in a letter to Leonhard Euler on 7 June 1742. It states that every even whole number greater than 2 is the sum of two prime numbers. The conjecture has been shown to hold for all integers less than 4×10^{18}, but remains unproven despite considerable effort [3].

In this paper, we will prove the above two conjectures.

· **Theorem 1.** For any positive integer d, there are infinitely many prime gaps of size $2d$.

· **Theorem 2.** Every even number greater than 2 is the sum of two prime numbers.

Here is a brief introduction to the main ideas of proofs.

In the study of $a+b$ problems, the $P_\omega(x,z)$ type sieve function is commonly used. Since Brun obtained $9+9$, many research results on $a+b$ type propositions have corresponding forms of twin prime number problem. For example, the Brun-Buchstab sieve method for deriving the $5+5$ problem can also be used to prove with almost the same complexity that there are infinite positive integers n such that the number of prime factors for n and $n+2$ does not exceed 5. But the complexity of these two problems shows a significant difference when the Selberg sieve is used to estimate the upper bound of $P_w(x,z)$. At this point, the two problems can be linked together through the monotonic principle in the sieve method.

The abstract form of the sieve method is usually referred to as

$$S(\mathcal{A},\mathcal{P}) := \mathcal{A} \setminus \bigcup_{p\in\mathcal{P}} \mathcal{A}_p,$$

where $\mathcal{A}$ is a set of integers, $\mathcal{P}$ is a set of prime numbers, and $\mathcal{A}_p$ is a subset of all elements in $\mathcal{A}$ that can be divisible by p. It is easy to see from the Inclusion-Exclusion Principle that

$$\#S(\mathcal{A},\mathcal{P}) = \sum_{\mathcal{Q}\subseteq\mathcal{P}} (-1)^{\#\mathcal{Q}} \#\mathcal{A}_{\mathcal{Q}},$$

For any subset $\mathcal{Q}$ of $\mathcal{P}$,

$$\mathcal{A}_{\mathcal{Q}} := \bigcap_{p\in\mathcal{Q}} \mathcal{A}_p.$$

It can be seen that the sieve method is essentially calculating the number of remaining elements in the Difference of a set and the Union. The basic problem of the sieve method is to estimate the upper bound and positive lower bound of the sieve function (if any).

In typical scenarios, the modern definition of the sieve function is

$$S\left(\mathcal{A},\mathcal{P},z\right):=\left\{a\in\mathcal{A}:\ \forall p|P\left(z\right),\ p\nmid a\right\},$$

where

$$P\left(z\right)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p.$$

It is easy to see that

$$\#S\left(\mathcal{A},\mathcal{P},z\right)=\sum_{\substack{a\in\mathcal{A}\\ \left(a,P\left(z\right)\right)=1}}1,$$

That is to calculate the number of elements in $\mathcal{A}$ that are coprime with $P\left(z\right)$. So when using the sieve method to study twin prime numbers and the Goldbach problem, that is $\mathcal{A}=\left\{i\left(i-2\right):\ i\leq w\right\}$ and $\mathcal{A}=\left\{i\left(w-i\right):\ i\leq w\right\}$.

The form of the sieve function on a continuous interval will be like

$$S\left(0,P;P\right):=\left\{0\leq a<P:\ \left(a,P\right)=1\right\},$$

where

$$P=\prod_{p\in\mathcal{P}}p,$$

$\mathcal{P}$ is composed of the first n odd prime numbers

$$\mathcal{P}:=\left\{p_1,p_2,p_3,\cdots,p_n\right\}.$$

When n is sufficiently large, for any positive integer d, if h and $h-d$ are both elements of $S\left(0,P;P\right)$, then there must exist two odd numbers q_1 and q_2 in $S\left(0,P;P\right)$ that are coprime with P and with a gap of size $2d$, such as:

$$q_1=2h-P,$$

$$q_2=2\left(h-d\right)-P.$$

It is easy to know that a sufficient condition for q_1 and q_2 to be prime numbers is that their values are both on the interval $\left[p_n,p_n^2+2p_n\right]$.

Defining product functions

$$v\left(a\right)=\prod_{\substack{p\in\mathcal{P}\\ a\equiv0\left(p\right)\vee a\equiv d\left(p\right)}}p.$$

Then, for the problem of the gap between prime numbers, there is a sieve function that removes multiple congruence classes

$$S_2\left(0, P; P, v\left(a\right)\right) := \left\{0 \leq a < P: \ \left(v\left(a\right), P\right) = 1\right\}.$$

In this way, the problem of the gap between two prime numbers is transformed into the problem of the distribution of elements in the S_2 sieve. Considering the gap between adjacent elements in S_2 sieve, if the maximum gap between adjacent elements in S_2 sieve is not greater than $\frac{p_n^2}{2}$, then there must be at least one element h^* in S_2 sieve, so that the values of $q_1^* = 2h^* - P$ and $q_2^* = 2\left(h^* - d\right) - P$ are both within the interval $\left[p_n, p_n^2 + 2p_n\right]$.

The problem of the sum of two prime numbers is similar. Simply replace the $(h - d)$ in the q_2 expression with $(d - h)$, and replace $-P$ with $+P$, then we can obtain that the two elements q_1 and q_2 satisfy $q_1 + q_2 = 2d$. But this constraint is more stringent on the maximum gap between adjacent elements in S_2, to ensure that such a prime pair always exists continuously for any d.

So we unified the sum of two prime numbers problem and the gap between prime numbers problem into the minimum upper bound problem of the gaps between adjacent elements in the S_2 sieve.

Certainly, we can also describe this same problem in a more intuitive set form.

For any positive integer d, take a sufficiently large prime p_n, where p_n is the n-th odd prime. [∗1]

Let the set $\mathbb{H}$ denotes all integers without factors $p_1, p_2, \cdots, p_n$.

$$\mathbb{H} = \left\{h: \ \left(\forall p \in \left\{p_1, p_2, \cdots, p_n\right\}\right) \left(p \nmid |h|\right)\right\} \qquad [*2]$$

For any element h belongs to $\mathbb{H}$, if $(h - d)$ also belongs to $\mathbb{H}$, there must be two odd pseudo primes [∗3] q_1 and q_2 with a gap of size $2d$ belonging to $\mathbb{H}$, such as:

$$q_1 = 2h - \mathcal{T},$$
$$q_2 = 2\left(h - d\right) - \mathcal{T},$$

where

$$\mathcal{T} = \prod_{p \in \left\{p_1, p_2, \cdots, p_n\right\}} p.$$

Then the sufficient condition for them to be real prime numbers is in the domain $\left[p_n, p_n^2 + 2p_n\right]$.

Let $\mathbb{H}^*$ be the set of overlapping pseudo primes [∗4], composed of all elements that meet the above conditions.

$$\mathbb{H}^* = \left\{h: \ h \in \mathbb{H} \wedge \left(h - d\right) \in \mathbb{H}\right\}.$$

Now let's consider the gaps between adjacent elements belong to $\mathbb{H}^*$. Obviously, if the maximum gaps between adjacent elements belong to $\mathbb{H}^*$ are less than $\frac{p_n^2}{2}$, there will be at least one element h^* belongs to $\mathbb{H}$, so that q_1 and q_2 are both in the domain $\left[p_n, p_n^2 + 2p_n\right]$, because the range is greater than p_n^2.

