Another Values of the Barnes Function and Formulas

Denis GALLET
Rectorat de DIJON
2 G, rue Général Delaborde 21000 DIJON (France)
densg71@gmail.com

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Abstract

In the continuity of my precedent paper Values of Barnes Function (1), this time I talk about for the Barnes function at unsual points as $G(1/8)$, $G(3/8)$ or $G(5/12)$ for example.

In the same time, I give several formulas and so we can evaluate easily elementary values of Barnes function.I give eight conjectural integral formulas and we see several applications, in particulary Wallis product.

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G\left(1+z\right)=\left(2\,\pi\right)^{\frac{z}{2}}\,\mathrm{e}^{-\frac{z\,(1+z)}{2}-\frac{\gamma\,z^2}{2}}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{k}\mathrm{e}^{-z+\frac{z^2}{2\,k}} \qquad (2)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \qquad (3)$$

$$G\left(1+z\right) = G\left(z\right)\Gamma\left(z\right) \qquad (4)$$

$$\log\left(G\left(1+z\right)\right) = \frac{z\log\left(2\,\pi\right)}{2}-\frac{z\left(1+z\right)}{2}+z\log\left(\Gamma\left(1+z\right)\right)-\int_0^z\log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t \qquad (5)$$

$$\int_0^z\log\left(\Gamma\left(t+1\right)\right)\,\mathrm{d}t = \frac{z\log\left(2\,\pi\right)}{2}-\frac{z\left(1+z\right)}{2}+z\log\left(\Gamma\left(1+z\right)\right)-\log\left(G\left(z\right)\right)-\log\left(\Gamma\left(z\right)\right) \qquad (6)$$

3 List of Formulas

Let A be the Glaisher–Kinkelin's constant (7),K be the Catalan's constant (8) and $\Psi\left(1,\frac{1}{3}\right)$ is the trigamma function at 1/3 (9).

If t is positive integer then $\log(G(t)) = (t-1)\log(\Gamma(t)) - [\sum_{v=2}^{t}[-\log(-1+v)+\log(-1+v)\,v]]$

If t = k+(1/2) where k is 0 or positive integer then $\log(G(t)) = (t-1)\log(\Gamma(t)) + \frac{1}{8} - \frac{3\log(A)}{2} + \frac{\log(2)}{24}$
$$-[\sum_{v=1}^{t}\left(v-\frac{1}{2}\right)\log\left(v-\frac{1}{2}\right)]$$

If t = k+(1/4) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) + \frac{3}{32} - \frac{9\log(A)}{8} - \frac{K}{4\pi}$
$$-[\sum_{v=1}^{t}\left(v-\frac{3}{4}\right)\log\left(v-\frac{3}{4}\right)]$$

If t = k+(3/4) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) + \frac{3}{32} - \frac{9\log(A)}{8} + \frac{K}{4\pi}$
$$-[\sum_{v=1}^{t}\left(v-\frac{1}{4}\right)\log\left(v-\frac{1}{4}\right)]$$

If t = k+(1/3) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) + \frac{\pi\sqrt{3}}{54} - \frac{\sqrt{3}\Psi\left(1,\frac{1}{3}\right)}{36\pi}$
$$+\frac{1}{9} - \frac{4\log(A)}{3} + \frac{\log(3)}{72} - [\sum_{v=1}^{t}\left(v-\frac{2}{3}\right)\log\left(v-\frac{2}{3}\right)]$$

If t = k+(2/3) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) - \frac{\pi\sqrt{3}}{54} + \frac{\sqrt{3}\Psi\left(1,\frac{1}{3}\right)}{36\pi}$
$$+\frac{1}{9} - \frac{4\log(A)}{3} + \frac{\log(3)}{72} - [\sum_{v=1}^{t}\left(v-\frac{1}{3}\right)\log\left(v-\frac{1}{3}\right)]$$

If t = k+(1/6) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) + \frac{\pi\sqrt{3}}{36} - \frac{\sqrt{3}\Psi\left(1,\frac{1}{3}\right)}{24\pi}$
$$+\frac{5}{72} - \frac{5\log(A)}{6} - \frac{\log(2)}{72} - \frac{\log(3)}{144} - [\sum_{v=1}^{t}\left(v-\frac{5}{6}\right)\log\left(v-\frac{5}{6}\right)]$$

