Scale-Invariant Conformal Waves

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Abstract

Investigating conformal metrics on (pseudo-) Riemannian spaces, a 'scale-invariant' choice for the Lagrange density leads to homogeneous d'Alembert eqations which allow for source-free wave phenomena in any number of dimensions.

This suggests to apply a scale-invariant action principle rather than the Hilbert-Einstein action to general relativity to also find general, non-conformal solutions.

Keywords: General Relativity, Action Principle

1 Notation

To reduce the number of letters used for indices, the same letter may be used more than once when unambiguous, like in

$$g^{gg}\Gamma_{ggc} = \tfrac{1}{2}\, g^{gg} g_{gg,c}\,,$$
$$g^{gg}\Gamma_{agg} = g^{gg}\left(g_{ag,g} - \tfrac{1}{2}\, g_{gg,a}\right)\,.$$

No distinction is made between greek and latin letters for indices. They are always understood as 4-dimensional.

2 Cartesian Conformal Map

With a constant metric being represented by any square root of the identity matrix,

$$\eta_{ab} \;=\; \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \;=\; \eta^{ab}\,, \qquad \det(\eta_{ab}) = \pm 1\,,$$

define a positive real function $E(x^\mu)$ from an arbitrary logarithm function $\alpha(x^\mu)$,

$$E:\ \mathbb{R}^n \to \mathbb{R}_+\,, \quad \alpha:\ \mathbb{R}^n \to \mathbb{R}\,,$$
$$x^\mu \;\mapsto\; E(x^\mu) \;:=\; e^{\alpha(x^\mu)}\,.$$

Metric Tensor

From a metric tensor, describing a conformal metric and constructed as

$$g_{ab} \;:=\; E^2\,\eta_{ab} \;=\; \begin{bmatrix} \pm e^{2\alpha} & & \\ & \ddots & \\ & & \pm e^{2\alpha} \end{bmatrix} \quad \Leftrightarrow \quad g^{ab} \;:=\; E^{-2}\,\eta^{ab}\,,$$

we find metric derivatives,

$$\begin{aligned} g_{ab,c} &= \partial_c\, e^{2\alpha}\;\eta_{ab} \;=\; 2\alpha_{,c}\; e^{2\alpha}\;\eta_{ab} \\ &= 2\,E^2\,\eta_{ab}\alpha_{,c}\,, \end{aligned}$$

and the Christoffel symbol of the first kind,

$$\begin{aligned} \Gamma_{abc} &= \tfrac{1}{2}\left(g_{bc,a} + g_{ac,b} - g_{ab,c}\right) \\ &= E^2\left(\eta_{ab}\alpha_{,c} + \eta_{ac}\alpha_{,b} - \eta_{bc}\alpha_{,a}\right). \end{aligned}$$

Metric Connection

From that we get the Christoffel symbol of the second kind,

$$\begin{aligned} \Gamma^{a}{}_{bc} &= g^{a\alpha}\Gamma_{\alpha bc} \;=\; E^{-2}\,\eta^{a\alpha}\Gamma_{\alpha bc} \\ &= \delta^{a}{}_{b}\alpha_{,c} + \delta^{a}{}_{c}\alpha_{,b} - \eta^{a\alpha}\alpha_{,\alpha}\eta_{bc}\,, \end{aligned} \tag{1}$$

with the particular contractions

$$\Gamma^{\delta}{}_{\delta c} \;=\; n\alpha_{,c}\,, \tag{2}$$

$$\Gamma^{a}{}_{gg}g^{gg} \;=\; E^{-2}\,\Gamma^{a}{}_{\eta\eta}\eta^{\eta\eta} \;=\; -(n-2)\,E^{-2}\,\eta^{a\mu}\alpha_{,\mu}\,. \tag{3}$$

Covariant Derivatives

From (1), the covariant derivative of an arbitrary covector V_a is given by

$$\begin{aligned} \nabla_d V_a &= V_{a;d} \;=\; V_{a,d} - V_\gamma\,\Gamma^{\gamma}{}_{ad} \\ &= V_{a,d} - V_a\,\alpha_{,d} - V_d\,\alpha_{,a} + V_\eta\,\alpha_{,\eta}\eta^{\eta\eta}\eta_{ad}\,, \end{aligned}$$

so the contracted second covariant derivative, that is, the Laplace operator (or d'Alembert operator, in four-dimensional spacetime) of the logarithmic conformal potential is then

$$\begin{aligned} \nabla_\mu\nabla^\mu\alpha &= \alpha_{;gg}\,g^{gg} \;=\; E^{-2}\,\alpha_{;\eta\eta}\,\eta^{\eta\eta} \\ &= E^{-2}\,\left(\alpha_{,\eta\eta} + (n-2)\alpha_{,\eta}\alpha_{,\eta}\right)\,\eta^{\eta\eta}\,. \end{aligned} \tag{4}$$

From this follows, that in $n = 2$ dimensions the covariant Laplacian is identical to the second partial derivative,

$$n = 2 \qquad \Rightarrow \qquad \alpha_{;gg}\, g^{gg} \;=\; \alpha_{,gg}\, g^{gg}\,. \tag{5}$$