The case of sums of two primes is similar, except that $(h-d)$ will be replaced by $(d-h)$ and the condition of maximum gaps between adjacent elements belong to $\mathbb{H}^*$ must be less than $\frac{p_n^2}{8}$.

Therefore, the core of this proofs is that the upper bound of the maximum gaps between overlapping pseudo primes must be less than $\frac{p_n^2}{8}$. By estimating the maximum length of consecutive elements in the complement set of $\mathbb{H}^*$, we will prove that it holds when p_n is greater than 2096.

Remark 1.
[∗1] in other words, as long as d is sufficiently small, such as $d = 1$, then p_n can be arbitrary. Actually, $p_n > 2d$ will be enough.
[∗2] for example, for $p_n = 5$, $\mathbb{H}$={...,-4,-2,-1,1,2,4,7,8,11,13,14,16,17,...}.
[∗3] pseudo prime means that it contains no factors $p_1, p_2, \cdots, p_n$.
[∗4] overlapping pseudo prime means that element h and its corresponding element $(h-d)$ are both pseudo primes in set $\mathbb{H}$.

2. **Notation and definitions**

Notation.

$a,\ b,\ c,\ d,\ h,\ i,\ j,\ k,\ m,\ n,\ q,\ t,\ w,\ u$: integers.
p: a prime number.
p_t: the t-th odd prime number p_1=3, p_2=5, etc.
$a \mid d$ means a is a divisor of d.
$a \nmid d$ means a does not divide d.
x: variable.
$\lfloor x \rfloor$ means the largest integer which does not exceed x.
$\lceil x \rceil$ means the least integer not less than x.
$\binom{d}{a}$ means d choose a; the binomail coefficient $\dfrac{d!}{a!\,(d-a)!}$.
$\mathbb{A}$: an abstract field for function parameter.
$\mathbb{Z}$: the field of integers.
$\mathbb{M}^\circ$: the base set of $p_1, p_2, \cdots, p_n$.
$\mathbb{M}_i$: infinite set generated by elements of $\mathbb{M}^\circ$ with offset i.
$\mathbb{M}_{i\cup j}$ means $\mathbb{M}_i \cup \mathbb{M}_j$.
$\mathbb{A}\,[a,b)$ means $\mathbb{A} \cap [a,b)$.
$|\mathbb{A}|$ denote the cardinality of set $\mathbb{A}$.
$\lambda\,(\mathbb{A}, d)$: generate a new set by adding d to each element of set $\mathbb{A}$.
$T\,(a)$: product function.
$\chi\,(a, \mathbb{A})$: use 0 or 1 to indicate whether a belongs to $\mathbb{A}$.

$\Lambda(d)$: the von Mangoldt function.
$\theta(x)$: the first Chebyshev function.
$\psi(x)$: the second Chebyshev function.
$(a_1, a_2, a_3, \cdots)$, $(\cdots)$: ordered arrays.
$\rho((a_1, a_2, \cdots))$, $\vartheta((\cdots))$: custom functions for lemma declaration.
$\mu((a_1, a_2, \cdots), m)$: a custom function for proving lemma.
$\mathcal{J}(p)$, $\mathcal{K}(p)$, $\mathcal{S}(w)$: custom functions for proving lemma.
$\varrho(x)$: a custom function, we will prove that it is less than 1.
η: used to denote the gaps of overlapping pseudo primes.
$L_i(a, t)$: used to estimate η .
$\mathcal{T}$, $\mathcal{H}$: custom sets.
$v(\mathcal{H}_1, \mathcal{H}_2, \cdots)$: defined to assist in estimating $L_i(a, t)$.
$(f(x))'$ means $f'(x)$, that is the derivative of $f(x)$.
$\exp\{\cdots\}$: exponential function.
$\inf\{\cdots\}$: greatest lower bound.
$\sup\{\cdots\}$: least upper bound.

Definition 1. For $n \geq 1$,

$$\mathbb{M}^\circ = \{p_1, p_2, \cdots, p_n\}.$$

Definition 2. For any i,

$$\mathbb{M}_i = \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + i\}.$$

Definition 3. For any i and j,

$$\mathbb{M}_{i \cup j} = \mathbb{M}_i \cup \mathbb{M}_j.$$

Definition 4. Let λ be the function, defined by

$$\lambda(\mathbb{A}, d) = \{m : m = a + d \wedge a \in \mathbb{A}\}.$$

Definition 5. For any a,

$$T(a) = \prod_{m \in \mathbb{M}^\circ} (m - a).$$

Definition 6. Let the function χ be given by

$$\chi(a, \mathbb{A}) = \begin{cases} 1 & \text{if } a \in \mathbb{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7. The von Mangoldt function Λ is defined by

$$\Lambda(d) = \begin{cases} \ln p & \text{if } d = p^k \wedge k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The unique factorization property of the natural numbers implies

$$\ln d = \sum_{a|d} \Lambda(a),$$

the sum is taken over all integers a that divide d [7].

Definition 8. The first Chebyshev function $\theta(x)$ is defined by

$$\theta(x) = \sum_{p \le x} \ln p,$$

where the sum is over primes $p \le x$ [2].

Definition 9. The second Chebyshev function $\psi(x)$ is defined similarly

$$\psi(x) = \sum_{k \in \mathbb{N}} \sum_{p^k \le x} \ln p = \sum_{d \le x} \Lambda(d),$$

with the sum extending over all prime powers not exceeding x [2].

3. Lemmas

In this section we introduce a number of prerequisite results, some of them given here may not be in the strongest forms, but they are adequate for the proofs of **Theorems 1** and **2**.

Lemma 1. $(\forall i, j)\,(\ \mathbb{M}_j = \lambda(\mathbb{M}_i,\ j-i)\ )$.

Proof. By Definition 2 and Definition 4, we obtain

$$\begin{aligned} \mathbb{M}_j &= \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + j\} \\ &= \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + i + (j-i)\} \\ &= \lambda(\mathbb{M}_i,\ j-i). \end{aligned}$$

Lemma 2. $(\forall i, h, a)\,(\ \chi(h,\ \mathbb{M}_i) = \chi(h+a,\ \lambda(\mathbb{M}_i,\ a)) = \chi(h+a,\ \mathbb{M}_{i+a})\ )$.

