Consequences of Planck constant for Relativity

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Abstract

It is known that the existence of Planck length and time contradicts the Lorentz-FitzGerald length contraction and time dilation of special relativity. After showing that the solution of this paradox leaves the spacetime transformations undetermined, it is shown that determining the transformations necessitates a new fundamental equation that governs the local amount of spacetime contraction/dilation.

Keywords— length contraction, Planck length, time dilation, Planck time, Lorentz transformations, stretch

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1 Introduction

It is known that the existence of Planck length and time is in contradiction with fundamental results of special relativity, i.e. *Lorentz-FitzGerald contraction* (length contraction) and time dilation[1, 2, 3, 4, 5]. In a previous attempt[6] I approached the problem from a blindly-formal perspective, but Nature can be more subtle than what formal reasoning can achieve. Although that attempt of mine is not satisfactory to myself, I (and for that matter nobody) have not still come up with a completely satisfactory solution, leaving [6] the only work that at least gets *some* transformations. Here I take a fresh perspective, free from preconceptions and prejudices shaped by attempts of doubly special relativity and loop quantum gravity; I follow another approach that is completely satisfactory but of more radical nature, making the work in need of a fundamental hypothesis about *stretch factor*[7] of two spacetimes. Rather surprisingly via the stretch factor, the work will prove to have a profound connection with the expansion of universe.

There are three crucial requirements for kinematics of *any* proper theory of quantum gravity

1. Invariance of Planck Length
2. Invariance of Planck Time
3. Invariance of Planck Mass

It is improbable for a theory that focuses on only one of these requirements to be able to satisfy others as well; the conditions must be satisfied simultaneously. This is one reason that the existing approaches of *Doubly Special Relativity* are not satisfactory, as they seem confused as to which of these conditions they must satisfy; some take energy, some mass, some length, etc.

The first two requirements are kinematical, hence it is perfectly conceivable that a merely kinematical theory can satisfy them. The third requirement on the other hand, as far as special relativity is concerned seems to be a dynamical one, for special relativity talks about energy, mass and momentum only when it enters laws of motion via its definition of four-momentum

$$p^\mu = mU^\mu,$$

which is the first appearance of mass in special relativity. But it is possible to treat the problem of Planck mass (energy) as a kinematical one as well[1]. To demonstrate our method we focus at first on satisfying the first two requirements. The third will be dealt with in a separate section.

The essential contradiction of special relativity and *existence*[2] of Planck

[1]From this perspective, using

$$M' = \gamma M,$$

it is readily seen that the problem of mass has a similar solution, however, as the notion of 'relativistic mass' is rather controversial and is not derived from coordinate transformations, I decided to deal with it separately.

[2]The importance of this *existence* seems underestimated to me by previous attempts of solving this paradox. The establishment tends to think that without *full* theory of Maxwell's electromagnetism, one cannot arrive at special relativity, while it can be an interesting practice

length and time is that special relativity via its length contraction

$$L' = \gamma L, \tag{1}$$

does not leave Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$

invariant. Similarly due to time dilation

$$T' = \gamma T, \tag{2}$$

Planck time

$$t_P = \sqrt{\frac{\hbar G}{c^5}}$$

is not left invariant.

2 The fundamental hypothesis

There is *but one* simple straightforward way to resolve this contradiction if we are to avoid adding an extra dimension: we must modify length contraction and time dilation in the following manner

$$\begin{cases} L' = L\gamma\zeta(L/l_P), \text{ s.t. } \zeta(1) = 1/\gamma \\ T' = T\gamma\xi(T/t_P), \text{ s.t. } \xi(1) = 1/\gamma, \end{cases} \tag{3}$$

for unknown sufficiently-smooth functions $\zeta, \xi : \mathbb{R} \to \mathbb{R}$. As we expect to recover length contraction and time dilation of special relativity (1),(2) for lengths $L \gg l_P$ and time durations $T \gg t_P$, we must impose the condition

$$\begin{cases} \lim_{x\to\infty} \zeta(x) = 1, \\ \lim_{y\to\infty} \xi(y) = 1. \end{cases} \tag{4}$$

