Proof of Firoozbakht's conjecture

Kouji Takaki

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Abstract

We showed that the following inequality of Firoozbakht's conjecture holds when $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds.

$$\log(p_{n+1}) - \log(p_n) < \log(p_n)/n$$

Moreover, in other case, the following inequality holds because the derivative function of $\log(x)$ decreases monotonically for $x > 0$.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) < \log(n+1) - \log(n)$$

We showed that the inequality of this conjecture is obtained by this inequality when $p_{n+1} - p_n \geqq \log(p_n)$ holds. From the above, we proved that Firoozbakht's conjecture is true.

Contents

1. Introduction

In number theory, Firoozbakht's conjecture (or the Firoozbakht conjecture) is a conjecture about the distribution of prime numbers. It is named after the Iranian mathematician Farideh Firoozbakht from the University of Isfahan who stated it first in 1982. The conjecture states that $\sqrt[n]{p_n}$ (where p_n is the nth prime) is a strictly decreasing function of n, i.e.,

$$\sqrt[n+1]{p_{n+1}} < \sqrt[n]{p_n} \text{ for all } n \geqq 1$$

Equivalently,

$$p_{n+1} < p_n^{1+1/n} \text{ for all } n \geqq 1$$

(Quoted from Wikipedia)

2. Proof

When we write log in this paper, log refers to natural logarithm. Let n be a positive integer. We will prove that the following inequality holds for any n.

$$p_{n+1} < p_n^{1+1/n} \ \dots (1)$$

I When $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds

$\log(p_{n+1})/\log(p_n) < 1 + 1/p_n$

Since $p_n > n$ holds,

$\log(p_{n+1})/\log(p_n) < 1 + 1/n$

holds. This inequality accords with the inequality (1).

Ⅱ When $\log(p_{n+1}) - \log(p_n) \geqq \log(p_n)/p_n$ holds

ⅰ When $p_{n+1} - p_n < \log(p_n)$ holds

$p_{n+1}/p_n < 1 + \log(p_n)/p_n$

$\log(p_{n+1}) - \log(p_n) < \log(1 + \log(p_n)/p_n) < \log(p_n)/p_n$

The case of ⅰ does not exist since this inequality is contrary to the condition of Ⅱ.

ⅱ When $p_{n+1} - p_n \geqq \log(p_n)$ holds

Let x be a real number. Let $f(x) = \log(x)$. $f'(x) = 1/x$ and $f''(x) = -1/x^2$ hold. The derivative function of $f(x)$ is a monotonically decreasing function for $x > 0$ since $f''(x) < 0$ holds for $x > 0$. The following inequalities hold for all n where $n \geqq 1$ holds because $f'(x) > 0$ and $f''(x) < 0$ hold for $x > 0$ and $p_{n+1} - p_n \geqq 1$ holds.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) < \log(n+1) - \log(n) \ \ldots (2)$$

Let $F_n(x) = (\log(x) - \log(p_n))/(x - p_n)$. We suppose $x > p_n$ holds. Let $\mathbb{R}$ be the set of real numbers, A_n be the subset of $\mathbb{R}$ consisting of all elements taken by $F_n(x)$ when $x > p_n$ holds and u_n be an upper bound of A_n. When $p_{n+1} \geqq p_n + \log(p_n)$ holds, u_n is the value of $F_n(x)$ at $x = p_{n+1} = p_n + \log(p_n)$ since it is clear from the graph of $f(x)$ that the function $F_n(x)$ decreases monotonically. Therefore,

$$u_n = (\log(p_{n+1}) - \log(p_n))/\log(p_n)$$

holds. And u_n is lesser than the value of the right side of the inequality (2) for any n since $f'(x)$ decreases monotonically, $p_n > n$ holds and the distance between p_n and p_{n+1}, $\log(p_n)$ is greater than the one between n and $n+1$ for $n \geqq 2$. Therefore, the following inequalities hold for $n \geqq 2$.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) \leqq u_n < \log(n+1) - \log(n)$$

$$(\log(p_{n+1}) - \log(p_n))/\log(p_n) < \log(n+1) - \log(n)$$

$$\log(p_{n+1})/\log(p_n) < 1 + \log(1 + 1/n) < 1 + 1/n$$

$$\log(p_{n+1})/\log(p_n) < 1 + 1/n \ \ldots (3)$$

This inequality coincides with the inequality (1). When $n = 1$ holds, the inequality (3) holds since $\log(p_{n+1})/\log(p_n) = 1.5849 \ldots < 2$ holds.

From the above, it is proved that Firoozbakht's conjecture is true since the inequality (1) holds for all n where $n \geqq 1$ holds. (Q.E.D.)

3. Acknowledgement

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4. References

[1] Wikipedia https://en.wikipedia.org/wiki/Firoozbakht%27s_conjecture

[2] Wikipedia https://en.wikipedia.org/wiki/Prime_number_theorem