Proof. Let us suppose

$$\chi(h,\ \mathbb{M}_i) = 1,$$

then

$$(\exists k_0 \in \mathbb{Z} \wedge m_0 \in \mathbb{M}^\circ)\,(\ k_0 m_0 + i = h\ ).$$

And by Lemma 1,

$$\lambda\left(\mathbb{M}_i,\ a\right)=\mathbb{M}_{i+a}.$$

Hence,

$$\begin{aligned}\chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right) &= \chi\left(h+a,\ \mathbb{M}_{i+a}\right)\\ &= \chi\left(\left(k_0m_0+i\right)+a,\ \mathbb{M}_{i+a}\right)\\ &= \chi\left(k_0m_0+\left(i+a\right),\ \mathbb{M}_{i+a}\right)\\ &= 1.\end{aligned}$$

Otherwise,

$$\chi\left(h,\ \mathbb{M}_i\right)=0,$$

then

$$\left(\forall k\in\mathbb{Z}\wedge m\in\mathbb{M}^{\circ}\right)\left(\ km+i\neq h\ \right).$$

Hence,

$$h+a\neq km+\left(i+a\right),$$

i.e.

$$\chi\left(h+a,\ \mathbb{M}_{i+a}\right)=\chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right)=0.$$

So that

$$\left(\forall i,h,a\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+a,\ \lambda\left(\mathbb{M}_i,\ a\right)\right)=\chi\left(h+a,\mathbb{M}_{i+a}\right)\right).$$

Lemma 3. $\left(\forall i,h\wedge m\in\mathbb{M}_0\right)\left(\ \chi\left(m\left(h-i\right)+i,\ \mathbb{M}_i\right)=1\ \right).$

Proof. Obviously,

$$\left(\exists k_0\in\mathbb{Z}\wedge m_0\in\mathbb{M}^{\circ}\right)\left(\ k_0m_0+0=m\ \right).$$

Let

$$k_1=k_0\left(h-i\right),$$

then

$$m\left(h-i\right)+i\ =k_1m_0+i.$$

So that

$$\chi\left(m\left(h-i\right)+i,\ \mathbb{M}_i\right)=\ \chi\left(k_1m_0+i,\ \mathbb{M}_i\right)=1.$$

Lemma 4. $\left(\forall i,h\wedge u\notin\mathbb{M}_0\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)\ \right).$

Proof. Suppose that

$$\chi\left(h,\ \mathbb{M}_i\right)=1,$$

then

$$\left(\exists k_0\in\mathbb{Z}\wedge m_0\in\mathbb{M}^{\circ}\right)\left(\ k_0m_0+i=h\ \right).$$

Hence,

$$\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)=\chi\left(\left(uk_0\right)m_0+i,\ \mathbb{M}_i\right)=1.$$

Otherwise,

$$\chi\left(h,\ \mathbb{M}_i\right)=0,$$

then

$$\left(\forall k_0\in\mathbb{Z}\wedge m_0\in\mathbb{M}^\circ\right)\left(\ k_0m_0+i\neq h\ \right).$$

Noting that

$$\left(\forall k_1\in\mathbb{Z}\wedge m_1\in\mathbb{M}^\circ\right)\left(\ k_1m_1+0\neq u\ \right).$$

Combining the both, we have

$$\left(\forall k_2\in\mathbb{Z}\wedge m_2\in\mathbb{M}^\circ\right)\left(\ k_2m_2\neq u\left(h-i\right)\ \right).$$

Thus,

$$u\left(h-i\right)+i\neq k_2m_2+i,$$

i.e.

$$\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)=0.$$

So that

$$\left(\forall i,h\wedge u\notin\mathbb{M}_0\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(u\left(h-i\right)+i,\ \mathbb{M}_i\right)\ \right).$$

Remark 2. A stronger conclusion is that

$$\{m:\ 0\leq m<T\left(0\right)\wedge m\notin\mathbb{M}_0\}$$

is a multiplicative group of integers modulo $T\left(0\right)$. It will not be proved here because this conclusion is not used in the proofs of this paper.

Lemma 5. $\left(\forall i,h,d\right)\left(\ \chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+dT\left(0\right),\ \mathbb{M}_i\right)\ \right).$

Proof. By Lemma 2,

$$\chi\left(h,\ \mathbb{M}_i\right)=\chi\left(h+dT\left(0\right),\ \mathbb{M}_{i+dT(0)}\right),$$

and

$$\mathbb{M}_{i+dT(0)}=\lambda\left(\mathbb{M}_i,\ dT\left(0\right)\right)=\bigcup_{\substack{k\in\mathbb{Z}\\ m\in\mathbb{M}^\circ}}\{km+i+dT\left(0\right)\}.$$

By the Definition 5,

$$T\left(0\right)=\prod_{m\in\mathbb{M}^\circ}m.$$

This implies that

$$\left(\forall k\in\mathbb{Z}\wedge m\in\mathbb{M}^\circ\right)\left(\left(\exists k_0\in\mathbb{Z}\right)\left(\ km+dT\left(0\right)=k_0m\ \right)\right)$$

Combining this with above,

$$\mathbb{M}_{i+dT(0)} = \bigcup_{\substack{k \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{km + i + dT(0)\} = \bigcup_{\substack{k_0 \in \mathbb{Z} \\ m \in \mathbb{M}^\circ}} \{k_0 m + i\} = \mathbb{M}_i. \tag{3.1}$$

Hence,

$$\chi(h,\ \mathbb{M}_i) = \chi\left(h + dT(0),\ \mathbb{M}_{i+dT(0)}\right) = \chi(h + dT(0),\ \mathbb{M}_i).$$

Remark 3. So we can see that $\mathbb{M}_i$ is periodic and its period is $T(0)$.

LEMMA 6. $(\forall i, j, h, d)\,(\ \chi(h,\ \mathbb{M}_{i\cup j}) = \chi(h + dT(0),\ \mathbb{M}_{i\cup j})\ )$.

Proof. By Lemma 5 we have

$$\chi(h,\ \mathbb{M}_i) = \chi(h + dT(0),\ \mathbb{M}_i),$$

and

$$\chi(h,\ \mathbb{M}_j) = \chi(h + dT(0),\ \mathbb{M}_j).$$

It is easy to see that

$$\begin{aligned}\chi(h + dT(0),\ \mathbb{M}_{i\cup j}) &= \chi(h + dT(0),\ \mathbb{M}_i) \circledast \chi(h + dT(0),\ \mathbb{M}_j) \\ &= \chi(h,\ \mathbb{M}_i) \circledast \chi(h,\ \mathbb{M}_j) \\ &= \chi(h,\ \mathbb{M}_{i\cup j}),\end{aligned}$$

where we do not need to know exactly what operator $\circledast$ does.

Remark 4. We can also prove it by the truth table.

$\chi(h, \mathbb{M}_i)$	$\chi(h, \mathbb{M}_j)$	$\chi(h + dT(0), \mathbb{M}_i)$	$\chi(h + dT(0), \mathbb{M}_j)$	$\chi(h, \mathbb{M}_{i\cup j})$	$\chi(h + dT(0), \mathbb{M}_{i\cup j})$
0	0	0	0	0	0
0	1	0	1	1	1
1	0	1	0	1	1
1	1	1	1	1	1

So $\mathbb{M}_{i\cup j}$ and $\mathbb{M}_i$ have the same period.

LEMMA 7. $|\mathbb{M}_0\,[0, T(0))| = T(0) - T(1)$.