If t = k+(5/6) where k is 0 or positive integer then $\log(G(t)) = \log(\Gamma(t))(t-1) - \frac{\pi\sqrt{3}}{36} + \frac{\sqrt{3}\Psi\left(1,\frac{1}{3}\right)}{24\pi}$
$$+\frac{5}{72} - \frac{5\log(A)}{6} - \frac{\log(2)}{72} - \frac{\log(3)}{144} - [\sum_{v=1}^{t}\left(v-\frac{1}{6}\right)\log\left(v-\frac{1}{6}\right)]$$

4 Hypothesis around the integral $\int_0^t \log(\Gamma(t+1))\ \mathrm{d}t$ and the sum $\sum_{k=1}^{\infty} \frac{(1-\zeta(2k+1))q^{2k+2}}{(k+1)(2k+1)}$

We consider the integral $\int_0^t \log(\Gamma(t+1))\ \mathrm{d}t$

So the closed form in general is $\int_0^t \log(\Gamma(t+1))\ \mathrm{d}t = \mathit{constant} +$

$y * \log(A) + x * \log(\pi) + f +$several terms in logarithms$+ z * \frac{K}{\pi}$

Where f is a complex function in terms of trigamma and constant is a real number.

When the bound t is between -1 and 0, we can directly calculate some terms and we have

$\int_0^t \log(\Gamma(t+1))\ \mathrm{d}t = (-6t^2 - 6t) * \log(A) + \frac{t}{2} * \log(\pi) + f +$several

terms in logarithms$+ z * \frac{K}{\pi}$.(There are no constant)

About the polynom $-6t^2 - 6t$: I see successively $\int_0^{-1} \log(\Gamma(t+1))\ \mathrm{d}t$

then $\int_0^{-1/2} \log(\Gamma(t+1))\ \mathrm{d}t$ then $\int_0^{-1/3} \log(\Gamma(t+1))\ \mathrm{d}t$ then $\int_0^{-2/3} \log(\Gamma(t+1))\ \mathrm{d}t$ then $\int_0^{-1/4} \log(\Gamma(t+1))\ \mathrm{d}t$

and just I see that the polynom give the correct value in $\log(A)$.

Now about the sum $\sum_{k=1}^{\infty} \frac{(1-\zeta(2k+1))q^{2k+2}}{(k+1)(2k+1)}$ (10)

Especially, there are no term in $\log(\pi)$ in the final closed form.

When we have a sum where we find a term of $\zeta(2k+a)$ where a is a odd positiv or negativ integer.

Just I see in the final closed form of the sum, we have no trigamma function and no Catalan's constant.

5 Expressions of G(1/8), G(3/8),G(5/8),G(7/8)

First I find $\log\left(G\left(\frac{7}{8}\right)\right)$, it's easy and we use the formula (5) if z = -1/8 we have:

$$\log\left(G\left(\frac{7}{8}\right)\right) = \frac{7}{128} - \frac{5\log(2)}{32} - \frac{3\log(\pi)}{16} - \int_0^{-1/8}\log(\Gamma(t+1))\,dt - \frac{\log\left(1+\sqrt{2}\right)}{16} + \frac{\log\left(\Gamma\left(\frac{1}{8}\right)\right)}{8}$$

Now we search $\log\left(G\left(\frac{1}{8}\right)\right)$ and I use for example:

$\int_0^t \pi x\cot(\pi x)\,dx = t\log(2\pi) + \log\left(\frac{G(1-t)}{G(1+t)}\right)$

When t = 1/8 and with Maple I find the closed form of the integral and finally:

$$\log\left(G\left(\frac{1}{8}\right)\right) = \frac{7}{128} - \frac{\log(2)}{16} - \frac{\log(\pi)}{16} - \int_0^{-1/8}\log(\Gamma(t+1))\,dt - \frac{K}{8\pi} - \frac{7\log\left(\Gamma\left(\frac{1}{8}\right)\right)}{8} + \frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right) + \left(2\pi^2+16K\right)\sqrt{2} + 2\pi^2}{64\pi}$$

Now we search $\log\left(G\left(\frac{3}{8}\right)\right)$ and $\log\left(G\left(\frac{5}{8}\right)\right)$

