Leibnizian Mathematics

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SYNOPSIS

An example is given to investigate the consequences of replacing the Ontological assumption that points exist (and that solids, surfaces and lines are formed from points) with the alternative Ontological assumption that solids and areas and lines exist (and that points are the places in Space where lines end). This example is then used to motivate the introduction of the Euclidean and the Leibnizian Cosmologies with their resulting Euclidean and Leibnizian Mathematics. Using the Leibnizian Cosmology, the Rule of L'Hospital is reinterpreted to define Infinitesimal Numbers and analysis of the Riemann Integral to define Infinitesimals. The introduction of Infinitesimal Numbers necessitates the extension of the number concept for use in Leibnizian Mathematics by defining the Cauchy Numbers which consist of the Infinitesimal Numbers, the Rated Numbers, and the Infinite Numbers.

PROLOGUE
Ontology the Bedrock of Knowledge

This document is an exposition of a second fundamental structure in Metamathematics based on different fundamental assumptions about space. This structure extends the intuitive ideas of Leibniz and consolidates them with ideas that later followed from his work. But the intuition of Leibniz reached back more than two thousand years to the ontological assumptions that were made by the philosophers in ancient Greece at the very beginning of Mathematics.

Metamathematics and Metaphysics share the same fundamental assumptions about Space. Ontology is that part of Metaphysics that concerns itself with our knowledge of fundamental issues such as what we know, how we know it and how we express our knowledge. As far as Space is concerned, these questions morph into “What is Space”, “How do we experience Space” and “How do we describe Space”.

Space exists: A moment’s reflection will convince the reader that this is not an answer to the question “What is Space?” but is an answer to “What does it mean to exist?”. For any entity to exist, means that it must manifest itself in space and any abstract entity relates to such an existing entity. Space is thus the primary given of everything that we know and therefore it does not derive its properties from anything else either by deduction or by induction.

Space is continuous: We experience[1] different objects to exist in different places in space and therefore Space separates whatever exists in it; and it does so without gaps or overlapping.

Space is continuing: Events in Space can follow on each other. We call this evolution of events “time” and experience it as continuous with no jumps or pauses.

We perceive Space as extent. Because Space is the ultimate “given” of physical reality, it is not derived from anything else and therefore its properties can only be described but cannot be explained. The obvious terms to use when describing Space are therefore those used to describe the properties of God: without beginning or end, changeless, timeless, omnipresent, the source of All. This point of view requires that time is not a property of Space, but that Space and Time form a single entity and that Space/Time must also be continuous and continuing.

Seen like this, Space is perceived as not mere emptiness in which objects occur, but that everything that exists is a property of Space[2]. The basic assumptions about

[1] Human experience of Space and whatever is in it, is called here “Perceived Reality” – Reality as perceived by our senses. It is not possible to know what reality is.

[2] The view of Parmenides that “Everything is one” may cause this dependency to be reciprocal.

existence of the Cosmos are therefore the assumptions we make about Space, and these assumptions are interpretations made as best as we can from our experience.

These assumptions are Metaphysical and are at the heart of Ontology. They form the bedrock on which the foundations of all structured knowledge rest.

But the assumptions and accompanying descriptions of the properties of Space are all extractions from our experience and as such are unreliable and their accuracy have many sources of uncertainty. Two of these sources of doubt are firstly that these conclusions are based on the limited, biased, and error prone observation by our senses and, secondly, that these conclusions are formed using inductive logic and not deductive logic (See APPENDIX 1 for an identification of the terms).

Since Space is the ultimate “given”, its properties can only be described by guessing possible relationships between whatever is observed. There are no objective means to identify “true” guesses (because Space can only be compared to itself) and thus many of these guesses are not falsifiable. Different Ontological assumptions to describe Space could, and should, thus be made; all formulated as axioms. These axioms then define utilitarian models (In the sense of APPENDIX 1). This is the way that Ontology forms the bedrock on which the foundations of all descriptions of the various interpretations of our experience of reality rest – including that of Mathematics.

PROSPECTUS

Space is and Space is being. The discrete and unchanging properties of space are described exquisitely by what is called here Euclidean Mathematics. But the continuous and continuing aspects are handled somewhat awkwardly (as per Zeno's paradox of the arrow) by using the concept of limit. The continuous and continuing aspects of Space are described by what is called here Leibnizian Mathematics wherein the discrete and unchanging aspects are almost completely ignored.

The current paradigm of Mathematics is called "Abstract Mathematics" because its basic properties are not realisable in perceived reality but require an abstract space wherein it is possible to perform an infinite number of operations to completion (As is described in paragraph 4 below). This paradigm of Mathematics is extended by renaming Abstract Mathematics as Euclidean Mathematics and then adding Leibnizian Mathematics, as defined below, to the paradigm.

In Euclidean Mathematics the basic assumptions about Space are the Axiom of Euclid: "There exists a piece of Space with no extent called a point" and: "All spatial entities are formed from points". Abstract Mathematics could thus by right be called "Euclidean Mathematics". What is called the "Foundations of Mathematics" is a logical structure designed to combine the consequences of these assumptions into one coherent approach and therefore should by right be called the "Foundations of Euclidean Mathematics".

Note that Space is complete in Euclidean Mathematics because a nested set of lines of which the lengths tend to zero can have a single point (of which the length is exactly zero) as limit.

When Leibniz founded Calculus, he posited that a nested set of lines of which the lengths tend to zero will have an infinitesimal (a discrete spatial entity of which the length is not exactly zero) as a limit. A model in which this is the case is incompatible[3] with a model in which Euclidean points exist because a set of nested lines of which the lengths tend to zero cannot have two different spatial limits and the sequence of numbers formed by the lengths of these intervals cannot have a limit different from zero. This assumption can therefore only be valid once an additional system of Ontological assumptions for Space is formed in which points do not exist as spatial entities. This leads to what could rightly be called "Leibnizian Mathematics". Its basic assumptions are firstly the axiom of Parmenides "All spatial

[3] As stated here, this incompatibility is a result of the way in which Leibniz introduced the concept of infinitesimal. But the actual reason why these two models cannot be unified is the assumption in Euclidean Mathematics that all spatial entities are formed from points (See paragraphs 4 and 8 below as well as APPENDIX 3).

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entities have extent" and secondly that "Any piece of space can always be divided and the total extent of the two parts equals the extent of the original".

Because there are no points as spatial entities in Leibnizian Mathematics (to serve as limits for sequences of nested lines of which the lengths converge to zero), space is not complete. In the absence of points a nested set of lines of which the lengths tend to zero can therefore have no spatial limit so that it is open ended and is therefore merely "never ending ".

The contribution of L'Hospital, who came after Leibniz, has at its core the insight that an argument about continuity should not be a local argument, but should be about what happens in a neighbourhood of a point instead of what happens at the point itself - as Leibniz assumed. Since the work of L'Hospital is embedded in Euclidean Mathematics the consequences of this advance are well hidden, but once uncovered leads to explicit definitions for infinitesimal numbers and infinitesimals. With this change of character infinitesimal numbers can describe rates which are non-material in character. It also illuminates Zeno's paradox of the arrow.

None of the concepts and results from Euclidean Mathematics that are consequences of the assumption that spatial entities are formed from points (see paragraph 4 below) are valid in Leibnizian Mathematics, and conversely all results from Leibnizian Mathematics that depends on the absence of points are not valid in Euclidean Mathematics.

In Leibnizian Mathematics a Cauchy sequence of numbers is open ended and is never ending[4]. It belongs to an equivalence class of Cauchy sequences and as such may specify a value if it is set up to converge to that number. This follows from the analysis in APPENDIX 2. In Leibnizian Mathematics a real number is the equivalence class to which a never-ending decimal fraction belongs. This analysis also indicates that in Euclidean Mathematics a never-ending decimal fraction, as used in Leibnizian Mathematics, is not a valid concept but that it is merely an incomplete infinite decimal fraction.

In Leibnizian Mathematics the word "infinite" reverts to its original Latin meaning of "not bounded", and "infinity" is the fictional place where infinite straight lines end.

None of the concepts like "Infinity is a number larger than all real numbers", "More than countable" and others have the meaning in Leibnizian Mathematics that they have in Euclidean Mathematics. These meanings are limited to Euclidian Mathematics because they are all consequences of the existence and properties of points (See APPENDIX 2 and Paragraph 4).

[4] In Leibnizian Mathematics a "Never-ending sequence" is perceived as in the original spirit of Peano: (1) There is a first term (2) Any given term but the first has a unique preceding term (3) Any given term has a unique following term.

TO THE READER

The purpose of this document is to extend the current paradigm of Mathematics - called here Euclidean Mathematics - by addition of Leibnizian Mathematics which is based on work of L'Hospital and Riemann that followed after Leibniz introduced the concept of "Infinitesimal". Leibniz introduced this concept by replacing a point (a spatial entity of zero extend) in his argument by an infinitesimal (a spatial entity of almost zero extent[5]) as the limit of a sequence of nested intervals of which the lengths converge to zero.

The core characteristics of Leibnizian Mathematics and of Euclidean Mathematics that are relevant to this process are stated in paragraph one and in paragraph four respectively.

The development of the ideas published in is document is published in [1] which is the text of the book with title AN ALTERNATIVE MODEL FOR SPACE by the author of this document. This book is not for sale at present, but its text can be freely downloaded from viXra. No other references are given because all the necessary background should be in the repertoire of any Mathematician or Theoretical Physicist with post graduate education.

The background to this research is given in the foreword to this book. Please note that this book is a collection of documents representing the evolution of the author's thinking over a span of almost forty years and its contents were written primarily as an effort to structure and preserve thoughts and not to communicate insights. It begins in part three by trying to prove that Cantor's Diagonal Theorem is a fallacy and progresses over time to formulating complementary assumptions to those of Abstract Mathematics in part one. Further progress in understanding is reflected in this exposition – the main improvement being that the reasoning in this exposition starts at the logical heart of the argument. This progression in thinking therefore also depicts an evolution in the author's paradigm of Mathematics.

This exposition is written from the perspective that Mathematics is not subjected to the Scientific Method but that it is a technique[6] of Ontology used to uncover the logical consequences of basic abductive assumptions by using symbolic reasoning. The body of Mathematics consists therefore of all the results obtained by application of this Ontological technique to various abductive assumptions. Mathematics is therefore invented and not discovered. Because of this the Euclidean Cosmology and the Leibnizian Cosmology, as described here, are complementary and are not adversarial alternatives. Thus, Euclidean Mathematics (which is basically discreet) and Leibnizian Mathematics (which is basically continuous) complement each other.

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[5] It turns out that Leibniz' introduction of an infinitesimal as the limit of ever shortening lines causes the length of an infinitesimal to be a "fat zero" since, according to paragraph eight, only countable many of them are required to form a line of unit length. For points, each of length exactly zero, more than countable many are required for this.

[6] This is a more sophisticated way of stating the well-known definition: "Mathematics is that what Mathematicians do."

LEIBNIZIAN COSMOLOGY and LEIBNIZIAN MATHEMATICS

1. Introduction

"A straight line is the shortest distance between two points."

This is the popular definition of a straight line in Mathematics. However, the deeper importance of this definition lies in the assertion that *straight lines derive their properties from the properties of points*. Hence straight lines can only be introduced after the existence of points as fundamental entities for a model of space has been assumed.

Mann muss immer umkehren[7].

A straight line, as a special kind of line, can however be defined without requiring the existence of points or using the concept of distance [how this is done is described later].

Assume therefore, alternatively, that points as spatial entities are not chosen as fundamental entities but assume instead that lines are chosen to exist[8] as the fundamental entities for a model of space. A point then becomes merely the place in space where a line - the spatial entity - ends. In a model built on this assumption *points derive their properties from the properties of lines*.