Proof. It is easy to see that

$$|\mathbb{M}_0\,[0, T\,(0))| = \left|\bigcup_{m_0 \in \mathbb{M}_0[0,T(0))} m_0\right|$$

$$= \left|\bigcup_{(\forall k\in\mathbb{Z}\wedge m\in\mathbb{M}^\circ)(m_0=km\wedge m_0\in[0,T(0)))} m_0\right|$$

$$= T\,(0)\left(\begin{array}{c} \sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{1}{m_1}\right) - \sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{1}{m_1m_2}\right) + \\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{1}{m_1m_2m_3}\right) - \\ \cdots + (-1)^{n-1} \sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{1}{m_1m_2m_3\cdots m_n}\right) \end{array}\right).$$

Then the alternating series can be reduced to showing that

$$|\mathbb{M}_0\,[0, T\,(0))| = T\,(0)\left(1 - \frac{T\,(1)}{T\,(0)}\right) = T\,(0) - T\,(1)\,.$$

LEMMA 8. $(\forall i, a)\,(\ |\mathbb{M}_i\,[a, a + T\,(0))| = T\,(0) - T\,(1)\ )$.

Proof. By Lemma 2,

$$\begin{aligned} |\mathbb{M}_i\,[a, a + T\,(0))| &= |\lambda\,(\mathbb{M}_i,\,-a)\,[\mathrm{a} - \mathrm{a}, \mathrm{a} + T\,(0) - \mathrm{a})| \\ &= |\mathbb{M}_{i-a}\,[0, T\,(0))| \\ &= |\mathbb{M}_0\,[0, T\,(0))|\,. \end{aligned}$$

By Lemma 7,

$$|\mathbb{M}_i\,[a, a + T\,(0))| = |\mathbb{M}_0\,[0, T\,(0))| = T\,(0) - T\,(1)\,.$$

LEMMA 9. $(\forall i, j, a)\,(\ |\mathbb{M}_{i\cup j}\,[a, a + T\,(0))| \leq T\,(0) -\ T\,(2) < T\,(0)\ )$.

Proof. If

$$(\exists k \in \mathbb{Z})\,(\ j = i + kT\,(0)\ )\,,$$

then (by (3.1))

$$\begin{aligned}\left|\mathbb{M}_{i\cup j}\left[a,a+T\left(0\right)\right)\right| &= \left|\mathbb{M}_{i\cup i}\left[a,a+T\left(0\right)\right)\right| \\ &= \left|\mathbb{M}_{i}\left[a,a+T\left(0\right)\right)\right| \\ &= T\left(0\right)-T\left(1\right).\end{aligned}$$

Otherwise, let us suppose

$$\left(\forall m\in\mathbb{M}_0\right)\left(\ j-i\not\equiv 0\left(mod\ m\right)\ \right).$$

It is similar to the proof of Lemma 7, we have

$$\left|\mathbb{M}_{i\cup j}\left[0,T\left(0\right)\right)\right| =$$

$$T\left(0\right)\left(\begin{array}{c}\sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{2^1}{m_1}\right)-\sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{2^2}{m_1m_2}\right)+\\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{2^3}{m_1m_2m_3}\right)-\\ \cdots+\left(-1\right)^{n-1}\sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{2^n}{m_1m_2m_3\cdots m_n}\right)\end{array}\right).$$

Then the alternating series can be reduced to showing that

$$\left|\mathbb{M}_{i\cup j}\left[0,T\left(0\right)\right)\right| = T\left(0\right)\left(1-\frac{T\left(2\right)}{T\left(0\right)}\right) = T\left(0\right)-T\left(2\right).$$

For the opposite case, there is at least one $m\in\mathbb{M}^\circ$ such that the coefficient of each term containing m in the above alternating series is divided by 2.
The reason is that

$$\left(\exists m\in\mathbb{M}^\circ\right)\left(\ \{km+j:k\in\mathbb{Z}\}=\{km+i:k\in\mathbb{Z}\}\ \right).$$

Therefore,

$$\left|\mathbb{M}_{i\cup j}\left[0,T\left(0\right)\right)\right| < T\left(0\right)\left(1-\frac{T\left(2\right)}{T\left(0\right)}\right) = T\left(0\right)-T\left(2\right),$$

when

$$\left(\exists m\in\mathbb{M}_0\right)\left(\ j-i\equiv 0\left(mod\ m\right)\ \right).$$

Obviously,

$$T\left(0\right)>T\left(1\right)>T\left(2\right)>0.$$

Combining with the above, we have

$$\left|\mathbb{M}_{i\cup j}\left[0,T\left(0\right)\right)\right| \le T\left(0\right)-\ T\left(2\right) < T\left(0\right).$$

By Lemma 6, $\mathbb{M}_{i\cup j}$ is periodic with $T(0)$, and considering Lemma 8, we can get

$$|\mathbb{M}_{i\cup j}[a, a+T(0))| = |\mathbb{M}_{i\cup j}[0, T(0))| \leq T(0) - T(2) < T(0).$$

Lemma 10. $(\exists\eta > 0)\left((\forall i, j, a)\left(\ |\mathbb{M}_{i\cup j}[a, a+\eta)| < \eta\ \right)\right).$

Proof. By Lemma 9, there are at least $T(2)$ numbers in any range $T(0)$ that make

$$\chi(h,\ \mathbb{M}_{i\cup j}) = 0,$$

where

$$h \in [a, a+T(0)).$$

It can also be expressed as

$$(\forall i, j, a)\left(\left(\sum_{h\in[a,a+T(0))\wedge\chi(h,\mathbb{M}_{i\cup j})=0} 1\right) \geq T(2) > 0\right)$$

So that

$$0 < \eta \leq T(0).$$

On the basis of Lemma 10 we have

Lemma 11. $(\forall i, j, a)\left((\exists\, h \in [a, a+\eta))\left(\ \chi(h,\ \mathbb{M}_i) = \chi(h+j-i,\ \mathbb{M}_i) = 0\ \right)\right).$

Proof. By Lemma 10,

$$(\forall i, j, \mathrm{a})\left((\exists h_0 \in [a, a+\eta))\left(\ \chi(h_0,\ \mathbb{M}_{i\cup j}) = 0\ \right)\right),$$

so that

$$\chi(h_0,\ \mathbb{M}_i) = \chi(h_0,\ \mathbb{M}_j) = 0.$$

By Lemma 2,

$$\chi(h_0,\ \mathbb{M}_j) = \chi(h_0+j-i,\ \mathbb{M}_i).$$

Therefore,

$$\chi(h_0,\ \mathbb{M}_i) = \chi(h_0+j-i,\ \mathbb{M}_i) = 0.$$

Lemma 12. *For* $t \geq 1$,

$$(\forall m_1, m_2, m_3, \cdots, m_t \in \mathbb{M}^\circ)\left(\sum_{\delta\in\mathcal{T}} \rho(\delta) \leq \sum_{\delta\in\mathcal{T}} \vartheta(\delta)\right)$$

where

$$\rho((a_1, a_2, a_3, \cdots))$$

$$= \begin{cases} 0 & (\exists j > i \geq 1) \left(a_i = a_j \wedge a_i \nmid (j-i) \right), \\ \displaystyle\prod_{\substack{m \in \mathbb{M}^\circ \\ m | (a_1 a_2 a_3 \cdots)}} \frac{1}{m} & \textit{otherwise.} \end{cases}$$

and

$$\vartheta \left((a_1, a_2, a_3, \cdots) \right) = \frac{1}{a_1 a_2 a_3 \cdots},$$

and

$$\mathcal{T} = \bigcup_{\{h_1,h_2,h_3,\cdots,h_t\}=\{1,2,3,\cdots,t\}} \left\{ (m_{h_1}, m_{h_2}, m_{h_3}, \cdots, m_{h_t}) \right\}.$$