Our task now is to find functions ζ and ξ. It might be expected that determining coordinate transformations will determine ζ and ξ as well, but it turns out that coordinate transformations give no useful information about these functions. To see this, we assume that the new coordinate transformations are given by

$$\begin{cases} x' = f(x,t) \\ t' = g(x,t), \end{cases} \tag{5}$$

where f, g are arbitrary functions[3] and contrary to special relativity, not necessarily linear. Now to apply (3), note that

$$L' = x'_2 - x'_1 = f(x_2, t_2) - f(x_1, t_1),$$

to see that once (Spinoza's) God says 'let there *exist* a constant of nature which is *a* velocity', one day after publication of Newton's Principia you can derive the full kinematics of special relativity without knowing what light is. Similarly, once (Spinoza's) God says 'let there *exist* a constant of nature which is *a* length/time' you should be able to reconcile the existence of Planck length and time with special relativity a day after publication of Einstein's *Zur Elektrodynamik bewegter Körper*. No artificial talks of 'Hopf algebras', etc. is necessary.

[3]For simplicity and without loss of generality we work in 1+1 dimensions in this paper.

$$T' = t_2' - t_1' = g(x_2, t_2) - g(x_1, t_1),$$

Application of (3) now yields

$$\begin{cases} f(x_2, t_2) - f(x_1, t_1) = \gamma(x_2 - x_1)\zeta(x_2 - x_1) \\ g(x_2, t_2) - g(x_1, t_1) = \gamma(t_2 - t_1)\xi(t_2 - t_1) \end{cases} \tag{6}$$

In the first equation divide by $x_2 - x_1$ and let $x_2 \to x_1$ and in the second, divide by $t_2 - t_1$ and let $t_2 \to t_1$,

$$\lim_{x_2 \to x_1} \frac{f(x_2, t_2) - f(x_1, t_1)}{x_2 - x_1} = \gamma\zeta(0)$$

$$\lim_{t_2 \to t_1} \frac{g(x_2, t_2) - g(x_1, t_1)}{t_2 - t_1} = \gamma\xi(0),$$

giving

$$\frac{\partial f(x, t)}{\partial x} = \gamma\zeta(0), \tag{7}$$

and

$$\frac{\partial g(x, t)}{\partial t} = \gamma\xi(0). \tag{8}$$

We now apply the principle of constancy of velocity of light,

$$\frac{dx}{dt} = \frac{dx'}{dt'} = c,$$

since

$$\begin{cases} dx' = f_x dx + f_t dt \\ dt' = g_x dx + g_t dt, \end{cases}$$

$$\frac{dx'}{dt'} = \frac{\gamma\zeta(0)dx + f_t dt}{g_x dx + \gamma\xi(0)dt} = \frac{\gamma\zeta(0)cdt + f_t dt}{g_x cdt + \gamma\xi(0)dt} = \frac{\gamma\zeta(0)c + f_t}{g_x c + \gamma\xi(0)} = c,$$

we arrive at

$$\gamma\zeta(0)c + f_t = g_x c^2 + \gamma\xi(0)c. \tag{9}$$

As we see, functions ζ and ξ cannot be determined from coordinate transformations. *There must be some condition on (3) itself, using which one can determine ζ and ξ*, and then, coordinate transformations. It is thus evident that *we need some differential equations for the functions L' and T'* in (3). There is however, one further condition that we must impose so that we will get proper transformations,

$$\zeta(0), \xi(0) \neq 0, \tag{10}$$

in accord with the obvious expectation that ζ and ξ cannot be linear functions.

