Proof of a Combinatorial Identity

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Abstract

In this present paper we will show you some interesting identity involving combinatorial symbols and a proof of it as a theorem. The theorem was a discovery from the times when I was studying Calculus at USAC/CUNOC University in Quetzaltenango, Guatemala around 2004 year.

Theorem 1. (Danilo Chavez 2004) If $n,k \in (\mathrm{N} \cup 0)$, $n \geqq 0$, $k \geqq 0$, $n \geqq$ k then

$$\binom{k}{k} + \binom{k+1}{k} + ... + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

Proof. By mathematical induction. If n=0 then k=0

$$\binom{0}{0} = \frac{0!}{0!0!} = 1 = \frac{1!}{1!0!} = \binom{1}{1}$$

If n=1, we have two possibilities: k=0 or k=1

$$\binom{0}{0} + \binom{1}{0} = \frac{0!}{0!0!} + \frac{1!}{0!1!} = 2 = \frac{2!}{1!1!} = \binom{2}{1}$$

$$\binom{1}{1} = \frac{1!}{1!0!} = 1 = \frac{2!}{2!0!} = \binom{2}{2}$$

So, n=0 and n=1 are covered. Supposing n=r we have

$$\binom{k}{k}+\binom{k+1}{k}+...+\binom{r-1}{k}+\binom{r}{k}=\binom{r+1}{k+1}$$

It must be for n= r+1

$$\binom{k}{k}+\binom{k+1}{k}+...+\binom{r-1}{k}+\binom{r}{k}+\binom{r+1}{k}=\binom{r+1}{k+1}+\binom{r+1}{k}$$

We are looking for an expression like this

$$\binom{r+1}{k+1}+\binom{r+1}{k}=\binom{r+2}{k+1}$$

Let's start

$$\binom{r+1}{k+1}+\binom{r+1}{k}=\frac{(r+1)!}{(k+1)!(r-k)!}+\frac{(r+1)!}{k!(r-k+1)!}$$

$$=(r+1)!(\frac{1}{(k+1)!(r-k)!}+\frac{1}{k!(r-k+1)!})$$

$$=(r+1)!(\frac{k!(r-k+1)!+(k+1)!(r-k)!}{(k+1)!(r-k)!k!(r-k+1)!})$$

$$=\frac{(r+1)!}{(k+1)!(r-k)!k!(r-k+1)!}(k!(r-k+1)!+(k+1)!(r-k)!)$$

$$=\frac{(r+1)!k!(r-k)!}{(k+1)!(r-k)!k!(r-k+1)!}((r-k+1)+(k+1))$$

$$=\frac{(r+1)!(r+2)}{(k+1)!(r-k+1)!}=\frac{(r+2)!}{(k+1)!(r-k+1)!}=\binom{r+2}{k+1}$$

Quod erat demonstrandum. □