Cubic Tetrahedra – part 1

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Abstract

Tetrahedral numbers have a well-defined construction but are not the only way of building tetrahedra from numbers, as is evidenced in studies we will briefly consider as a motivation for looking at the following proposed construction of number shapes, this being a new perspective on a pattern often written in a slightly different context, to a different stacking of numbers. We will go on to look at a few properties of shapes built from this novel construction, including a few specific shapes built from prime numbers, taking us neatly into some parabolic equations with prime-heavy cartesian coordinates.

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Introduction

Although we will consider patterns written about in detail by many authors before, this is intended to be an original work and not a derivative work. This paper looks to report a short investigation by a single researcher with limited mathematical background, and as such will be short on deep commentary, claims, conjectures and world-ending conclusions. Through a simple reporting of findings, the author hopes to provide material and momentum for new studies into the construction described herein. A second investigation into these shapes has been started, but in the interest of keeping this report clean and focused, these will be included in a follow-up report.

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Background, research and rationale

This investigation follows a previous look at Observing the Movement of Prime Numbers in Prime-based Number Shapes (viXra:2012.0157), in which the nature of primes above 2 being odd numbers was used to explore how combinatorics might be used to view primes in a system of numbers where odd numbers were in a predictable place (within Wacław Sierpiński's famous odds and evens pattern imbedded in Pascal's/Pingala's triangle of the binomial coefficient), and then viewed in a number shape where odd and even numbers had been separated. This following report should be a lot simpler to read than its predecessor but comes off of some of the patterns in that report suggesting to the author that tetrahedra are perhaps under-studied.

There are a number of ways of graphing a tetrahedron in cartesian space. One such way is to graph coordinates of the tetrahedron's vertices with

(+1, +1, +1);

(−1, −1, +1);

(−1, +1, −1);

(+1, −1, −1).

This yields a tetrahedron with edge-length $2\sqrt{2}$, centered at the origin.

(Davis, D., Dods, V., Traub, C., & Yang, J. (2017). Geodesics on the regular tetrahedron and the cube. Discrete Mathematics, 340(1), 3183-3196.)

In a 3D implicit graph, two regular tetrahedra can be drawn using a similar ternary pattern of polarities. Graphing 0=abs(+x +y +z), 0=abs(−x −y +z), 0=abs(−x +y −z) and 0=abs(+x, −y −z) gives us the shape shown in figure 1. Using the absolute value allows us to not have to graph eight graphs, where we might look at a ternary pattern of polarities, +++,++-,+-+,+--,-++,-+-,--+,---.

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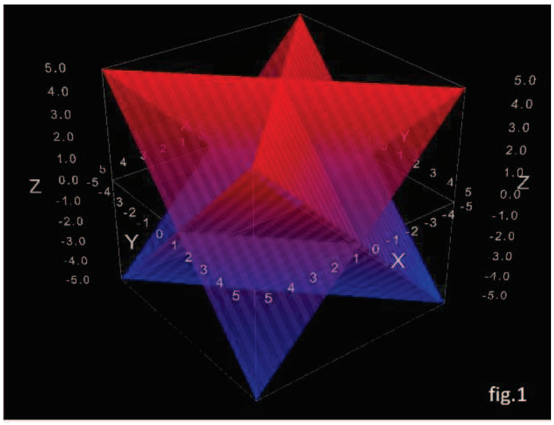
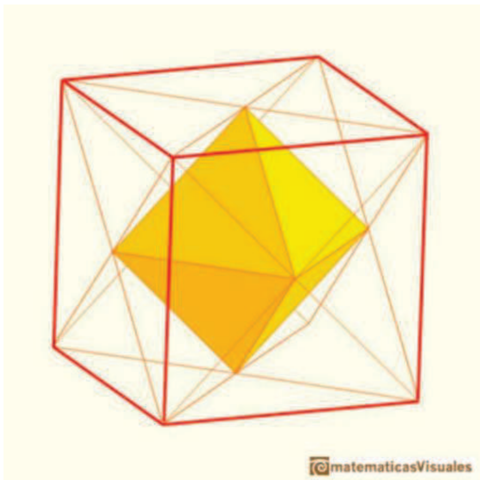


fig.1

Another way of embedding two regular tetrahedra in cubic space was explored in this graph, where we also look at how such shapes embedded in cubes can start to be packed in a cubic matrix: String (desmos.com) (Edited from an open-source spinning cube file with no traceable author)

It should be noted that these two graphings show two tetrahedra sharing the same space, where we might consider that their interaction form an octahedron. Consequently, this shape is normally termed as a compound polyhedron of an octahedron and eight tetrahedra, or of a cube and an octahedron, and packed together in a tetrahedral-octahedral honeycomb. (Coxeter, H. S. M. (1954). Regular honeycombs in elliptic space. Proceedings of the London Mathematical Society, 3(1), 471-501.)