Note that space is complete when points are introduced as spatial entities: a line of which the length goes down to zero will have a point as limit. If points are not introduced as spatial entities, then space is not complete and a line of which the length goes down to zero will have no limit but will become shorter and shorter and be merely a never-ending sequence of ever shortening lines.

It turns out that a model for Mathematics based on the assumption that points exist as fundamental entities is essentially discrete while the model for Mathematics based on the assumption that solids and areas and lines exist as fundamental entities is essentially continuous.

2. Basic Construction

All lines in this construction are straight. Let AB be a given line and let CD be a line of unit length defining the scale. According to Euclidean Geometry any straight line parallel to one side of a triangle divides the other two sides in the same ratio. Draw a line perpendicular to CD passing through C. On this line mark ten sublines CE_1,

[7] "One must always invert": C G J Jacobi 1804 to 1851

[8] This will later be properly identified as a basic assumption of the Leibnizian Cosmology and as such this introductory example illustrates Leibnizian Mathematics. The validity of the arguments and references of this example is therefore restricted to Leibnizian Mathematics.

E_1E_2, … ,E_9E_{10} all of the same length, and then draw the line $E_{10}D$. Lines through E_1, E_2, .. ,E_9 drawn parallel to the line $E_{10}D$ then divides the line CD into ten parts, the length of each equal to one tenth of the unit.

Starting at A measure lines of length CD along AB until a piece shorter than CD is left over. Let the number of full lengths be n_0. Repeat this process on the piece that is left over, but now using a line of length one tenth of CD as constructed above. Let the number of full such lengths now be n_1. The piece left over now is shorter than one tenth of CD, so that the decimal number $n_0.n_1$ is within one tenth of the length of AB.

Repeat this process. In this way the decimal number $n_0.n_1n_2 \ldots n_m$ is found after m steps, and this number is within 10^{-m} of the length of AB.

In this way the terms of the Cauchy sequence (n_0; $n_0.n_1$; $n_0.n_1n_2$; ……) are generated one after the other. As a simpler notation this Cauchy sequence can be written as the never-ending decimal fraction $n_0.n_1n_2$ …. with the understanding that the n^{th} term of the Cauchy sequence results when this decimal is truncated after the n^{th} digit.

This Cauchy sequence belongs to an equivalence class of Cauchy sequences. This class is a real number, and traditionally it is also indicated by the notation $n_0.n_1n_2$ … but, in this case, it is referred to as an infinite decimal fraction (See APPENDIX 2 for a discussion of this distinction).

Hence a **never-ending** decimal fraction is constructed by this process and this never-ending decimal is but another way of writing a Cauchy sequence. [To emphasise: the words “never-ending decimal fraction” shall refer to the Cauchy sequence and the phrase “infinite decimal fraction” shall refer here to the equivalence class to which the Cauchy sequence belongs - even though the word “infinite” does not have a precise meaning with the current assumptions.]

By the above construction, every given line therefore has an associated never-ending decimal fraction to describe its length, and two lines of different lengths cannot be described by the same never-ending decimal fraction. Thus, there is a one-to-one mapping of straight lines onto a subset of the set of never-ending decimal fractions and therefore the cardinality of the set of all lines is less than the cardinality of the set of never-ending decimal fractions.

However, according to Theorem 2 of APPENDIX 2, the set of never-ending decimal fractions is countable so that there are countable many straight lines falling on an axis.

the value of a real number as marked on an axis is represented by the length of the line from the origin to where the number is marked. The cardinality of the real numbers on the axis is therefore equal to the cardinality of the set of such lines along the axis. Hence there can only be countable many real numbers in this model.

This can also be shown by using a non-geometrical argument; but it should be stressed that this argument is only valid when points are not defined as spatial entities and therefore it illustrates a difference between Euclidean- and Leibnizian Mathematics:

The limit of every never-ending decimal fraction is the equivalence class to which it belongs and that is a real number. But, as shown in Theorem 1 of APPENDIX 2, when points are not defined as spatial entities every such equivalence class contains at least one never-ending decimal fraction unique to that class, so that the cardinality of the set of equivalence classes is less than the cardinality of the set of never-ending decimal fractions (Due to the non-linearity of this exposition, this argument - which would have been wrong if points in the Euclidean sense were assumed to exist - is discussed in more detail elsewhere). Thus, there can only be countable many real numbers in the number system associated with the model for Space derived here by using as fundamental assumption that straight lines (and not points) exist as fundamental spatial entities.

Note that a discrepancy like this is quite common in Mathematics when different but equally valid assumptions lead to different but equally valid conclusions - for instance, the difference in the sum of the internal angles of a triangle between Euclidean and Riemannian geometry.

3. Space

Space Is and Space is Being - without beginning and without end.

Ontologically, space is **the** fundamental "given" and thus its properties can only be named but cannot be explained.

We experience space as extent.

It is useful to consider an example - say visualising an apple - to formulate how the properties of space are perceived[9] from the perspective of this exposition:

- A piece of space, like the space occupied by the apple, can be isolated and has a non-zero volume.
- A solid extends in three directions.
- The skin of the apple is the interface between the inside of the apple and the rest of space.
- A solid can always be cut in two and the sum of the volumes of the two pieces equals the volume of the original solid.
- The interface between two abutting solids is a surface.
- A surface extends in two directions.
- A surface has an area.
- When the apple is cut again, a second surface is generated and the interface between these two surfaces is a line.
- A line extends in one direction.
- A line has a length.

[9] As remarked before, perceived space is how we as humans experience Space via our senses. It is not possible to know reality.

- If a severed part of the apple is cut again a third surface is formed as well as two new lines.
- The interface between any two of these lines is a point.
- A point extends in no direction and is a place in space.
- A piece of space, like the apple, can be rotated.
- A piece of space, like the apple, can be translated.

This list is a short abstraction from experience and as such forms part of the “Model”, called here perceived reality, that we develop in our minds from experience to describe Space.

In addition to these observations, the logical basis of this exposition is stated in more detail in APPENDIX 1.

As mentioned in the Prologue, even though the items in the list above form part of the perceived reality of the world that we live in and is shared by all human beings, there are mainly two reasons why we dare not accept any one of the above conclusions about space as “true”.

The first cause is best looked at from the perspective of Rene Descartes. He is universally known for stating “Cogito ergo sum” – “I think therefore I am”. He then proceeded to reject the reality of what he can see while awake on the grounds that he can see the same images in his dreams.

But we currently live more than three centuries after Descartes, and we now realise that we do not live in our bodies (as was tacitly assumed by Descartes) but that we are our bodies. His dictum above has now changed to “Sum ergo cogito” because it turned out that we can only think what our brains are able to think and experience only what our senses can detect – and the selection of these abilities happened through evolution over millions of years and was based on the criterium “need to survive” and not on the requirement “need to know and understand”. Amongst many other factors, our perception of space and our insight into whatever happens there are therefore limited to the scale at which we experience distance, speed, time, and mass.

The second cause is the limitation imposed by logical induction: Our senses present our minds only with circumstantial evidence about Reality. But no unconditionally true conclusion about the cause of an event can be drawn from such circumstantial evidence using logical induction.

This is true in general for Ontology – even though the Scientific Method partially compensates for it. How the scientific method is interpreted here is described in more detail in APPENDIX 1, where it is shown that this method turns vague inductive arguments into systems of deductive arguments by clearly stating the relevant assumptions and then improving on these assumptions as more experience is gained.

Here such systems of deductive arguments are called Models and, since they are about the nature of Space, they are here referred to as Cosmologies.

4. The Euclidean Cosmology

Euclid defined a point as “That which has no extent”.

The current model for space is formed by using the above definition as an axiom and thus it became the basic assumption of this model:

Axiom

There exists a piece of space with no extent called a point.

Note that the **definition** above is a description of a point so that it can be recognised when found, while the **axiom** states the assumption that a point is a spatial entity, and that its existence is independent of whether it can be found or not.

This axiom is used to form a model by using the following Ontological assumptions to describe our concept of Space:

- A solid is formed by lumping points together.
- A surface is a single layer of points.
- A line is a string of points.

This model of Space can aptly be called the **Euclidean Cosmology** because it is formulated from Euclidian Geometry, and Mathematics based on this Cosmology can be called **Euclidean Mathematics**.

Having no extent (in contrast to extending in no direction) implies that if A is a point, then d(A) = 0 where d is the maximum diameter of a solid or an area, or the length of a line.

Let Z be an index set and let

$$\{A_\alpha | \alpha \epsilon Z\}$$

be the set of points that form a line of unit length when strung together.

Let

$$D = \sum_{\propto \epsilon Z} d(A_\alpha)$$

If Z is a finite set, then D is a finite sum of zeros and is therefore zero.

If Z is a countable set, then D is the limit of the partial sums which are all zero. Hence D is zero.

However, for the points to form the unit interval requires that D=1. The cardinality of the set of points forming a line of non-zero length must therefore of necessity be **more than countable.**

This argument has two implications:

- **There exist more than countable many points in this model**.
  Thus, if the real line is introduced by associating a real number with every point on an axis, then there exist more than countable many real numbers in the Euclidean cosmology. This was shown algebraically by Cantor in his diagonal proof.
- **Infinitely many operations can be performed to completion.**
  The operation above is a sum and **not** a limit[10] (This is discussed in more detail in APPENDIX 2). Infinitely many additions must therefore be made to reach the final value of exactly one for the length of the line. Hence this is an implicit introduction of the essence of the Axiom of Choice into Abstract Mathematics and in so doing it validates the logical structure of Cantor's diagonal argument:

Cantor, in his diagonal proof, assumes that a list of all real numbers between zero and one can be made and he then shows that there exists a real number that cannot be in the list. He concludes from this that there are more than countable many real numbers. However, another possibility is that real numbers cannot be listed at all – meaning that a list of a single real number cannot be made. The second point above shows that the infinitely many digits forming a real number can all be determined in the Euclidean Cosmology. It can consequently be listed – albeit not in perceived reality. It therefore eliminates the alternative conclusion from Cantor's proof by introducing an alternative reality, called 'Abstract', where a symbol consisting of infinitely many digits is possible. This alternative reality is the reality where Euclidean Mathematics is valid and where a "never-ending" decimal fraction has merely the status of a truncated infinite decimal fraction (because it lacks part of the information) and therefore cannot be used in an argument.

These are the main fundamental properties of Abstract (Euclidean) Mathematics that are pertinent to this exposition.

Euclidean Mathematics consists of all conclusions that can be logically drawn using the basic assumptions of Euclidean Cosmology. Since this is a Model forming an alternative reality, such conclusions do not accurately describe perceived reality, but form a structure that can be used as a commentary to be compared to perceived reality.

The Euclidean Cosmology is essentially discrete. In this cosmology the "limit" concept creates a link between the continuous and the discrete. This is because, in Euclidean Cosmology, a point is a spatial entity and is therefore available to be the spatial limit of a set of nested intervals of which the lengths tend to zero:

[10] The limit is investigated in paragraph 7.2 and the result discussed in paragraph 8

Let (D;E) be the notation for a straight line from point D to point E. Let A be the midpoint of all the nested straight lines $\{(B_n;C_n);\ n=1;\ 2;\ 3\ldots\}$ such that $d(B_n;A) = d(A;C_n) = n^{-1}$. Then this is a set of nested intervals focussed on the point A. Completeness of the line then implies that

$$\lim_{n\to\infty}(B_n; C_n) = A$$

where A is a single point and d(A)=0.

Note that the number zero in this last equation is a real number and therefore it is the equivalence class of all Cauchy sequences that converge to zero.

5. The Leibnizian Cosmology

The philosopher Parmenides never proposed a Model for space of his own, but his work points to the unsatisfactory way in which points in the Euclidean Cosmology describe continuous events in perceived reality. His ideas are best expounded by commenting on the paradox of Achilles and the tortoise as posed by his pupil Zeno. (The paradox of the arrow is relevant to the continuity of Space/time.)