3 Dynamics of stretch space

3.1 Length contraction/time dilation are geodesics

Physical reasoning or intuition seems to have little if nothing to offer for finding such an equation. We therefore here appeal to formal reasoning and inductive inference. To that end observe that (1) and (2) can be written as

$$\frac{d^2L'}{dL^2} = 0, \quad L'(0) = 0, \tag{11}$$

$$\frac{d^2T'}{dT^2} = 0, \quad T'(0) = 0, \tag{12}$$

meaning that we are dealing with equations of the same form as that of geodesic equation. As dL'/dL and dT'/dT are in fact *stretch* factors[7] of Euclidean space of lengths and times, length contraction and time dilation of special relativity are geodesics of 'stretch space'.

3.2 Towards an equation

We are looking for a differential equation which reduces to (11) and (12) when $L \gg l_P$, $T \gg t_P$. We must be cautious now, for without a firm physical guide one can easily be led astray. First to make quantities dimensionless appropriate for a differential equation, let

$$\lambda := \frac{L}{l_P},$$

$$\tau := \frac{T}{t_P}.$$

There are infinitely many ways to write such an equation. Any equation of the form

$$\frac{d^2\lambda'}{d\lambda^2} = f(\lambda), \quad \lim_{\lambda\to\infty} f(\lambda) = 0,$$

or

$$\frac{d^2\lambda'}{d\lambda^2} = f(\lambda'), \quad \lim_{\lambda'\to\infty} f(\lambda') = 0,$$

would do. Evidently we are now in need of a physical principle which can kill arbitrariness. Let us begin with the simplest case of a *Cauchy-Euler equation*

$$\frac{d^2\lambda'}{d\lambda^2} = \frac{1}{\lambda^n}, \quad n \in \mathbb{N} \tag{13}$$

and

$$\frac{d^2\tau'}{d\tau^2} = \frac{1}{\tau^m}, \quad m \in \mathbb{N} \tag{14}$$

Again we take the simplest choice $n = m = 1$,

$$\frac{d^2\lambda'}{d\lambda^2} = \frac{1}{\lambda} \tag{15}$$

and

$$\frac{d^2\tau'}{d\tau^2} = \frac{1}{\tau}. \tag{16}$$

Let us now proceed to solve our hypothesized equations (15) and (16). Observe that these equations are in fact (the simplest) *Cauchy-Euler equation*

$$xy''(x) = 1.$$

Accordingly the solutions of (15) and (16) is given by

$$\begin{cases} L' = A\lambda + l_P \lambda \log \lambda + B \\ T' = E\tau + t_P \tau \log \tau + F, \end{cases}$$

but we do not need to go further to see this solution is utterly wrong. The term $l_P \lambda \log \lambda$ goes to infinity as $\lambda \to \infty$. Also this solution cannot be factorized in the form required by (3). What is to be done? There is only one solution and that is to find a clear physical principle.

3.3 Principle of Stationary Stretch

We begin by the principle that is equivalent to (1) and (2). Observe that length contraction of special relativity results from

$$\delta \int \sqrt{dL^2 + dL'^2} = 0,$$

similarly for time dilation

$$\delta \int \sqrt{dT^2 + dT'^2} = 0.$$

This observation shows that there exist clear physical principles underlying length contraction and time dilation.

3.4 The fundamental equation

In fact (11) and (12) suggest themselves to be the left-hand-sides of an $F = ma$–like equation. It is perfectly consistent to consider the following table of analogies

L'	L
x	t

meaning that the following formal equivalence holds

$$\begin{cases} L' = \gamma L \\ x = vt. \end{cases} \tag{17}$$

These observations, together with the fact that (13) and (14) did not work, suggest that the right-hand-side of our desired equation must be formally equivalent to Newton's Law of Gravitation. Having (17) in mind we are led to

$$\frac{d^2\lambda'}{d\lambda^2} = \frac{1}{\lambda'^2},$$

returning to L', L and T', T variables, we have

$$\boxed{\frac{d^2L'}{dL^2} = \frac{l_P}{L'^2}} \tag{18}$$

and similarly for time

$$\boxed{\frac{d^2T'}{dT^2} = \frac{t_P}{T'^2}} \tag{19}$$

With these equations at hand, unknown functions ζ, ξ in (3) are now determined in principle.

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