We will come to a sequence of numbers relevant to this shape packing later in this paper.

It might be then noted that none of the tetrahedra in these graphs or links are sitting in the orientation we might be used to seeing as being the construction of tetrahedral numbers, as shown in figure 2.

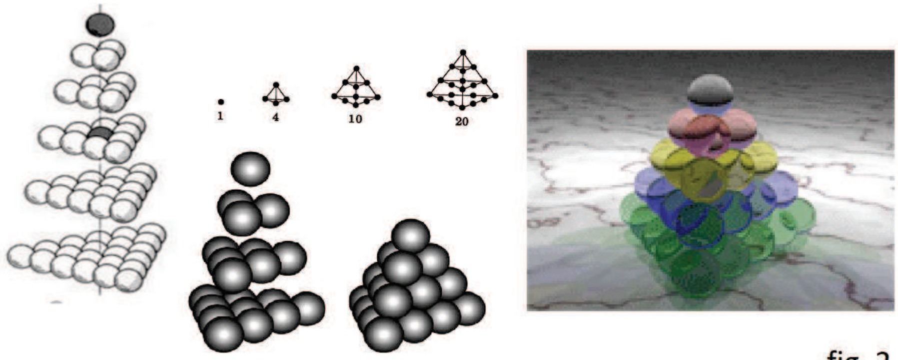


fig. 2

Consequently, it was decided to explore numbers constructed by placing cubes together, as illustrated in figure. 3, in line with how we view numbers stacking in 3D Cartesian xyz graphs. The notation included in figure 3 will be explained in the following section. We will look at some patterns of prime numbers in these '3D pixellated' tetrahedra (referred to as cubic tetrahedra here-on-in).