In this paradox, both Achilles and the tortoise are modelled as points. (In the time of Zeno, the only way to specify a point was to use a geometric construction.) Now, more than two millennia later, in the Euclidean Cosmology a point can be specified as an infinite decimal fraction. The words "Achilles can never pass the tortoise" can now be replaced by "It cannot be known where Achilles will pass the tortoise". In the Euclidean Cosmology, where infinitely many operations can be performed, the classic refutation of Zeno's paradox proves both the existence and the identification of the point - albeit in an abstract space.

But if the scale of the paradox is changed by replacing Achilles with a high-energy proton and the tortoise by a lower-energy proton, then the Uncertainty Principle of Heisenberg implies that neither the positions nor the speeds of the particles can be known with arbitrary precision ***in principle***. The distance between them is therefore never known. But the faster particle must move past the slower so that, even though the distance between them is indeterminate, the distance between them should first tend to something that must in some way be equivalent to zero and then increase again. This implies that the Euclidean Cosmology is a very poor model for modelling motion at that scale.

In contrast, the idea of Parmenides that nothing can come into being where it did not exist before implies that the particles cannot be modelled as points but should be modelled as being continuous and extending through space[11]. They therefore could have been next to each other in some sense throughout the whole process. This is reason enough to honour Parmenides by:

[11] Modelling them as waves does just that. The Parmenides axiom given here is a non-quantum alternative.

The Parmenides Axiom: All spatial entities have non-zero extent.

The **Leibnizian Cosmology** can then be formulated as:

- Solids and surfaces and lines exist and are spatial entities.
- Any spatial entity can be divided and the total extent of the two parts equals the extent of the original entity.

As before, Mathematics based on this Cosmology can be called **Leibnizian Mathematics**.

6. Defining a straight line without using points

In the Leibnizian Cosmology lines exist and lines can be translated and rotated.

A straight line can be defined as:

Definition: A **straight line** is a line such that any part of the line can be translated without rotation to cover any other part of the line of the same length.

Definition: The **distance** between two places in space is the length of a straight line such that its end points coincide with these places.

A straight line describes **Direction** in space.

The difference in direction of two straight lines describes **Rotation**.

With this definition of a straight line the construction in the introduction is a valid never-ending geometrical process **in the Leibnizian Cosmology**.

7. Leibniz

It turns out that what is now called "Non-standard Analysis" in Euclidean Mathematics, is the "Standard Analysis" in Leibnizian Mathematics.

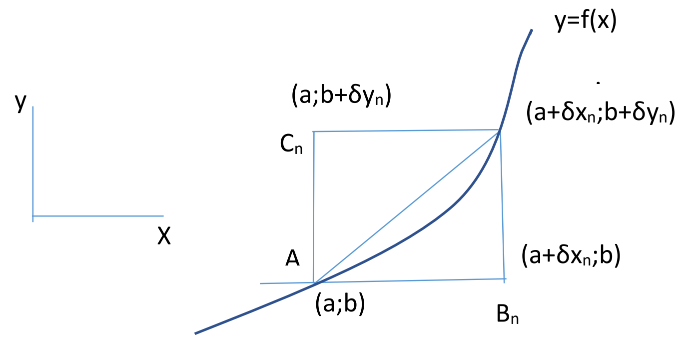
7.1 Infinitesimal numbers

The advent of Newton's Laws made it necessary to study continuous processes in mathematical detail with renewed intensity. Newton invented fluxions and Leibniz introduced the concept of infinitesimals.

Consider the graph of the function y = f(x) and let A be a point on this graph.

Let { δx_n ; n=1,2,3,..} be a Cauchy sequence such that $\delta x_n \to 0$ and $\lim_{n\to\infty} \delta x_n = 0$.

In accordance with the distinction introduced in paragraph 2, the first of these expressions states that this Cauchy sequence is equivalent to the null sequence (0; 0; 0; …..) while the second states that the limit of this sequence is the real number zero; i.e. the limit is the equivalence class of all Cauchy sequences that are equivalent to the null sequence.



Let $\delta y_n = f(a+\delta x_n)-f(a)$. Assume that the function y=f(x) is smooth enough so that the sequence { δy_n ; n=1,2,3,..} is a Cauchy sequence such that $\delta y_n \to 0$ and $\lim_{n\to\infty} \delta\, y_n = 0$

Referring to the figure, in the Euclidean Cosmology the limit of the set of nested intervals { AB_n ; n=1, 2, 3, …} is the single point A and hence d(A) = 0.

The gradient of the tangent to the graph of y=f(x) at x=a is then

$$\lim_{n\to\infty} \frac{\delta y_n}{\delta x_n} = \lim_{n\to\infty} \frac{d(AC_n)}{d(AB_n)}$$

This is of the form ${}^{0}/_{0}$ which is indefinite (i.e. undefined) in the Euclidean Cosmology because division by the real number zero is not allowed.

This created a problem for Leibniz in the sense that he needed a number equivalent to zero for which this division would make sense.

But Dedekind[12] studied the Real numbers only a century and a half later and, even then, there were still no real numbers equivalent to zero. Leibniz then solved the problem of not being able to divide by zero by boldly inventing a new type of number as well as a new type of spatial entity. The Spatial entity he called an “Infinitesimal” which is pidgin Latin for “The little thing at infinity”. The numeric entity is called an “Infinitesimal number”, but unluckily is also referred to simply as an “infinitesimal” in common language. Leibniz assumed that this infinitesimal number was equivalent to zero in the sense that it can be a limit for the lengths of the intervals, but that division by such a number was defined.

[12] Richard Dedekind (1831-1916)

For infinitesimal numbers he introduced the notation of adding a “d” before a variable name. Thus:

$$\delta x_n \to dx \text{ and } \delta y_n \to dy$$

Note that dx and dy are not real numbers and hence cannot be limits for the two Cauchy sequences.

This extension to Mathematics by Leibniz is not acceptable in Euclidean Mathematics.

When Leibniz assumed that a sequence of numbers that converges to zero has an infinitesimal number of non-(absolutely)zero magnitude as limit, this assumption implied that the associated sequence of nested intervals must have a spatial entity of which the length is a non-(absolutely)zero number as limit, and hence cannot be a point. Thus, in the Euclidean Cosmology, infinitesimal numbers cannot be defined by this argument since their introduction means that the sequences of numbers and of intervals each must have two different limits. This is logically not acceptable and is a contradiction of the basic assumptions of Euclidean Mathematics.

Hence an Analysis based on infinitesimal numbers in the Euclidean Cosmology is called “Non-Standard Analysis” and cannot have a valid numerical or geometrical interpretation.

But, in the alternative model for space based on the existence of lines (in which points do not exist as spatial entities) the set of lines of which the lengths tend to zero has no limit. Associating an infinitesimal as a “limit” for such a sequence is therefore valid in this model and the first of the above objections disappears.

However, the objection to the requirement that the set of numbers converging to zero must have two limits remains and is only resolved by the argument of L’Hospital. His approach changes the argument from a local argument at the point where the indefinite form occurs into an argument that involves a neighbourhood of the point. Since this argument does not involve the limit of the Cauchy sequence of numbers but the sequence of numbers itself, it allows the number-concept in Mathematics to be extended by defining the Cauchy numbers, and thereby to resolve the second objection as well.

It thus validates the approach of Leibniz for creating this alternative to the Euclidean Cosmology.

Note that when L’Hospital realised that the value assigned to an indefinite form is not to be determined by what happened at the point A where it occurred but by what happened in a neighbourhood of that point, he introduced the need for continuity into the argument. He thus opened the door for the development of Topology to describe continuity in Euclidean Mathematics.

To extend the work of Leibniz by using the insight of L’Hospital, one should bear in mind that L’Hospital formulated his rule in Euclidean Mathematics and therefore used the limit-concept. Since this concept does not occur in Leibnizian Mathematics the Rule of L'Hospital needs to be first transcribed into an alternative form that does not use the limit concept:

Let f(x) and g(x) be two sufficiently smooth functions of x and let them be such that f(a)=0 and g(a)=0. The function $h(x) = {f(x)}/{g(x)}$ is then of the indefinite form ${0}/{0}$ at x=a. The rule devised by L'Hospital assigns a value to h(a) by requiring that h should be continuous at x=a. He then stated this rule as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Let $\{\delta_n ; n=1; 2; \ldots\}$ be a Cauchy sequence such that $\delta_n \to 0$. If $a_n = a + \delta_n$ then $a_n \to a$ and therefore $f(a_n) \to f(a)=0$ and $g(a_n) \to g(a)=0$.

Thus both $\{f(a_n)\}$ and $\{g(a_n)\}$ are Cauchy sequences equivalent to zero.

The above rule of L'Hospital then states that the Cauchy sequence

$$\left\{\frac{f(a_n)}{g(a_n)} ; n = 1; 2; \ldots\right\}$$

belongs to the equivalence class (real number) $\frac{f\prime(a)}{g\prime(a)}$.

Transcription of the derivatives above shows that the Cauchy sequence

$$\left\{\frac{f(a_n)/\delta_n}{g(a_n)/\delta_n} ; n = 1; 2; \ldots.\right\}$$

also belongs to this equivalence class.

For use in the Leibnizian Mathematics the rule can now be restated as:

The Rule of L'Hospital:

The two Cauchy sequences

$$\left\{\frac{f(a_n)}{g(a_n)} ; n = 1; 2; \ldots\right\}$$

and

$$\left\{\frac{f(a_n)/\delta_n}{g(a_n)/\delta_n} ; n = 1; 2; \ldots.\right\}$$

both belong to the same equivalence class and this equivalence class is the real number assigned as value to the indefinite form.

DefinitionThe symbol “=” will mean “belong to the same equivalence class[13] ”.

Then, using curly brackets to denote sequences, the rule of L’Hospital can be written as

$$\left\{\frac{f(a_n)}{g(a_n)}\right\} = \left\{\frac{f(a_n)/\delta_n}{g(a_n)/\delta_n}\right\} \qquad [A]$$

This enables the extension of the number concept by adding Cauchy Numbers:

Definition

A **Cauchy Number**, a, is defined as the Cauchy sequence of rational numbers:

$$a = (a_1 ; a_2 ; a_3 ; \ldots.)$$

with the following algebraic operations:

Addition: $a+b = \{a_n\} + \{b_n\} = \{a_n + b_n\}$

Subtraction: $a - b = \{a_n\} - \{b_n\} = \{a_n - b_n\}$

Multiplication: $axb = \{a_n\}x\{b_n\} = \{a_nxb_n\}$

Division: $\frac{a}{b} = \frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$ provided none of the numbers $\{b_n\}$ is zero.

note that these are the rules for the four basic operations on truncated decimal fractions.

The rational numbers a_n forming a Cauchy number will be called its **components**.

Note that the Cauchy numbers are the sequences of rational numbers that form the equivalence classes that are the real numbers, and as such form a refinement of the number concept.

This formulation of the rule of L’Hospital can loosely be interpreted as a comparison of the rates of convergence of the two sequences $\{f(a_n)\}$ and $\{g(a_n)\}$ to assign a value to the quotient $h(x) = {}^{f(x)}\!/_{g(x)}$ so as to make it continuous at the place where it becomes indefinite.

The intuitive introduction of the concept of infinitesimal numbers can now be formalised:

[13] The same meaning as for Rational Numbers.

Definition

An **infinitesimal number** is a Cauchy number that converges to zero.

Equivalently one could say:

"An infinitesimal number belongs to the real number zero" or "is equivalent to the null sequence {0; 0; 0; ….}"

In the spirit of Leibniz, when a Cauchy number "a" is an infinitesimal number, the notation "da" will be used to indicate this special property.