We know that $\sum_{k=1}^{\infty}\frac{(1-\zeta(2k+1))q^{2k+2}}{(k+1)(2k+1)} = (\gamma-2)q^2 - \frac{1}{6} + 2\log(A) + \zeta(1,-1,2-q) + \zeta(1,-1,2+q)$

And using the relation $\zeta(1,-1,t) = \frac{1}{12} - \log(A) - \log(G(t)) + (t-1)\log(\Gamma(t))$

I have $\sum_{k=1}^{\infty}\frac{1-\zeta(2k+1)}{(k+1)(2k+1)}\left(\frac{1}{8}\right)^{2k+2} = \frac{\gamma}{64} - \frac{9}{64} + 2\int_0^{-1/8}\log(\Gamma(t+1))\,dt + \frac{\log(\pi)}{8} + \frac{\sqrt{2}\Psi\left(1,\frac{1}{8}\right) + \left(-2\pi^2-16K\right)\sqrt{2} - 2\pi^2 + 8K}{64\pi} + \frac{7\log(7)}{8} - \frac{600\log(2)}{96} + \frac{9\log(3)}{4}$

I know that the partial closed form of $\int_0^{-1/8}\log(\Gamma(t+1))\,dt$:

$-\frac{\log(\pi)}{16} + \frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right) + \left(2\pi^2+16K\right)\sqrt{2} + 2\pi^2 - 8K}{128\pi}$

Remember in the final closed form of the sum there are no trigamma function, no $\log(\pi)$ and no term in Catalan's constant.

Using the relation in the paper Polygamma Function of Negativ Order, page 5:

$$\zeta\left(1,-1,\tfrac{5}{8}\right)-\zeta\left(1,-1,\tfrac{3}{8}\right)=\frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(2\,\pi^2+16\,K\right)\sqrt{2}+2\,\pi^2+8\,K}{64\,\pi}$$

Using the relation $\zeta(1,-1,t)=\frac{1}{12}-\log(A)-\log(G(t))+(t-1)\log(\Gamma(t))$ and the relation (5)

Equivalently $\int_0^{5/8}\log(\Gamma(t+1))\;\mathrm{d}t-\int_0^{3/8}\log(\Gamma(t+1))\;\mathrm{d}t=\frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(2\,\pi^2+16\,K\right)\sqrt{2}+2\,\pi^2+8\,K}{64\,\pi}-\frac{5\,\log(2)}{8}+\frac{\log(\pi)}{8}-\frac{1}{4}+\frac{5\,\log(5)}{8}-\frac{3\,\log(3)}{8}$

In the same time $\sum_{k=1}^{\infty}\frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)}\left(\frac{3}{8}\right)^{2\,k+2}=\frac{9\,\gamma}{64}-\frac{43}{96}+2\,\log(A)+\zeta\left(1,-1,\frac{5}{8}\right)+\frac{5\,\log(5/8)}{8}+\zeta\left(1,-1,\frac{3}{8}\right)+\frac{3\,\log(3/8)}{8}+\frac{11\,\log\left(\frac{11}{8}\right)}{8}$

$\sum_{k=1}^{\infty}\frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)}\left(\frac{3}{8}\right)^{2\,k+2}=\int_0^{5/8}\log(\Gamma(t+1))\;\mathrm{d}t+\int_0^{3/8}\log(\Gamma(t+1))\;\mathrm{d}t+\frac{9\,\gamma}{64}+\frac{31}{64}-\frac{\log(\pi)}{2}-\frac{37\,\log(2)}{8}+\frac{11\,\log(11)}{8}$

Now we search $\int_0^{5/8}\log(\Gamma(t+1))\;\mathrm{d}t+\int_0^{3/8}\log(\Gamma(t+1))\;\mathrm{d}t$

The closed form of $\sum_{k=1}^{\infty}\frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)}\left(\frac{3}{8}\right)^{2\,k+2}$ look like as the closed form of $\sum_{k=1}^{\infty}\frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)}\left(\frac{1}{8}\right)^{2\,k+2}$

The beginning of $\int_0^{5/8}\log(\Gamma(t+1))\;\mathrm{d}t+\int_0^{3/8}\log(\Gamma(t+1))\;\mathrm{d}t$ is:

$$\frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(2\,\pi^2+16\,K\right)\sqrt{2}+2\,\pi^2-8\,K}{64\,\pi}-2\int_0^{-1/8}\log(\Gamma(t+1))\;\mathrm{d}t$$

Or

$$\frac{\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(-2\,\pi^2-16\,K\right)\sqrt{2}-2\,\pi^2+8\,K}{64\,\pi}+2\int_0^{-1/8}\log(\Gamma(t+1))\;\mathrm{d}t$$

We have two possibilities but I choose the first expression.

