ON THE SERIES REPRESENTATION OF SUM OF POSITIVE DIVISORS FUNCTION

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ABSTRACT. If m divides n then $\sin(n\pi/m) = 0$. By counting number of zeros of $\sin(n\pi/m)$ for a given $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we can find the total number of divisors that n has and in this way, we can construct a series representation of the *Number-of-divisors function*, $\mathcal{S}(n)$. Similarly, we can find a closed-form of another important integer-valued function in Number Theory, *Sum-of-divisors function*, $\sigma(n)$. After constructing series representation of these functions we can resolve a well known conjectures in Number Theory – the *Riemann Conjecture*. To conclude the Riemann conjecture we use *Robin's inequality* which sets an upper-limit of $\sigma(n)$ for $n > 5040$, if Riemann conjecture is true. This method can be trivially extended to the other higher-order divisor functions. To construct these series representations we have explored *Matsubara* technique which is commonly used in Condensed Matter Physics to perform various sum over integer index with a contour integral.

1. INTRODUCTION

$$m|n \implies \sin(\tfrac{n\pi}{m}) = 0 \quad \forall n, m \in \mathbb{Z} \tag{1}$$

A magical journey started from the observation in Eq 1. Using this property we can construct a series representation of *all* divisor functions. Here our main focus is two of them – *Sum of divisor function* $\sigma(n)$ and *Divisor function* $\mathcal{S}(n)$. Ironically we will start with the most difficult one, $\sigma(n)$. This effort can be easily extended to other divisor functions, which we will mention briefly. Once these are available in our disposal we can address a well known conjecture in Number Theory – the *Riemann Conjecture*. For neatness we will leave most of the elaborate details in Appendices. Except for the Robin's inequality we have not mentioned any references *deliberately*, since they were all learned from online sources like Wikipedia, Mathematics Stack Exchange and seems to be standard academic topic.

2. SERIES REPRESENTATION OF THE SUM-OF-DIVISORS FUNCTION

Formally, one can express the sum-of-divisors function $\sigma(n)$ in the following manner:

$$\sigma(n) \equiv \sum_{m|n} m \tag{2}$$

$$\Rightarrow \sigma(n) = 2\sum_{m=1}^{n} m\Theta\left[\sin^2(n\pi/m)\right] \tag{3}$$

where the *flipped* Heaviside step function $\Theta(x)$ is defined below.

$$\Theta(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases} \tag{4}$$

An integral representation of this step function usually given by:

$$\Theta(x) = \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} \frac{e^{i\psi}}{\psi + ix} \tag{5}$$

The above improper integral has to be considered as the Cauchy principal value. Instead of this conventional form, we will use the following form with an added factor which is inevitably needed for the future purpose.

$$\Theta\left[\sin^2(n\pi/m)\right] = \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} \frac{e^{i\psi}}{\psi + i(2m\pi)^6 \sin^2(n\pi/m)} \tag{6}$$

where the contour is closed in the upper half of complex plane. Now we will analyse behaviour of the poles. There are three types of poles for different values of m. When $m < n$ and $m|n$ the pole is at the origin and when m does not divides n the pole is on the negative imaginary axis. Also, when $m > n$ the pole is again on the negative imaginary axis and its behaviour in the limit m goes to infinity is:

$$\lim_{m\to\infty} (2m\pi)^2 \sin^2(n\pi/m) \tag{7}$$

$$= \lim_{x\to 0} (2\pi)^2 \frac{\sin^2(n\pi x)}{x^2} \tag{8}$$

$$= 4\pi^4 n^2 \lim_{x\to 0} \frac{\sin^2(n\pi x)}{(n\pi x)^2} \tag{9}$$

$$= 4\pi^4 n^2 \tag{10}$$

$$\Rightarrow \lim_{m\to\infty} (2m\pi)^6 \sin^2(n\pi/m) = 4\pi^4 n^2 \lim_{m\to\infty} (2m\pi)^4 \tag{11}$$

So the pole moves toward negative infinity on the imaginary axis as m goes to positive or negative infinity and hence it is safe to use the contour as the pole will never come back to the origin and we will not end up miss-counting total number of divisors. Not only that in this way there is *no messy crowd of poles near the origin.*

Since eventually we are going to extend the sum over m in Eq 3 to $\{[-n, n]\backslash 0\}$ and to restrict it to only positive values, we introduce Heaviside step $H(x)$ function as:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{12}$$

For our need we are going to use following integral representation of the step function:

$$H(2m\pi) = \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} \frac{e^{i\tau}}{2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon} \tag{13}$$

It has poles at

$$\tau = i\epsilon \frac{1 + (2m\pi)^2}{1 + 2m\pi} \tag{14}$$

Since $\text{sign}(1 + 2m\pi) = \text{sign}(m)\ \forall m \in \{\mathbb{Z}\backslash 0\}$, only $m > 0$ term will contribute to the integration in Eq 13 after closing the contour in the upper-half of complex plane. The primary reason for choosing this non-trivial representation is, its real part of the denominator is always greater than zero. It is worthy mentioning that the integrand is well defined on the interval $(-\infty, \infty)$.

From Eq 3, Eq 6 and Eq 13 we have

$$\sigma(n) = 2 \sum_{m=-n}^{n}{}' \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} \frac{mH(2m\pi)\, e^{i\psi}}{\psi + i(2m\pi)^6 \sin^2(n\pi/m)} \tag{15}$$

where $\sum_m'$ means $m \neq 0$. Now we will perform many manipulations on Eq 15 and to maintain the uninterrupted flow of understanding we will leave explanations in the Appendix A.

$$\sigma(n) = \int_{-\infty}^{\infty} \frac{d\psi}{\pi i} e^{i\psi} \sum_{m=-n}^{n}{}' \frac{mH(2m\pi)}{\psi + i(2m\pi)^6 \sin^2(n\pi/m)} \quad \text{[see Note 1]} \tag{16}$$

$$\Rightarrow \sigma(n) = \int_{-\infty}^{\infty} \frac{d\psi}{\pi i} e^{i\psi} \sum_{m=-n}^{n}{}' \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} \frac{e^{i\tau}}{2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon} \times \frac{m}{\psi + i(2m\pi)^6 \sin^2(n\pi/m)} \tag{17}$$

$$\Rightarrow \sigma(n) = -\int_{-\infty}^{\infty} \frac{d\psi}{\pi} e^{i\psi} \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} e^{i\tau} \sum_{m=-n}^{n}{}' \frac{1}{2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon} \times \frac{m}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m)} \tag{18}$$

$$\Rightarrow \sigma(n) = -\int_{-\infty}^{\infty} \frac{d\psi}{\pi} e^{i\psi} \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} e^{i\tau} \sum_{m=-\infty}^{\infty}{}' \frac{1}{2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon} \times \tag{19}$$

$$\frac{m}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m)} \quad \text{[see Note 2]} \tag{20}$$

Now we look at the sum over m separately

$$\sum_{m=-\infty}^{\infty}{}' \frac{1}{2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon} \times \frac{m}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m)} \tag{21}$$

Since the real part of denominator in the first fraction is always greater than zero we can exponentiate it in the following way:

$$\sum_{m=-\infty}^{\infty}{}' \int_{0}^{\infty} dy\, e^{-y\left[2m\pi i\tau + \epsilon(2m\pi)^2 + i\tau + \epsilon\right]} \frac{m}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m)} \tag{22}$$

$$\Rightarrow \int_{0}^{\infty} dy\, e^{-y(\epsilon + i\tau)} \sum_{m=-\infty}^{\infty}{}' \frac{me^{-y\left[\epsilon(2m\pi)^2 + 2m\pi i\tau\right]}}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m)} \quad \text{[see Appendix B]} \tag{23}$$

$$\Rightarrow \int_{0}^{\infty} dy\, e^{-y(\epsilon + i\tau)} \lim_{\delta\to 0+} \sum_{m=-\infty}^{\infty}{}' \frac{me^{-y\left[\epsilon(2m\pi)^2 + 2m\pi i\tau\right]} \operatorname{sech}[(2m\pi\delta)^6]}{-i\psi + (2m\pi)^6 \sin^2(n\pi/m) + \delta} \tag{24}$$

The sum over m is evidently convergent. But while applying Matsubara technique to perform this sum, we need to extend this expression to the entire complex plane where divergence may appear in certain directions. To remediate this artificial divergence we introduce a suitable convergence factor $\operatorname{sech}[(2m\pi\delta)^6]$. The term in the numerator may diverge as e^{z^2}. But the denominator after exponentiation may contribute to the divergence as e^{z^4} (see Eq 11) whereas $\operatorname{sech}(z^6)$ goes to zero as e^{z^6} in every direction of the complex plane.