The Cauchy Numbers need to be extended so that they are closed under division by infinitesimal numbers:

Definition

An **Infinite Number** is a sequence of rational numbers that diverge to infinity.

The Cauchy Numbers can now be divided into three classes[14]:

- The Infinitesimal numbers that are sequences converging to zero.
- The Rated Numbers that are sequences belonging to the other equivalence classes of sequences.
- The Infinite Numbers that are unbounded sequences.

Returning to L'Hospital:

Applying the notation introduced above for Cauchy numbers, one gets the infinitesimal numbers:

$dx = \{\delta x_n\}$; $dy = \{\delta y_n\}$; $df = \{f(a_n)\}$; $dg = \{g(a_n)\}$

The expression [A] can then be rewritten as

$$\frac{df}{dg} = \frac{df}{dx} \Big/ \frac{dg}{dx} = \frac{df}{dx} \cdot \frac{dx}{dg}$$

Hence the transcribed rule of L'Hospital reduces to the Chain Rule for differentiation in Leibnizian Mathematics.

Note that equality of Cauchy Numbers merely implies that they belong to the same equivalence class and thus does not imply term-wise equality. However, term-wise equality does imply equality of Cauchy Numbers.

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[14] See also APPENDIX 4

7.2 Infinitesimals

The Riemann integral addresses the indefinite form $\infty\cdot 0$ and a reinterpretation of this theory leads to a suitable definition of an infinitesimal (which is a spatial entity).

Consider the area of a square of which the sides are of unit length. Its area is equal to one and thus:

$$1 = \int_0^1 1 \cdot dx$$

Starting with the whole interval as partition zero, form successive partitions by dividing each interval of the previous partition into three equal parts. In this way the n^{th} partition will consist of 3^n intervals, each of length 3^{-n}. If $x = a_i^n$ is at the centre of the i^{th} interval of the n^{th} partition, then

$$a_i^n = \frac{2i-1}{2} 3^{-n} \quad \text{For } i = 1, 2, 3, \ldots, 3^n \text{ and } n = 0, 1, 2, \ldots. \qquad [A]$$

But the length of the whole interval is the sum of the lengths of the parts so that:

$$1 = \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad \text{for } n=0, 1, 2, \ldots.$$

Since this sum is the same for all values of n

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad [B]$$

Thus, the right-hand side is in some vague way a kind of multiple (that goes to infinity) of interval lengths (that all go to zero), and it is therefore some kind of indefinite form $\infty \cdot 0$

But these partitions have two specific properties that can be shown rigorously to be true, but can be easily seen by drawing three lines of unit length below each other and marking the partitions on them:

1) When a point is in the middle of one part of a partition, it will be in the middle of a part for all subsequent partitions. A set of nested intervals can therefore be formed by selecting from consecutive partitions all intervals having the same midpoint. The lengths of these intervals converge to zero while all these selected intervals remain symmetric about their common midpoint. In the Euclidean Cosmology the set of intervals will have this point as limit while in the Leibnizian Cosmology the set is focussed on that point.

   Definition:
   A nested set of intervals of which the lengths converge to zero is called an **infinitesimal**.

   This definition extends directly to areas and solids.

2) The set of all infinitesimals as defined above is a directed set where the pre-order is defined by: A>B is true when the first interval of the infinitesimal B is contained in an interval that forms part of A. The infinitesimal that has the whole unit interval as first partition is the first element in this directed set. An integral can then be defined as a net on the directed set of infinitesimals.

8. Aspects of Leibnizian Mathematics

It is not possible to unify Euclidean Mathematics and Leibnizian Mathematics into a single system. The fundamental reason for this is that Euclidean Mathematics is a model built to act in an alternative reality where the real number “infinite” exists and where an infinite number of operations can be performed to completion. Leibnizian Mathematics is a model built to act in a reality that more closely resemble perceived reality and where only never-ending sequences of operations can be performed[15].

Euclidean Mathematics is suitable to study situations that are fundamentally discrete, and Leibnizian Mathematics is suitable to study situations that are fundamentally continuous. This means, for example, that Leibnizian Mathematics is unsuitable to study Lebesgue integrals because the concept of open and closed intervals is not part of Leibnizian Mathematics. Euclidean Mathematics is suitable to study Lebesgue integrals but is unsuitable to study Riemann integrals which are based on infinitesimals that are formed from continuous lines. This last remark can be illuminated by the following (non-rigorous) argument:

One can accept that the infinitesimals forming the sums [B] in paragraph 7 above should in Euclidean Mathematics converge[16] to their midpoints and that one can assume that the convergence of the sums to the limit is sufficiently uniform so that the limit of the sums can be inverted to be the sum of the limits. But only points of the form [A] can be limits for the infinitesimals. (This is because, for any rational number that is not of the form [A] and any given infinitesimal, a number ε>0 can be found so that the number will differ by more than ε from the centre of the infinitesimal.) In this case the limit would be the sum of the lengths of all the possible points a_i^n as in [A]. These points are all rational and hence form a countable set. In this specific case the lengths of countable many points (all zeros) will therefore sum to one, contradicting paragraph four.

But sequences of rational numbers and nested sets of lines as mathematical entities form part of both models. Use of infinitesimal numbers lends a fine structure to Euclidean Mathematics, but the ominous presence of infinity restricts their use to

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[15] In Euclidean Mathematics the noun “Infinity” refers to a real number larger than all other numbers. In Leibnizian Mathematics the noun “infinity” refers to the fictional place where infinite Cauchy numbers are focussed. (See APPENDIX 4)

[16] Alternatively: a net can be defined on the directed set of infinitesimals by associating the point at which it is focussed with every infinitesimal. This sets up a function from the directed set into the points that form the unit interval. All these points represent rational numbers and as such form a countable set.

indefinite forms. A large part of Numerical Mathematics is devoted to the study of rated numbers.

Leibnizian Mathematics is a model that grew out of Calculus. This makes this model suitable for arguments in Calculus. It simplifies situations – like in the above work of L'Hospital – where it is much simpler to embed an argument in Leibnizian Mathematics than in Euclidean Mathematics.

Leibnizian Mathematics simplify arguments by avoiding the limit-concept. This is illustrated by the following non-rigorous argument to illuminate this line of thought:

8.1 The Fundamental Theorem of Calculus

Since this is merely a discussion to illustrate the principle, the argument is presented for the simple case where the limits of the interval of integration are zero and one and the directed set of infinitesimals is the one used in paragraph 7 above. Set $\delta_n = 3^{-n}$ for ease of notation.

Let F(x) be a suitably smooth function of x. Then, for any given value of n:

$F(1) – F(0) = F(3^{-n}) – F(0) + F(2x3^{-n}) – F(3^{-n}) + \ldots + F(1) – F([3^n – 1]x3^{-n})$

$$= \sum_{i=0}^{n-1}\left(F\left(a_i^n + \tfrac{1}{2}\delta_n\right) - F\left(a_i^n - \tfrac{1}{2}\delta_n\right)\right)$$

$$= \sum_{i=0}^{n-1}\left(F\left(a_i^n + \tfrac{1}{2}\delta_n\right) - F\left(a_i^n - \tfrac{1}{2}\delta_n\right)\right)\frac{\delta_n}{\delta_n}$$

$$= \sum_{i=0}^{n-1}\left(\frac{F\left(a_i^n+\frac{1}{2}\delta_n\right) - F\left(a_i^n-\frac{1}{2}\delta_n\right)}{\delta_n}\,\delta_n\right)$$

As was noted in paragraph 7, once a number a_i^n appears in the centre of a part of a partition, that number, albeit with different indexes, will be in the centre of all intervals forming the infinitesimal of which that part is one of the nested lines. This implies that every infinitesimal is fully identified by the common centre of all the lines forming that infinitesimal. It is also the place on the X-axis where the infinitesimal is focussed.

Let A be the set of all numbers of the form a_i^n (i = 1, 2, 3, …, 3^n and n = 0, 1, 2, ….) and let A_n be that subset of A formed by all numbers of A that is at the centre of any part of length 3^{-n}.

Therefore, the above expression can be rewritten as

$$F(1) - F(0) = \sum_{\propto\in A_n}\frac{F\left(\propto + \frac{1}{2}\delta_n\right) - F\left(\propto - \frac{1}{2}\delta_n\right)}{\delta_n}\,\delta_n$$

The right-hand side of this equation is a rational number for every value of n and is therefore a Cauchy number associated with the directed set of partitions. Hence the partitions form a net that maps onto a Cauchy number that belongs to some equivalence class of sequences.

But, written as Cauchy numbers:

$$\frac{dF}{dx}(\propto) = \left\{\frac{F\left(\propto + \frac{1}{2}\delta_n\right) \; - \; F\left(\propto - \frac{1}{2}\delta_n\right)}{\delta_n} : n = 1; 2; 3; \ldots.\right\}$$

and

$$dx = \{\delta_n : n = 1; 2; 3; \ldots.\}$$

so that each term in the sum above is the nth component of the product of Cauchy numbers

$$\frac{dF}{dx}(\propto) \cdot dx$$

If f(x) is the derivative of F(x) then

$$f(\propto) = \left\{\frac{F\left(\propto + \frac{1}{2}\delta_n\right) \; - \; F\left(\propto - \frac{1}{2}\delta_n\right)}{\delta_n} : n = 1; 2; 3; \ldots.\right\}$$

Hence every term in the sum is the nth component of the product $f(\propto) \cdot dx$

Therefore, because the symbol "=" denotes "belong to the same equivalence class"

$$F(1) - F(0) = \sum_{\propto \in A_n} f(\propto_n) \cdot \delta_n$$

The equivalence class to which the right-hand side of this equation belongs is denoted by the symbol

$$\int_0^1 f(x)dx$$

8.2 The Dirac δ-Function

Use of infinite Cauchy numbers in Leibnizian Mathematics can be illustrated by looking at the derivative of the Heaviside function H(x) at the value x=a when it is given that H(x) = 0 if x < a and H(x) = 1 if x > a. (A value for H(x) at x = a cannot be prescribed because of the absence of the concept of "point" as a spatial entity.)

Let dx = {δ_1; δ_2; δ_3; …..} be an infinitesimal number. Then the following set of nested intervals focussed at a is an infinitesimal:

$$\left\{\left(a - \frac{\delta_n}{2}; a + \frac{\delta_n}{2}\right); n = 1, 2, 3, \ldots\right\}$$

The derivative of H(x) at x = a is the Cauchy number

$$\frac{dH}{dx}(a) = \left\{ \frac{H\left(a + \frac{\delta_n}{2}\right) - H\left(a - \frac{\delta_n}{2}\right)}{\delta_n} \right\}$$

so that

$$\frac{dH}{dx}(a) = \frac{1}{dx}$$

and

$$\frac{dH}{dx}(x) = 0$$

for all other values of x.

In Leibnizian Mathematics Dirac's δ-distribution therefore becomes an ordinary piecewise function involving an infinite Cauchy number.

9. Concluding remarks

Remark 1

In Euclidean Mathematics a set of nested intervals of which the lengths converge to zero has a point as limit, while the set of intervals itself forms an infinitesimal in Leibnizian Mathematics. The original objection to the introduction of infinitesimals by Leibniz - that such a sequence of intervals cannot have two limits - has thus been resolved.

In Euclidean Mathematics the limit of a Cauchy sequence is the equivalence class to which the sequence belongs, while in Leibnizian Mathematics a Cauchy number is the sequence itself and it belongs to the equivalence class that is the real number which is the limit in Euclidean Mathematics.

Since Euclidean Mathematics and Leibnizian Mathematics use different aspects of sequences of numbers it seems to be possible to unify the two systems by including infinitesimals and infinitesimal numbers, as defined in Leibnizian Mathematics, in Euclidean Mathematics. This has been discussed in some detail above, where it was pointed out that Euclidean Mathematics cannot support infinite numbers.