Now we calculate missing terms at the right.

But the problem that we calculate terms in $\log(A)$ for a bound t between 0 and 1 and so the polynom is $-6\,t^2+6\,t$, for terms in $\log(\pi)$ the polynom is $\frac{t}{2}$ and there are a constant who is $-t$.And when the bound is a/b where b and a both positiv integer, a smaller than b and a prime there are a term who is $\frac{a\log(a)}{b}$.

For $\int_0^{5/8}\log(\Gamma(t+1))\;\mathrm{d}t$ we have $\frac{5\,\log(\pi)}{16}+\frac{5\,\log(5)}{8}-\frac{5}{8}+\frac{45\,\log(A)}{32}$

For $\int_0^{3/8}\log(\Gamma(t+1))\,\mathrm{d}t$ we have $\frac{3\log(\pi)}{16}+\frac{3\log(3)}{8}-\frac{3}{8}+\frac{45\log(A)}{32}$

For $\int_0^{-1/8}\log(\Gamma(t+1))\,\mathrm{d}t$ we have $-\frac{\log(\pi)}{16}+\frac{21\log(A)}{32}$

I make the calcul $\frac{5\log(\pi)}{16}+\frac{5\log(5)}{8}-\frac{5}{8}+\frac{45\log(A)}{32}+\frac{3\log(\pi)}{16}+\frac{3\log(3)}{8}-\frac{3}{8}+\frac{45\log(A)}{32}+2\,[-\frac{\log(\pi)}{16}+\frac{21\log(A)}{32}]$

So I find $-1+\frac{33\log(A)}{8}+\frac{5\log(5)}{8}+\frac{3\log(\pi)}{8}+\frac{3\log(3)}{8}$

Now using the command identify with Maple, I have the missing term in $\log(2)$ and finally the closed form of $\int_0^{5/8}\log(\Gamma(t+1))\,\mathrm{d}t+\int_0^{3/8}\log(\Gamma(t+1))\,\mathrm{d}t$ is:

$-1-\frac{251\log(2)}{96}+\frac{33\log(A)}{8}+\frac{5\log(5)}{8}+\frac{3\log(\pi)}{8}+\frac{3\log(3)}{8}+\frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(2\pi^2+16K\right)\sqrt{2}+2\pi^2-8K}{64\pi}-2\int_0^{-1/8}\log(\Gamma(t+1))\,\mathrm{d}t$

Now I have $\int_0^{5/8}\log(\Gamma(t+1))\,\mathrm{d}t+\int_0^{3/8}\log(\Gamma(t+1))\,\mathrm{d}t$ and $\int_0^{5/8}\log(\Gamma(t+1))\,\mathrm{d}t-\int_0^{3/8}\log(\Gamma(t+1))\,\mathrm{d}t$

So I find respectively the two integrals and consequently $\log\left(G\left(\frac{3}{8}\right)\right)$ and $\log\left(G\left(\frac{5}{8}\right)\right)$

6 List of conjectural formulas

$$\log\left(G\left(\frac{1}{8}\right)\right)=\frac{7}{128}-\frac{\log(2)}{16}-\frac{\log(\pi)}{16}-\int_0^{-1/8}\log(\Gamma(t+1))\,\mathrm{d}t-\frac{K}{8\pi}-\frac{7\log\left(\Gamma\left(\frac{1}{8}\right)\right)}{8}+\frac{-\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(2\pi^2+16K\right)\sqrt{2}+2\pi^2}{64\pi}$$