Again positive definiteness of real part of the denominator allows exponentiation

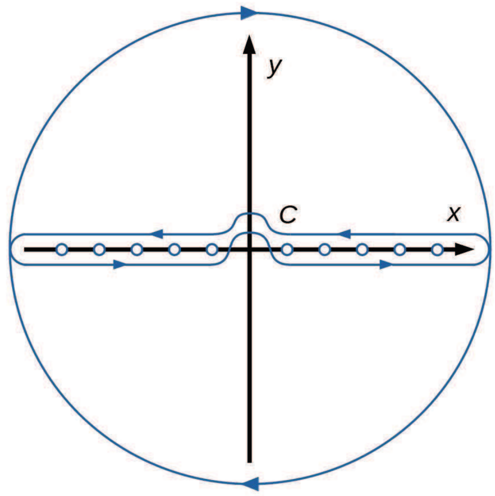

FIGURE 1. C is the contour to evaluate the sum in Eq 27 over real *frequency* with the Matsubara technique. Since zero is excluded in the sum we take a detour to avoid the origin. The clockwise big circle is the contour being added to it and then it becomes a contour C' around the origin.

$$\int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \lim_{\delta\to 0+} \sum_{m=-\infty}^{\infty}{}' m e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau\right]} \operatorname{sech}[(2m\pi\delta)^6] \int_0^\infty dx\, e^{-x\left[-i\psi+(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)+\delta\right]} \tag{25}$$

$$\Rightarrow \int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \lim_{\delta\to 0+} \int_0^\infty dx\, e^{-x(\delta-i\psi)} \sum_{m=-\infty}^{\infty}{}' m e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau\right]} \operatorname{sech}[(2m\pi\delta)^6] e^{-x(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)} \tag{26}$$

See Appendix B for the last step. Now we separate out the summation part and perform it using the Matsubara technique and it fulfils all the required conditions [see Appendix C].

$$F(n) = \sum_{m=-\infty}^{\infty}{}' m e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau\right]} \operatorname{sech}[(2m\pi\delta)^6] e^{-x(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)} \tag{27}$$

$$\Rightarrow F(n) = \sum_{m=-\infty}^{\infty}{}' \frac{2m\pi}{2\pi} e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau\right]} \operatorname{sech}\left[(2m\pi\delta)^6\right] e^{-x(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)} \tag{28}$$

To perform the sum, the factor that we are going to use is:

$$\frac{1}{e^{iz}-1} \tag{29}$$

since it has poles on the real axis, particularly at $z = 2m\pi$. Residue at $z = 2m\pi$ is:

$$\lim_{z\to 2m\pi} \frac{z-2m\pi}{e^{iz}-1} = \lim_{z\to 2m\pi} \frac{z-2m\pi}{e^{i(z-2m\pi)}e^{2m\pi i}-1} = \lim_{u\to 0} \frac{u}{e^{iu}-1} = \lim_{u\to 0} \frac{1}{ie^{iu}} = -i \tag{30}$$

In the last step we have used the L'Hospital rule. Now we rewrite Eq 28, for clarity as:

$$F(n) = \frac{i}{2\pi} \sum_{m=-\infty}^{\infty}{}' (-i)\, 2m\pi\, e^{-y\left[\epsilon(2m\pi)^2 + 2m\pi i\tau\right]} \operatorname{sech}\left[(2m\pi\delta)^6\right] e^{-x(2m\pi)^6 \sin^2\left(\frac{2n\pi^2}{2\pi m}\right)} \tag{31}$$

$$\Rightarrow F(n) = \frac{i}{2\pi} \oint_C \frac{dz}{2\pi i} \frac{z}{e^{iz}-1} e^{-y\left[\epsilon z^2 + i\tau z\right]} \operatorname{sech}(z^6\delta^6) e^{-xz^6 \sin^2(\theta/z)} \tag{32}$$

where $\theta = 2n\pi^2$. The contour used in Eq 32 is shown in the Fig 1. After adding a contour integral over a large circle of radius r and $r \to \infty$, which evaluates to be zero, the contour integral (C') becomes a contour integral around the origin.

$$F(n) = -\frac{1}{2\pi} \oint_{C'} \frac{dz}{2\pi i} \frac{iz}{e^{iz}-1} e^{-y\left[\epsilon z^2 + i\tau z\right]} \operatorname{sech}(z^6\delta^6) e^{-xz^6 \sin^2(\theta/z)} \tag{33}$$

The extra negative sign appears because of the clock-wise contour C' around the origin. So to evaluate this contour integral we have to find residue at the origin and we find it by the Laurrent expansion of individual factors around the origin and they are given below consecutively. It is worth mentioning that there are another contributions from the $\operatorname{sech}(z^6\delta^6)$ factor, but it can be shown (see Appendix D) that in the limit $\delta \to 0+$ it vanishes.

2.1. **Residue.** For the first factor we have:

$$\frac{iz}{e^{iz}-1} = \sum_{r=0}^{\infty} B_r \frac{(iz)^r}{r!} \tag{34}$$

The above expansion follows from the exponential generating function definition of Bernoulli number and radius of convergence is $|z| < \pi$. Here $r = 0, 1, 2, 4, 6, \ldots$ and $B_1 = -1/2$ (According to the convention in Wikipedia this is B_r^-). For the second factor we have:

$$e^{-y\left[\epsilon z^2 + i\tau z\right]} = \sum_{s=0}^{\infty} \frac{(-y)^s}{s!} \left[\epsilon z^2 + i\tau z\right]^s \tag{35}$$

$$\Rightarrow e^{-y\left[\epsilon z^2 + i\tau z\right]} = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-y)^s}{s!} \binom{s}{t} (\epsilon z^2)^{s-t} (i\tau z)^t \tag{36}$$

For the third factor, using the known expansion of $\operatorname{sech}(z)$ around the origin we have:

$$\operatorname{sech}(z^6\delta^6) = \sum_{u=0}^{\infty} \frac{E_{2u}}{(2u)!} (z\delta)^{6u}, \qquad \left|z^6\delta^6\right| < \pi/2 \tag{37}$$

For sufficiently small δ both the radius convergence in Eq 34 and Eq 44 can be satisfied without any *contradiction.*
Now we find expansion of the last factor in the following manner:

$$e^{-xz^6 \sin^2(\theta/z)} = \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} z^{6m} \sin^{2m}(\theta/z) \tag{38}$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} z^{6m} \left[\tfrac{1}{2i}(e^{i\theta/z} - e^{-i\theta/z})\right]^{2m} \tag{39}$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \frac{z^{6m}}{(2i)^{2m}} \sum_{p=0}^{2m} (-1)^p e^{i(2m-p)\theta/z} e^{-ip\theta/z} \binom{2m}{p} \tag{40}$$

$$\Rightarrow e^{-xz^6\sin^2(\theta/z)} = \sum_{m=0}^{\infty}\sum_{p=0}^{2m} \frac{x^m z^{6m}(-1)^p}{4^m\, m!} e^{i(2m-2p)\theta/z}\binom{2m}{p} \tag{41}$$

$$\Rightarrow e^{-xz^6\sin^2(\theta/z)} = \sum_{m,q=0}^{\infty}\sum_{p=0}^{2m} \frac{x^m z^{6m}(-1)^p}{4^m\, m!}\frac{i^q\theta^q(2m-2p)^q}{z^q\, q!}\binom{2m}{p} \tag{42}$$

In the last line we have interchanged the order of p and q sum since terms in the sum are individually convergent. But the order of the sum over m and q *can not be interchanged* because of the same reason of convergence. Here we introduce a polynomial $G(q,m)$ to compactly express the above expression and its definition is given below (for more details and properties of this polynomial see Appendix G)

$$G(q,m) = \sum_{p=0}^{2m}(-1)^p(2m-2p)^q\binom{2m}{p} \tag{43}$$

With this we have:

$$e^{-xz^6\sin^2(\theta/z)} = \sum_{m,q=0}^{\infty}\frac{x^m z^{6m}}{4^m\, m!}\frac{i^q\theta^q}{z^q\, q!}G(q,m) \tag{44}$$

Combining all from Eq 33 and Eq 34, Eq 35, Eq 37, Eq 44 we have:

$$-\frac{1}{2\pi}\sum_{\substack{r,u,s,\\ m,q=0}}^{\infty}\sum_{t=0}^{s}\frac{B_r i^r}{r!}z^r\frac{E_{2u}}{(2u)!}(z\delta)^{6u}\frac{(-y)^s}{s!}\binom{s}{t}(i\tau z)^t(\epsilon z^2)^{s-t}\frac{x^m z^{6m}}{m!\,4^m}\frac{i^q\theta^q}{q!\,z^q}G(q,m) \tag{45}$$

$$= -\frac{1}{2\pi}\sum_{\substack{r,u,s,\\ m,q=0}}^{\infty}\sum_{t=0}^{s}\frac{B_r i^r}{r!}\frac{E_{2u}\delta^{6u}}{(2u)!}\frac{(-y)^s}{s!}\binom{s}{t}(i\tau)^t\epsilon^{s-t}\frac{x^m z^{6m}}{m!\,4^m}\frac{i^q\theta^q}{q!\,z^q}G(q,m)z^{r+6u+2s-t+6m-q} \tag{46}$$

To find the residue at the origin we extract coefficient of z^{-1} and which gives:

$$r+6u+2s-t+6m-q = -1 \tag{47}$$

$$\Rightarrow q = 1+r+6u+2s-t+6m \tag{48}$$

So finally, after performing the sum we have a closed form of $F(n)$ as:

$$F(n) = -\frac{1}{2\pi}\sum_{\substack{r,u,s,\\ m=0}}^{\infty}\sum_{t=0}^{s}\frac{B_r i^r}{r!}\frac{E_{2u}\delta^{6u}}{(2u)!}\frac{(-y)^s}{s!}\binom{s}{t}(i\tau)^t\epsilon^{s-t}\frac{x^m}{m!\,4^m}\frac{i^q\theta^q}{q!}G(q,m) \tag{49}$$

where q is given by the Eq 48. It is worthy mentioning that there is contribution from the pole of sech term but it can be shown in the limit $\delta\to 0+$ it vanishes, see Appendix for more details.