But the impossibility of uniting the two models is most clearly exposed by the diagonal proof of Cantor's Theorem.

Cantor's proof assumes that it is possible to make a list of all infinite decimal fractions between 0 and 1. His proof then shows that there exists an infinite decimal fraction that does not belong to the list. But there are two equally valid possible conclusions to be drawn from this.

The first option is that of Cantor: it is not possible to make a list of **all** infinite decimal fractions (and hence there must be more than countable many infinite decimal fractions) This tacitly assumes that infinite decimal fractions can be listed.

The second option is that it is not possible to make a list of infinite decimal fractions **at all**. In this case a list consisting of a single infinite decimal fraction cannot be made.

In Paragraph 4 it is shown that (because of the assumption in Euclidean Mathematics that a line is a string of points) an infinite number of operations can be performed to completion in this cosmology. Therefore, an infinite string of digits exists as an abstract entity, and it can therefore be listed in some abstract space. Thus, Cantor's diagonal proof is correct in the abstract Space of Euclidean Mathematics. But it is clearly impossible to make a list of a single infinite decimal fraction in perceived reality or in Leibnizian Mathematics where only never-ending decimal numbers are possible.

Since it is not possible in Leibnizian Cosmology to perform an infinite number of operations to completion, the second possible conclusion in Cantor's proof is viable and the resulting logical structure of Cantor's proof is unacceptable.

Remark 2

To emphasise: in Euclidean Mathematics a real number has three avatars namely an equivalence class of Cauchy sequences, a point on an axis, and an infinite decimal fraction, while in the Leibnizian Mathematics a real number is an equivalence class of Cauchy numbers or the length of a line (i.e. the place on the axis where a line from the origin ends). In the Leibnizian Cosmology the real line exists but is not formed from points. Only countable many points, as endpoints of lines, exist.

Various Remarks:

Comparison of the axioms of the Euclidean Cosmology and those of the Leibnizian Cosmology shows the impact on Ontology:

In the Euclidean Cosmology all spatial entities with extent are formed by combining more than countable many points, each of zero extent. In the Leibnizian Cosmology all spatial entities have non-zero extent and can be analysed by indefinitely dividing them into smaller entities of the same kind.

These differences offer an option as to how we think about Space. For example, according to the Euclidean Cosmology, Space must be infinite because never-ending is not allowed. In the Leibnizian Cosmology Space must be never-ending because infinite is not allowed.

In the Leibnizian Cosmology all discrete spatial entities have extent and are therefore fundamentally continua. Thus, if a particle should be defined as being an infinitesimal (instead of as being a point), it can extend throughout all of space while being focussed at where it is observed. This offers an alternative way to model particle-particle and particle-wave relationships as is discussed in the variant of the paradox of Achilles and the tortoise above. Reformulation of the paradox as the motion of two protons gives credence to the ideas of Parmenides because the two protons would have been “next” to each other (extending through the same space) during the whole process. Motion becomes an illusion because it is merely a change in where an infinitesimal is focussed, and nothing comes into being where it did not exist before.

APPENDIX 1: Philosophy

1.1 Background Logic

The material of this appendix should be familiar, in some way or another, to all intellectuals and academics. It is inserted here to set a baseline and to fix the terminology.

Academic tradition originated in Academia in ancient Greece and was founded by philosophers who insisted on rigorous logic. Syllogisms are used in logical arguments. Summarising:

- **Deduction**: The sure (forward) way of inference, e.g., 1) When it rains it is wet 2) It is raining 3) Hence it is wet.
- **Induction**: The unsure (backwards) way of inference, e.g., 1) when it rains it is wet 2) It is wet 3) Hence it may be raining.
- **Abduction**[17]: an inspired guess. E.g., when it was hot and humid in old Rome and the air got muggy, a fever spread that was assumed to be caused by the "bad-air" (Mal aria).

Note that any logical argument must necessarily start with abduction – a first statement (also called a *principle,* a *hypothesis* or an *axiom*) that is accepted as true – for example "When it rains it is wet".

The time before the enlightenment (roughly before 1750) is called philosophical antiquity. During this era, in serious arguments, a statement was taken to be either a truth or a lie. Therefore, when the Greeks defined an axiom as a "self-evident truth" they implied firstly that it is an abductive statement because nothing else is evidence for it (this means that it does not follow deductively or inductively from any other statement), and secondly that it is true (it is not a lie).

After the enlightenment philosophical thinking developed to the point where the initial abductive statement is now always considered to be true (in order to start the sequence of statements in a logical argument) but that it is only true as far as the argument is concerned and that the result of the argument is only true modulo the truth of this initial assumption, e.g., "All philosophers are mad".

Note that all our knowledge are the results of inductive arguments: Everything we know is the result of what we consider to be the cause of the information gathered by our senses and transmitted to our brains (all knowledge are therefore conclusions that were drawn from evidence gathered by our senses). This is best summarised in the quote from Socrates:

The only true wisdom is in knowing you know nothing.

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[17] The Greeks assumed that "Truth" could be discovered by debate. Nowadays the less strict term "abduction" is used.

This quote is usually interpreted as “The only true wisdom is knowing that you know **nothing**” and you should therefore be humble. For Ontology, it is better to read this quote as “The only true wisdom is knowing that you **know** nothing” and therefore you will forever only see shadows on a wall. What we call “knowledge” is, in this sense, only an attempt to approximate knowledge by imposing a structure on our ignorance.

1.2 The scientific method

The scientific method is a technique to aid the managing of this structure, and it emulates how a newly born human acquires a vocabulary when it must start from not knowing anything and then to become a talking individual.

When it is born it cannot speak and has no vocabulary. Its brain is however able to identify and remember sequences of sounds that tend to occur regularly among other such strings of sounds. It will also come to realise that often when it hears the sound “Mamma” its other senses make it aware of a soft warm creature that envelops it in an aura of love and care, and which overflows with a sweet nurturing liquid. Jumping to the conclusion that this sequence of sounds is associated with this creature creates the first word of its vocabulary out of nothingness.

Anybody who ever learned to read goes through the same process when encountering an unknown word while reading. On the first encounter the meaning of the word is guessed from the context, and this guess is refined at every future encounter.

In the same way the scientific method splits the inductive process of accumulating knowledge into a sequence of deductive arguments as is done above when refining the understanding of the meaning of a word. A new deductive process in such a sequence is started whenever evidence appears which implies that the abductive first assumption of the current deductive argument cannot deductively explain this new evidence. The current abductive assumption is then replaced by a new abductive assumption which is such that the deductive argument still yields all the relevant results of the old deductive argument, but in addition explains the new evidence. In this process an ever better understanding of the cause of the observed evidence is formed without ever finding the “true” cause – or even ever progressing to the stage of “beyond reasonable doubt” which is the phrase used by the legal profession. In this scheme the abductive assumption is often called a hypothesis and in a specific case the argument is sometimes called a model.

One should note that this creates a tree of hypothesis since the results of preceding arguments can be used in later arguments. Thus, whenever a discrepancy occurs, it is necessary to identify the relevant hypothesis in the line of assumptions that will cause the argument to explain the new evidence without contradicting any of the old evidence. The set of all trees of assumptions (and the evidence supporting them) is the structure imposed on our ignorance and forms the body of Science.

It is important to realise that only disciplines which are based on assumptions that are verifiable by experiment should use the scientific method. This immediately rules out religious systems and the assumptions about Space as discussed in the prologue.

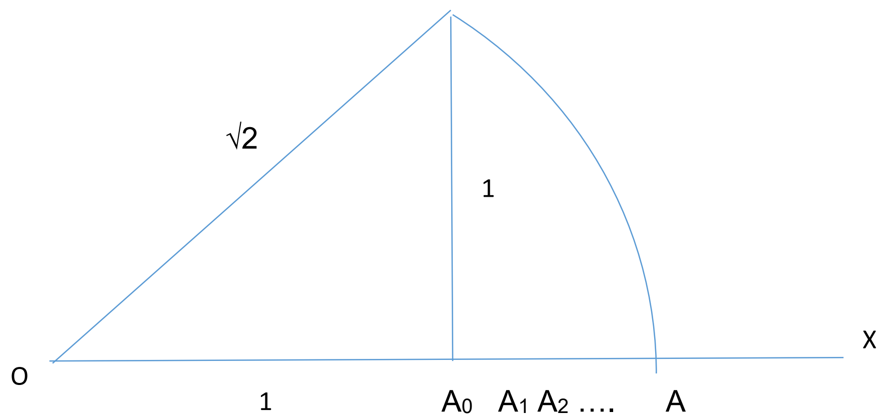
It has become fashionable to refer to human beings as “Story telling Animals”, which is but another way of stating that humans are capable of symbolic reasoning. For example, in a story an abstract concept like “honour” is explained via the conduct of an “honourable” character in a story. A story is used as a vehicle for this type of communication because it creates an “alternative reality” that is related to perceived reality, but much simplified and tailored to the needs of the narrative.

Science (i.e., Structured knowledge) is such a story and the scientific method is a way to manage the setting of the story to improve the quality of its simplified view of perceived reality. In the narrative presented here Mathematics is viewed as a technique of Ontology that can be used to explore the consequences of abductive assumptions by way of arguing using symbols. This last sentence can be reformulated as that “Mathematics is a symbolic story that is told in an alternative reality in which the ontological assumptions are assumed to be true”.

In Paragraph 4 it is shown that the ontological assumptions of the Euclidean Cosmology require an “alternative reality” in which infinitely many operations are performed to completion and “infinity” is a number larger than all other numbers. This reality is an abstracted form of perceived reality and Euclidean Mathematics is consequently called “Abstract Mathematics”. The ontological assumptions of Leibnizian Cosmology do not require infinitely many operations (in the above sense) to be performed but requires only never-ending sequences of operations – a property without meaning in Euclidean Mathematics. Hence the alternative realities of Euclidean- and Leibnizian Mathematics are mutually exclusive in this aspect.

APPENDIX 2: Real numbers

Using Pythagoras' Theorem for a right triangle of which both catheti are of length one, the hypotenuse - and hence also the line OA - has the length √2.



2.1 Arithmetic

There is a simple and well-known argument to show that the length of the line OA cannot be a rational number. But this length can be approximated from below to any degree of accuracy by the same type of argument as that used in Paragraph 2:

Because $(\sqrt{2})^2 = 2$ and because $1^2 = 1$ and $2^2 = 4$, the first approximation to the length is $a_1 = 1$. But 2^2 is larger than the required value, so that the next approximation must be somewhere between 1 and 2. The following list of numbers can then be tried as candidates for the next approximation:

$$1.1^2 = 1.21 \;;\; 1.2^2 = 1.44 \;;\; 1.3^2 = 1.69 \;;\; 1.4^2 = 1.96 \;;\; 1.5^2 = 2.25$$

Since 2.25 is larger than 2, the value of the number that is to be used is somewhere between 1.4 and 1.5 and the number $a_2 = 1.4$ differs by less than 10^{-1} from the targeted value while still being less than the number itself.

This process can be repeated indefinitely and yields a sequence of rational numbers:

$(a_0; a_1; a_2; a_3; \ldots)$

$= (1 \;;\; 1.4 \;;\; 1.41 \;;\; 1.414 \;;\; 1.4142 \;;\; 1.41424 \;;\; \ldots)$

These numbers can be marked on the X-axis as A_0, A_1, A_2, …

The n^{th} term of this sequence of numbers, a_n, differs by less than 10^{-n} from the length of OA so that the length of the line $(A_n;A)$ is less than 10^{-n}.

The set of lines { (A_n;A) ; n=0, 1, 2, …} is therefore a set of nested lines of which the lengths go down to zero.