$$\log\left(G\left(\frac{3}{8}\right)\right)=\frac{15}{128}+\frac{11\log(2)}{192}-\frac{\log(\pi)}{4}+\int_0^{-1/8}\log(\Gamma(t+1))\,\mathrm{d}t-\frac{33\log(A)}{16}+\frac{5\log\left(1+\sqrt{2}\right)}{16}+\frac{K}{8\pi}-\frac{5\log\left(\frac{\Gamma(1/8)}{\Gamma(1/4)}\right)}{8}$$

$$\log\left(G\left(\frac{5}{8}\right)\right)=\frac{15}{128}-\frac{43\log(2)}{192}-\frac{\log(\pi)}{8}+\int_0^{-1/8}\log(\Gamma(t+1))\,\mathrm{d}t-\frac{33\log(A)}{16}+\frac{3\log\left(\frac{\Gamma(1/8)}{\Gamma(1/4)}\right)}{8}+\frac{\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(-2\pi^2-16K\right)\sqrt{2}-2\pi^2}{64\pi}$$

$$\log\left(G\left(\frac{7}{8}\right)\right) = \frac{7}{128} - \frac{5\,\log(2)}{32} - \frac{3\,\log(\pi)}{16} - \int_0^{-1/8} \log(\Gamma(t+1))\,\mathrm{d}t - \frac{\log\left(1+\sqrt{2}\right)}{16} + \frac{\log\left(\Gamma\left(\frac{1}{8}\right)\right)}{8}$$

Now we search G(1/12),G(5/12),G(7/12) and G(11/12): it's the same principle but just we work with the sum $\sum_{k=1}^{\infty} \frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)} \left(\frac{1}{12}\right)^{2\,k+2}$ and $\sum_{k=1}^{\infty} \frac{1-\zeta(2\,k+1)}{(k+1)(2\,k+1)} \left(\frac{5}{12}\right)^{2\,k+2}$.

7 List of conjectural formulas

$$\log\left(G\left(\frac{1}{12}\right)\right) = \frac{11}{288} + \frac{3\,\log(2)}{16} - \frac{11\,\log(3)}{32} + \frac{5\,\log(\pi)}{12} - \int_0^{-1/12} \log(\Gamma(t+1))\,\mathrm{d}t$$
$$-\frac{11\,\log\left(1+\sqrt{3}\right)}{24} - \frac{K}{3\,\pi} - \frac{11\,\log(\Gamma(1/3)\,\Gamma(1/4))}{12} + \frac{\left(-3\,\Psi(1,1/3)+2\,\pi^2\right)\sqrt{3}}{144\,\pi}$$

$$\log\left(G\left(\frac{5}{12}\right)\right) = \frac{35}{288} - \frac{19\,\log(2)}{48} + \frac{5\,\log(3)}{72} - \frac{\log(\pi)}{4} + \int_0^{-1/12} \log(\Gamma(t+1))\,\mathrm{d}t$$
$$+\frac{7\,\log\left(1+\sqrt{3}\right)}{24} - \frac{23\,\log(A)}{12} + \frac{7\,\log\left(\frac{\Gamma(1/3)}{\Gamma(1/4)}\right)}{12} + \frac{\sqrt{3}\left(3\,\Psi(1,1/3)-2\,\pi^2\right)}{144\,\pi}$$

$$\log\left(G\left(\frac{7}{12}\right)\right) = \frac{35}{288} - \frac{13\,\log(2)}{48} - \frac{\log(3)}{18} - \frac{\log(\pi)}{6} + \int_0^{-1/12} \log(\Gamma(t+1))\,\mathrm{d}t$$
$$+\frac{5\,\log\left(1+\sqrt{3}\right)}{24} - \frac{23\,\log(A)}{12} + \frac{K}{3\,\pi} - \frac{5\,\log\left(\frac{\Gamma(1/3)}{\Gamma(1/4)}\right)}{12}$$

$$\log\left(G\left(\frac{11}{12}\right)\right) = \frac{11}{288} - \frac{5\,\log(2)}{48} + \frac{\log(3)}{32} - \frac{\log(\pi)}{6} - \int_0^{-1/12} \log(\Gamma(t+1))\,\mathrm{d}t$$
$$-\frac{\log\left(1+\sqrt{3}\right)}{24} + \frac{\log\left(\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{4}\right)\right)}{12}$$

