Einstein Field Equations and Geodesic Equation Paradox for a Gravitational Plane Wave Pulse Colliding with a Mass

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Abstract

We consider a gravitational plane wave pulse colliding with a point mass. The path of the mass can be determined using the Einstein field equations. We expect, for small mass, that this path to be approximately a geodesic. We show this need not be the case.

1 Introduction

For a system of just a gravitational plane wave pulse and no mass let the metric be $\widetilde{g}_{\mu\nu}(t-x)$ having $\widetilde{g}_{\mu\nu}(t-x) = \eta_{\mu\nu}$ for $x > t$. Require $\widetilde{g}_{\mu\nu}(t-x)$ satisfy the Einstein field equations. For a system of a point mass M at rest at the origin and no wave let the metric be $\widehat{g}_{\mu\nu}(\mathbf{x})$. Require $\widehat{g}_{\mu\nu}(\mathbf{x})$ satisfies the Einstein field equations. Now consider a system of gravitational plane wave pulse colliding with M. Let $g_{\mu\nu}(t,\mathbf{x})$ be the metric of the combined system of colliding wave and M. Require $g_{\mu\nu}(t,\mathbf{x})$ satisfy the Einstein field equations. Define

$$\widetilde{h}_{\mu\nu}(t-x) = \widetilde{g}_{\mu\nu}(t-x) - \eta_{\mu\nu} \tag{1}$$
$$\widehat{h}_{\mu\nu}(\mathbf{x}) = \widehat{g}_{\mu\nu}(\mathbf{x}) - \eta_{\mu\nu} \tag{2}$$
$$h_{\mu\nu}(t,\mathbf{x}) = g_{\mu\nu}(t,\mathbf{x}) - \eta_{\mu\nu} \tag{3}$$

Let $\widetilde{h}(t-x), \widehat{h}(\mathbf{x}), h(t,\mathbf{x})$ represent $\widetilde{h}_{\mu\nu}(t-x), \widehat{h}_{\mu\nu}(\mathbf{x}), h_{\mu\nu}(t,\mathbf{x})$ or first or second order partial derivatives of $\widetilde{h}_{\mu\nu}(t-x), \widehat{h}_{\mu\nu}(\mathbf{x}), h_{\mu\nu}(t,\mathbf{x})$ respectively.

The exact Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \tag{4}$$

can be written [1]

$$R^{(1)}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\alpha}{}_{\alpha} = -8\pi(T_{\mu\nu} + t_{\mu\nu}) \tag{5}$$

where

$$t_{\mu\nu} = \frac{1}{8\pi}\Big[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha}{}_{\alpha} - R^{(1)}{}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)\alpha}{}_{\alpha}\Big] \tag{6}$$

and

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\Big[\frac{\partial^2 h^{\alpha}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 h^{\alpha}_{\mu}}{\partial x^{\alpha}\partial x^{\nu}} - \frac{\partial^2 h^{\alpha}_{\nu}}{\partial x^{\alpha}\partial x^{\mu}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\alpha}\partial x_{\alpha}}\Big] \tag{7}$$

Indices on $h_{\mu\nu}, R^{(1)}{}_{\mu\nu}$, and $\partial/\partial x^{\alpha}$ are raised and lowered with η's. For example $h^{\nu}{}_{\mu} = \eta^{\nu\alpha}h_{\alpha\mu}$ and $\partial/\partial x_{\alpha} = \eta^{\alpha\beta}\partial/\partial x^{\beta}$. Computing $t_{\mu\nu}$ in a power series in h we have

$$t_{\mu\nu} = \frac{1}{8\pi}\Big[-\frac{1}{2}h_{\mu\nu}R^{(1)\alpha}{}_{\alpha} + \frac{1}{2}\eta_{\mu\nu}h^{\alpha\beta}R^{(1)}{}_{\alpha\beta} + R^{(2)}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}R^{(2)}{}_{\alpha\beta}\Big] + \mathcal{O}(h^3) \tag{8}$$