It has become customary to write the above sequence as the infinite decimal fraction

1.41424 …

with the understanding that the n^{th} term of the sequence is obtained by truncating this string of digits after the n^{th} digit, and where "infinite" has the appropriate meaning as described below.

It is convenient to refer to the above infinite decimal fraction as a Cauchy Base to distinguish it from all other Cauchy sequences that are equivalent to it.

2.2 Euclidean Mathematics

To make it possible to assign a discrete numerical value to the length of the hypotenuse of this triangle, it must be assumed that the value of the irrational number assigned to √2 is such that the square root function f(x)=√x is continuous at the point x=2. This will be the case if this value should be the limit of the sequence of rational numbers (a_0; a_1; a_2; a_3; ….).

According to the definition of an axis in Euclidean Mathematics, these numbers correspond to the points (A_0; A_1; A_2; A_3; ….) so that the set of nested intervals {(A_n;A)} is convergent in the Euclidean Topology. Because the axis is complete in Euclidean Mathematics, this set of intervals has the point A as limit.

The infinite decimal fraction 1.41424 ….. is then chosen as a numeral for the irrational number √2. In this case the string may not be truncated – since it would then refer to one of the rational numbers a_n – and the complete string must perforce be used as a numeral for √2. Therefore, in Euclidean Mathematics, the term "Infinite number" has to mean "a number larger than all other numbers" and "infinite decimal fraction" means that it has an infinite number of digits. Note that according to Paragraph 4 this is a valid choice of numeral, albeit not valid in perceived reality but only in abstract Euclidean space.

In Euclidean Mathematics the term "real number" has three equivalent meanings:

1) A real number is an infinite decimal fraction.
2) A real number is a point on an axis.
3) A real number is an equivalence class of Cauchy sequences.

2.3 Leibnizian Mathematics

The set of lines {(A_n; A); n=0, 1, 2, …} is a set of nested lines of which the lengths go down to zero. But because points as spatial entities are not present in the Leibnizian Cosmology space is not complete and this set of lines cannot - and therefore does not - have a limit. As pointed out in Paragraph 2 this implies that in Leibnizian Mathematics the set of nested lines is a never-ending set of lines focussed on the end A of the line OA. Because there is no discrete spatial object at the end A of the line OA to assign a value to, this implies that it is not required to have the concept "a number larger than all other numbers" in Leibnizian Mathematics, but only the concept "never-ending". In Leibnizian Mathematics the word "infinite" therefore means "Never-ending".

The concept "never-ending" is a property of sequences in perceived reality that is best described in the spirit of Peano[18] as:

(1) The sequence has a first element.

(2) Any given element except the first element has a unique predecessor.

(3) Any given element has a unique successor.

Note

The concept "Never-ending" cannot exist in Euclidean Mathematics because, according to Paragraph 4, more than countably many operations can be performed to completion in abstract space. A never-ending sequence is therefore merely an incomplete infinite sequence and has the same status as a truncated decimal fraction in Arithmetic. Neither can the concept "a number larger than all other numbers" exist in Leibnizian Mathematics because there an infinite number of operations cannot be done to completion.

Theorem 1 (Valid only in the Leibnizian Cosmology)

In Leibnizian Mathematics every equivalence class of Cauchy sequences contains at least one infinite decimal fraction unique to that class.

Proof

Let $a > 0$ be any equivalence class of Cauchy sequences and let $\{a_n\}$ be any Cauchy sequence with positive terms in a. Let A_n denote the rational number obtained by truncating a_n after digit number n.

In the definition of a Cauchy sequence, set $\varepsilon_n = 10^{-n}$. Then there exists a natural number N such that $|a_r - a_s| < 10^{-n}$ for all $r,s \geq N$. Thus all the decimal fractions in the subsequence $\{a_s : s \geq N\}$ can only differ after digit number n, and by an amount of at most 10^{-n}.

Setting $\varepsilon_0 = 1$ (= 10^{-0}), the integral part of all terms of $\{a_n\}$ for $n > N_0$ is obtained. When this is subtracted from all these terms, a Cauchy sequence is obtained with all terms less than one. Assume therefore that the Cauchy sequence $\{a_n\}$ is such a sequence.

[18] Giuseppe Peano; 1858 - 1932

The sequence {A_1; A_2; A_3; …} is therefore a Cauchy sequence for which the terms are all truncations of the same infinite decimal fraction.

The only exception that can occur is when a is a rational number less than one and therefore its representation as an infinite decimal fraction may result in becoming either an infinite string of zeros or of nines. In the case of repeating nines this will manifest as that $A_{n+1} < A_n$ for some value of n. This can be righted by setting A_{n+1} equal to A_n with a zero appended. This will result in generating an equivalent infinite string of zeros.

Every equivalence class of Cauchy sequences therefore contains at least one never-ending decimal fraction. But two equivalence classes of Cauchy sequences cannot share a Cauchy sequence because of transitivity so that this infinite decimal fraction is unique to the number a.

■

It can therefore be concluded that there exists a one-to-many mapping of the real numbers into the set of infinite decimal fractions.

Theorem 2 (Valid only in the Leibnizian Cosmology)

The set of never-ending decimal fractions is countable.

Proof

(This proof mimics the proof that the set of rational numbers is countable)

Write the digits 4, 5, 6, 7, 8, 9, 0, 1, 2, 3 on 10 consecutive lines.

Repeat this group another 9 times so that 100 lines are filled with ten of these groups of ten digits.

Add second digits to the lines: a 1 to the first group, and from 2 to 0 respectively to every consecutive group. All 100 possible permutations of two digits are now listed, the top one being the first two digits of the decimal part of √2.

Take this 100x2 array and repeat it 9 times so that 1000 lines now have two digits. Append the next digit of √2 to the first 100, and then proceed cyclically as above, adding the other nine digits to the next nine groups. All 1000 possible permutations of three digits are now listed, the first line being the first three digits of √2.

Repeating this process, a never-ending two-dimensional array is constructed containing as rows all possible permutations of the ten digits, the first row being the fractional part of √2.

A never-ending list of all possible never-ending decimal fractions between zero and one is thus constructed, and the theorem is proved.

■

The set of all Cauchy sequences with the limit A forms an equivalence class in the Euclidean Topology, and such an equivalence class is then defined as a real number. There is a one-to-one order preserving homeomorphism between infinite decimal fractions and points on the axis in the Euclidean Topology – Therefore the term “Real Line”.

APPENDIX 3: Excerpt from the reference

This appendix is an extract from pages 28 to 40 of [1], edited to conform to the narrative of this document.

3.1 Models for space

All mathematicians agree that space is infinitely divisible. In Euclidean Mathematics the word “infinite” means “a number larger than all other numbers” and thus it is possible to divide an interval until a single point is left - as is done in the Dedekind Cut when the endpoint of an interval is removed to turn the interval from being closed to being open in the Euclidean Topology. In Leibnizian Mathematics the word “Infinite” means “never-ending” so that a piece of space can be divided over and over without end and all the pieces are of non-zero extent.

3.2 Introduction: The basic construction

Start with any line and cut it at a convenient place. This creates a discontinuity that defines a place in space. This place in space can be indicated by the ends of either of the two half-lines formed. Name this place in space as O and call it the origin.

With the choice of a suitable scale, it is then possible to construct an interval of which the length is any given rational number by using the same technique as described in Paragraph 2 where it was used to construct a line of one tenth of the gauge length. Hence, choosing the original line as axis and keeping the usual rules for axes in mind, a line OA of length a can be constructed for any given positive rational number a. As per convention, the interval formed by the line OA on the X-axis can be indicated by (0;a) i.e. the line (or vector) starting at x=0 and ending at x=a. Note that this argument is solely about lines and their lengths so that the concept of “point” is not involved and that therefore the concept of open and closed intervals is absent as well. Thus x=0 means “at the origin” and x=a means “at the end A of line OA of length a”

Let δ be a positive rational number that is less than a. Construct the lines OB and OC of lengths a-δ and a+δ in the same direction as OA. The line BC forms the interval (a-δ; a+δ) of length 2δ on the X-axis with its centre at x=a. Intervals like these are then used in the following construction:

Consider the unit interval (or line) from 0 to 1 on the X-axis. Thus d(0;1) = 1 where d is the distance between the two ends of the interval. This unit interval can now be partitioned repeatedly, i.e. cut up into shorter lines, by forming sub-intervals in the following way:

Starting with the whole interval as partition zero, form successive partitions by dividing each interval of the previous partition into three equal parts. In this way the

n^{th} partition will consist of 3^n intervals, each of length 3^{-n}. If $x = a_i^n$ is at the centre of the i^{th} interval of the n^{th} partition, then

$$a_i^n = \frac{2i-1}{2} 3^{-n} \quad \text{For } i = 1, 2, 3, \ldots, 3^n \text{ and } n = 0, 1, 2, \ldots. \qquad [1.0.1]$$

But the length of the whole interval is the sum of the lengths of the parts so that:

$$1 = \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} , a_i^n + \tfrac{1}{2}3^{-n}) \qquad \text{for } n=0, 1, 2, \ldots. \qquad [1.0.2]$$

Therefore

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} , a_i^n + \tfrac{1}{2}3^{-n}) \qquad [1.0.3]$$

The equation [1.0.3] is based solely on the property of space that any given line can always be sub-divided.

There are two fundamentally different ways to handle the infinite divisibility of space. The first way is a consequence of what was decided in Athens at about 500 B.C. It is a process of synthesising space and is studied directly below. It is referred to here as the approach of Euclid. The other is a process of analysing space, as is done above. The above construction follows the approach of Leibniz at about 1700 A.D.

3.3 Euclid

The basic assumptions of the Mathematical sciences as practiced today were made in ancient Greece. At that time the concept of "point" as a spatial entity was introduced by philosophers who were fundamentally geometers. A notable exception was the philosopher Parmenides of Elea who argued about the nature of motion as well (change and coming into being) and thus, in today's classification, was fundamentally a Physicist.

The Athenians looked at volumes, areas and lines that extend in three, two and one directions and then extended this triplet to include a fourth entity. This entity was defined as "That which has no extent".

In Paragraph 4 it is argued that this is not a definition, but should be treated as an axiom: "There exists a piece of space with no extent", and that in this axiom the word "exists" has its literal meaning, namely that a point is a spatial entity, i.e. it is a piece of space.

The introduction of this concept not only gave a tool to refer to a specific place in space but made space complete in the sense that any nested sequence of solids, surfaces or lines of which the volumes, areas or lengths converge to zero in such a way that their maximum diameters converge to zero, will have a spatial object (a point) as limit.

For intervals as used here this means that, if a is any point, the sequence of nested intervals

$$\{(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) : n = 0,1,2,3\ldots\}$$

has an interval consisting of the single point a as limit. This is because a belongs to all these intervals and any other given point will eventually fall outside one of the intervals. Thus

$$d\left[\lim_{n\to\infty}(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n})\right] = d(a)$$

Also

$$d(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) = 3^{-n}$$

So that

$$\lim_{n\to\infty} d(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) = d(a) \qquad [1.1.1]$$

Here d(a) is the length of a single point and thus d(a) = 0.

But a second assumption about space was made by the philosophers in Greece. This assumption was that space is formed (synthesised) from points. This implies that any interval consists of a string of points.

This is a second fundamental assumption about the nature of space (The first was that space is infinitely divisible). On a philosophical level this assumption means that anything that is continuous is formed from discrete entities. This assumption will be called the **Euclidean Cosmology**.

This second assumption is the reason why the concept of "more than countable" had to be introduced into Mathematics:

Let Γ be an index set and let $B = \{a_\beta : \beta \in \Gamma\}$ be the set of points forming the unit interval. Each α_β is a point and thus $d(\alpha_\beta) = 0$. But, according to the second assumption above, the total length of the interval must be the **sum** (not the limit) of the lengths of all the points forming the interval:

$$1 = \sum_{\beta\in\Gamma} d(a_\beta) \qquad [1.1.2]$$

(Where the notation is not standard but is hopefully self-explanatory.)

