Exploring the Christoffel Symbols

Anamitra Palit

Physicist, Freelancer

P154 Motijheel Avenue, Flat C4,Kolkata 700074,India

Cell number:+919163892336

Palit.anamitra@gmail.com

Abstract

The brief article explores the Christoffel symbols starting from its transformation rules and by such an exploration demonstrates the fact that only linear transformations are possible. It derives that Christoffel symbols behave like rank three tensors[with one upper index and two lower ones] since the term standing in the way of such behavior vanishes.

Keywords: Christoffel Symbols, Four Acceleration, Riemann Tensor

Introduction

The transformation of the Christoffel symbol is considered first. From there we derive that transformation are necessarily linear and that the Chrstoffel symbol should transform like rank three tensors with one upper index.

Derivation

Transformation of the Christoffel Symbols[1]

$$\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \bar{\Gamma}^n{}_{jk} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \Gamma^m{}_{pq} \quad (1)$$

[In the last line $n, p$and q are dummy indices]

For $j \neq k$ in (1)we have,

$$\frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^s} \left[\bar{\Gamma}^n{}_{jk} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \Gamma^m{}_{pq} \right]$$

[In the summation above a specific value of j could be k]

[In the last j is a dummy index in the last line asides$n, p$and q]

$$\Rightarrow \frac{\partial \bar{x}^j}{\partial x^s} \frac{\partial}{\partial \bar{x}^j} \left(\frac{\partial x^m}{\partial \bar{x}^k} \right) = \frac{\partial \bar{x}^j}{\partial x^s} \left[\bar{\Gamma}^n{}_{jk} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \bar{\Gamma}^m{}_{pq} \right]$$

$$\Rightarrow \frac{\partial}{\partial x^s}\left(\frac{\partial x^m}{\partial \bar{x}^k}\right) = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\bar{\Gamma}^m{}_{pq}\right]$$

$$\Rightarrow \frac{\partial^2 x^m}{\partial x^s \partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right] \; (2)$$

Since the partial differential operator commutes we have,

$$\Rightarrow \frac{\partial^2 x^m}{\partial \bar{x}^k \partial x^s} = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right]$$

$$\Rightarrow \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^m}{\partial x^s}\right) = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right]$$

$$\Rightarrow \frac{\partial \delta^m{}_s}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right]$$

$$\Rightarrow 0 = \frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right]$$

$$\Rightarrow 0 = \frac{dx^s}{d\tau}\frac{\partial \bar{x}^j}{\partial x^s}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right]$$

$$\Rightarrow 0 = \frac{d\bar{x}^j}{d\tau}\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq}\right] \; (3)$$

In equation (3) $\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\bar{\Gamma}^m{}_{pq}\right]$ is a point function while the quadruplet $\left\{\frac{d\bar{x}^j}{d\tau} j = 1,2,3,4\right\}$ is a path function.We do have an infinitude of $\frac{d\bar{x}^j}{d\tau}$for every $\left[\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\bar{\Gamma}^m{}_{pq}\right]$.Therefore from equation (3) we infer that for $j \neq k$,

$$\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\bar{\Gamma}^m{}_{pq} = 0 \; (4)$$

From (1) and (4),we have,

$$\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = 0 \; (5)$$

Though equation (5) has been derived assuming $j \neq k$ at the outset it is valid for both $j \neq k$and $j = k$

as we shall soon see.

From $\frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^m}{\partial \bar{x}^k}\right) = 0$, $\frac{\partial x^m}{\partial \bar{x}^k}$ is independent of $\bar{x}^{j\neq k}$.In view of that let us consider $\frac{\partial x^m}{\partial \bar{x}^k} = f(x^k) \neq constant$ and hence $\frac{\partial^2 x^m}{\partial \bar{x}^{k^2}} \neq 0$.To analyze the situation we consider the coordinate curves of $\bar{x}^\mu$ which are parallel to each other.If the $\bar{x}^j$ curves are orthogonal to these curves then $\bar{x}^k$will not change along these curves and $\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = 0$.Now with the same unbarred system[coordinate curves not changing there] we replace the orthogonal system of the barred system by a non orthogonal one so that $\bar{x}^k$varies long the $\bar{x}^j$curves. The new $\bar{x}^j$ curves are orthogonal with respect to the $\bar{x}^k$curves. Now $\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} \neq 0$and equation (5) gets disrupted. Therefore $\frac{\partial x^m}{\partial \bar{x}^k} = constant$,independent of $\bar{x}^k$ asides being independent of $\bar{x}^j$:we do have $\frac{\partial^2 x^m}{\partial \bar{x}^{k^2}} = 0$ that is (5) holds for i=j[asides for $j \neq k$]