2.2. **Further simplification.** With this and from Eq 26 we have

$$-\int_0^{\infty} dy\, e^{-y(\epsilon+i\tau)}\lim_{\delta\to 0+}\int_0^{\infty}dx\, e^{-x(\delta-i\psi)}\frac{1}{2\pi}\sum_{\substack{r,u,s,\\ m=0}}^{\infty}\sum_{t=0}^{s}\frac{x^m}{m!}\cdots \tag{50}$$

$$= -\int_0^{\infty} dy\, e^{-y(\epsilon+i\tau)}\lim_{\delta\to 0+}\frac{1}{2\pi}\sum_{\substack{r,u,s,\\ m=0}}^{\infty}\sum_{t=0}^{s}\int_0^{\infty}dx\, e^{-x(\delta-i\psi)}\frac{x^m}{m!}\cdots \tag{51}$$

Justify the interchange here. We use the following relation to perform the x integration

$$\int_0^{\infty}dx\, e^{-\alpha x}x^m = \frac{(-1)^m m!}{\alpha^{m+1}} \tag{52}$$

where $\Re(\alpha) > 0$ and we have

$$-\int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \lim_{\delta\to 0+} \frac{1}{2\pi} \sum_{\substack{r,u,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \frac{(-1)^m}{(\delta - i\psi)^{m+1}} \frac{E_{2u}\delta^{6u}}{(2u)!} \cdots \tag{53}$$

$$= -\int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \lim_{\delta\to 0+} \sum_{u=0}^{\infty} \frac{(-1)^m}{(\delta - i\psi)^{m+1}} \frac{E_{2u}\delta^{6u}}{(2u)!} \cdots \tag{54}$$

After taking the limit $\delta \to 0+$ only $u = 0$ term will survive. So we have,

$$-\int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \frac{(-1)^m}{(-i\psi)^{m+1}} \frac{B_r i^r}{r!} \frac{(-y)^s}{s!} \binom{s}{t} (i\tau)^t \epsilon^{s-t} \frac{i^q\theta^q}{4^m q!} E_0 G(q,m) \tag{55}$$

$$= \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \int_0^\infty dy\, e^{-y(\epsilon+i\tau)} \frac{(-y)^s}{s!} \frac{B_r i^r}{r!} \binom{s}{t} (i\tau)^t \epsilon^{s-t} \frac{i^q\theta^q}{4^m q!} \frac{G(q,m)}{(i\psi)^{m+1}} \quad [\because E_0 = 1] \tag{56}$$

$$= \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \frac{1}{(\epsilon + i\tau)^{s+1}} \frac{B_r i^r}{r!} \binom{s}{t} (i\tau)^t \epsilon^{s-t} \frac{i^q\theta^q}{4^m q!} \frac{G(q,m)}{(i\psi)^{m+1}} \quad [\text{using Eq } 52] \tag{57}$$

$$= \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \frac{(i\tau)^t \epsilon^{s-t}}{(\epsilon + i\tau)^{s+1}} \binom{s}{t} \frac{B_r i^r}{r!} \frac{i^q\theta^q}{4^m q!} \frac{G(q,m)}{(i\psi)^{m+1}} \tag{58}$$

Now, from Eq 20 we have $\sigma(n)$ as:

$$\sigma(n) = -\int_{-\infty}^{\infty} \frac{d\psi}{\pi} e^{i\psi} \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} e^{i\tau} \frac{1}{2\pi} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \frac{(i\tau)^t \epsilon^{s-t}}{(\epsilon + i\tau)^{s+1}} \binom{s}{t} \frac{B_r i^r}{r!} \frac{i^q\theta^q}{4^m q!} \frac{G(q,m)}{(i\psi)^{m+1}} \tag{59}$$

$$\Rightarrow \sigma(n) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\psi}{\pi} e^{i\psi} \lim_{\epsilon\to 0+} \sum_{\substack{r,s,\\ m=0}}^{\infty} \sum_{t=0}^{s} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi i} \frac{e^{i\tau}(i\tau)^t \epsilon^{s-t}}{(\epsilon + i\tau)^{s+1}} \binom{s}{t} \frac{B_r i^r}{r!} \frac{i^q\theta^q}{4^m q!} \frac{G(q,m)}{(i\psi)^{m+1}} \tag{60}$$

justify the interchange here. The integration over τ can be performed in the following manner. Since $t \in [0, s]$ the integral is well defined within the integration limits $(-\infty, \infty)$. We observe that there is a pole of order s at $\tau = i\epsilon$, hence we find the residue at $\tau = i\epsilon$ as, upto a factor $\frac{1}{s!}$,

$$\left. \frac{d^s}{d\tau^s} e^{i\tau} (i\tau)^t \right|_{\tau = i\epsilon} \tag{61}$$

Using the Leibniz rule for differentiation we have:

$$\frac{d^s}{d\tau^s}e^{i\tau}(i\tau)^t\bigg|_{\tau=i\epsilon} = i^t\sum_{j=0}^{s}\binom{s}{j}\left(e^{i\tau}\right)^{(s-j)}\left(\tau^t\right)^{(j)}\bigg|_{\tau=i\epsilon} \tag{62}$$

$$\Rightarrow \frac{d^s}{d\tau^s}e^{i\tau}(i\tau)^t\bigg|_{\tau=i\epsilon} = i^t\sum_{j=0}^{t}\binom{s}{j}i^{s-j}e^{i\tau}t(t-1)\cdots(t-j+1)\tau^{t-j}\bigg|_{\tau=i\epsilon} \tag{63}$$

$$\Rightarrow \frac{d^s}{d\tau^s}e^{i\tau}(i\tau)^t\bigg|_{\tau=i\epsilon} = i^t\sum_{j=0}^{t}\binom{s}{j}i^{s-j}e^{-\epsilon}t(t-1)\cdots(t-j+1)(i\epsilon)^{t-j} \tag{64}$$

$$\Rightarrow \frac{d^s}{d\tau^s}e^{i\tau}(i\tau)^t\bigg|_{\tau=i\epsilon} = (-1)^t i^s\sum_{j=0}^{t}\binom{s}{j}(-1)^j t(t-1)\cdots(t-j+1)e^{-\epsilon}\epsilon^{t-j} \tag{65}$$

After taking the limit $\epsilon \to 0+$ only $j = t$ term will survive, hence we have:

$$\lim_{\epsilon\to 0+}\frac{d^s}{d\tau^s}e^{i\tau}(i\tau)^t\bigg|_{\tau=i\epsilon} = i^s\binom{s}{t} \tag{66}$$

But there is another term on which taking the limit $\epsilon \to 0+$, non-vanishing terms are corresponds to $t = s$. So, we have

$$\sigma(n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\psi}{\pi}e^{i\psi}\sum_{\substack{r,s,\\ m=0}}^{\infty}\frac{B_r i^r}{r!}\frac{i^s}{s!}\frac{i^q\theta^q}{4^m q!}\frac{G(q,m)}{(i\psi)^{m+1}} \tag{67}$$

$$\Rightarrow \sigma(n) = \frac{1}{\pi i}\sum_{\substack{r,s,\\ m=0}}^{\infty}\int_{-\infty}^{\infty}\frac{d\psi}{2\pi}\frac{e^{i\psi}}{(i\psi)^{m+1}}\frac{B_r i^r}{r!}\frac{i^s}{s!}\frac{i^q\theta^q}{4^m q!}G(q,m) \tag{68}$$

where $q = 1 + r + s + 6m$. *Justify the interchange.* The integration over ψ in case of $m \geq 0$ can be performed in the following way. Since there is a pole of higher-order at origin on the contour $(-\infty,\infty)$, say $\mathcal{D}$, we deform the contour by creating a semi-circle centred at the origin of radius ϵ and finally we take the limit $\epsilon \to 0$.