In this sum every $d(\alpha_\beta)$ is zero. But the sum of a finite number of zeros is zero, and the sum of a countable number of zeros, being the limit of its partial sums, must then also be zero. This implies that Γ cannot be either a finite or a countable index set. Thus, Γ must be more than countable. This means that it requires more than countable many zeros to add up to the non-zero number one.

Equation {1.1.1] is also true for any one of the real numbers α_β.

Thus equation [1.1.2] becomes

$$1 = \sum_{\beta}^{\Gamma} \lim_{n\to\infty} \mathrm{d}(a_\beta - \tfrac{1}{2}3^{-n}, a_\beta + \tfrac{1}{2}3^{-n}) \qquad [1.1.3]$$

Note that the assumption that a line is a string of points is therefore an implicit introduction of the axiom of choice into the Euclidean Cosmology: To perform the sum of zeros, consecutive points from the interval needs to be chosen one by one and for each a zero must be added to the sum. Moreover, when the last point is processed, the interval is completely deconstructed, and the sum is exactly one. Thus, it is not only possible to perform an infinite number of operations to completion, but in the end the line is discrete and identified.

3.4 Leibniz: Integrals

Leibniz and Newton both developed tools to study motion. To describe the rates at which bodies moved and accelerated Newton used fluxions and Leibniz used infinitesimals. Although Newton's notation is still sometimes used in Mechanics, the notation introduced by Leibniz survived in general use.

The word "infinitesimal" is pidgin Latin for "a little thing at infinity" and, even today when physicists use the word, the intended meaning is "a number that is zero, but not completely so".

Leibniz introduced the notation

$$\int_a^b f(x)dx$$

for his process to determine the area under the graph of y=f(x) between the values x=a and x=b. This is called an integral and dx is an infinitesimal number (and hence f(x)dx is an infinitesimal number too) and the integral sign is an elongated "S" to denote an infinite (but also not completely so) sum of infinitesimals. Therefore, an integral can be imagined to be an (almost) infinite sum of (almost) zeros, and hence it is alike to the indefinite form $\infty \cdot 0$ as studied by L'Hospital. This aspect is discussed in more detail in the following paragraphs.

The evaluation of integrals like these is done using Riemann sums. To do this the interval (a,b) on the X-axis is partitioned into sub-intervals and the areas of rectangles, each with one part of the partition as base and with its height equal to the function value at some point of that part of the partition, are summed to approximate the area under the curve. The required area under the curve is then the limit of these sums as the number of parts in the partition tends to infinity in such a way that the length of each part tends to zero.

To come to an understanding of the way that Leibniz was thinking, one notes that Leibniz introduced the concept of "infinitesimal" to study integrals and rates of movement. It is therefore advantageous to look at a simple integral in order to get a grip on his thinking about this concept. The following is the simplest possible integral. It is to determine the area under the line f(x)=1 between the values x=0 and x=1: hence it is the area of a square of side length one.

Thus:

$$\int_0^1 1 \cdot dx = 1$$

In this case the height of all the rectangles in the Riemann sum is one, and the Riemann sum is not an approximation anymore but is exactly one for all possible partitions. Thus, using the set of partitions as described in section 1.0 the integral becomes:

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} 1 \cdot d(a_i^n - \tfrac{1}{2}3^{-n} \text{ , } a_i^n + \tfrac{1}{2}3^{-n}) \qquad [1.2.1]$$

Which turns out to be a restatement of equation [1.0.3].

This is an extremely simple set of partitions, but as such they are suitable to study the process of refining partitions:

Drawing a few lines of unit length and filling in the partitions for n=0, n=1, n=2… the following conclusions become obvious (but can be shown rigorously to be true):

- Once x=a_i^n is in the middle of a part of a partition for some value of n and some value of i, it will be in the middle of a part of a partition for all subsequent values of n and the consequential value of i.

- This property can be used to generate sequences of nested intervals using parts from consecutive partitions as n increases, e.g.
  {($\frac{1}{2} - \frac{1}{2}3^{-n}$, $\frac{1}{2}$ + $\frac{1}{2}3^{-n}$) ; n=0, 1, 2, ….} ;
  {($\frac{1}{6} - \frac{1}{2}3^{-n}$, $\frac{1}{6}$ + $\frac{1}{2}3^{-n}$) ; n=1, 2, 3, ….} ;
  {($\frac{5}{6} - \frac{1}{2}3^{-n}$, $\frac{5}{6}$ + $\frac{1}{2}3^{-n}$) ; n=1, 2, 3, ….} ;

  e.t.c

Definition

A sequence of nested volumes, areas, or lines of which the maximum diameters converge to zero is called an **infinitesimal volume**, - **area** or - **line** (or simply an "**infinitesimal**" as is the common practice)

3.5 The Leibnizean cosmology

There are two things to notice about expression 1.2.1:

The first is that the sum ranges over non-zero quantities – no matter how large the value of n. Therefore, there is no need here to introduce the concept "more than countable".

The second is that the limit is taken for the sum, and not for the partitions. Therefore, there is also no need to introduce the concept of "point" as a property of space.

Hence, this leads to the

Leibnizean Cosmology:

Any continuum, no matter how big or how small, is composed of smaller continua, no matter how small[19].

Also included in the assumptions of the Leibnizian Cosmology is the Axiom of Parmenides:

All spatial entities have extent.

This implies that a point, as a spatial entity, cannot form part of the Leibnizean Cosmology because a point cannot be divided into two parts of non-zero extent.

Without the concept of "point" as a spatial entity, any discrete object has extent and is a continuum. Thus, on the philosophical level, in the Leibnizean Cosmology whatever is discrete is formed from continuous entities.

3.6 Points and infinitesimals

The concept of "point" is central to the Euclidean Cosmology, but it has no role as a spatial entity in the Leibnizean Cosmology. However, "point" is a handy word to use to refer to a "place in space".

The way that the concept of "point" was introduced above, was done on purpose in such a way that the point was at the end of a given line. In Geometry the endpoints of lines, or vectors, are handy discontinuities to indicate places in space.

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[19] It is fun to pun the well-known quip about fleas:

Big fleas have little fleas upon their backs to bite them,

and little fleas have lesser fleas, and so *ad infinitum*

As

Big space has little space that sum to what is in it,

And little space has lesser space and so on without limit.

Note

While discussing the expression on the right in paragraph 1.1, a point (as per the Euclidean Cosmology) was identified with the following limit:

$$\lim_{n\to\infty}(a - \frac{1}{2}3^{-n}\,, a + \frac{1}{2}3^{-n})$$

This limit is an interval of length zero consisting of the single point a, and as such is a spatial entity. This point a, being a limit for the set of nested intervals of which the maximum diameters converge to zero, indicates that space is complete in the Euclidean Cosmology.

The intervals appearing in this expression form a nested sequence of intervals, focussed at the endpoint a of the line (0;a) (as per the Leibnizean Cosmology).

Thus, in the Euclidean Cosmology this limit exists as the spatial point a, but in the Leibnizean Cosmology there is no corresponding spatial entity that can serve as a limit. Thus, the sequence of nested intervals is a never-ending sequence that is focussed at a. Space is therefore not complete in the Leibnizean cosmology.

Thus, an infinitesimal is a never-ending sequence of volumes, surfaces or intervals for which no limit exists.

In the Leibnizean Cosmology there is no harm in retaining the word “point” (as a shortening of “endpoint of a vector”) to have available for use when working with numbers; as long as it is used only to indicate the place in space where a line ends and is **not** used as a building block for a continuum. Therefore, in the Leibnizean Cosmology, a point is simply the place in space where a line ends and thus a point is a property of a line and not a property of space.

Note that every infinitesimal is focussed at a place in space (which can be indicated by a discontinuity like the end of a line).

3.7 Leibniz, Riemann and L’Hospital

These three are the originators of all the ideas formulated here, but they never realised that they could move away from the Euclidean Cosmology. A look at a possible analysis of their thinking is in order. The ultimate goal of this is to motivate the extension of the number system that is in use in the Mathematical Sciences to include the Cauchy Numbers of which the infinitesimal numbers form a subset.

Leibniz

Leibniz and Newton lived in the same era. That was the time when Newton’s Laws were formulated during a renewed interest in Mechanics due to the introduction of the heliocentric model for our solar system. The aim was to describe and predict the motion of particles and bodies and both Newton and Leibniz developed tools to study Mechanics.

To describe the motion of a particle Leibniz[20] had to find ways to determine the gradient of the tangent to the graph of the function y = f(x) at the point x=a in the XOY–plane and the area under the graph of y=f(x) between the values x=a and x=b.

To find the gradient of the tangent, he started with a sequence of numbers

$\{\delta x_n$; n=1; 2; 3 …}

that converges to zero. He used this sequence to generate two sequences of nested intervals

{(a , a+δx_n) ; n=1; 2; 3 …}

each of length δx_n and

{(f(a) , f(a+δx_n)) ; n=1; 2; 3 … }

each of length δy_n = f(a+δx_n) - f(a) using the usual sign conventions.

The gradient of the tangent at x=a is then

$$\lim_{n\to\infty} \frac{\delta y_n}{\delta x_n}$$

This limit is of the indefinite form 0/0.

At this point Leibniz must have realised that $\lim_{n\to\infty} \delta x_n$ and $\lim_{n\to\infty} \delta y_n$ cannot be the number zero of the Euclidean Cosmology because division by zero is not defined. He therefore defined a new class of numbers to augment the number zero and called them the Infinitesimal numbers.

Leibniz introduced the notation dx and dy to denote the infinitesimal numbers that are the limits of the above sequences $\{\delta x_n\}$ and $\{\delta y_n\}$. Thus

$$\lim_{n\to\infty} \frac{\delta y_n}{\delta x_n} = \frac{dy}{dx}$$

Note

This new kind of number was defined as a purely abstract concept by Leibniz. In the present paradigm of the Mathematical Sciences there are no numerals for infinitesimal numbers, but they are visualised by some as numbers that are small enough to be the limits of sequences that converge to zero but are still large enough to use in calculations. The limits of the sets of nested intervals above can then also not be points but must be new spatial entities called infinitesimals which are visualised as intervals of almost zero length. However, with our awareness of the existence of transfinite numbers in the paradigm of Mathematics, these new

[20] Gottfried Wilhelm (von) Leibniz 1/7/1646 – 14/11/1716

infinitesimal numbers are immediately recognised as a separate class of cisfinite[21] numbers.

The introduction of infinitesimals and infinitesimal numbers created a structural discrepancy in the Euclidean Cosmology: While zero and infinitesimal numbers can comfortably co-exist in a number system, points and infinitesimals are two different types of spatial entities, and a sequence of nested intervals of which the lengths tend to zero cannot have two different spatial entities as limits[22].

The only way in which this conflict can be resolved is to formulate a second alternative model for Space in which a nested sequence of intervals of which the lengths converge to zero does not need to have a point as limit. This alternative model for Space is **the Leibnizean Cosmology.**

Another reason why Leibniz would have wanted to have an infinitesimal of length dx available to use as an interval of length almost zero at the point x on the X-axis, is to be able to calculate the area under the graph of the function y=f(x) between the values x=a and x=b. In this case, the product f(x)dx is an infinitesimal representing the area of a rectangle based on the infinitesimal and with height equal to the value of the function at that place on the axis. An “almost infinite” sum of these “almost zero” areas will then be the required area under the graph. Leibniz used the notation

$$\int_a^b f(x)dx$$

for this process. The integral sign is an elongated “S” to indicate this “almost infinite” sum.

This operation is an indefinite form of type $\infty \cdot 0$.

