A probabilistic solution for the Syracuse conjecture

Imad El ghazi*

2021

Abstract

We prove the veracity of the Syracuse conjecture by establishing that starting from an arbitrary positive integer diffrent of 1 and 4, the Syracuse process will never comeback to any positive integer reached before and then we conclude by using a probabilistic approach.

Classification : MSC: 11A25

1 Introduction

The SYRACUSE conjecture is an idea introduced by Lothar Collatz in 1937. It is also known as the $3n + 1$ problem and has been studied by many mathematicians as J.J. O'Connor, J.J.Robertson, E.F. in [1] and T.Tao in [2], since its first appearance.
We consider the following operation on an arbitrary positive integer l:

- If l is even, divide it by two.
- If the l is odd, triple it and add one.

The Collatz (or Syracuse) conjecture is: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

We can also understand this process by the following:
If l is a positive even integer (when l is a positive odd integer we get to the even case by tripling l and adding one to the result of the last multiplication) we

* **author e-mail address: elimadimad@gmail.com**

divide it by 2 until we get an odd number, this last one we triple it and we add one, or we continue dividing l by two, until we get to 1. This last case is possible just when l is of the form $l = 2^n$ with $n \in \mathbb{N}^*$. In fact when l is odd by tripling it and adding one, what we do is trying to get to an even number of the form $3l + 1 = 2^n$ ($n \in \mathbb{N}$ an even integer). Of course, half the numbers of the form 2^n can be written $3k + 1$, k been a positive odd integer, the other half is of the form $3k - 1$.

The Syracuse process can be modeled as a random variable taking its values in the set of positive integers (strictly supeior to 1) without any possibility to return to a positive integer reached before.

Using this random walk modelization of the Syracuse process and by a geometric distribution argument we prove that the Syracuse conjecture is true.

2 Main results

We first prove the following proposition that will be necessary to prove the lemma coming after.

Proposition 2.1. *For all $(m, n) \in \mathbb{N}^2$ such that $(m, n) \neq (1, 0)$, $(m, n) \neq (2, 1)$ and $2^m - 3^n > 0$, we have*

$$2^m - 3^n \neq 1.$$

Proof. For all $(m, n) \in \mathbb{N}^2$ such that $m > 2$ and $2^m - 3^n > 0$, we have,

$$2^m - 3^n > 0, \textit{implies that}, \; m > n \times \frac{ln3}{ln2}$$

then there exists $\alpha > 0$ such that

$$m = n \times \frac{ln3}{ln2} + \alpha$$

We can always find $n_0 \in \mathbb{N}^*$ and $n_0 > 1$ such that

$$\alpha > \frac{1}{ln2} \times \frac{1}{3^{n_0}}$$

and we also can find $\beta > 0$ such that

$$\alpha > \beta > \frac{1}{ln2} \times \frac{1}{3^{n_0}}.$$

Then for all $n \geq n_0$ we have

$$\alpha > \beta > \frac{1}{ln2} \times \frac{1}{3^n}.$$

We do know that the function $x - \ln(x+1)$ is strictly increasing and then strictly positive for all $x > 0$. If we choose $x = \frac{1}{3^n}$ then $\frac{1}{3^n} > \ln(1 + \frac{1}{3^n})$.

Finally we have the folowing inequalities

$$\alpha > \beta > \frac{1}{ln2} \times \frac{1}{3^n} > \frac{1}{ln2} \times \ln(1 + \frac{1}{3^n}).$$

Then

$$\frac{2^m}{3^n} = 2^m \times 2^{-n \times \frac{ln3}{ln2}} = 2^\alpha > 2^\beta,$$

and since

$$\beta > \frac{1}{ln2} \times \ln(1 + \frac{1}{3^n})$$

then

$$2^\beta > 1 + \frac{1}{3^n}$$

therfore

$$\frac{2^m}{3^n} > 1 + \frac{1}{3^n}$$

which leads to $2^m > 3^n + 1$ for all $n \geq n_0$ and $n < \frac{\ln(2^m-1)}{\ln 3}$ (because when $n > \frac{\ln(2^m-1)}{\ln 3}$ we have $2^m < 3^n + 1$).

In the other hand, the set $\{n \in \mathbb{N}^*, \mathit{such\,that}\, n < n_0\}$ is non empty because $n_0 > 1$, then for all $n < n_0$ since $3^n < 3^{n_0}$, we have $2^m > 3^n + 1$.

The case $n_0 = \frac{\ln(2^m-1)}{\ln 3}$ is impossible because we will have $2^m = 3^{n_0} + 1$ but we already know that $2^m > 3^{n_0} + 1$.

Hence for all $n \in \mathbb{N}$ either $2^m > 3^n + 1$ or $2^m < 3^n + 1$ when $n > \frac{\ln(2^m-1)}{\ln 3}$.