where

$$
\begin{aligned}
R^{(2)}{}_{\mu\nu} = & - \frac{1}{2}h^{\alpha\beta}\left[\frac{\partial^2 h_{\alpha\beta}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\alpha}\right] \\
& + \frac{1}{4}\left[2\frac{\partial h^\beta{}_\sigma}{\partial x^\beta} - \frac{\partial h^\sigma{}_\sigma}{\partial x^\beta}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x^\nu} + \frac{\partial h^\sigma{}_\nu}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x_\sigma}\right] \\
& - \frac{1}{4}\left[\frac{\partial h_{\sigma\nu}}{\partial x^\alpha} + \frac{\partial h_{\sigma\alpha}}{\partial x^\nu} - \frac{\partial h_{\alpha\nu}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\alpha} + \frac{\partial h^{\sigma\alpha}}{\partial x^\mu} - \frac{\partial h^\alpha{}_\mu}{\partial x_\sigma}\right] \qquad (9)
\end{aligned}
$$

The first term of $t_{\mu\nu}$ is then quadratic in h.

2 Plane gravitational wave pulse metric

Define $u = t - x$ and let the metric $\widetilde{g}_{\mu\nu}(u)$ be [2]

$$
ds^2 = -dt^2 + dx^2 + [L(u)]^2 e^{2\beta(u)} dy^2 + [L(u)]^2 e^{-2\beta(u)} dz^2 \qquad (10)
$$

having $\widetilde{g}_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ and

$$
\frac{d^2 L}{du^2}(u) + \left[\frac{d\beta}{du}(u)\right]^2 L(u) = 0 \qquad (11)
$$

This metric will satisfy $R_{\mu\nu} = 0$. It is the metric of a gravitational plane wave pulse. Let $L(0) = 1$ and $\beta \neq 0$. We then have by (11) that $L(u)$ will decrease and become zero at some point $u_0 > 0$. Consequently $\widetilde{g}_{22}(u) > 0$ for $u < u_0$. Now choose a $\beta(u)$ so that the resulting graviational plane wave pulse has a bound B such that $|\widetilde{h}(u)| < B$.

3 Proper Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1+\cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation [3] having

$$
\begin{aligned}
t &= t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta && (12) \\
x &= 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta && (13) \\
y &= 2t'\cot\theta - 2x'\cot\theta + y' && (14) \\
z &= z' && (15)
\end{aligned}
$$

By (12) and (13) we have $u = t - x = t' - x' = u'$. Transforming (10) to t', x', y', z' coordinates we get by (12)-(15) a metric $\widetilde{g}'_{\mu\nu}(u')$

$$
\begin{aligned}
ds^2 = & \Big\{-1 - 4[1 - g_{22}(u')]\cot^2\theta\Big\}dt'^2 + 8[1 - g_{22}(u')]\cot^2\theta dt'dx' \\
& + \Big\{1 - 4[1 - g_{22}(u')]\cot^2\theta\Big\}dx'^2 - 4[1 - g_{22}(u')]\cot\theta dt'dy' \\
& + 4[1 - g_{22}(u')]\cot\theta dx'dy' + g_{22}(u')dy'^2 + g_{33}(u')dz'^2 \qquad (16)
\end{aligned}
$$

The metric $\widetilde{g}'_{\mu\nu}(u')$ satisfying $R'_{\mu\nu}(u') = 0$ and $\widetilde{g}'_{\mu\nu}(u') = \eta_{\mu\nu}$ for $u' < 0$ is then also the metric of a gravitational plane wave pulse. Since $|h(u)| < B$ there is then a B' such that $|h'(u')| < B'$.