Riemann

Riemann approach uses repeated partitioning of an interval into subintervals to generate repeated approximations to the value of the integral. The limit of these approximations was the value of the integral.

The value of his insight is that he moved away from the sum of the limits, as Leibniz did, to the limit of the sums. But he failed to notice the relationship between the various parts of consecutive partitions so that he never noticed the possibility of defining infinitesimals as sequences of intervals instead of merely intervals like Leibniz did.

Once the sequences of intervals that are called infinitesimals in the Leibnizean Cosmology is recognised, it is but a small step to realise that they can be organised to form a directed set by introducing a pre-order:

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[21] A cisfinite number is defined here as a non-negative number that is equivalent to zero.

[22] The proper study of Topology happened only after the death of Leibniz so that he probably was not aware of this discrepancy.

Let A and B be two infinitesimals. Then A > B means that all intervals forming the infinitesimal B are subintervals of an interval that is a part of the infinitesimal A

In the set of partitions of the basic construction of SECTION 1.0 the first infinitesimal of the directed set of infinitesimals is

$$A = \{(\tfrac{1}{2} - \tfrac{1}{2}3^{-n}, \tfrac{1}{2} + \tfrac{1}{2}3^{-n}) ; n=0, 1, 2, \ldots\} \qquad (a)$$

The second and third members of the directed set are

$$B1 = \{(\tfrac{1}{6} - \tfrac{1}{2}3^{-n}, \tfrac{1}{6} + \tfrac{1}{2}3^{-n}) ; n=1, 2, 3, \ldots\} \qquad (b)$$

and

$$B2 = \{(\tfrac{5}{6} - \tfrac{1}{2}3^{-n}, \tfrac{5}{6} + \tfrac{1}{2}3^{-n}) ; n=1, 2, 3, \ldots\} ; \qquad (c)$$

Thus A > B1 and A > B2 but B1 and B2 are not comparable, e.t.c.

The integral is then defined as a net defined on this directed set into the real numbers like in DIVISION 3 of [1] where the net is into the Cauchy numbers that are still to be defined below.

L'Hospital

L'Hospital studied the indefinite form 0/0 that is formed from the quotient of two functions h and g at the point x=a when both h(a)=0 and g(a)=0.

The rule of L'Hospital then states that

$$\lim_{x\to a}\frac{h(x)}{g(x)} = \frac{h'(a)}{g'(a)} \qquad [A]$$

Let $\{\delta_n\}$ be a sequence of rational numbers that converges to zero. Let $a_n = a + \delta_n$. Then the sequence of numbers X = $\{a_n\}$ is a Cauchy sequence that converges to a and the lines (a, a_n) form a nested sequence of intervals of which the lengths converge to zero (Thus the set of intervals is an infinitesimal focussed at a).

Also, if the functions h and g are smooth enough, then H = $\{h(a_n)\}$ and G = $\{g(a_n)\}$ are two Cauchy sequences of numbers that converge to zero because both h and g are zero at x=a.

Using this notation, the Rule of L'Hospital [A] can be stated as

$$\lim_{n\to\infty}\frac{h(a_n)}{g(a_n)} = \frac{\lim_{n\to\infty}\frac{h(a_n)}{\delta_n}}{\lim_{n\to\infty}\frac{g(a_n)}{\delta_n}} \qquad [B]$$

Stated like this, the rule of L'Hospital points out a way to find numerals for infinitesimal numbers by the following argument:

The equality [B] is an equality of real numbers. This means that the equivalence class of Cauchy sequences on the left-hand side is the same as the equivalence class of Cauchy sequences on the right-hand side.

The equivalence class formed by the left-hand side is the limit of the Cauchy sequence

$$\{\frac{h(a_n)}{g(a_n)};\ \mathrm{n} = 1,2,\dots\}$$

and hence this Cauchy sequence belongs to the equivalence class.

But this Cauchy sequence is a comparison of the rates of convergence of the two Cauchy sequences

$$H = \{h(a_n): n = 1,2,3,\dots\}$$

And

$$G = \{g(a_n): n = 1,2,3,\dots\}$$

Which both converge to zero.

It is now clear that if division of these two Cauchy sequences of rational numbers is defined as term-wise division (provided that not more than a finite number of the values of g is zero) then:

$$\frac{H}{G} = \left\{\frac{h(a_n)}{g(a_n)} : n = 1,2,3,..\right\}$$

Let X be the sequence

$$X = \{\delta_n: n = 1,2,3,\dots\}$$

Then, twice using the same argument as above, the two limits appearing in then right-hand side

$$\lim_{n\to\infty} \frac{h(a_n)}{\delta_n} \quad \text{and} \quad \lim_{n\to\infty} \frac{g(a_n)}{\delta_n}$$

Become

$$\frac{H}{X} \quad \text{and} \quad \frac{G}{X}$$

Note that the sign "=" that means "is the same equivalence class" when used with real numbers has the meaning "belong to the same equivalence class" when used with Cauchy sequences in the present context.

Then the rule of L'Hospital can be written as

$$\frac{H}{G} = {\frac{H}{X}}\Big/{\frac{G}{X}}$$

In this form the rule of L'Hospital states that the rates of convergence of two Cauchy sequences that converge to zero can be obtained by comparing their rates of convergence relative to a "gauge" Cauchy sequence that converges to zero. He thereby transformed the process of determining rates of convergence to algebraic operations on rates of convergence that can be known by differentiation:

Notice that comparison of the two forms of the right-hand side implies that

$$\frac{H}{X} = \frac{dh}{dx} \quad \text{and} \quad \frac{G}{X} = \frac{dg}{dx}$$

So that the infinitesimal numbers dh, dg and dx can be defined as the Cauchy sequences

dh = $\{h(a_n)\}$; dg = $\{g(a_n)\}$ and dx = $\{\delta_n\}$

which all converge to zero: i.e. they are all equivalent to 0 = {0; 0 ;0; ….}

The above motivates the following definition of the **Cauchy Numbers**:

Definition

The Cauchy sequences that are the elements of the equivalence classes that form the real numbers, are defined as the Cauchy Numbers and the arithmetical operations on Cauchy numbers are performed component wise.

The Cauchy numbers that are equivalent to zero are the *Infinitesimal Numbers*.

To make the Cauchy numbers closed under division the *Infinite Cauchy Numbers* are defined as the members of the equivalence class of unbounded sequences.

The Cauchy numbers that belong to the equivalence classes forming the other real numbers are called *Rated Numbers*.

With this definition, the component-wise application of the arithmetical operations implies that the limiting process in the rule of L'Hospital can be replaced with an arithmetical process. In this notation the rule of L'Hospital can be derived as follows:

It was shown above that because h(a)=0 and g(a)=0 the Cauchy numbers dh, dg as well as dx are all infinitesimal numbers. Therefore, at x=a

$$\frac{df}{dg} = \frac{df}{dg} \cdot \frac{dx}{dx} = \frac{df}{dx} \Big/ \frac{dg}{dx}$$

As noted above the sign "=" means "belong to the same equivalence class" when used with Cauchy numbers. With this and the definition of the arithmetical operations for Cauchy numbers in mind, the above operation is validated by the fact that the rational numbers that are the components of the two equivalent Cauchy numbers are asymptotically equal.

One should also note that these operations are not limited to infinitesimal numbers only but are valid for all Cauchy numbers. For example, the division of a rated number by an infinitesimal number will result in an infinite Cauchy number (an unbounded sequence).

APPENDIX 4: An illuminating compactification

All four arithmetical operations can be performed as long as the Cauchy numbers used in division do not have more than a finite number of zero components – i.e. from some point on they do not contain any zeroes.

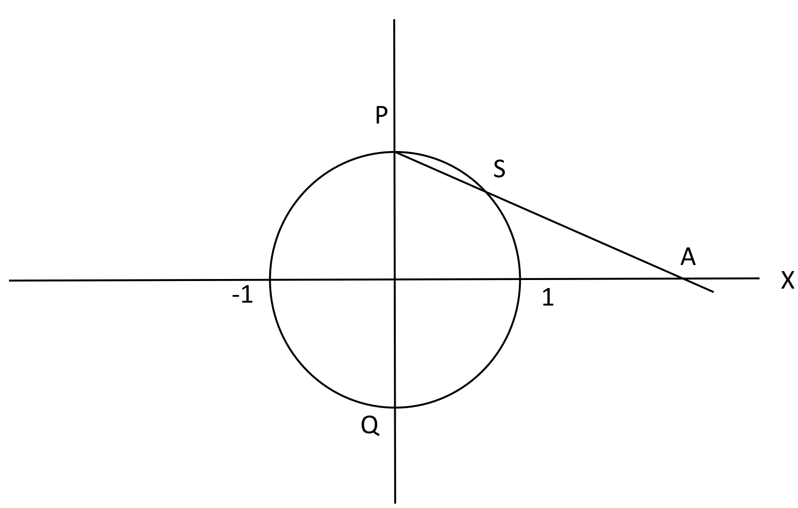
Three classes of Cauchy numbers have been defined:

- The **infinitesimal numbers**. These are Cauchy numbers equivalent to zero, indicated as the class A.
- The **infinite numbers**. These are sequences of numbers of which the magnitude of the terms increases without limit, indicated as the class B.
- The **rated numbers**. These are Cauchy numbers belonging to the other equivalence classes of Cauchy numbers, indicated as the class C.

Any sequence of numbers that does not belong to any of these classes can be called a **meandering** sequence.

The rule of L'Hospital indicates that when two Cauchy numbers are multiplied or divided the result can move from class to class. The transition most often used is when the quotient of two infinitesimal numbers becomes a rated number. The traditional use of this transition is referred to as differentiation.

The nature of these classes of Cauchy numbers becomes clearer when one emulates the compactification of the complex plane as was done by Lars Ahlfors[23] when he introduced his 'point at infinity'.



[23] Lars V Ahlfors 1907 - 1996

In the following description the word “point” has its usual meaning when viewed from the Euclidean Cosmology. In the Leibnizian Cosmology it refers to a place in space where an infinitesimal is focussed or where a line ends.

The horizontal line in the figure is the real line and the circle is a unit circle centred at the origin. A line drawn from the topmost point P of the circle to any point A on the real line then maps that point onto a point S of the circle. The rightmost and leftmost points of the circle are the points +1 and -1 and they map onto themselves. The origin maps onto the lowermost point Q while the topmost point P corresponds to the ‘point at infinity’ for the Euclidean Cosmology and for the Leibnizian Cosmology it refers to the fictional place where the infinite Cauchy sequences are focussed.

With a metric topology of ‘length of arc’ on the circle, the three classes of Cauchy numbers defined above correspond to classes of Cauchy sequences converging in this topology to points on this circle. Infinitesimals are Cauchy sequences that converge to Q, the image of zero. Rated Cauchy numbers are sequences that converge to all other points of the circle apart from the topmost, and infinite numbers correspond to the Cauchy sequences that converge to the topmost point P. In this sense the infinite numbers also form an equivalence class, and this equivalence class can be called **infinity**. This validates the term ‘Infinite Cauchy Numbers’ and allows ‘infinity’ to act like a real number in both cosmologies.

The consequence of all this is that ‘infinity’ acquires a fine-structure and thus need not to be avoided any more in Leibnizian Mathematics (apart from division by the null sequence). For example, the Dirac delta function (the derivative of the Heaviside function) is an ordinary piecewise function when using the Cauchy numbers and has an infinite number as value at the point of discontinuity.

Thus, each point on this circle corresponds to an equivalence class of Cauchy numbers and can be called, as is traditional, a real number- but with care.

REFERENCE

[1] Adriaan van der Walt; AN ALTERNATIVE MODEL FOR SPACE (2019)

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This book is currently not for sale, but its text has been posted on viXra as the PDF file ***From CANTOR to LEIBNIZ*** and can be freely downloaded from there.