$$\int_{\mathcal{D}}\frac{d\psi}{2\pi i}\frac{e^{i\psi}}{\psi^{m+1}} = \lim_{\epsilon\to 0}\int_{\mathcal{D}_\epsilon}\frac{d\psi}{2\pi i}\frac{e^{i\psi}}{\psi^{m+1}} = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\left[-\frac{e^{i\psi}}{m\psi^m}\bigg|_{-\infty}^{\infty} + \frac{i}{m}\int_{\mathcal{D}_\epsilon}d\psi\,\frac{e^{i\psi}}{\psi^m}\right] \tag{69}$$

$$\Rightarrow \int_{\mathcal{D}}\frac{d\psi}{2\pi i}\frac{e^{i\psi}}{\psi^{m+1}} = \frac{i}{m}\lim_{\epsilon\to 0}\int_{\mathcal{D}_\epsilon}\frac{d\psi}{2\pi i}\frac{e^{i\psi}}{\psi^m} = \cdots = \frac{i^m}{m!}\lim_{\epsilon\to 0}\int_{\mathcal{D}_\epsilon}\frac{d\psi}{2\pi i}\frac{e^{i\psi}}{\psi} = \frac{i^m}{2\,m!} \tag{70}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{d\psi}{2\pi}\frac{e^{i\psi}}{(i\psi)^{m+1}} = \frac{1}{2}\frac{1}{m!} \tag{71}$$

In the last step we have closed the contour in the upper-half of complex plane and used residue theorem to obtain the result. With this finally we have a closed form of $\sigma(n)$ as

$$\boxed{\sigma(n) = \frac{1}{2\pi i}\sum_{r,s,m=0}^{\infty}\frac{B_r i^r}{r!}\frac{i^s}{s!}\frac{G(q,m)}{4^m\,m!}\frac{i^q\theta^q}{q!}} \tag{72}$$

where $q = 1 + r + s + 6m$ and $\theta = 2n\pi^2$. Since q has to be even, $r + s$ has to be odd and hence reality of $\sigma(n)$ is established. With the same spirit we can also find, a similar closed form expression for the divisor function $\mathcal{S}(n)$ (see Section 4) and its generalisation to the higher orders.

2.3. **Generalisation to higher orders.** At this point it is natural to extend this result to higher order sum-of-positive divisors function. Here we mention for odd orders and for even orders see Section 4. The modifications to Eq 72 can be guessed from Eq 15 and 47. For the $(1+2j)^{\text{th}}$ order, $\forall j \in \mathbb{Z}$, required changes to Eq 47 is:

$$2j + r + 2(3+j)u + 2s - t + 2(3+j)m - q = -1 \tag{73}$$

$$\Rightarrow q = 1 + 2j + r + 2(3+j)u + 2s - t + 2(3+j)m \tag{74}$$

After all the intermediate manipulation, finally we have

$$q = 1 + 2j + r + s + 2(3+j)m \tag{75}$$

So, $\sigma_{1+2j}(n)$ is given by the following expression

$$\boxed{\sigma_{1+2j}(n) = \frac{1}{(2\pi)^{1+2j}i} \sum_{r,s,m=0}^{\infty} \frac{B_r i^r}{r!} \frac{i^s}{s!} \frac{G(q,m)}{4^m\, m!} \frac{i^q \theta^q}{q!}} \tag{76}$$

where q is given by the Eq 75.

3. Robin's Inequality and Riemann Conjecture

To prove Riemann Conjecture we follow two theorems of Guy Robin[1] and its statements are given below.

Theorem 1 (Theorem 1 of Robin[1])**.** *If the Riemann hypothesis is true, then for each* $n \geq 5041$

$$\sigma(n) \leq e^{\gamma} n \log\log n \tag{77}$$

where γ *is Euler–Mascheroni constant.*

The above inequality is now known as Robin's inequality after his work. This inequality is known to fail for 27 numbers (sequence A067698 in the OEIS): 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040.

Theorem 2 (Proposition 1 of Section 4 of Robin[1])**.** *If the Riemann hypothesis is false, then there exist constants* $0 < \beta < 1/2$ *and* C *such that*

$$\sigma(n) \geq e^{\gamma} n \log\log n + \frac{Cn \log\log n}{(\log n)^{\beta}} \tag{78}$$

holds for infinitely many n*.*

The combined implication of these two theorems is, if Robin's inequality fails for $n > 5041$ it must fail for *infinitely many* integers. Since the Riemann hypothesis is true and Robin's inequality fails for few $n > 5041$ is not possible *simultaneously*.

[1] G. Robin, "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann", Journal de Mathématiques Pures et Appliquées 63 (1984), pp. 187–213.

3.1. **Extension of $\sigma(n)$ to real argument.** Before we apply Robin's inequality we need to extend $\sigma(n)$ in Eq 72 to $n \in \mathbb{R}$. Since there is no unique way to do this for any general integer valued function, this we do trivially, by extending the domain of n from $\mathbb{Z}$ to $\mathbb{R}$. The nature of extended function can be assessed from the Eq 15. If n is not an integer then only $m = 1$ term will be contributing in the sum of Eq 15. So, $\sigma(n)$ assumes maximum value when $n \in \mathbb{Z}$ when at least two values of $m = \{1, n\}$ contributes and hence the extended $\sigma(n)$ is *under the envelop* of the original one. So if $\sigma(n)$ satisfy Robin's inequality for integer n it will also hold for $n \in \mathbb{R}$ and vice-versa for this form of extension.

3.2. **Convergence nature of the series representation of $\sigma(n)$.** Before we proceed here we quickly explore some properties of the expansion coefficient of $\sigma(n)$. From Eq 72 we have:

$$\sigma(n) = \frac{1}{2\pi i} \sum_{r,s,m=0}^{\infty} \frac{B_r i^r}{r!} \frac{i^s}{s!} \frac{G(q,m)}{4^m\, m!} \frac{i^q \theta^q}{q!} \tag{79}$$

$$\Rightarrow \sigma(n) = \frac{1}{2\pi i} \sum_{r,s,m=0}^{\infty} \frac{B_r}{r!s!} \frac{G(q,m)}{4^m\, m!} \frac{\theta^q}{q!} i^{q+r+s} \tag{80}$$

$$\Rightarrow \sigma(n) = \frac{1}{2\pi i} \sum_{r,s,m=0}^{\infty} \frac{B_r}{r!s!} \frac{G(q,m)}{4^m\, m!} \frac{\theta^q}{q!} i^{2q-1-6m} \qquad [\because q = 1 + r + s + 6m] \tag{81}$$

$$\Rightarrow \sigma(n) = -\frac{1}{2\pi} \sum_{r,s,m=0}^{\infty} \frac{B_r}{r!s!} \frac{i^{-6m} G(q,m)}{4^m\, m!} \frac{i^{2q}\theta^q}{q!} \tag{82}$$

$$\Rightarrow \sigma(n) = -\frac{1}{2\pi} \sum_{r,s,m=0}^{\infty} \frac{B_r}{r!s!} \frac{(-1)^m G(q,m)}{4^m\, m!} \frac{\theta^q}{q!} \qquad [\because q \text{ is even}] \tag{83}$$

Now we will rearrange the sum in the following way. Since $q = 1 + r + s + 6m \Rightarrow m \in [0, \lfloor \frac{q-1}{6} \rfloor]$ and hence $r \in [0, q-1-6m]$. *Before we rearrange we have to justify it.* So we have:

$$\sigma(n) = -\frac{1}{2\pi} \sum_{q=0}^{\infty} \sum_{m=0}^{\lfloor \frac{q-1}{6} \rfloor} \sum_{r=0}^{q-1-6m} \frac{(2\pi^2)^q}{q!} \frac{(-1)^m G(q,m)}{4^m\, m!} \frac{B_r}{r!s!} \theta^q = \sum_{q=0}^{\infty} a_q n^q \tag{84}$$

where $s = q - 1 - 6m - r$ and a_q is given by:

$$a_q = -\frac{(2\pi^2)^q}{2\pi\, q!} \sum_{m=0}^{\lfloor \frac{q-1}{6} \rfloor} \sum_{r=0}^{q-1-6m} \frac{(-1)^m G(q,m)}{4^m\, m!} \frac{B_r}{r!s!} \tag{85}$$

$$\Rightarrow a_q = -\frac{(2\pi^2)^q}{2\pi\, q!} \sum_{m=0}^{\lfloor \frac{q-1}{6} \rfloor} \frac{(-1)^m G(q,m)}{4^m\, m!} \sum_{r=0}^{q-1-6m} \frac{B_r}{r!s!} \tag{86}$$

$$\Rightarrow a_q = -\frac{(2\pi^2)^q}{2\pi\, q!} \sum_{m=0}^{\lfloor \frac{q-1}{6} \rfloor} \frac{(-1)^m G(q,m) S(q,m)}{4^m\, m!} \tag{87}$$

where

$$S(q,m) = \sum_{r=0}^{q-1-6m} \frac{B_r}{r!(q-1-6m-r)!} \tag{88}$$

Now we will review convergence nature of $\sigma(n)$. Our attempt will be to show the series is absolutely convergent. From the Appendix G we have the following results.

