On the Converse of Lagrange's Theorem: An Elementary Proof

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Abstract: The smallest counterexample to the converse of Lagrange's Theorem is the alternating group A_4 , consisting of twelve elements. We offer an elementary proof that A_4 has no subgroup of order 6.

With an appropriate nod to the elegant and concise proof by Joseph Gallian [1], we offer the following.

Theorem: The alternating group A_4 has no subgroup of order 6.

proof: Suppose that H is a subgroup of A_4 and that $|H| = 6$. The identity and the three elements of order 2 in A_4 constitute a Klein 4-group. If H contains two elements of order 2, then by closure H contains this Klein 4-group. But this is a contradiction to Lagrange's Theorem. So H contains at most one element of order 2.
Now suppose that H contains two subgroups of order 3. Without loss of generality, say that H contains $\langle(1,2,3)\rangle = \{(1),(1,2,3),(1,3,2)\}$ and $\langle(1,2,4)\rangle = \{(1),(1,2,4),(1,4,2)\}$. Then, by closure, H also contains $(1,2,3)(1,2,4) = (1,3)(2,4)$ and $(1,2,4)(1,2,3) = (1,4)(2,3)$. As shown above, this is a contradiction. So H can contain at most one subgroup of order 3.
We have established that our subgroup H can contain no more than four elements of A_4. This is a contradiction. No such subgroup exists. Q.E.D.

Commentary: While not as elegant as Gallian's, this proof is arguably slightly more elementary in that it does not even mention cosets (although they are essential to the proof of Lagrange's Theorem). Notably, this proof uses Lagrange's Theorem in demonstrating the falsehood of its converse. Finally, we find it interesting that this proof reaches a contradiction by showing that H cannot contain enough elements while Gallian's proof shows that it must contain too many. Such is the nature of where one can arrive when arguing from a false premise.

1. Joseph A. Gallian (1993) On the Converse of Lagrange's Theorem, Mathematics Magazine, 66:1, 23, DOI: 10.1080/0025570X.1993.11996067

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