Proof. Let

$$(\forall \omega \geq 1) \left(m_{t+\omega} = p_{n+\omega} \right),$$

and

$$\mathcal{T}(n) = \bigcup_{\{h_1,h_2,h_3,\cdots,h_n\}=\{1,2,3,\cdots,n\}} \{(m_{h_1}, m_{h_2}, m_{h_3}, \cdots, m_{h_n})\}.$$

Then

$$\begin{aligned} \left(\sum_{\delta \in \mathcal{T} \wedge \rho(\delta) \neq 0} 1 \right) / |\mathcal{T}| &\leq \lim_{n \to \infty} \left(\sum_{\delta \in \mathcal{T}(n) \wedge \rho(\delta) \neq 0} 1 \right) / |\mathcal{T}(n)| \\ &= \left(\prod_{m \in \mathbb{M}^\circ \wedge m | (m_1 m_2 m_3 \cdots m_t)} m \right) / (m_1 m_2 m_3 \cdots m_t). \end{aligned}$$

So that

$$\begin{aligned} \sum_{\delta \in \mathcal{T}} \rho(\delta) &= \left(\sum_{\delta \in \mathcal{T} \wedge \rho(\delta) \neq 0} 1 \right) \left(\prod_{m \in \mathbb{M}^\circ \wedge m | (m_1 m_2 m_3 \cdots m_t)} \frac{1}{m} \right) \\ &\leq \frac{|\mathcal{T}|}{m_1 m_2 m_3 \cdots m_t} = \sum_{\delta \in \mathcal{T}} \vartheta(\delta). \end{aligned}$$

On the basis of Lemma 12 we have

Lemma 13. *For $t \geq 1$,*

$$\sum_{\delta \in \mathcal{T}} \rho(\delta) \leq \sum_{\delta \in \mathcal{T}} \vartheta(\delta)$$

where

$$\mathcal{T} = \bigcup_{\substack{m_1 \in \mathbb{M}^\circ \\ m_2 \in \mathbb{M}^\circ \\ \vdots \\ m_t \in \mathbb{M}^\circ}} \{(m_1, m_2, m_3, \cdots, m_t)\}.$$

Proof. Let

$$\mathcal{J}(p) = \sum_{d \in [1,t]} \begin{cases} 0 & m_d \neq p, \\ 1 & m_d = p. \end{cases}$$

and

$$s = \sum_{\substack{h_1 \geq 0 \wedge h_2 \geq 0 \wedge \cdots \wedge h_n \geq 0 \\ h_1 + h_2 + \cdots + h_n = t}} 1,$$

and

$$\{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_s\} = \bigcup_{\substack{h_1 \geq 0 \wedge h_2 \geq 0 \wedge \cdots \wedge h_n \geq 0 \\ h_1 + h_2 + \cdots + h_n = t}} \left(\left\{ \bigcup_{\substack{m_1 \in \mathbb{M}^\circ \\ m_2 \in \mathbb{M}^\circ \\ \vdots \\ m_t \in \mathbb{M}^\circ \\ (\forall d \in [1,n]) (\mathcal{J}(p_d) = h_d)}} \{(m_1, m_2, m_3, \cdots, m_t)\} \right\} \right).$$

It is easy to see that

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_s,$$

and

$$(\forall s \geq j > i \geq 1) (\mathcal{T}_j \cap \mathcal{T}_i = \phi).$$

Combining this with Lemma 12, we have

$$\sum_{\delta \in \mathcal{T}} \rho(\delta) = \sum_{d \in [1,s]} \sum_{\delta_d \in \mathcal{T}_d} \rho(\delta_d) \leq \sum_{d \in [1,s]} \sum_{\delta_d \in \mathcal{T}_d} \vartheta(\delta_d) = \sum_{\delta \in \mathcal{T}} \vartheta(\delta).$$

Similarly, for

$$(\forall \omega \geq 1)\left(m_{t+\omega}=p_{n+\omega}\right),$$

and

$$\mathcal{T}(n)=\bigcup_{\{h_1,h_2,h_3,\cdots,h_n\}=\{1,2,3,\cdots,n\}}\{(m_{h_1},m_{h_2},m_{h_3},\cdots,m_{h_n})\},$$

we also have

$$\sum_{\delta\in\mathcal{T}(n)}\rho(\delta)\leq\sum_{\delta\in\mathcal{T}(n)}\vartheta(\delta).$$

LEMMA 14. $(\forall x \geq 3)$

$$\left(\prod_{2<p\leq x}\left(1-2p^{-1}\right)\geq\frac{0.4}{\ln^2 x}\right).$$

Proof. By Mertens' second theorem [5],

$$\sum_{p\leq x}\left(p^{-1}\right)=\ln\ln x+M+O\left(1/\ln x\right).$$

The value of M is approximately [4]

$$M\approx 0.261497212847642784\cdots.$$

For $p>2$,

$$\sum_{2<p\leq x}\left(p^{-1}\right)=\ln\ln x+M'+O\left(1/\ln x\right).$$

The value of M' is approximately

$$M'\approx -0.238502787152357217\cdots.$$

Since

$$\begin{aligned}\left|\ln\left(1-2p^{-1}\right)+2p^{-1}\right| &= \left|\int_1^{1-2p^{-1}}\left(t^{-1}-1\right)dt\right| \\ &= \left|\frac{2}{p}-\frac{2}{p}-\frac{2^2}{2p^2}-\frac{2^3}{3p^3}-\frac{2^4}{4p^4}-\cdots\right| \\ &< \frac{2^2}{2p^2}+\frac{2^3}{2p^3}+\frac{2^4}{2p^4}+\cdots \\ &= \frac{2}{p\left(p-2\right)},\end{aligned}$$

and

$$\sum_{p>2}\frac{2}{p\left(p-2\right)}$$

is convergent, the series

$$\sum_{p>2}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)$$

must be convergent. Because the series

$$\sum_{p>2}\left(p^{-1}\right)$$

is divergent and so the product

$$\prod_{p>2}\left(1-2p^{-1}\right)$$

must diverge also (to zero). We can deduce that

$$\begin{aligned}
&\sum_{2<p\le x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)\\
&\quad=\sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)-\sum_{p>x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)\\
&\quad=\sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)+O\left(\sum_{p>x}\left(\frac{1}{p\left(p-2\right)}\right)\right)\\
&\quad=\sum_{p}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)+O\left(x^{-1}\right),
\end{aligned}$$

$$\begin{aligned}
&\ln\left(\prod_{2<p\le x}\left(1-2p^{-1}\right)\right)\\
&=-2\sum_{p\le x}\left(p^{-1}\right)+\sum_{2<p\le x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)\\
&=-2\ln\ln x-2M'+\sum_{2<p\le x}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)+O\left(\ln^{-1}x\right).
\end{aligned}$$

It's known from numerical calculation

$$\sum_{p>2}\left(\ln\left(1-2p^{-1}\right)+2p^{-1}\right)\approx-0.660393386913.$$

Combining with the above, we can crudely estimate

$$\prod_{2<p\le x}\left(1-2p^{-1}\right)\ge\frac{0.4}{\ln^{2}x}$$

through numerical analysis.