i) $G(q,m) \geq 0$

ii) $G(q,m) < 2^q m^q 4^m$

and from Appendix H we have:

$$S(q,m) = \begin{cases} 1/2 & \text{if } q-1-6m=1 \\ 0 & \text{otherwise} \end{cases} \tag{89}$$

Now not all m will give non-zero contribution to the sum. Rather for some q there is only one m such that the condition $q-1-6m=1$ is satisfied. Hence it is natural to re-index the sum in Eq 84, in the following way.

$$\sigma(n) = \sum_{q=0}^{\infty} a_q n^{6q+2} \tag{90}$$

where the new a_q is given by:

$$a_q = -\frac{(2\pi^2)^{6q+2}}{2\pi(6q+2)!}\frac{(-1)^q G(6q+2,q)S(6q+2,q)}{4^q\, q!} \tag{91}$$

$$\Rightarrow a_q = -\frac{(2\pi^2)^{6q+2}}{4\pi}\frac{(-1)^q G(6q+2,q)}{4^q\, q!(6q+2)!} \tag{92}$$

We can find approximate upper bound of a_q as:

$$a_q = -\frac{(2\pi^2)^{6q+2}}{4\pi}\frac{(-1)^q G(6q+2,q)}{4^q\, q!(6q+2)!} \tag{93}$$

$$\Rightarrow |a_q| \leq \frac{(2\pi^2)^{6q+2}}{4\pi}\frac{(2q)^{6q+2}4^q}{4^q\, q!(6q+2)!} \tag{94}$$

$$\Rightarrow |a_q| \leq \frac{1}{4\pi}\frac{(4\pi^2)^{6q+2}q^{6q+2}}{q!(6q+2)!} \tag{95}$$

Now we will perform the ratio test on the series in Eq 90

$$\lim_{q\to\infty}\frac{|a_{q+1}|n^{6(q+1)+2}}{|a_q|n^{6q+2}} \tag{96}$$

$$= \lim_{q\to\infty} n\frac{|a_{q+1}|}{|a_q|} \tag{97}$$

$$= n\lim_{q\to\infty}\frac{(4\pi^2)^{6(q+1)+2}q^{6(q+1)+2}q!(6q+2)!}{(4\pi^2)^{6q+2}q^{6q+2}(q+1)![6(q+1)+2]!} \tag{98}$$

$$= (4\pi^2)^6 n\lim_{q\to\infty}\frac{q^6(6q+2)^{6q+2+1/2}e^{6(q+1)+2}}{e^{6q+2}(q+1)[6(q+1)+2]^{6(q+1)+2+1/2}} \tag{99}$$

$$= (4\pi^2 e)^6 n\lim_{q\to\infty}\frac{q^6(6q+2)^{6q+2+1/2}}{(q+1)[6(q+1)+2]^{6(q+1)+2+1/2}} \tag{100}$$

$$= (4\pi^2 e)^6 n\lim_{q\to\infty}\frac{(6q+2)^{6q+2+1/2}}{[6(q+1)+2]^{6q+2+1/2}}\frac{q^6}{(q+1)[6(q+1)+2]^6} \tag{101}$$

$$= (4\pi^2 e)^6 n\lim_{q\to\infty}\left[1+\tfrac{6}{6q+2}\right]^{6q+2}\lim_{q\to\infty}\left[1+\tfrac{6}{6q+2}\right]^{1/2}\lim_{q\to\infty}\left[6+\tfrac{8}{q}\right]^6\lim_{q\to\infty}\frac{1}{q+1} \tag{102}$$

$$= (4\pi^2 e)^6 n\lim_{x\to\infty}\left[1+\tfrac{6}{x}\right]^x\lim_{y\to\infty}\left[1+\tfrac{6}{y}\right]^{1/2}\lim_{z\to\infty}\left[6+\tfrac{8}{z}\right]^6\lim_{q\to\infty}\frac{1}{q} \tag{103}$$

$$= (4\pi^2 e)^6 n\cdot e^6\cdot 1\cdot 6\lim_{q\to\infty}\frac{1}{q} = 6(4\pi^2 e^2)^6 n\lim_{q\to\infty}\frac{1}{q} = 0 \tag{104}$$

So the series in Eq 90 converges *absolutely*.

3.3. **Existence of upper bound of $\sigma(n)$.** Now let $n = e^{e^u}$, if Riemann hypothesis is true then from Eq 77 we have

$$\sigma(e^{e^u}) < e^{\gamma} e^{e^u} u \tag{105}$$

$$\Rightarrow \sum_q a_q [e^{e^u}]^{6q+2} < e^{\gamma} e^{e^u} u \tag{106}$$

$$\Rightarrow \sum_q a_q e^{(6q+2)e^u} < e^{\gamma} e^{e^u} u \tag{107}$$

$$\Rightarrow \sum_{q,r} a_q \frac{(6q+2)^r e^{ru}}{r!} < e^{\gamma} u \sum_r \frac{e^{ru}}{r!} \tag{108}$$

$$\Rightarrow \sum_{q,r,s} \frac{a_q (6q+2)^r r^s}{r!s!} u^s < e^{\gamma} \sum_{r,s} \frac{r^s}{r!s!} u^{s+1} \tag{109}$$

Since the series $\sigma(n)$ is absolutely convergent we can reorder the summation and we have:

$$\sum_{r,s} \frac{r^s}{r!s!} \left[\sum_{q=1}^{\infty} a_q (6q+2)^r \right] u^s < \sum_{r,s} \frac{r^s}{r!s!} \left[e^{\gamma} u \right] u^s \tag{110}$$

Since $\sigma(n) \geq 0$ it is guaranteed from the above equation that $\sum_{q=1}^{\infty} a_q (6q+2)^r$ is has to be positive. Now if we can show the upper bound of the series $\sum_{q=0}^{\infty} a_q (6q+2)^r$ is a *finite for all r* then there exists a u_0 such that:

$$\sum_{q=1}^{\infty} a_q (6q+2)^r < e^{\gamma} u_0 \tag{111}$$

and hence Robin's inequality can be satisfied with the upper limit of n as $n_0 = e^{e^{u_0}}$. The proof that $n_0 = 5041$ can be omitted for the following reason. If $n_0 > 5041$ then Robin's inequality fails for finite number of points $n \in (5041, n_0)$ which is not in accord with the Theorem $1, 2$. So, a proof of a *finite* n_0 is sufficient to prove the Riemann hypothesis.

The series that we have in our hand is:

$$\sum_{q=0}^{\infty} a_q (6q+2)^r \tag{112}$$

$$\Rightarrow -\frac{1}{4\pi} \sum_{q=0}^{\infty} (-1)^q (2\pi^2)^{6q+2} \frac{G(6q+2, q)}{4^q \, q!(6q+2)!} (6q+2)^r \tag{113}$$

$$\Rightarrow -\frac{1}{4\pi} \sum_{q=0}^{\infty} (-1)^q \frac{(2\pi^2)^{6q+2}}{q!} \frac{(6q+2)^r}{(6q+2)!} \frac{\partial^{6q+2}}{\partial z^{6q+2}} \sinh^{2q} z \bigg|_{z=0} \tag{114}$$

For the last step see Eq G5. To analyse the above infinite series, let's look at the following expansion:

$$\exp\left[-\left(2\pi^2\right)^6 e^{6z} \right] = \sum_{q=0}^{\infty} (-1)^q \frac{\left(2\pi^2\right)^{6q}}{q!} e^{6qz} \tag{115}$$

$$\Rightarrow \left(2\pi^2\right)^2 e^{2z} \exp\left[-\left(2\pi^2\right)^6 e^{6z} \right] = \sum_{q=0}^{\infty} (-1)^q \frac{\left(2\pi^2\right)^{6q+2}}{q!} e^{(6q+2)z} \tag{116}$$

To incorporate the last factor in Eq 114 we use the Cauchy's integral formula:

$$\frac{1}{2\pi i}\oint_C \frac{f(\psi)}{\psi^{n+1}}d\psi = \frac{1}{n!}f^n(0) \tag{117}$$

where the contour C is a simple closed curve around the origin and we have the following modification.

$$\frac{(2\pi^2)^2}{\psi^3}e^{2z}\exp\left[-\frac{(2\pi^2)^6}{\psi^6}e^{6z}\sinh^2\psi\right] = \sum_{q=0}^{\infty}(-1)^q\frac{(2\pi^2)^{6q+2}}{q!}e^{(6q+2)z}\frac{\sinh^{2q}\psi}{\psi^{6q+3}} \tag{118}$$

$$\Rightarrow\ \mathcal{M}(z) \equiv e^{2z}\oint_C d\psi\frac{(2\pi^2)^2}{\psi^3}\exp\left[-\frac{(2\pi^2)^6}{\psi^6}e^{6z}\sinh^2\psi\right]$$
$$= \sum_{q=0}^{\infty}(-1)^q\frac{(2\pi^2)^{6q+2}}{q!}\frac{e^{(6q+2)z}}{(6q+2)!}\frac{\partial^{6q+2}}{\partial\psi^{6q+2}}\sinh^{2q}\psi\bigg|_{\psi=0} \tag{119}$$

