Peacocks and the Zeta distributions

Imad El ghazi

November 2021

Abstract

We prove in this short paper that the stochastic process defined by:

$$Y_t := \frac{X_{t+1}}{\mathbb{E}\left[X_{t+1}\right]},\ t \geq a > 1,$$

is an increasing process for the convex order, where X_t a random variable taking values in $\mathbb{N}$ with probability $\mathbb{P}(X_t = n) = \frac{n^{-t}}{\zeta(t)}$ and $\zeta(t) = \sum\limits_{k=1}^{+\infty} \frac{1}{k^t},\ \ \forall t > 1.$

1 Introduction

The notion of increasing process for the convex order, (PCOC, acronym of the french expression, *Processus Croissant pour l'Ordre Convexe*) has been deeply studied in [2]. This type of stochastic processes is quiet interesting in the financial options markets.
The main example of PCOC was introduced by Carr, Ewald and Xiao in [1]. Let $(B_s,\ s \geq 0)$ be a Brownian motion started from 0 and $(N_s := \exp^{B_s - \frac{s}{2}},\ s \geq 0)$ then,

$$X_t := \frac{1}{t}\int_0^t N_s ds,\ t \geq 0$$

is a PCOC.

The other attractive property satisfited by the PCOCs is ulustrited by the Kellerer Theorem [3] establishing the relationship with the martingales theory.

2 Peacocks and 1−martingales

Definition 2.1. *A process $(X_t,\, t \geq 0)$ is a peacock if the following conditions are verified:*

i) $|X_t|$ *is integrable, i.e., for every* $t \geq 0$, $\mathbb{E}\left[|X_t|\right] < \infty$.

ii) For every convex $\mathcal{C}^2$*-function* $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$, *such that* Ψ'' *has a compact support, the function* $\mathbb{E}\left[\Psi(X_t)\right]$ *is increasing with respect to* t.

Proposition 2.1. *(Proposition 1.3 [2])*
Let $(X_t,\, t \geq 0)$ *be a real valued process satisfying the following hypotheses:*

i) the process $(X_t,\, t \geq 0)$ *is a.s. continuous on* $[0, +\infty[$ *and differentiable on* $]0, +\infty[$*, its derivative being denoted by* $\frac{\partial X_t}{\partial t}$

ii) for every $a > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,a]} |X_t|\right] < \infty$$

and for every $0 < a < b$,

$$\mathbb{E}\left[\sup_{t\in[a,b]} \left|\frac{\partial X_t}{\partial t}\right|\right] < \infty.$$

Then, the process $(X_t,\, t \geq 0)$ *is a peacock if and only if the two following properties hold:*

a) $\mathbb{E}\left[X_t\right]$ *does not depend on* $t \geq 0$,

b) for every real c *and* $t > 0$:

$$\mathbb{E}\left[1_{\{X_t \geq c\}} \frac{\partial X_t}{\partial t}\right] \geq 0.$$

Definition 2.2. *A process* $(X_t,\, t \geq 0)$ *is a* 1*-martingale if there exists a martingale* $(M_t,\, t \geq 0)$*, not necessarily defined on the same probability space, such that for every fixed* $t \geq 0$:

$$X_t \overset{law}{=} M_t$$

Theorem 2.1. *(H.G. Kellerer [3]). The following properties are equivalent:*

1) $(X_t,\, t \geq 0)$ *is a peacock.*

2) $(X_t,\, t \geq 0)$ *is a* 1*-martingale.*

3 Peacocks and the Zeta laws

Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{P}_t)$ a probability space, such that $\mathbb{P}_t$ is the Zeta probability law of parameter $t > 1$ the law on $\mathbb{N}^*$ wich assigns the mass $\frac{n^{-t}}{\zeta(t)}$ to the point n, i.e, $\mathbb{P}_t(x = n) = \frac{n^{-t}}{\zeta(t)}$ where

$$\zeta(t) = \sum_{k=1}^{+\infty} \frac{1}{k^t}$$

is the Riemann Zeta function.

Let suppose the hypothetical experience consesting of picking a number $n \in \mathbb{N}^*$ at each instant t (supposed to be strictly superior to 1), with probability $\frac{n^{-t}}{\zeta(t)}$. The resulting process will be denoted $(X_t)_{t>1}$.