LEMMA 15. $(\forall x \geq 1)$

$$\left(\frac{\lfloor x \rfloor !}{\left(\lfloor \frac{x}{2} \rfloor ! \right)^2} < 6^{\frac{x}{2}} \right).$$

Proof. If $\lfloor x \rfloor = 2k$ is even, then

$$\frac{\lfloor x \rfloor !}{\left(\lfloor \frac{x}{2} \rfloor ! \right)^2} = \left(\begin{array}{c} 2k \\ k \end{array} \right) \leq 2^{2k} = 4^{\frac{x}{2}} \leq 2^{2k+1} \left(1 + \frac{1}{2} \right)^k < 6^{\frac{x}{2}},$$

because it's the largest binomail coefficient in the binomail expansion of $(1+1)^{2k}$. Otherwise, $\lfloor x \rfloor = 2k+1$ is odd, then

$$\frac{\lfloor x \rfloor !}{\left(\lfloor \frac{x}{2} \rfloor ! \right)^2} = \left(\begin{array}{c} 2k+1 \\ k \end{array} \right) (k+1) \leq 2^{2k} (k+1) \leq 2^{2k+1} \left(1 + \frac{1}{2} \right)^k < 6^{\frac{x}{2}}.$$

LEMMA 16. *Upper bounds exist for both $\theta(x)$ and $\psi(x)$ that*

$$(\forall x \geq 1) \left(\ \theta(x) \leq \psi(x) < x \ln 6 \ \right).$$

Proof. By Definition 9, we have

$$\ln(\lfloor x \rfloor !) = \psi(x) + \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) + \psi\left(\frac{x}{4}\right) + \cdots.$$

Changing x to $\frac{x}{2}$, and inserting -2ln $\left(\lfloor \frac{x}{2} \rfloor !\right)$ into the above equation we obtain

$$\ln(\lfloor x \rfloor !) - 2\ln\left(\lfloor \frac{x}{2} \rfloor !\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \psi\left(\frac{x}{4}\right) + \cdots.$$

It is obvious that

$$\psi(x) \geq \psi\left(\frac{x}{2}\right) \geq \psi\left(\frac{x}{3}\right) \geq \psi\left(\frac{x}{4}\right) \geq \cdots,$$

so that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \ln(\lfloor x \rfloor !) - 2\ln\left(\lfloor \frac{x}{2} \rfloor !\right) = \ln\left(\frac{\lfloor x \rfloor !}{\left(\lfloor \frac{x}{2} \rfloor ! \right)^2} \right).$$

Combining this with Lemma 15, we can get

$$\psi(x) - \psi\left(\frac{x}{2}\right) < \left(\frac{x}{2}\right) \ln 6.$$

Changing x to $\frac{x}{2}, \frac{x}{4}, \frac{x}{8}, \cdots$, we have

$$\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) < \left(\frac{x}{4}\right)\ln 6,$$
$$\psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) < \left(\frac{x}{8}\right)\ln 6,$$
$$\psi\left(\frac{x}{8}\right) - \psi\left(\frac{x}{16}\right) < \left(\frac{x}{16}\right)\ln 6,$$
$$\vdots$$

Adding up them all, we have

$$\psi(x) < x\ln 6.$$

It is easy to see that the relationship between $\theta(x)$ and $\psi(x)$ is given by

$$\psi(x) = \sum_{d\geq 1}\theta\left(x^{\frac{1}{d}}\right).$$

There is the fact that

$$\theta(x) \leq \psi(x) < x\ln 6.$$

LEMMA 17. *For $x \geq 3$, let*

$$\varrho(x) = \left(\prod_{2<p\leq x} p\right)\left(1 - \prod_{2<p\leq x}\left(1-2p^{-1}\right)\right)^{\left(\frac{\ln^2 x}{0.4}\right)x\ln 6},$$

then $\varrho(x) < 1$.

Proof. By Lemma 16,

$$\ln\left(\prod_{2<p\leq x} p\right) < \theta(x) < x\ln 6,$$

thus

$$\left(\prod_{2<p\leq x} p\right) < \exp\{x\ln 6\}.$$

By Lemma 14,

$$\left(1 - \prod_{2<p\leq x}\left(1-2p^{-1}\right)\right) \leq \left(1 - \frac{0.4}{\ln^2 x}\right).$$

Combining these results with numerical analysis we obtain

$$\begin{aligned}\varrho(x) &< \exp\{x\ln 6\}\left(1-\frac{0.4}{\ln^2 x}\right)^{\left(\frac{\ln^2 x}{0.4}\right)x\ln 6}\\ &< \exp\{x\ln 6\}\exp\{-x\ln 6\}\\ &= 1.\end{aligned}$$

4. Estimation of $L_2(a,t)$ and η

In this section we estimate $L_2(a,t)$ and η.
First, for $t \geq 0$, let

$$L_1(a,t) = \{m : \{m, m+1, m+2, \cdots, m+t\} \subseteq \mathbb{M}_i[a, a+T(0)+t)\}.$$

We can see that for each element in $L_1(a,t)$, it denotes that there are $(t+1)$ consecutive elements in $\mathbb{M}_i[a, a+T(0)+t)$. We have

$$|L_1(a,t)| \leq T(0)\left(1-\frac{T(1)}{T(0)}\right)^{t+1}.$$

Proof. Considering the proof of Lemma 7 and Lemma 8, and combining this with Lemma 12 and Lemma 13, we have

$$|L_1(a,t)| = \left|\bigcap_{w\in[0,t]}\mathbb{M}_{i+w}[a, a+T(0))\right| =$$

$$\left|\bigcap_{w\in[0,t]}\bigcup_{m\in\mathbb{M}^\circ}\{km+i+w\}[a, a+T(0))\right| =$$

$$\left|\bigcup_{\substack{m_0\in\mathbb{M}^\circ\\ m_1\in\mathbb{M}^\circ\\ \vdots\\ m_t\in\mathbb{M}^\circ}}\bigcap_{w\in[0,t]}\{km_w+i+w\}[a, a+T(0))\right| \leq$$

$$T(0)\left(\begin{array}{c}\sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{1}{m_1}\right)-\sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{1}{m_1m_2}\right)+\\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{1}{m_1m_2m_3}\right)-\\ \cdots+(-1)^{n-1}\sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{1}{m_1m_2m_3\cdots m_n}\right)\end{array}\right)^{t+1}$$

$$=T(0)\left(1-\frac{T(1)}{T(0)}\right)^{t+1}.$$

i.e.

$$|L_1(a,t)|\le T(0)\left(1-\frac{T(1)}{T(0)}\right)^{t+1}.$$

Because according to Lemma 12 and Lemma 13, we can see that the count of a specific $t+1$ consecutive elements appearing in the range $T(0)$ is not greater than the value characterized by the function ϑ.

$$T(0)\sum\rho(\delta)\le T(0)\sum\vartheta(\delta).$$

Next, let us look at the case of

$$L_2(a,t)=\{m:\{m,m+1,m+2,\cdots,m+t\}\subseteq\mathbb{M}_{i\cup j}[a,a+T(0)+t)\}.$$

We can also see that for each element in $L_2(a,t)$, it denotes that there are $(t+1)$ consecutive elements in $\mathbb{M}_{i\cup j}[a,a+T(0)+t)$.