Using Cauchy-Riemann condition it is easy to show that $\mathcal{M}(z)$ is an analytical function at $z=0$. For this consider an open neighborhood U at $z=0$. Since the integrand in the definition of $\mathcal{M}(z)$ and its partial derivatives w.r.t $Re(z)$ and $Im(z)$ is continuous in the region $U\times C$, while using the Cauchy-Riemann condition we can interchange partial differentiations and the contour integration. Hence $\mathcal{M}(z)$ has derivatives of *all orders*, so we have:

$$\frac{\partial^r\mathcal{M}(z)}{\partial z^r}\bigg|_{z=0} = \sum_{q=0}^{\infty}(-1)^q\frac{(2\pi^2)^{6q+2}}{q!}\frac{(6q+2)^r}{(6q+2)!}\frac{\partial^{6q+2}}{\partial\psi^{6q+2}}\sinh^{2q}\psi\bigg|_{\psi=0} \tag{120}$$

RHS in the above equation is exactly the summation appears in Eq . Let $\Lambda = \sup\{|{}^{\partial^r\mathcal{M}(z)}/_{\partial z^r}|_{z=0},\ r\in\mathbb{Z}\}$. Since $\mathcal{M}(z)$ is analytic at $z=0$, its all derivatives are *finite* so Λ is finite too. Then we have:

$$\frac{\partial^r\mathcal{M}(z)}{\partial z^r}\bigg|_{z=0} \leq \Lambda \quad \forall r\in\mathbb{Z} \tag{121}$$

This implies $\sum_{q=0}^{\infty}a_q(6q+2)^r$ is a bounded function of r and hence the *Riemann conjecture is proved to be true.*

4. Series Representation of the number-of-divisors Function

By counting number of zeros of the function $\sin(n\pi/m)$ we can find a series representation of the divisor function $\mathcal{S}_n$. We can express $\mathcal{S}_n$ in the following manner:

$$\mathcal{S}_n \equiv \sum_{m|n} = \sum_{m=-n}^{n}{}' \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} \frac{e^{i\psi}}{\psi + i(2m\pi)^4 \sin^2(n\pi/m)} \tag{122}$$

$$= \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} e^{i\psi} \sum_{m=-n}^{n}{}' \frac{1}{\psi + i(2m\pi)^4 \sin^2(n\pi/m)} \tag{123}$$

$$= \int_{-\infty}^{\infty} \frac{d\psi}{2\pi i} e^{i\psi} \sum_{m=-n}^{n}{}' \lim_{\delta\to 0+} \frac{\operatorname{sech}\left[(2m\pi\delta)^4\right]}{\psi + i(2m\pi)^4 \sin^2(n\pi/m) + i\delta} \tag{124}$$

$$= -\int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta\to 0+} \sum_{m=-\infty}^{\infty}{}' \frac{\operatorname{sech}\left[(2m\pi\delta)^4\right]}{-i\psi + (2m\pi)^4 \sin^2(n\pi/m) + \delta} \tag{125}$$

$$= -\int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta\to 0+} \sum_{m=-\infty}^{\infty}{}' \operatorname{sech}\left[(2m\pi\delta)^4\right] \int_0^{\infty} dx\, e^{-x\left[-i\psi + (2m\pi)^4 \sin^2(n\pi/m) + \delta\right]} \tag{126}$$

$$\Rightarrow \mathcal{S}_n = -\int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta\to 0+} \int_0^{\infty} dx\, e^{-x(\delta - i\psi)} \sum_{m=-\infty}^{\infty}{}' \operatorname{sech}\left[(2m\pi\delta)^4\right] e^{-x(2m\pi)^4 \sin^2(n\pi/m)} \tag{127}$$

Separate out the sum over m:

$$F_n = \sum_{m=-\infty}^{\infty}{}' \operatorname{sech}\left[(2m\pi\delta)^4\right] e^{-x(2m\pi)^4 \sin^2(n\pi/m)} \tag{128}$$

$$\Rightarrow F_n = \sum_{m=-\infty}^{\infty}{}' \operatorname{sech}\left[(2m\pi\delta)^4\right] e^{-x(2m\pi)^4 \sin^2(2n\pi^2/2m\pi)} \tag{129}$$

We are going to use the same factor $\frac{1}{e^{iz}-1}$ to implement Matsubara technique. We re-express F_n in this way:

$$F_n = i \sum_{m\neq 0} (-i) \operatorname{sech}\left[(2m\pi\delta)^4\right] e^{-x(2m\pi)^4 \sin^2(2n\pi^2/2m\pi)}$$
$$\Rightarrow F_n = i \oint \frac{dz}{2\pi i} \frac{1}{e^{iz}-1} \operatorname{sech}(z^4\delta^4) e^{-xz^4 \sin^2(\theta/z)} \tag{130}$$

where $\theta = 2n\pi^2$.

We find residue at the origin by Laurrent expansion of individual factors around the origin and from the previous analysis here we only mention them

$$\frac{1}{e^{iz}-1} = \frac{1}{iz} \frac{iz}{e^{iz}-1} = \sum_{r=0}^{\infty} B_r \frac{(iz)^{r-1}}{r!}, \qquad |z| < \pi \tag{131}$$

$$\operatorname{sech}(z^4\delta^4) = \sum_{u=0}^{\infty} \frac{E_{2u}}{(2u)!} (z\delta)^{4u}, \qquad \left|z^4\delta^4\right| < \pi/2 \tag{132}$$

$$e^{-x\sin^2(\theta/z) z^4} = \sum_{m,q=0}^{\infty} \frac{x^m z^{4m}}{4^m m!} \frac{i^q \theta^q}{z^q q!} G(q,m) \tag{133}$$

Combining all we have:

$$i \sum_{r,m,q,u=0}^{\infty} B_r \frac{i^{r-1}}{r!} z^{r-1} \frac{E_{2u}}{(2u)!} (z\delta)^{4u} \frac{x^m z^{4m}}{4^m m!} \frac{i^q \theta^q}{q!\, z^q} G(q,m) \tag{134}$$

$$= i \sum_{r,m,q,u=0}^{\infty} B_r \frac{i^{r-1}}{r!} \frac{E_{2u}\delta^{4u}}{(2u)!} \frac{x^m}{4^m m!} \frac{i^q \theta^q}{q!} G(q,m) z^{r+4m+4u-q-1} \tag{135}$$

To find residue we extract coefficient of z^{-1} which gives:

$$r + 4m + 4u - q - 1 = -1 \tag{136}$$

$$\Rightarrow q = r + 4m + 4u \tag{137}$$

Clockwise contour around the origin gives an extra −ve sign and we have:

$$F_n = -i \sum_{r,m,u=0}^{\infty} B_r \frac{i^{r-1}}{r!} \frac{E_{2u}\delta^{4u}}{(2u)!} \frac{x^m}{4^m m!} \frac{i^q \theta^q}{q!} \tag{138}$$

where $q = r + 4m + 4u$. With this $\mathcal{S}_n$ is given by:

$$\mathcal{S}_n = i \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta \to 0+} \int_0^{\infty} dx\, e^{-x(\delta - i\psi)} \sum_{r,m,u=0}^{\infty} \frac{x^m}{m!} \cdots \tag{139}$$

$$\Rightarrow \mathcal{S}_n = i \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta \to 0+} \sum_{r,m,u=0}^{\infty} \int_0^{\infty} dx\, e^{-x(\delta - i\psi)} \frac{x^m}{m!} \cdots \tag{140}$$

We use following relation to perform the x integration, like in the previous section:

$$\int_0^{\infty} dx\, e^{-\alpha x} x^m = \frac{(-1)^m m!}{\alpha^{m+1}} \tag{141}$$

and we have:

$$\mathcal{S}_n = i \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \lim_{\delta \to 0+} \sum_{r,m,u=0}^{\infty} \frac{(-1)^m}{[\delta - i\psi]^{m+1}} \cdots \frac{E_{2u}\delta^{4u}}{(2u)!} \tag{142}$$

$$= i \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \sum_{r,m=0}^{\infty} \lim_{\delta \to 0+} \sum_{u=0}^{\infty} \frac{(-1)^m}{[\delta - i\psi]^{m+1}} \cdots \frac{E_{2u}\delta^{4u}}{(2u)!} \tag{143}$$

$$= i \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \sum_{r,m=0}^{\infty} \frac{(-1)^m}{(-i\psi)^{m+1}} B_r \frac{i^{r-1}}{r!} \frac{1}{4^m} \frac{i^q \theta^q}{q!} G(q,m) E_0 \tag{144}$$

$$= -iE_0 \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} e^{i\psi} \sum_{r,m=0}^{\infty} \frac{1}{(i\psi)^{m+1}} B_r \frac{i^{r-1}}{r!} \frac{1}{4^m} \frac{i^q \theta^q}{q!} G(q,m) \tag{145}$$

$$= -\sum_{r,m=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi}{2\pi} \frac{e^{i\psi}}{(i\psi)^{m+1}} B_r \frac{i^r}{r!} \frac{1}{4^m} \frac{i^q \theta^q}{q!} G(q,m) \qquad [\because E_0 = 1\,] \tag{146}$$