4 Geodesic curve

The curve

$$t'(\lambda) = (1 + 2\cot^2\theta)\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{17}$$

$$x'(\lambda) = 2\cot^2\theta\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{18}$$

$$y'(\lambda) = -2\cot\theta\lambda + 2\cot\theta \int_0^\lambda \frac{dw}{g_{22}(w)} \tag{19}$$

$$z'(\lambda) = 0 \tag{20}$$

satisfies the geodesic equation for the metric $\widetilde{g}'_{\mu\nu}(u')$ and so is a geodesic curve. Now $g_{22}(u) = 1$ for $u < 0$ so we have $t'(\lambda) = \lambda, x'(\lambda) = y'(\lambda) = z'(\lambda) = 0$ for $\lambda < 0$. Choose θ so that $\cot\theta \neq 0$. We then have by (17)-(20), since the integral goes to positive infinity as $\lambda \to u_0$, that $t'(\lambda) \to -\infty$ as $\lambda \to u_0$. Let λ_1 be large negative hence $t'(\lambda_1)$ is large negative. From (17)-(20) there is a $\lambda_2 > 0$ such that $t'(\lambda_2) = t'(\lambda_1)$. We then have points

$$p_1 = \Big(t'(\lambda_1), \mathbf{x}'(\lambda_1)\Big) = (\lambda_1, 0) \qquad p_2 = \Big(t'(\lambda_2), \mathbf{x}'(\lambda_2)\Big) = \Big(\lambda_1, \mathbf{x}'(\lambda_2)\Big) \tag{21}$$

are on the geodesic and λ_1 is large negative. Also $\lambda_1 - u_0 < x'(\lambda_2) < \lambda_1$.

5 Approximate solution

We have for $x' > t'$ that $\widetilde{h}' = 0$ hence $\widetilde{h}'\widehat{h}' = 0$ for $x' > t'$. Now for large $|\mathbf{x}'|$ we have $\widehat{h}'(\mathbf{x}')$ is small hence for $x' < t'$ and t' large negative $\widehat{h}'$ is small. From section (3) there is a B' such that $|\widetilde{h}'(u')| < B'$. Consequently $\widetilde{h}'\widehat{h}'$ is small for $x' < t'$ and t' large negative. We can conclude $\widetilde{h}'\widehat{h}'$ is small for $t' < t'_0$ where t'_0 is large negative. The cross terms of the Einstein field equations involving factors $\widetilde{h}'\widehat{h}'$ will then be small for $t' < t'_0$. We then have $\widetilde{h}'_{\mu\nu} + \widehat{h}'_{\mu\nu}$ for $t' < t'_0$ will approximately satisfy (5) expressed in prime coordinates and with $T'_{\mu\nu} = 0$ for a point mass. Consequently for $t' < t'_0$

$$h'_{\mu\nu}(t', \mathbf{x}') \approx \widetilde{h}'_{\mu\nu}(t' - x') + \widehat{h}'_{\mu\nu}(\mathbf{x}') \tag{22}$$

6 Contradiction

As the mass of M goes to zero that the path of M approaches the geodesic (17)-(20). Let mass of M be small. There is then a point $p_3 = (\lambda_1, \mathbf{x}'_3)$ on the path of M close to p_2 hence $\mathbf{x}'_3$ is close to $\mathbf{x}'(\lambda_2)$. Now $\widetilde{h}'_{\mu\nu}(p_3)$ and $\widehat{h}'_{\mu\nu}(p_3)$ are finite and λ_1 is large negative hence by (22) $h'_{\mu\nu}(p_3)$ is finite. Now p_3 is a point of the path of M and since M is a point mass we have $h'_{\mu\nu}(p_3)$ is not finite. From the Einstein field equations we get $h'_{\mu\nu}(p_3)$ is finite but from the geodesic equation we get $h'_{\mu\nu}(p_3)$ is not finite. This is a contradiction.

References

[1] S. Weinberg, Gravitation and Cosmology

[2] C. Misner, K. Thorne, J. Wheeler, Gravitation p.957

[3] K. De Paepe, Physics Essays, June 2009

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