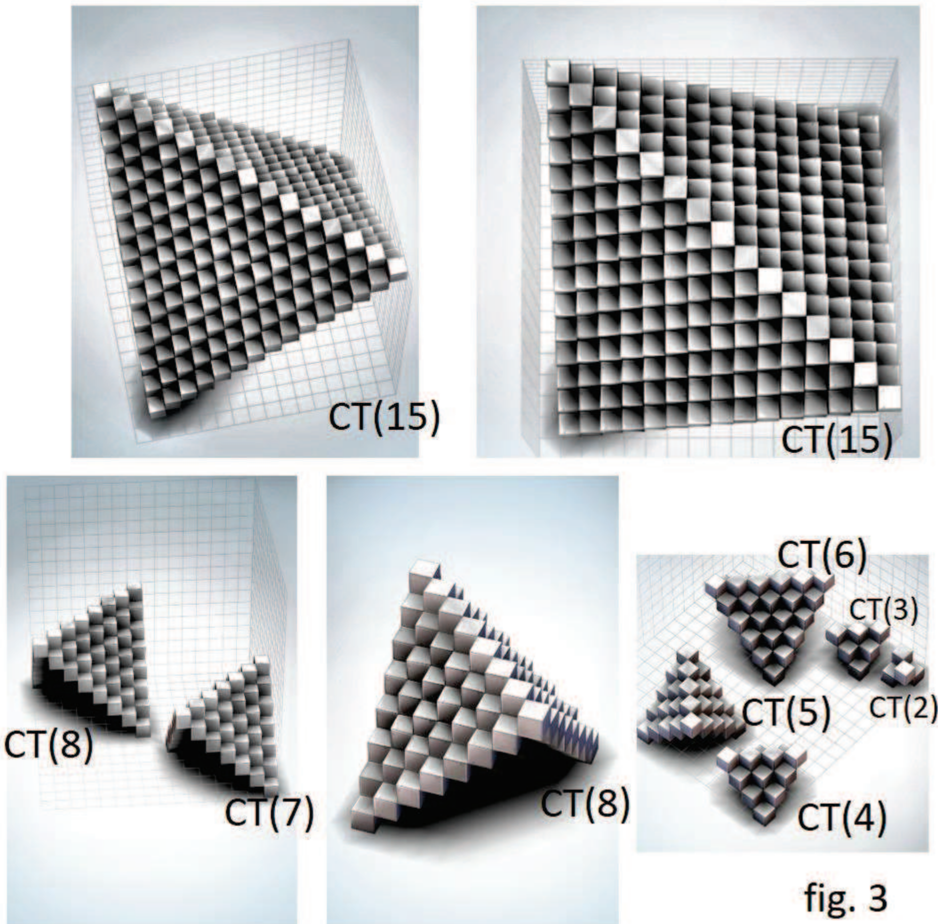


fig. 3

Cubic Tetrahedra – part 1

Looking at cubic tetrahedra with a side length of 1 to 16 cubes

It is not intended to say for such number packing as we are looking at that spheres could not be used, but for ease of illustration we will here use cubes, where we might similarly pack spheres. For this reason, this paper will stop short of looking at line lengths as algebraic numbers based on the dimensions of cubes, instead we will just consider the number of cubes used to build these cubic tetrahedra. We will notate this as CT(n) being the number of cubes forming a line between any two vertices of the cubic tetrahedra (but more specifically the bottom line), as illustrated in figure 3. We will also look at the different layers of these shapes, so to give notation to this, we will use CT(n,*l*) where *l* denotes the *l*th layer up from the bottom line of n being *l*=1. This is illustrated in figure 4, showing layer 5 of a cubic tetrahedron with a side length of 16 cubes and layer 2 of a cubic tetrahedron with a side length of 4 cubes. Counting the cubes on these two layers, we then write CT(16,5)=104 cubes and CT(4,2)=8 cubes. We will notate the total amount of cubes in CT(n) as ΣCT(n), being the sum of all the cubes on all of the layers.

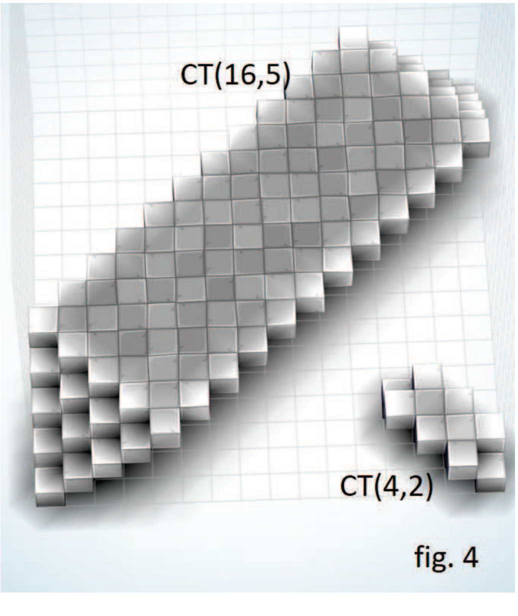


fig. 4

The amounts of cubes in each layer of CT(16) has been highlighted in figure 5 with a blue accent. This triangular table shows layer amounts for cubic tetrahedra with side lengths of 1 cube through to 16 cubes. CT(16,5)=104 has been highlighted with a darker shade of blue.

side length →

layers ↓

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CT(16,1)=16
2	5	8	11	14	17	20	23	26	29	32	35	38	41	44		CT(16,2)=44
3	8	13	18	23	28	33	38	43	48	53	58	63	68			CT(16,3)=68
4	11	18	25	32	39	46	53	60	67	74	81	88				CT(16,4)=88
5	14	23	32	41	50	59	68	77	86	95	104					CT(16,5)=104
6	17	28	39	50	61	72	83	94	105	116						CT(16,6)=116
7	20	33	46	59	72	85	98	111	124							CT(16,7)=124
8	23	38	53	68	83	98	113	128								CT(16,8)=128
9	26	43	60	77	94	111	128									CT(16,9)=128
10	29	48	67	86	105	124										CT(16,10)=124
11	32	53	74	95	116											CT(16,11)=116
12	35	58	81	104												CT(16,12)=104
13	38	63	88													CT(16,13)=88
14	41	68														CT(16,14)=68
15	44															CT(16,15)=44
16																CT(16,16)=16

fig.5

This table only goes up to CT(16) at this point as that's as big a cubic tetrahedron as the software used allowed. We will go on to look at how we might consider larger cubic tetrahedra, but for now we will look to see the amount of cubes in CT(n) for n = 1 to 16.