It is similar to the case of $L_1(a,t)$, combining this with Lemma 9 and Lemma 13, we have

$$|L_2(a,t+1)|\le$$

$$T(0)\left(\begin{array}{c}\sum\limits_{\substack{\{m_1\}\subseteq\mathbb{M}^\circ\\ |\{m_1\}|=1}}\left(\frac{2^1}{m_1}\right)-\sum\limits_{\substack{\{m_1,m_2\}\subseteq\mathbb{M}^\circ\\ m_1<m_2}}\left(\frac{2^2}{m_1m_2}\right)+\\ \sum\limits_{\substack{\{m_1,m_2,m_3\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3}}\left(\frac{2^3}{m_1m_2m_3}\right)-\\ \cdots+(-1)^{n-1}\sum\limits_{\substack{\{m_1,m_2,m_3,\cdots,m_n\}\subseteq\mathbb{M}^\circ\\ m_1<m_2<m_3<\cdots<m_n}}\left(\frac{2^n}{m_1m_2m_3\cdots m_n}\right)\end{array}\right)^{t+1}$$

$$=T(0)\left(1-\frac{T(2)}{T(0)}\right)^{t+1}.$$

Now we can deduce that

$$(\forall t \geq 0)\left(|L_2(a,t)| \leq T(0)\left(1-\frac{T(2)}{T(0)}\right)^{t+1} \right). \tag{4.1}$$

Considering the relationship between $L_2(a,t)$ and η (in Lemma 10, Lemma 11), we have

$$\eta \geq \inf\{m+1:\ m \geq 0 \wedge |L_2(a,m)| = 0\}, \tag{4.2}$$

according to the definition of $L_2(a,t)$.
For the next proof of theorems, we assume that there exists η that satisfies

$$\eta \leq \frac{p_n^2}{8}. \tag{4.3}$$

It requires

$$\left| L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| = 0.$$

By (4.1), we have

$$\left| L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| < T(0)\left(1-\frac{T(2)}{T(0)}\right)^{\lfloor\frac{p_n^2}{8}\rfloor}$$

$$= \left(\prod_{p\in\mathbb{M}^\circ} p\right)\left(1-\prod_{p\in\mathbb{M}^\circ}\left(1-2p^{-1}\right)\right)^{\lfloor\frac{p_n^2}{8}\rfloor}.$$

By Lemma 17, we know that we have

$$\left| L_2\left(a, \lfloor\frac{p_n^2}{8}\rfloor - 1\right)\right| < 1$$

when

$$\left(\frac{\ln^2 p_n}{0.4}\right) p_n \ln 6 < \lfloor\frac{p_n^2}{8}\rfloor. \tag{4.4}$$

Let

$$f(x) = \frac{x^2}{8} - \left(\frac{\ln^2 x}{0.4}\right) x \ln 6.$$

Then for $x \neq 0$,

$$\left(\frac{f(x)}{x}\right)' = \frac{1}{8} - \frac{2\ln 6 \ln x}{0.4x}.$$

We can easily get a crude result that

$$\left(\frac{f(x)}{x}\right)' > 0$$

when $x > 436$ through numerical analysis.
So that $f(x)$ is monotonically increasing when $x > 436$.

Next, the numerical analysis is continued, we can easily get another crude result that

$$f(x) > 0$$

when $x > 2096$.
Now we know that the condition (4.4) is satisfied when $p_n > 2096$.
Therefore, (4.3) holds for $p_n > 2096$.
i.e.

$$(\forall p_n > 2096)\left((\exists \eta)\left(\eta \leq \frac{p_n^2}{8}\right)\right).$$

5. Proof of theorems

We are now in the position to prove **Theorem 1** and **2**.
For n with $p_n \leq 2096$, we know that the theorems hold through computer verification.
Otherwise, we have

$$\eta \leq \frac{p_n^2}{8}.$$

Since

$$\eta \leq \frac{p_n^2}{8} < \frac{p_n^2+1}{2},$$

combining this with Lemma 11, we have

$$\left(\forall a, i, j \in \left[i+1, i+\frac{p_n+1}{2}\right)\right)$$

$$\left(\left(\exists h \in \left[a, a+\frac{p_n^2+1}{2}\right)\right)(\ \chi(h,\ \mathbb{M}_i) = \chi(h+j-i,\ \mathbb{M}_i) = 0\ )\right).$$

And let

$$a = i + \frac{T(0)+p_n}{2},$$

we have

$$\left(\forall i, j \in \left[1, \frac{p_n+1}{2}\right)\right)\left(\left(\exists h \in \left[i+\frac{T(0)+p_n}{2}, i+\frac{T(0)+p_n^2+p_n+1}{2}\right)\right)(\ \chi(h,\ \mathbb{M}_i) = \chi(h+j,\ \mathbb{M}_i) = 0\ )\right). \tag{5.1}$$

Then we can deduce that for every h in (5.1) satisfying the condition

$$(\ \chi(h,\ \mathbb{M}_i) = \chi(h+j,\ \mathbb{M}_i) = 0\ ),$$

so we have q_1 and q_2 are both prime numbers, defined by

$$q_1 = 2(h-i) - T(0),$$
$$q_2 = q_1 + 2j = 2(h-i) - T(0) + 2j.$$

Proof. Since

$$\chi\,(h,\ \mathbb{M}_i) = \chi\,(h + j,\ \mathbb{M}_i) = 0,$$

we have

$$\chi\,(h,\ \mathbb{M}_i) = \chi\,(h - i,\ \lambda\,(\mathbb{M}_i,\, 0 - i)) = \chi\,(h - i,\ \mathbb{M}_0) = 0.$$

Because the prime number 2 does not belong to $\mathbb{M}^\circ$,
by Lemma 4, we have

$$\chi\,(h,\ \mathbb{M}_i) = \chi\,(2\,(h - i) + i,\ \mathbb{M}_i) = 0.$$

Combining this with Lemma 5 we have

$$\chi\,(h,\ \mathbb{M}_i) = \chi\,(\,2\,(h - i) + i - T\,(0)\,,\ \mathbb{M}_i) = \chi\,(\,q_1,\ \mathbb{M}_0) = 0.$$

i.e.

$$(\forall m \in \mathbb{M}^\circ)\,(\ q_1 \not\equiv 0\,(mod\ m)\ )\,. \tag{5.2}$$