Solving equation(5)we understand

$$x^m = A\bar{x}^j + B\bar{x}^k + C(indpendent\ of\ x^j and\ x^k)$$

$$x^m = A\bar{x}^1 + B\bar{x}^2 + C\bar{x}^3 + D\bar{x}^4 + K\ (6)$$

[A ,B and C in (6) are independent of space time coordinates]

Suppose we took

$$x^m = A(\bar{x}^2, \bar{x}^3, \bar{x}^4)\bar{x}^1 + B\bar{x}^2 + C\bar{x}^3 + D\bar{x}^4 + K$$

then equation (5) is not being satisfied

Again if we took

$$x^m = A(\bar{x}^4)\bar{x}^1 + B\bar{x}^2 + C\bar{x}^3 + D\bar{x}^4 + K$$

equation (5) is not being satisfied for k=1,j=4

We do have a linear transformation given by (6). K becomes zero if one origin maps into the other[homogeneous transformations].Therefore we have a linear homogeneous transformation given by

$$x^m = A\bar{x}^1 + B\bar{x}^2 + C\bar{x}^3 + D\bar{x}^4\ (7)$$

for which the origins map into one another.Equation (5) holds irrespective of whether j and k are unequal or not.. The space time transformations expressed through (7) should stay unchanged irrespective of $j \neq k$ or $j = k$ .Because of equation (6) or (7) ,the linearity expressed by the, equation (5) is valid for $j = k$ asides for $j \neq k$. Considering (1) in the light of equation (5) irrespective of $j = k$ or $j \neq k$ we conclude that the Christoffel symbols are tenors.

Our current arguments are based on the fact that th general solution of $\frac{\partial^2 f}{\partial x \partial y} = 0$is $f(x, y) = Ax + By + C \Longrightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$

Alternative Method

We recall (5) for $j \neq i$

$$\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = 0 \Longrightarrow \frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^m}{\partial \bar{x}^k}\right) = 0 \ (8)$$

$$\frac{\partial \bar{x}^j}{\partial \tau}\frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^m}{\partial \bar{x}^k}\right) = 0$$

$$\frac{d}{d\tau}\left(\frac{\partial x^m}{\partial \bar{x}^k}\right) = 0 \ \ (9)$$

As we move along an arbitrary path passing through fixed point,say P, $\frac{\partial x^m}{\partial \bar{x}^k} = C[independent\ of\ \tau$and hence independent of x,yz and t. The constant value of $\frac{\partial x^m}{\partial \bar{x}^k}$ is equal to that of at P

For every m and k

$$x^m = C\bar{x}^k + D\left(\bar{x}^{i\neq k}\right)$$

$$x^m = C\bar{x}^1 + C\bar{x}^2 + C\bar{x}^3 + C\bar{x}^4 + K \ (10)$$

Consequently due to the linearity expressed by equation (10) , equation (5) is valid for j=k asides for $j \neq k$

Considering equation (5) in a general manner with (1) irrespective of $j \neq k$or $j = k$ we obtain,

$$\bar{\Gamma}^n{}_{jk}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq} = 0$$

$$\Longrightarrow \bar{\Gamma}^n{}_{jk}\frac{\partial \bar{x}^s}{\partial x^m}\frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial \bar{x}^s}{\partial x^m}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq} = 0$$

$$\Longrightarrow \bar{\Gamma}^n{}_{jk}\delta^s{}_n - \frac{\partial \bar{x}^s}{\partial x^m}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq} = 0$$

$$\Longrightarrow \bar{\Gamma}^s{}_{jk} - \frac{\partial \bar{x}^s}{\partial x^m}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m{}_{pq} = 0 \ (11)$$

Thus for $j \neq k, \Gamma^m{}_{pq} \leftrightarrow \bar{\Gamma}^n{}_{jk}$transforms as a rank three mixed tensor with one upper index.

Velocity Transformation

$$\frac{d\bar{x}^\mu}{d\tau} = \frac{\partial x^\mu}{\partial x^\alpha}\frac{dx^\alpha}{dx}$$

$$\frac{d\bar{x}^{\mu}}{d\tau} = M^{\mu}{}_{\alpha}\frac{dx^{\alpha}}{dx}$$

Since the transformations are linear $M^{\mu}{}_{\alpha}$are constants

Four acceleration [rank one tensor] is given by

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\alpha}}{d\tau}$$

The product of tensors being a tensor $\Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\alpha}}{d\tau}$ is a tensor[rank one contravaiant tensor]

Four acceleration and $\Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\alpha}}{d\tau}$ being a tensors the quadruplet $\left\{\frac{d^2x^{\alpha}}{d\tau^2}; \alpha = 1,2,3,4\right\}$ is unexpectedly tensor in curved space time.Incidentally $\frac{d^2x^{\alpha}}{d\tau^2}$ is a tensor in flat space time