□

Let l be a positive integer:

a- If l is an odd integer then the next odd integer will be reached after those two operations:
  - triple l and add one.
  - divide $3l + 1$ by 2 until we have the second odd integer.

b- If l is an even integer then the next even integer will be reached after those two operations:
  - divide l by 2 until we have the first odd number.
  - triple the odd number resulting from the first operation and add one.

We will call this passage from l supposed to be odd (even)to the next odd (even) integer a step.

Lemma 2.1. *For every positive integer l strictly superior to 1 and diffrent of 4, the Syracuse process starting from l will never return to l after $i \geq 1$ steps.*

Proof. We first suppose that l is a positive odd integer.
Let $m_j,\ \ j \in \{1, ..., i\}$ be the number of divisions by 2 after the $j-$ieth step.

After i steps, we have l_i the $i-$ieth odd number reached :

$$l_i = \frac{1}{2^{m_i}}(\frac{3}{2^{m_{i-1}}}(\frac{3}{2^{m_{i-2}}}(...(\frac{3}{2^{m_2}}(\frac{3}{2^{m_1}}(3l+1)+1)+1)...)+1)+1)$$

If the process returns (after i steps) to l then we have:

$$l \times 3^i = l \times 2^{\sum_{j=1}^{i} m_j} - 2^{\sum_{j=1}^{i-1} m_j} - 3 \times 2^{\sum_{j=1}^{i-2} m_j} - ... - 3^{i-2} \times 2^{m_1} - 3^{i-1} \quad (I)$$

If $i = 1$ then since l is the first odd positive integer reached (after one step) we have :

$$3l = 2^{m_1} l - 1$$

this leads to the equation:

$$l(2^{m_1} - 3) = 1$$

The last equation has a sens if and only if $l = 1$ and $m_1 = 2$ which is absurd because l is supposed to be strictly superior to 1.

If $i = 2$, the eaquation (I) becomes $3^2 l = 2^{m_1+m_2} l - 2^{m_1} - 3$ and hence

$$(2^{m_1+m_2} - 3^2)l = 2^{m_1} + 3$$

implies that $2^{m_1+m_2} - 3^2$ divides $2^{m_1} + 3$.

In the other hand we have

$$(2^{m_1} + 3)(2^{m_2} - 3) = 2^{m_1+m_2} - 3^2 - 3 \times 2^{m_1} + 3 \times 2^{m_2} \quad (A)$$

by multiplying both sides by l we have

$$l \times (2^{m_1} + 3)(2^{m_2} - 3) = 3 \times l \times (2^{m_2} - 2^{m_1}) + l \times (2^{m_1+m_2} - 3^2)$$

hence

$$l \times (2^{m_1} + 3)(2^{m_2} - 3) = 3 \times l \times (2^{m_2} - 2^{m_1}) + 2^{m_1} + 3$$

which implies that $2^{m_1} + 3$ divides $3 \times l \times (2^{m_2} - 2^{m_1})$, since $3 \wedge (2^{m_1} + 3) = 1$ and $2^{m_1} + 3 > l$ because $2^{m_1+m_2} - 3^2 > 1$ according to proposition 2.1.
Hence $2^{m_1} + 3$ divides $2^{m_2} - 2^{m_1}$ according to equation (A) then $2^{m_1} + 3$ divides

$2^{m_1+m_2}-3^2$ which is absurd because as we know $(2^{m_1+m_2}-3^2)l = 2^{m_1}+3$ and $l>1$ and $2^{m_1+m_2}-3^2>1$.

We use the same idea for $i\geq 3$, the equation (I) becomes

$$l\times(2^{\sum_{j=1}^{i}m_j}-3^i) = 2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}.$$

In the other hand we have
$(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})(2^{m_i}-3) = 2^{\sum_{j=1}^{i}m_j}-3^i+2^{m_i}\times(3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})-3\times(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1})$ (B).

By multiplying both sides by l we have
$l\times(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})(2^{m_i}-3) = l\times(2^{\sum_{j=1}^{i}m_j}-3^i)+l\times 2^{m_i}\times(3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})-l\times 3\times(2^{\sum_{j=1}^{i-1}m_j}+l\times 3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1})$

hence
$l\times(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})(2^{m_i}-3) = 2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}+l\times 2^{m_i}\times(3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})-3\times l\times(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1})$

which implies that $2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}$ divides $2^{m_i}\times(3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1})-3\times(2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1})$ since $2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}>l$ because $2^{\sum_{j=1}^{i-1}m_j}-3^i>1$ acording to proposition 2.1.