By adding together the layer totals in CT(16), as illustrated in figure 5, we can see that ΣCT(16)=CT(16,1)+CT(16,2)+CT(16,3)…+CT(16,15)+(CT(16,16)=1376 cubes.

Lets take a smaller cubic tetrahedron, with a side length of 5 cubes.

ΣCT(5)=CT(5,1)+CT(5,2)+CT(5,3)+CT(5,4)+CT(5,5)=5+11+13+11+5=45 cubes.

We can now look at the total amount of cubes involved in building cubic tetrahedra with side lengths 1-16 cubes. They are as follows:

Cubic Tetrahedra – part 1

ΣCT(1)	1	1
ΣCT(2)	2+2	4
ΣCT(3)	3+5+3	11
ΣCT(4)	4+8+8+4	24
ΣCT(5)	5+11+13+11+5	45
ΣCT(6)	6+14+18+18+14+6	76
ΣCT(7)	7+17+23+25+23+17+7	119
ΣCT(8)	8+20+28+32+32+28+20+8	176
ΣCT(9)	9+23+33+39+41+39+33+23+9	249
ΣCT(10)	10+26+38+46+50+50+46+38+26+10	340
ΣCT(11)	11+29+43+53+59+61+59+53+43+29+11	451
ΣCT(12)	12+32+48+60+68+72+72+68+60+48+32+12	584
ΣCT(13)	13+35+53+67+77+83+85+83+77+67+53+35+13	741
ΣCT(14)	14+38+58+74+86+94+98+98+94+86+74+58+38+14	924
ΣCT(15)	15+41+63+81+95+105+111+113+111+105+95+81+63+41+15	1135
ΣCT(16)	16+44+68+88+104+116+124+128+128+124+116+104+88+68+44+16	1376

This sequence of numbers is logged in the OEIS as entry A006527 and with the equation a(n) = (n^3 + 2*n)/3. A comment on this OEIS entry by Dr Jason Pruski notes that this sequence relates to the number of unit tetrahedra contained in an n-scale tetrahedron composed of a tetrahedral-octahedral honeycomb.

Looking at larger cubic tetrahedra

In order to look at larger cubic tetrahedra, without having to build them in 3D software and count and add the layers as we did above, we will go back to the triangular table in figure 5 to see if we can divine some rules that can be extrapolated. This was achieved by taking lines of numbers and searching them in WolframAlpha.

side length →

layers																
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	a(n) = 1*n + 1.
2	5	8	11	14	17	20	23	26	29	32	35	38	41	44		a(n) = 3*n + 2.
3	8	13	18	23	28	33	38	43	48	53	58	63	68			a(n) = 5*n + 3.
4	11	18	25	32	39	46	53	60	67	74	81	88				a(n) = 7*n + 4.
5	14	23	32	41	50	59	68	77	86	95	104					a(n) = 9*n + 5.
6	17	28	39	50	61	72	83	94	105	116						a(n) = 11*n + 6.
7	20	33	46	59	72	85	98	111	124							a(n) = 13*n + 7.
8	23	38	53	68	83	98	113	128								a(n) = 15*n + 8.
9	26	43	60	77	94	111	128									a(n) = 17*n + 9.
10	29	48	67	86	105	124										a(n) = 19*n + 10.
11	32	53	74	95	116											a(n) = 21*n + 11.
12	35	58	81	104												a(n) = 23*n + 12.
13	38	63	88													a(n) = 25*n + 13.
14	41	68														a(n) = 27*n + 14.
15	44															a(n) = 29*n + 15.
16																a(n) = 31*n + 16.
a(n) = 1*n + 1.	a(n) = 3*n + 2.	a(n) = 5*n + 3.	a(n) = 7*n + 4.	a(n) = 9*n + 5.	a(n) = 11*n + 6.	a(n) = 13*n + 7.	a(n) = 15*n + 8.	a(n) = 17*n + 9.	a(n) = 19*n + 10.	a(n) = 21*n + 11.	a(n) = 23*n + 12.	a(n) = 25*n + 13.	a(n) = 27*n + 14.	a(n) = 29*n + 15.	a(n) = 31*n + 16.	