Similarly, we have

$$\chi\,(h + j,\ \mathbb{M}_i) = \chi\,(\,2\,(h - i) + i - T\,(0) + 2j,\ \mathbb{M}_i) = \chi\,(q_2,\ \mathbb{M}_0) = 0.$$

i.e.

$$(\forall m \in \mathbb{M}^\circ)\,(\ q_2 \not\equiv 0\,(mod\ m)\ )\,. \tag{5.3}$$

Noting that the domain of h, we can deduce

$$q_1 \in [p_n,\ p_n\,(p_n + 1)]\,,$$
$$q_2 \in [p_n + 2,\ p_n\,(p_n + 2)]\,.$$

Obviously,

$$T\,(0) \not\equiv 0\,(mod\ 2)\,,$$
$$q_1 \not\equiv 0\,(mod\ 2)\,,$$
$$q_2 \not\equiv 0\,(mod\ 2)\,.$$

And $\mathbb{M}^\circ$ contains all odd primes not greater than p_n, so that

$$\forall w \in [p_n,\ p_n\,(p_n + 2)]\,,$$

if w is not a prime number, there must be

$$(\exists m \in (\mathbb{M}^\circ \cup \{2\}))\,(\ w \equiv 0\,(mod\ m)\ )\,.$$

Thus, combined with (5.2) and (5.3), q_1 and q_2 must be prime numbers.
This implies that

for every $p_s > 2096$, there must be primes p_a and p_b between p_s and $p_s^2 + 2p_s$,

$$\left(\forall d \in \left[1, \frac{p_s + 1}{2}\right)\right)\,(\ p_a - p_b = 2d\ )\,.$$

i.e.

$$(\forall p_s > 2096)\left(\left(\exists p_a, p_b \in \left[p_s, p_s^2 + 2p_s\right]\right)\right.$$
$$\left.\left(\left(\forall d \in \left[1, \frac{p_s+1}{2}\right)\right)\left(\ p_a - p_b = 2d\ \right)\right)\right).$$

Since there are infinite primes, we can conclude that for any positive integer d, there are infinitely many prime gaps of size $2d$. This proves **Theorem 1**.

Next, let us transform the problem of gaps between primes into the problem of sums of two primes.

Let

$$a = i + \frac{(T\,(0) + p_1)}{2}.$$

Since

$$\eta \le \frac{p_n^2}{8},$$

combining this with Lemma 10, we have

$$\left(\forall i, j \in \left[i + p_1 + \lceil \frac{p_{\mathrm{n}}^2}{8} \rceil, i + p_1 + \lfloor \frac{p_n^2}{2} \rfloor\right)\right)\left(\left(\exists h \in \left[i + \frac{T\,(0) + p_1}{2},\right.\right.\right.$$
$$\left.\left.\left. i + \frac{T\,(0) + p_1}{2} + \lceil \frac{p_{\mathrm{n}}^2}{8} \rceil\right)\right)\ \left(\ \chi\,(h,\ \mathbb{M}_{i \cup j}) = 0\ \right)\right). \tag{5.4}$$

Then we can deduce that for every h in (5.4) satisfying the condition

$$\left(\ \chi\,(h,\ \mathbb{M}_{i \cup j}) = 0\ \right),$$

so we have q_1 and q_2 are both prime numbers, defined by

$$q_1 = 2\,(h - i) - T\,(0)\,,$$
$$q_2 = 2\,(j - h) + T\,(0)\,.$$

Proof. By the condition,

$$\chi\,(h,\ \mathbb{M}_i) = \chi\,(h,\ \mathbb{M}_{\mathrm{j}}) = 0.$$

Then it is similar to the proof of **Theorem 1**,

$$\chi\,(q_1,\ \mathbb{M}_0) = \chi\,(2\,(h - i)\,,\ \mathbb{M}_0) = \chi\,(h - i,\ \mathbb{M}_0) = \chi\,(h,\ \mathbb{M}_i) = 0,$$
$$\chi\,(q_2,\ \mathbb{M}_0) = \chi\,(2\,(j - h)\,,\ \mathbb{M}_0) = \chi\,(h - j,\ \mathbb{M}_0) = \chi\,(h,\ \mathbb{M}_j) = 0.$$

It is easy to see that

$$(\forall m \in (\mathbb{M}^\circ \cup \{2\}))\left(\ q_1 \not\equiv 0\,(mod\ m)\ \right),$$

$$(\forall m \in (\mathbb{M}^\circ \cup \{2\}))\left(\ q_2 \not\equiv 0\,(mod\ m)\ \right).$$

Noting that the domain of h, we can deduce

$$q_1 \in \left[p_1,\ 2\lceil\frac{p_{\mathrm{n}}^2}{8}\rceil + p_1\right),$$

$$q_2 \in \left[p_1,\ p_n^2 + p_1\right).$$

So q_1 and q_2 are both prime numbers.
Now let us look at the domain of $(q_1 + q_2)$,

$$q_1 + q_2 = 2\,(j - i) \in \left[2p_1 + 2\lceil\frac{p_n^2}{8}\rceil,\ 2p_1 + 2\lfloor\frac{p_n^2}{2}\rfloor\right).$$

This implies that

for every $p_s > 2096$, there must be primes p_a and p_b between p_1 and $p_s^2 + p_1$,

$$\left(\forall d \in \left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\ (\ p_a + p_b = 2d\ ).$$

i.e.

$$(\forall p_s > 2096)\left((\exists p_a, p_b \in [p_1, p_s^2 + p_1])\left(\left(\forall d \in \left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\right.\right.\right.\right.$$
$$\left.\left.\left.\left. p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\ (\ p_a + p_b = 2d\ )\right)\right). \tag{5.5}$$

By Bertrand-Chebyshev theorem [1], we have

$$p_{s+1} < 2p_s,$$

then

$$\frac{p_{s+1}^2}{8} < \frac{p_s^2}{2},$$

so

$$(\forall s > 1)\left(\left[p_1 + \lceil\frac{p_s^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right) \cap \left[p_1 + \lceil\frac{p_{s+1}^2}{8}\rceil, p_1 + \lfloor\frac{p_{s+1}^2}{2}\rfloor\right) \neq \phi\right).$$

Combining this with (5.5), we can conclude that

$$(\forall p_s > 2096)\left((\exists p_a, p_b)\left(\left(\forall d \in \left[p_1 + \lceil\frac{p_u^2}{8}\rceil,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\ (\ p_a + p_b = 2d\ )\right)\right),$$

where p_u is the smallest prime number greater than 2096, that is, 2099.
It is easy to get

$$p_1 + \lceil\frac{p_u^2}{8}\rceil = 3 + 550726 = 550729.$$

i.e.

$$(\forall p_s > 2096)\left((\exists p_a, p_b)\left(\left(\forall d \in \left[550729,\ p_1 + \lfloor\frac{p_s^2}{2}\rfloor\right)\right)\ (\ p_a + p_b = 2d\ )\right)\right).$$

While the results of $d \in [1, 550729)$ can be obtained by computer-aided verification.

Since there are infinite primes, we can conclude that every even number greater than 2 is the sum of two prime numbers. This proves **Theorem 2**.

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