Since $\frac{d^2x^{\alpha}}{d\tau^2}$ is a tensor

$$\frac{d^2\bar{x}^{\mu}}{d\tau^2} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{d^2x^{\alpha}}{d\tau^2}$$

$$\frac{d^2\bar{x}^{\mu}}{d\tau^2} = M^{\mu}{}_{\alpha}\frac{d^2x^{\alpha}}{d\tau^2}$$

[For linear transformations $M^{\mu}{}_{\alpha}$ are constants]

Impact on the Riemann Tensor

The transformation of the Riemann tensor is considered in the light of equation (5)

$$\frac{\partial^2 x^m}{\partial\bar{x}^j\partial\bar{x}^k} = 0$$

Transformation of the Riemann Tensor[2]:

$$\bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta}$$

$$\frac{\partial}{\partial\bar{x}^{\zeta}}\bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\beta}}{\partial\bar{x}^{\zeta}\partial\bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta} + \frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial}{\partial\bar{x}^{\zeta}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta}\right)$$

$$\frac{\partial}{\partial\bar{x}^{\zeta}}\bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial}{\partial\bar{x}^{\zeta}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta}\right)$$

$$\frac{\partial}{\partial\bar{x}^{\zeta}}\bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\gamma}}{\partial\bar{x}^{\zeta}\partial\bar{x}^{\rho}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta}\right) + \frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\rho}}\frac{\partial}{\partial\bar{x}^{\zeta}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\delta}}{\partial\bar{x}^{\sigma}}R^{\alpha}{}_{\beta\gamma\delta}\right)$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial}{\partial \bar{x}^{\zeta}} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} R^{\alpha}{}_{\beta\gamma\delta} \right)$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x^{\delta}}{\partial \bar{x}^{\zeta} \partial \bar{x}^{\sigma}} R^{\alpha}{}_{\beta\gamma\delta} \right) + \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial}{\partial \bar{x}^{\zeta}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} R^{\alpha}{}_{\beta\gamma\delta} \right)$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \frac{\partial}{\partial \bar{x}^{\zeta}} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} R^{\alpha}{}_{\beta\gamma\delta} \right)$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial}{\partial \bar{x}^{\zeta}} \left(R^{\alpha}{}_{\beta\gamma\delta} \right) + \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \frac{\partial^2 \bar{x}^{\mu}}{\partial \bar{x}^{\zeta} \partial x^{\alpha}} R^{\alpha}{}_{\beta\gamma\delta}$$

$$\frac{\partial^2 \bar{x}^{\mu}}{\partial \bar{x}^{\zeta} \partial x^{\alpha}} = \frac{\partial^2 \bar{x}^{\mu}}{\partial x^{\alpha} \partial \bar{x}^{\zeta}} = \frac{\partial}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\zeta}} = \frac{\partial}{\partial x^{\alpha}} \delta^{\mu}{}_{\zeta} = 0$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial}{\partial \bar{x}^{\zeta}} \left(R^{\alpha}{}_{\beta\gamma\delta} \right)$$

$$\frac{\partial}{\partial \bar{x}^{\zeta}} \bar{R}^{\mu}{}_{\nu\rho\sigma} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\sigma}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{m}}{\partial \bar{x}^{\zeta}} \frac{\partial}{\partial x^{m}} \left(R^{\alpha}{}_{\beta\gamma\delta} \right)$$

Thus the derivative of the Riemann tensor behaves like a tensor.

General Perspectives

In general,

$$\bar{A}^{\mu} = M^{\mu}{}_{\alpha} A^{\alpha}$$

Suppose we carry out a coordinate transformation ,say from Cartesian to spherical considering the fact that tensor transformations may be achieved between arbitrary systems. The transformation elements will not be constants contrary to what has been deduced.. A resolution to this contradiction would be to consider $A^{\alpha}$and consequently $\bar{A}^{\mu}$ as null tensors

$$\bar{g}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}$$

We transform between spherical and Cartesian systems. The transformation elements will not be constants. As before a resolution would be to consider the metric tensor as the null tensor. With that the Riemann tensor becomes the null tensor.

One should take note of the fat that the transformation elements are independent of the metric.The definition

$$\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} A^{\alpha}$$

is independent of the metric coefficients: $A^{\alpha}$and $\bar{A}^{\mu}$are the coordinate values and not he physical values[3] of the tensor components. Elements of the transformation matrix between two coordinate systems remain the same doesn't matter which manifold we are considering. As for an example the equations relating the Cartesian and the spherical systems are identical for the flat space time manifold and Schwarzschild geometry[or for any other geometry for that matter]. The transformation elements re space time dependent.

Conclusion

As claimed , considering the transformation of the Christoffel symbol we have derived that transformation have to b necessarily linear and that the Christoffel symbol are indeed tensers the relevant term preventing such a behavior reducing to zero.

References

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