Remarque 3.1. *The resultats of the experience are supposed to be independant, this implies that the resulting process $(X_t)_{t>1}$ is not a martingale.*

Theorem 3.1. *Let $(Y_t,\, t \geq a)$, $a > 1$, be the process defined by:*

$$Y_t := \frac{X_{t+1}}{\mathbb{E}\left[X_{t+1}\right]}$$

such that, $\mathbb{P}_t(X_t = n) = \frac{n^{-t}}{\zeta(t)}$, $n \in \mathbb{N}^*$ and $\zeta(t) = \sum\limits_{k=1}^{+\infty} \frac{1}{k^t}$ for every $t > 1$. Then $(Y_t,\, t \geq a)$ is a peacock.

Proof. We will prove that $(Y_t,\, t \geq a)$, $a > 1$, verifies the above Proposition.

Remark first that for every $t \geq a$ one has $\mathbb{E}\left[Y_t\right] = 1$ which means that $\mathbb{E}\left[Y_t\right]$ does not depend on t.

Recall that $t \longrightarrow n^{-t}$ and $t \longrightarrow \zeta(t)$ are $\mathcal{C}^\infty-$continous functions and that $\frac{1}{\zeta(t)}$ is well defined on $[a, +\infty[$ for every $a > 1$.

The continuity of $(Y_t,\, t \geq 1)$ follows from the Colmogorov criterion:

$$|Y_t - Y_s| = |\frac{X_{t+1}}{\mathbb{E}\left[X_{t+1}\right]} - \frac{X_{s+1}}{\mathbb{E}\left[X_{s+1}\right]}|$$

$$|Y_t - Y_s| \leq \max(X_{t+1}, X_{s+1})|\frac{1}{\mathbb{E}\left[X_{t+1}\right]} - \frac{1}{\mathbb{E}\left[X_{s+1}\right]}|$$

$$|Y_t - Y_s| \leq \max(X_{t+1}, X_{s+1})|\frac{\zeta(t+1)}{\zeta(t)} - \frac{\zeta(s+1)}{\zeta(s)}|$$

since $\frac{\zeta(t+1)}{\zeta(t)}$ is $\mathcal{C}^\infty$ then it is Lipschitien and hence there exists $K_1 > 0$ such that,

$$|\frac{\zeta(t+1)}{\zeta(t)} - \frac{\zeta(s+1)}{\zeta(s)}| \leq K_1|t-s|$$

let's choose $0 < \gamma < a-1$ then $t-\gamma > 1$ and $s-\gamma > 1$ and we have,

$$|Y_t - Y_s|^{1+\gamma} \leq \max(X_{t+1}, X_{s+1})^{1+\gamma} K_2 |t-s|^{1+\gamma}$$

and

$$\mathbb{E}\left[|Y_t - Y_s|^{1+\gamma}\right] \leq \mathbb{E}(\max(X_{t+1}, X_{s+1})^{1+\gamma}) K_2 |t-s|^{1+\gamma}$$

$$\mathbb{E}(|Y_t - Y_s|^{1+\gamma}) \leq (\frac{n^{1+\gamma} n^{-t-1}}{\zeta(t+1)} + \frac{m^{1+\gamma} m^{-s-1}}{\zeta(s+1)}) K_2 |t-s|^{1+\gamma}$$

$$\mathbb{E}(|Y_t - Y_s|^{1+\gamma}) \leq 2K_2 |t-s|^{1+\gamma}$$

because $\frac{n^{1+\gamma} n^{-t-1}}{\zeta(t+1)} < 1$.

For the defferenciability of $(Y_t,\, t \geq 1)$ we use again the Kolmogorov Criterion. To do that we define $(\frac{\partial Y_t}{\partial t},\, t \geq a)$ by,

$$\frac{\partial Y_t}{\partial t} = \lim_{\partial t \longrightarrow 0} \frac{Y_{t+\partial t} - Y_t}{\partial t}$$

$$|Y_t^{'} - Y_s^{'}| \leq \lim_{\partial t \longrightarrow 0} \frac{1}{\partial t} |Y_{t+\partial t} - Y_t - Y_{s+\partial t} + Y_s| \tag{1}$$

we denote $\phi(t) = (\frac{\zeta(t+1)}{\zeta(t)})')$ and hence

$$|Y_t^{'} - Y_s^{'}| \leq \max(X_{t+1}, X_{s+1}) |\phi(t) - \phi(s)|$$

$$|Y_t^{'} - Y_s^{'}| \leq \max(X_{t+1}, X_{s+1}) K_3 |t-s|.$$

Let's choose $0 < \gamma < a-1$ then $t-\gamma > 1$ and $s-\gamma > 1$,we have,

$$|Y_t^{'} - Y_s^{'}|^{1+\gamma} \leq \max(X_{t+1}, X_{s+1})^{1+\gamma} K_4 |t-s|^{1+\gamma}$$