8 Applications with Wallis product (11)

First example

Consider and calculate the closed form

$$\prod_{k=1}^{\infty} \left(\frac{(8\,k+5)^2\,(8\,k+1)\,(8\,k+7)}{(8\,k+4)^2\,(8\,k+2)\,(8\,k+8)}\right)^{k}$$

So we have $\frac{\left(G\left(\frac{13}{8}\right)\right)^2 G\left(\frac{9}{8}\right) G\left(\frac{15}{8}\right)}{\left(G\left(\frac{3}{2}\right)\right)^2 G\left(\frac{5}{4}\right) G(2)}$

We obtain

$$\Gamma\left(\frac{1}{4}\right)\left(\Gamma\left(\frac{1}{8}\right)\right)^{-2} 2^{\frac{3}{2}}\pi\left(1+\sqrt{2}\right)^{\frac{7}{16}} \mathrm{e}^{\frac{\sqrt{2}\Psi\left(1,\frac{1}{8}\right)+\left(-2\pi^2-16K\right)\sqrt{2}-2\pi^2+8K}{64\pi}}$$

Second example

Consider and calculate the closed form

$$\prod_{k=1}^{\infty}\left(\frac{(8k+1)^2(8k+3)^3(8k+4)}{(8k+2)^5(8k+5)}\right)^k$$

So we have $\frac{\left(G\left(\frac{9}{8}\right)\right)^2\left(G\left(\frac{11}{8}\right)\right)^3 G\left(\frac{3}{2}\right)}{\left(G\left(\frac{5}{4}\right)\right)^5 G\left(\frac{13}{8}\right)}$

We obtain

$$\left(\Gamma\left(\frac{1}{4}\right)\right)^{-3}\left(\Gamma\left(\frac{1}{8}\right)\right)^{2} 2^{-\frac{7}{16}}\pi^{\frac{1}{2}}\left(1+\sqrt{2}\right)^{-\frac{9}{16}} \mathrm{e}^{\frac{-3\sqrt{2}\Psi(1,1/8)+\left(6\pi^2+48K\right)\sqrt{2}+6\pi^2+88K}{64\pi}}$$

Third example

Consider and calculate the closed form

$$\prod_{k=1}^{\infty}\left(\frac{(12k+1)(12k+4)^2(12k+5)}{(12k+2)(12k+3)^2(12k+6)}\right)^k$$

So we have $\frac{G\left(\frac{13}{12}\right)\left(G\left(\frac{4}{3}\right)\right)^2 G\left(\frac{17}{12}\right)}{G\left(\frac{7}{6}\right)\left(G\left(\frac{5}{4}\right)\right)^2 G\left(\frac{3}{2}\right)}$

We obtain

$$3^{-\frac{7}{96}} 2^{\frac{23}{72}}\left(1+\sqrt{3}\right)^{-\frac{1}{6}} \mathrm{e}^{\frac{2\sqrt{3}\pi^2-3\sqrt{3}\Psi(1,1/3)+36K}{216\pi}}$$

Fourth example

Consider and calculate the closed form

$$\prod_{k=1}^{\infty} \left(\frac{(12\,k+1)^2\,(12\,k+5)^4\,(12\,k+6)}{(12\,k+2)\,(12\,k+3)^4\,(12\,k+7)^2} \right)^k$$

So we have $\frac{\left(G\left(\frac{13}{12}\right)\right)^2 \left(G\left(\frac{17}{12}\right)\right)^4 G\left(\frac{3}{2}\right)}{G\left(\frac{7}{6}\right)\left(G\left(\frac{5}{4}\right)\right)^4 \left(G\left(\frac{19}{12}\right)\right)^2}$

We obtain

$$\left(\Gamma\left(\frac{1}{4}\right)\right)^2 \left(\Gamma\left(\frac{1}{3}\right)\right)^{-3} 3^{-\frac{3}{8}} 2^{\frac{4}{9}} \pi^{\frac{1}{2}} \left(1+\sqrt{3}\right)^{-\frac{1}{6}} \mathrm{e}^{\frac{-2\,\sqrt{3}\pi^2+3\,\sqrt{3}\Psi(1,1/3)-12\,K}{36\,\pi}}$$

Remark and conclusion: this is a first approach with these particular values of the Barnes function and any improvement for these eight expressions is welcome.

9 References

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