Since the rules found were relatively simple, it was found that having the numbers in a spreadsheet and dragging them into new cells extrapolated the rules without having to manually type the rules in. In this fashion, the table in figure 5 was extrapolated to cover CT(1) to CT(77) cubes, and a mod based prime checking formula was used to look for primes in this larger table. In this way it was found that cubic tetrahedra CT(3), CT(5), CT(11) & CT(19) are built from layers with prime amounts of cubes, which did seem a remarkable property, especially for the two larger shapes. The construction of these shapes are shown below in figure 6.

side length →

layers																											
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	2
2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65	68	71	74	77	80	8
3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	1
4	11	18	25	32	39	46	53	60	67	74	81	88	95	102	109	116	123	130	137	144	151	158	165	172	179	186	1
5	14	23	32	41	50	59	68	77	86	95	104	113	122	131	140	149	158	167	176	185	194	203	212	221	230	239	2
6	17	28	39	50	61	72	83	94	105	116	127	138	149	160	171	182	193	204	215	226	237	248	259	270	281	292	3
7	20	33	46	59	72	85	98	111	124	137	150	163	176	189	202	215	228	241	254	267	280	293	306	319	332	345	3
8	23	38	53	68	83	98	113	128	143	158	173	188	203	218	233	248	263	278	293	308	323	338	353	368	383	398	4
9	26	43	60	77	94	111	128	145	162	179	196	213	230	247	264	281	298	315	332	349	366	383	400	417	434	451	4
10	29	48	67	86	105	124	143	162	181	200	219	238	257	276	295	314	333	352	371	390	409	428	447	466	485	504	5
11	32	53	74	95	116	137	158	179	200	221	242	263	284	305	326	347	368	389	410	431	452	473	494	515	536	557	5
12	35	58	81	104	127	150	173	196	219	242	265	288	311	334	357	380	403	426	449	472	495	518	541	564	587	610	6
13	38	63	88	113	138	163	188	213	238	263	288	313	338	363	388	413	438	463	488	513	538	563	588	613	638	663	6
14	41	68	95	122	149	176	203	230	257	284	311	338	365	392	419	446	473	500	527	554	581	608	635	662	689	716	7
15	44	73	102	131	160	189	218	247	276	305	334	363	392	421	450	479	508	537	566	595	624	653	682	711	740	769	7
16	47	78	109	140	171	202	233	264	295	326	357	388	419	450	481	512	543	574	605	636	667	698	729	760	791	822	8
17	50	83	116	149	182	215	248	281	314	347	380	413	446	479	512	545	578	611	644	677	710	743	776	809	842	875	9
18	53	88	123	158	193	228	263	298	333	368	403	438	473	508	543	578	613	648	683	718	753	788	[illegible]	[illegible]	[illegible]	[illegible]	9
19	56	93	130	167	204	241	278	315	352	389	426	463	500	537	574	611	648	685	722	759	796	833	8	[illegible]	[illegible]	[illegible]	#
20	59	98	137	176	215	254	293	332	371	410	449	488	527	566	605	644	683	722	761	800	839	878	917	956	995	###	#

fig.6

Looking at the larger two shapes CT(11) and CT(19), a search of the amount of cubes in each of their layers reveals that they have equations, and that these two sequences are parabolic.

Again using *l* to donate the layer number, we see that $CT(11,l) = (-2*l^2)+(24*l)-11$, and the equation for $CT(19,l) = (-2*l^2)+(40*l)-19$. To demonstrate this, we will look at CT(11,4) and CT(19,12), the 4th layer of a cubic tetrahedron with a side length of 11 cubes and the 12th layer of a cubic tetrahedron with a side length of 19 cubes.

Plugging $l=4$ into $-2*l^2+24*l-11 = (-2*4^2)+(24*4)-11 = (-32)+(96)-11 = 53$, which we can see in the table above is correct, that CT(11,4) has 53 cubes. Similarly, plugging $l=12$ into $(-2*l^2)+(40*l)-19 = (-2*12^2)+(40*12)-12 = (-288)+(480)-19 = 173$ is confirmed in the table above, that the 12th layer of CT(19) is composed of 173 cubes.

Plotting $y=(-2*x^2)+(24*x)-11$ and $y=(-2*x^2)+(40*x)-19$, we get the lines shown in figure 7. The x component of each coordinate relates to the line length of the two cubic tetrahedra CT(11) and CT(19), and the y coordinates report the prime number amounts of cubes in each of their layers.