Then according to equation (B), $2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}$ divides $2^{\sum_{j=1}^{i}m_j}-3^i$ which is absurd because $l\times(2^{\sum_{j=1}^{i}m_j}-3^i) = 2^{\sum_{j=1}^{i-1}m_j}+3\times 2^{\sum_{j=1}^{i-2}m_j}+...+3^{i-2}\times 2^{m_1}+3^{i-1}$ and $l>1$ and $2^{\sum_{j=1}^{i}m_j}-3^i>1$ according to proposition 2.1.

If l is even, let $r=\frac{l}{2^{m_1}}$ be the first odd number reached. If we suppose that the process returns to l after i steps then it will reach r again, which is absurd according to what precedes exept for $r=1$ and in this case $l=4$.

□

Remark 2.1. • *The lemma 2.1 conffrms that the only loops performed by the Srycuse process are:*

$$1\longrightarrow 4\longrightarrow 1$$

and

$$4 \longrightarrow 1 \longrightarrow 4$$

.

- *Let $l \neq 1$ be a positive odd integer, the lemma 2.1 states that starting from l the Syracuse process will never comeback to l. Let $(l_k)_{k\geq 1}$, $l_k \neq 1$ be the sequence of odd integers reached by the Syracuse poces starting from l. Each positive odd integer l_k can be considred as a starting point for the Syracuse process, then according to the lemma 2.1, the Syracuse process starting from l_k can never comeback to l_k. It follows that the Syracuse process starting from l can never comeback to any l_k, $k \geq 1$. It is then legitimate to consider the Syracuse process starting from an odd positive integer l as a drawing without replacement in the set of positive odd integers.*

Theorem 2.1. *Starting from an arbitrary positive integer the Syracuse process will always reach the value* 1*.*

Proof. According to the Lemma 2.1, starting from an integer l, the Syracuse process will never come back to l after $i \geq 1$ steps. Therefore starting from an arbitrary odd positive integer l, the Syracuse process can be assimilated to a random walk in the set of odd integers (without any possibility to comeback to any of the positive odd integers reached before), we will denote this random variable Y_l.

Remark 2.2. *When l is even then the first odd integer reached ($r = \frac{l}{2^{m_1}}$, $m \in \mathbb{N}^*$) will be the starting point of the random walk of the Syracuse process.*

Let Y_l be a random variable taking values in the set $\{s = 2k+1, k \in \mathbb{N}^*\}$, without coming back to any value reached before.

Let A be the set of positive odd integers of the forme $\frac{2^n-1}{3}$ for $n > 2$ such that n is even. Concretely :

$$A := \{\frac{2^n - 1}{3} \in \mathbb{N}/\ n \textit{ is even and } > 2.\}$$

Remark 2.3. *The arbitrary odd integer l is assumed not to belong to A.*

Consider the Bernoulli trial with two possible outcomes:

- "Failure" if $\{Y_l \in A\}$,
- "Success" if $\{Y_l \notin A\}$.

Let $0 \leq q \leq 1$ be the probability of the event "Success", then $1-q$ is the probability of the event "Failure". Since the set A is a non-empty (in fact it is an infinite) subset of the set of odd numbers, the probability $1-q$ is strictly superior to 0 and therfore $0 < q < 1$.

Consider now the random variable X_l, taking value in $\mathbb{N}^* \bigcup \{+\infty\}$ and representing the number of success of the previous Bernoulli trials, followed by the first failure. X_l has a geometric distribution $\mathbb{P}$ of parameter q, then:

$$\lim_{m \longrightarrow +\infty} \mathbb{P}(X_l = m) = \lim_{m \longleftarrow +\infty} q^{m-1}(1-q) = 0$$

So,

$$\lim_{m \longrightarrow +\infty} \mathbb{P}(X_l = m) = \mathbb{P}(X_l = \lim_{m \longrightarrow +\infty} m) = 0$$

and hence $\mathbb{P}(X_l = +\infty) = 0$. In other words $\mathbb{P}(X_l < +\infty) = 1$, i.e., the appearance of the first "failure" after a finite number of the previous mentioned Bernoulli trials , is a certain event.

This means that Y_l will necessarily reach a positive odd integer belonging to A, after a finite number of steps in the set of the odd numbers.

Once such a positive odd integer $s = \frac{2^{n_0}-1}{3}$ (for some positive even integer $n_0 > 2$) reached, the next operation in the Syracuse process is to multiply s by 3 and to add 1, then we get to the even integer 2^{n_0}, after n_0 divisions by 2, we get to the value 1.

According to what have been proved before, we deduce that starting from an arbitrary integer the Syracuse process will always reach the value 1.

□

References

[1] O'Connor, J.J.; Robertson, E.F.: "Lothar Collatz". St Andrews University School of Mathematics and Statistics, Scotland (2006).

[2] Tao,T.: Almost all Collatz orbits attain almost bounded values. Arxiv(2019)