Using the result from the previous section the integration over ψ is given by:

$$\int_{-\infty}^{\infty} \frac{d\psi}{2\pi} \frac{e^{i\psi}}{(i\psi)^{m+1}} = \frac{1}{2} \frac{1}{m!} \tag{147}$$

With this we have,

$$\mathcal{S}_n = -\frac{1}{2} \sum_{r,m=0}^{\infty} B_r \frac{i^r}{r!} \frac{1}{4^m} \frac{i^q \theta^q}{m!\, q!} G(q,m) \tag{148}$$

where B_r is r^{th} Bernoulli number, $\theta = 2n\pi^2$ and $q = r + 4m$. From Appendix G we know q has to be even for non-zero value and this implies r is also even. Hence we have:

$$\boxed{\mathcal{S}_n = -\frac{1}{2}\sum_{r,m=0}^{\infty} \frac{|B_r|}{r!} \frac{G(q,m) i^q}{4^m\, m!\, q!} \theta^q} \tag{149}$$

This closed form of $\mathcal{S}_n$ can be used to determine a number is prime or not numerically. As a by product of this effort we have obtained an quadrature to determine primeness of a number.

4.1. **Generalisation to higher order.** The modification to Eq 136 for the $2j^{\text{th}}$ order, $\forall j \in \mathbb{Z}$, is given below:

$$2j + r + 2(2+j)m + 4u - q - 1 = -1 \tag{150}$$

$$\Rightarrow q = 2j + r + 2(2+j)m + 4u \tag{151}$$

After performing the above steps we have:

$$q = 2j + r + 2(2+j)m \tag{152}$$

So, $\mathcal{S}_{2j}(n)$ is given as:

$$\boxed{\mathcal{S}_{2j}(n) = -\frac{1}{2(2\pi)^{2j}}\sum_{r,m=0}^{\infty} \frac{|B_r|}{r!} \frac{G(q,m) i^q}{4^m\, m!\, q!} \theta^q} \tag{153}$$

where q is given by the Eq 152.

Appendix A. Notes

1. Since each improper integral converges independently the summation and integration can be interchanged. Thanks Alan in MathStackExchange for providing this reasoning.

2. If $m > n$ then m does not divide n. Hence all such term will vanish after performing the ψ integration and we can extend limit of the summation over m to $(-\infty, \infty)$.

Appendix B.

To justify the interchange of summation and integration we invoke the *dominated convergence theorem* and we mention it here for the general readers.

The Dominated Convergence Theorem. *If $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of measurable functions which converge pointwise almost everywhere to f, and if there exists an integrable function g such that $|f_n(x)| \leq g(x)$ for all n and for all x, then f is integrable and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f \tag{B1}$$

Basically it tells when integration and taking limit can be interchanged. An infinite summation can be thought of limit to a finite sum, so we can use this theorem to justify the interchange in the following manner.

$$\sum_{n=0}^{\infty} \int_X f_n(x) = \lim_{k\to\infty} \sum_{n=0}^{k} \int_X f_n(x) = \lim_{k\to\infty} \int_X \sum_{n=0}^{k} f_n(x) \tag{B2}$$

So, if $\sum_{n=0}^{\infty} f_n(x)$ exists for all x and there is some integrable function $g(x)$ such that

$$\left| \sum_{n=0}^{k} f_n(x) \right| \leq g(x) \tag{B3}$$

for every k, then

$$\sum_{n=0}^{\infty} \int_X f_n(x) = \int_X \sum_{n=0}^{\infty} f_n(x) \tag{B4}$$

Now,

$$\left| \sum_{n=0}^{k} f_n(x) \right| \leq \sum_{n=0}^{k} |f_n(x)| \tag{B5}$$

$$\leq \sum_{n=0}^{\infty} |f_n(x)| \tag{B6}$$

so if the sum converges absolutely to an integrable function, then the integral and the summation can be exchanged. The sum we have in Eq 23 is:

$$\sum_{m=-\infty}^{\infty}{}' \left| \frac{m e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau+\epsilon+i\tau\right]}}{-i\psi+(2m\pi)^6\sin^2(n\pi/m)} \right| \tag{B7}$$

$$= 2\sum_{m=1}^{\infty} e^{-y\left[\epsilon(2m\pi)^2+\epsilon\right]} \left| \frac{m}{-i\psi+(2m\pi)^6\sin^2(n\pi/m)} \right| \tag{B8}$$

$$\leq e^{-\epsilon y}\sum_{m=1}^{\infty} \left| \frac{m}{-i\psi+(2m\pi)^6\sin^2(n\pi/m)} \right| \qquad [\because y\in[0,\infty)] \tag{B9}$$

$$\leq K e^{-\epsilon y} \tag{B10}$$

where K is some constant and this is an integrable function for $\epsilon > 0$ and over the domain $[0, \infty)$.

The sum we have in Eq 26 is:

$$\sum_{m=-\infty}^{\infty}{}' \left| e^{-x(\delta-i\psi)} m e^{-y\left[\epsilon(2m\pi)^2+2m\pi i\tau\right]} \operatorname{sech}[(2m\pi\delta)^6] e^{-x(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)} \right| \tag{B11}$$

$$= 2\sum_{m=1}^{\infty} e^{-x\delta} \left| m e^{-y\epsilon(2m\pi)^2} \operatorname{sech}[(2m\pi\delta)^6] e^{-x(2m\pi)^6\sin^2\left(\frac{n\pi}{m}\right)} \right| \tag{B12}$$

$$\leq 2e^{-x\delta}\sum_{m=1}^{\infty} \left| m e^{-y\epsilon(2m\pi)^2} \operatorname{sech}[(2m\pi\delta)^6] \right| \qquad [\because x\in[0,\infty)] \tag{B13}$$

$$\leq K e^{-x\delta} \tag{B14}$$

where K is some constant and this is an integrable function for $\delta > 0$ and over the domain $[0, \infty)$.

This part is mostly taken from an answer on this topic in Quora by Senia Sheydvasser, PhD in Mathematics.

Appendix C. Matsubara Technique

Appendix D.

Appendix E.

The reason for introducing this convergence factor is that when $\sin(n\pi/m) = 0$ still the numerator is positive definite real part which is necessary for the next manipulation. Now we justify the interchange.

Appendix F.

Appendix G. Properties of the polynomial $G(q,m)$

G.1. **Positivity.** Now we will show $G(q,m) \geq 0$ in the following way

$$G(q,m) = \sum_{p=0}^{2m} (-1)^p (2m-2p)^q \binom{2m}{p} \tag{G1}$$

$$\Rightarrow G(q,m) = \sum_{p=0}^{2m} (-1)^p \left. \frac{\partial^q}{\partial x^q} e^{(2m-2p)x} \right|_{x=0} \binom{2m}{p} \tag{G2}$$

$$\Rightarrow G(q,m) = \left. \frac{\partial^q}{\partial x^q} \sum_{p=0}^{2m} (-1)^p e^{(2m-2p)x} \binom{2m}{p} \right|_{x=0} \tag{G3}$$

$$\Rightarrow G(q,m) = \left. \frac{\partial^q}{\partial x^q} \left[e^x - e^{-x} \right]^{2m} \right|_{x=0} \tag{G4}$$

$$\Rightarrow G(q,m) = 4^m \left. \frac{\partial^q}{\partial x^q} \sinh^{2m} x \right|_{x=0} \tag{G5}$$

Since Taylor series expansion of $\sinh(x)$ about the origin contains only positive coefficients and hence $\sinh^{2m} x$ will also have positive expansion coefficient only. Hence $G(q,m) > 0 \ \forall q \geq 2m$. It can be easily inferred that if $q < 2m$ then $G(q,m) = 0$.