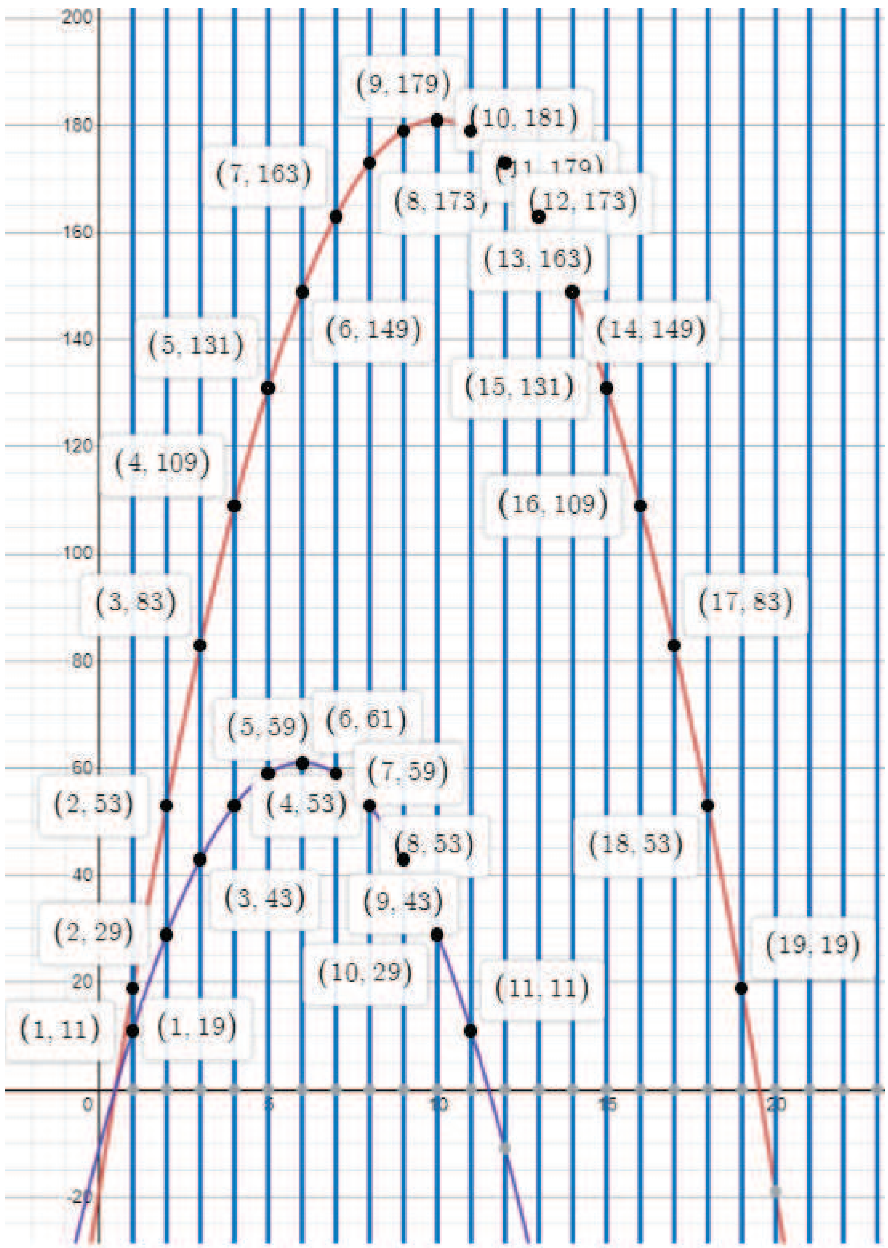


fig.7

The similarity of these two lines suggests that the relationship between other cubic tetrahedrons line lengths and the number of cubes in their layers will also plot parabolas. To see this, we will transform the two equations used as follows:

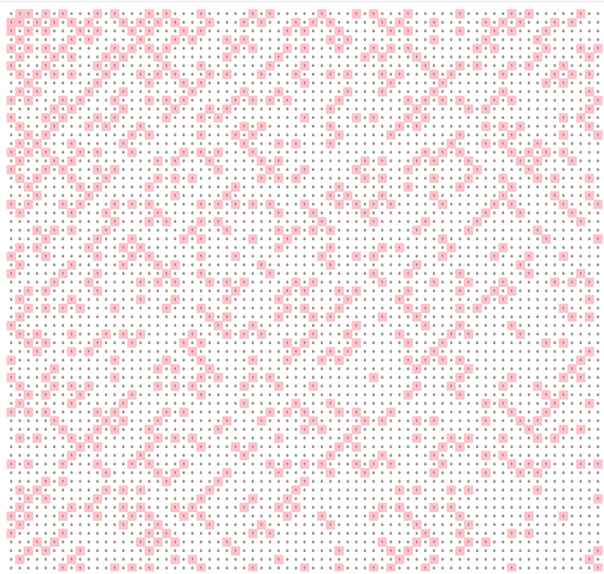
$$-2l^2 + 24l - 11 = \frac{11^2 + 1}{2} - 2\left(l - \frac{11^2 + 1}{2}\right)^2$$

$$-2l^2 + 40l - 19 = \frac{19^2 + 1}{2} - 2\left(l - \frac{19^2 + 1}{2}\right)^2$$

Exploring $CT(n,l) = \frac{n^2+1}{2} - 2(l - \frac{n^2+1}{2})^2$ in the following graph, using axis x=n and y=*l*, we see that the y coordinates match up with the layer tables for CT(n,*l*) we explored before in figures 5 and 6.

Parabolas of cubic tetrahedra (desmos.com)

There are other prime patterns in the table of CT(n,*l*), including that CT(13,1), CT(14,2), CT(15,3), CT(16,4)… starts a sequence of $2x^2 + 22x - 11$ with 10 primes. This is shown in the following table of cubic tetrahedra with side lengths 1 –68, and where prime numbers have been highlighted in a separate table using the Taylor series to report on primes. The question begs itself if larger cubic tetrahedra with all-prime layers might exist, and if such shapes might also have a prime value of ΣCT(n), but because of errors arising from how we use floating point arithmetic, MS Excel is not the program in which to try answer this question.



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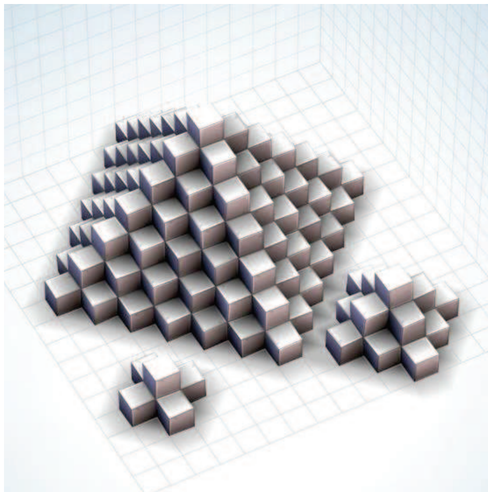
Conclusions and next steps

As much as it might be a redundant point that square numbers relate to triangular matrices as well as squares, and to then pick up on the term 'square numbers' being somewhat misleading, the point that sequence A006527 might have as much reason to be called 'tetrahedral numbers' as sequence A000292 could be viewed as redundant. This is surely a pedantic argument, based on the limited nature of language. When Sir Frederick Pollock raised questions about tetrahedral and pyramidal numbers, his questions were about a specific definition of tetrahedral numbers, a specific stacking of billiard balls. The point of this paper is that the tetrahedral number stacking he asked about is not the only stacking of cubic or spherical shaped numbers we can use to build tetrahedra, and that there may be merit in exploring alternative number stacking, especially when we already know we can build tetrahedra in Cartesian space with a cubic lattice of numbers. Similarly, there might also be answers in looking at a cubic stacking of numbers to build pyramids. It seems fair to say that there will either be answers in such investigations, or an insight into the difficulty already inherent in these questions, based as they are on conceptualizing spheres sitting on spheres in a way easily abandoned to look at ±x±y±z graphs.



The link between ternary patterns of operands and the coordinates of a tetrahedrons vertices might well point at there being links between other Platonic solids and use of axes more complex than ±x±y±z.

This report will end here as it is the author's intention to spin away from prime numbers and to explore geometrical manipulations of number stacking. These will start with considering the outer shell of CT(n), and look at how we might smooth it into a regular tetrahedron of side length n. It is hoped we might then find alternative and perhaps 'less expensive' ways of computing irrational numbers.