and,

$$\mathbb{E}(|Y_t^{'} - Y_s^{'}|^{1+\gamma}) \leq \mathbb{E}(\max(X_{t+1}, X_{s+1})^{1+\gamma})K_4|t-s|^{1+\gamma}$$

$$\mathbb{E}(|Y_t^{'} - Y_s^{'}|^{1+\gamma}) \leq (\frac{n^{1+\gamma}n^{-t-1}}{\zeta(t+1)} + \frac{m^{1+\gamma}m^{-s-1}}{\zeta(s+1)})K_4|t-s|^{1+\gamma}$$

$$\mathbb{E}(|Y_t^{'} - Y_s^{'}|^{1+\gamma}) \leq 2K_4|t-s|^{1+\gamma}$$

because $\frac{n^{1+\gamma}n^{-t-1}}{\zeta(t+1)} < 1$.

Let $c = \sup\limits_{t\in[a,b]}(Y_t)$, then,

$$c = \frac{m}{\mathbb{E}\left(X_{t_0+1}\right)} \qquad for\ some \quad t_0 \in [a,b]$$

it comes that,

$$\mathbb{E}(\sup_{t\in[a,b]}(Y_t)) = \frac{m \times m^{-t_0-1}}{\zeta(t_0)} \leq \frac{m^{-a}}{\zeta(t_a)} < +\infty$$

because $\lim\limits_{m\longrightarrow+\infty} \mathbb{E}(X_t = m) = 0$

For $\mathbb{E}(\sup\limits_{t\in[a,b]}|\frac{\partial Y_t}{\partial t}|)$ we have,

$$\sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}\frac{\partial Y_t}{\partial t}| = \sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}(\frac{Y_{t+\partial t} - Y_t}{\partial t})|$$

$$\sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}\frac{\partial Y_t}{\partial t}| \leq \sup_{t\in[a,b]}\lim_{\partial t\longrightarrow 0}|(\frac{Y_{t+\partial t} - Y_t}{\partial t})|$$

$$\sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}\frac{\partial Y_t}{\partial t}| \leq \sup(X_{t+1})\phi(t)$$

$$\sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}\frac{\partial Y_t}{\partial t}| \leq \sup(X_{t+1})\max_{t\in[a,b]}\phi(t)$$

$$\sup_{t\in[a,b]}|\lim_{\partial t\longrightarrow 0}\frac{\partial Y_t}{\partial t}| \leq K_5 \sup_{t\in[a,b]}(X_{t+1})$$

$$\mathbb{E}(\sup_{t\in[a,b]} |\frac{\partial Y_t}{\partial t}|) \leq K_5 \frac{m^{-t_0}}{\zeta(t_{0+1})} < +\infty$$

because

$$\lim_{m\longrightarrow+\infty} \mathbb{E}(X_t = m) = 0$$

and therfore,

$$\mathbb{E}(\sup_{t\in[a,b]} | \lim_{\partial t\longrightarrow 0} \frac{\partial Y_t}{\partial t}|) \leq K_5 \mathbb{E}(\sup_{t\in[a,b]} (X_{t+1})) < +\infty.$$

Finaly since $Y_t > 0$ then for every $c > 0$ we have,

$$\mathbb{E}\left(\frac{\partial Y_t}{\partial t} 1_{Y_t \geq c}\right) = \mathbb{E}\left(\lim_{\partial t\longrightarrow 0} \frac{1}{\partial t}(Y_{t+\partial t} - Y_t) 1_{Y_t\geq c}\right)$$

$$\mathbb{E}\left(\frac{\partial Y_t}{\partial t} 1_{Y_t \geq c}\right) = \lim_{\partial t\longrightarrow 0} \frac{1}{\partial t}\left(1 - \sum_{n\geq c}^{\infty} \frac{n^{-t}}{\mathbb{E}\left(X_{t+1}\right)\zeta(t+1)}\right) \geq 0$$

for every $c \in \mathbb{R}$ and every $t \geq a > 1$.

□

Corollary 3.1. *The process* $(Y_t,\, t \geq a)$, $a > 1$ *is a* $1-$*martingale, i.e., there exists a martingale* $(M_t,\, t \geq a)$, $a > 1$, *not necessarily defined on the same probability space, such that for every fixed* t*:*

$$Y_t \overset{law}{=} M_t$$

Proof. According to the above Theorem and the Kereller Theorem.

□

References

[1] Carr, P., Ewald, C.O., Xiao, Y.: On the qualitative effect of volatility and duration on price of Asian options. Finance Research Letters 5(3), 162-171 (September 2008)

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