G.2. **Dependence on q and m.** First we will show $G(q,m) = 0$ if q is odd. This can be proved very easily from the defining equation, Eq 43 and since $p \in [0, 2m]$ and $\binom{2m}{p} = \binom{2m}{2m-p}$, a sum over p of $(-1)^p (2m-2p)^q \binom{2m}{p}$ would be zero for odd q. From here also we can see that only even q gives non-zero values. $\forall m \ G(0,m) = 0$ and $\forall q \ G(q,0) = 0$. *For odd p individual term in the sum probably vanishes.*

G.3. **Approximate upper-bound.** Here we will find an approximate upper bound for $G(q,m)$.

$$G(q,m) = \sum_{p=0}^{2m} (-1)^p (2m-2p)^q \binom{2m}{p} \tag{G6}$$

$$\Rightarrow G(q,m) < \sum_{p=0}^{2m} (2m-2p)^q \binom{2m}{p} \quad [\because q \text{ is even}] \tag{G7}$$

$$\Rightarrow G(q,m) < (2m)^q \sum_{p=0}^{2m} \binom{2m}{p} \tag{G8}$$

$$\Rightarrow G(q,m) < (2m)^q 2^{2m} \tag{G9}$$

$$\Rightarrow G(q,m) < 2^q m^q 4^m \tag{G10}$$

G.4. **Recurrence relation.** From Eq we have

$$G(q,m+1)=4^{m+1}\frac{\partial^q}{\partial x^q}\sinh^{2m+2}x\Big|_{x=0} \tag{G11}$$

$$\Rightarrow G(q,m+1)=4^{m+1}\frac{\partial^q}{\partial x^q}\left[\sinh^2 x\cdot\sinh^{2m}x\right]_{x=0} \tag{G12}$$

$$\Rightarrow G(q,m+1)=4^{m+1}\sum_{p=0}^{q}\binom{q}{p}\left(\sinh^2 x\right)^{(p)}\left(\sinh^{2m}x\right)^{(q-p)}\Big|_{x=0} \tag{G13}$$

$$\Rightarrow G(q,m+1)=4^{m+1}\sum_{p=0}^{q}\binom{q}{p}\left(\sinh^2 x\right)^{(p)}\Big|_{x=0}\left(\sinh^{2m}x\right)^{(q-p)}\Big|_{x=0} \tag{G14}$$

Now,

$$\frac{\partial^p}{\partial x^p}\sinh^2 x\Big|_{x=0}=\begin{cases}2^{p-1} & p \text{ even}\\ 0 & \text{otherwise}\end{cases} \tag{G15}$$

Hence,

$$G(q,m+1)=2\sum_{p=0}^{q}\binom{q}{p}2^p4^m\frac{\partial^{q-2}}{\partial x^{q-2}}\sinh^{2m}x\Big|_{x=0} \tag{G16}$$

$$\Rightarrow G(q,m+1)=2\sum_{p=0}^{q}\binom{q}{2p}2^{2p}G(q-2p,m) \tag{G17}$$

G.5. **Solution of the recurrence relation.** (*Yet to be finished.*) From solution of the above recurrence relation we can compute the exact value of $G(q,m)$ in the following way

$$\frac{G(q,m+1)}{q!}=\frac{2}{q!}\sum_{p=0}^{q}\binom{q}{2p}2^{2p}G(q-2p,m) \tag{G18}$$

$$\Rightarrow \frac{G(q,m+1)}{q!}=\frac{2}{q!}\sum_{p=0}^{q}\frac{2^{2p}\,q!}{(2p)!(q-2p)!}G(q-2p,m) \tag{G19}$$

$$\Rightarrow \frac{G(q,m+1)}{q!}=2\sum_{p=0}^{q}\frac{2^{2p}}{(2p)!}\frac{G(q-2p,m)}{(q-2p)!} \tag{G20}$$

Now from Eq G5 we can explicitly compute $G(q,1)$ in the following way

$$G(q,1)=4\,\frac{\partial^q}{\partial x^q}\sinh^2 x\Big|_{x=0} \tag{G21}$$

$$\Rightarrow G(q,1)=2\,\frac{\partial^q}{\partial x^q}2\sinh^2 x\Big|_{x=0} \tag{G22}$$

$$\Rightarrow G(q,1)=2\,\frac{\partial^q}{\partial x^q}(\cosh 2x-1)\Big|_{x=0} \tag{G23}$$

$$\Rightarrow G(q,1)=2\,\frac{\partial^q}{\partial x^q}\cosh 2x\Big|_{x=0} \tag{G24}$$

$$\Rightarrow G(q,1)=2\,\frac{\partial^q}{\partial x^q}\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!}\Big|_{x=0} \tag{G25}$$

$$\Rightarrow G(q,1)=2^{q+1} \tag{G26}$$

It is worthy mentioning, $G(q,m)$ has striking similarity with Stirling number of the second kind.

Appendix H. Properties of the polynomial $S(q,m)$

Here we analyse the polynomial $S(q,m)$. From the definition we have:

$$S(q,m) = \sum_{r=0}^{q-1-6m} \frac{B_r}{r!(q-1-r-6m)!} \tag{H1}$$

$$\Rightarrow S(q,m) = \frac{1}{k!}\sum_{r=0}^{k} \frac{B_r k!}{r!(k-r)!} \quad \text{[where } k = q-1-6m\text{]} \tag{H2}$$

$$\Rightarrow S(q,m) = \frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r \tag{H3}$$

From the closed form expression of the sum of the m^{th} powers of the first n positive integers we have:

$$\frac{1}{m+1}\sum_{k=0}^{m} \binom{m+1}{k} B_k^+ n^{m+1-k} = \sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m \tag{H4}$$

Set $n = 1$ in the above equation and we have:

$$\frac{1}{m+1}\sum_{k=0}^{m} \binom{m+1}{k} B_k^+ = 1 \tag{H5}$$

Since q is even it implies k is always odd in Eq H3 and we have:

$$\frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = \frac{1}{k!}\sum_{r=0}^{k-1} \binom{k}{r} B_r^- + \frac{B_k^-}{k!} \tag{H6}$$

$$\Rightarrow \frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = \frac{k}{k!}B_1^- + \frac{1}{k!}{\sum_{r=0}^{k-1}}' \binom{k}{r} B_r^- + \frac{B_k^-}{k!} \tag{H7}$$

where $\sum_k'$ means $k \neq 1$. Now if $k > 1$ we have:

$$\frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = \frac{k}{k!}B_1^- + \frac{1}{k!}{\sum_{r=0}^{k-1}}' \binom{k}{r} B_r^+ + \frac{B_k^-}{k!} \tag{H8}$$

$$\Rightarrow \frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = \frac{k}{k!}(B_1^+ - 1) + \frac{1}{k!}{\sum_{r=0}^{k-1}}' \binom{k}{r} B_r^+ + \frac{B_k^-}{k!} \tag{H9}$$

$$\Rightarrow \frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = -\frac{k}{k!} + \frac{k}{k!}B_1^+ + \frac{1}{k!}{\sum_{r=0}^{k-1}}' \binom{k}{r} B_r^+ + \frac{B_k^-}{k!} \tag{H10}$$

$$\Rightarrow \frac{1}{k!}\sum_{r=0}^{k} \binom{k}{r} B_r^- = -\frac{k}{k!} + \frac{1}{k!}\sum_{r=0}^{k-1} \binom{k}{r} B_r^+ + \frac{B_k^-}{k!} \tag{H11}$$

But from Eq H5 we have:

$$\frac{1}{m}\sum_{k=0}^{m-1} \binom{m}{k} B_k^+ = 1 \tag{H12}$$

$$\Rightarrow \sum_{k=0}^{m-1} \binom{m}{k} B_k^+ = m \tag{H13}$$

Using this we have from Eq H3

$$\frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}B_r^- = \frac{B_k^-}{k!} \tag{H14}$$

From the properties of Bernoulli's number this sum vanishes for odd $k > 1$. For $k = 1$ we have

$$\sum_{r=0}^{k}\binom{k}{r}B_r^- = B_0^- + B_1^- = 1 - 1/2 = 1/2 \tag{H15}$$

So in essence what we have shown here is that, over the present domain $S(q,m) > 0$ and only one value of m will contribute for which $q - 1 - 6m = 1$ when summed over m.

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7. www.wikipedia.org - I suppose it requires no description.
8. www.math.stackexchange.com - I can not tell how many posted questions and answers I have gone through. Thanks a lot to all users.
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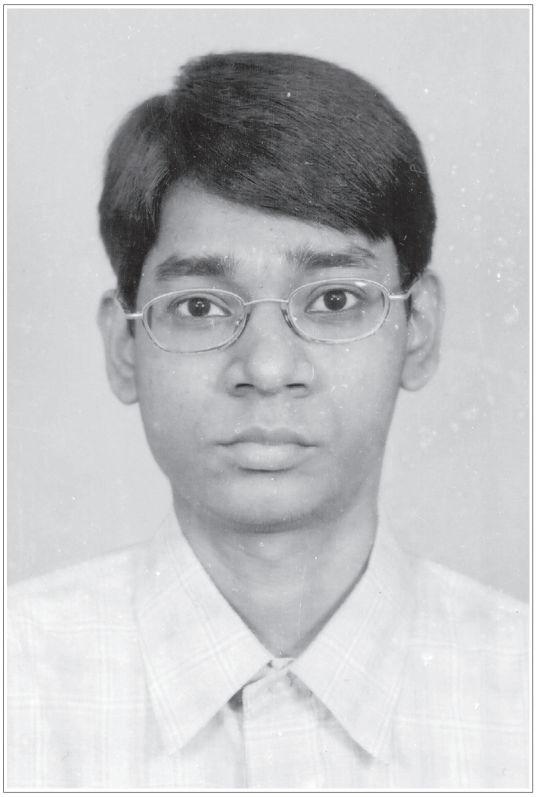

ARUP SAHA
(1984 - 2011)

This Work Is Dedicated To My Beloved Late Brother Mr. Arup Saha.