Connection Derivatives

From (2) and (3) we get the partial derivatives of the contracted connection,

$$\begin{aligned} \Gamma^{\lambda}{}_{\lambda c,d} &= n\,\alpha_{,cd}\,, \\ \Gamma^{a}{}_{gg,d}\, g^{gg} &= E^{-2}\,\Gamma^{a}{}_{\eta\eta,d}\,\eta^{\eta\eta} \;=\; -(n-2)\,E^{-2}\,\eta^{a\mu}\,\alpha_{,\mu d}\,. \end{aligned}$$

and their second contractions,

$$\Gamma^{\lambda}{}_{\lambda g,g}\, g^{gg} \;=\; n\,E^{-2}\,\alpha_{,\eta\eta}\,\eta^{\eta\eta}\,, \tag{6}$$

$$\Gamma^{\lambda}{}_{gg,\lambda}\, g^{gg} \;=\; E^{-2}\,\Gamma^{\lambda}{}_{\eta\eta,\lambda}\,\eta^{\eta\eta} \;=\; -(n-2)\,E^{-2}\,\alpha_{,\eta\eta}\,\eta^{\eta\eta}\,. \tag{7}$$

Connection Derivative Difference

So from (6) and (7), the fully contracted connection derivative difference is

$$\left(\Gamma^{\lambda}{}_{\lambda g,g} - \Gamma^{\lambda}{}_{gg,\lambda}\right) g^{gg} \;=\; 2\,(n-1)\,E^{-2}\,\alpha_{,\eta\eta}\,\eta^{\eta\eta}\,. \tag{8}$$

Connection Products

From (2) and (3) we get the fully contracted 'straight' connection product,

$$\Gamma^{\lambda}{}_{\lambda\gamma}\,\Gamma^{\gamma}{}_{gg}\, g^{gg} \;=\; -n\,(n-2)\,E^{-2}\,\alpha_{,\eta}\,\alpha_{,\eta}\,\eta^{\eta\eta}\,. \tag{9}$$

while the calculation of the 'crossed' connection product needs a more general calculation from (1),

$$\begin{aligned} \Gamma^{a}{}_{b\gamma}\,\Gamma^{\gamma}{}_{cd} &= \left(\delta^{a}{}_{b}\alpha_{,\gamma} + \delta^{a}{}_{\gamma}\alpha_{,b} - \eta^{a\rho}\alpha_{,\rho}\eta_{b\gamma}\right)\cdot\left(\delta^{\gamma}{}_{c}\alpha_{,d} + \delta^{\gamma}{}_{d}\alpha_{,c} - \eta^{\gamma\sigma}\alpha_{,\sigma}\eta_{cd}\right) \\ &= \delta^{a}{}_{b}\left(2\,\alpha_{,c}\alpha_{,d} - \alpha_{,\eta}\alpha_{,\eta}\eta^{\eta\eta}\eta_{cd}\right) + \left(\delta^{a}{}_{c}\alpha_{,b}\alpha_{,d} + \delta^{a}{}_{d}\alpha_{,b}\alpha_{,c}\right) - \eta^{a\alpha}\alpha_{,\alpha}\left(\alpha_{,c}\,\eta_{bd} + \alpha_{,d}\,\eta_{bc}\right)\,, \end{aligned}$$

contracting once,

$$\Gamma^{\lambda}{}_{b\gamma}\,\Gamma^{\gamma}{}_{c\lambda} \;=\; \left((n+2)\,\alpha_{,b}\,\alpha_{,c} - 2\,\alpha_{,\eta}\,\alpha_{,\eta}\,\eta^{\eta\eta}\,\eta_{bc}\right)\,E^{-2}\,,$$

and fully contracted,

$$\Gamma^{\lambda}{}_{g\gamma}\,\Gamma^{\gamma}{}_{g\lambda}\, g^{gg} \;=\; -(n-2)\,E^{-2}\,\alpha_{,\eta}\,\alpha_{,\eta}\,\eta^{\eta\eta}\,. \tag{10}$$

Connection Product Difference

From (9) and (10) we get the fully contracted connection product difference,

$$\left(\Gamma^{\lambda}{}_{\lambda\gamma}\,\Gamma^{\gamma}{}_{gg}-\Gamma^{\lambda}{}_{g\gamma}\,\Gamma^{\gamma}{}_{g\lambda}\right)\,g^{gg} \;=\; -(n-1)\,(n-2)\,E^{-2}\,\alpha_{,\eta}\,\alpha_{,\eta}\,\eta^{\eta\eta}\,. \tag{11}$$

Scalar Curvature

The fully contracted Riemann tensor is now obtained from (11) and (8),

$$\begin{aligned} R \;&=\; \Big(\left(\Gamma^{\lambda}{}_{\lambda\gamma}\,\Gamma^{\gamma}{}_{gg}-\Gamma^{\lambda}{}_{g\gamma}\,\Gamma^{\gamma}{}_{g\lambda}\right)-\left(\Gamma^{\lambda}{}_{\lambda g,g}-\Gamma^{\lambda}{}_{gg,\lambda}\right)\Big)\,g^{gg} \\ &=\; -\,(n-1)\,\big(\,2\,\alpha_{,\eta\eta}+(n-2)\alpha_{,\eta}\alpha_{,\eta}\,\big)\,\eta^{\eta\eta}E^{-2}\,, \end{aligned} \tag{12}$$

According to (4), the second partial derivative can be substituted by the covariant derivative,

$$\alpha_{;\eta\eta}\,\eta^{\eta\eta} \;=\; \left(\alpha_{,\eta\eta}-(n-2)\alpha_{,\eta}\alpha_{,\eta}\right)\,\eta^{\eta\eta}\,,$$

so (12) can be expressed with covariant derivatives,

$$R \;=\; -\,(n-1)\,\big(\,2\,\alpha_{;\eta\eta}-(n-2)\alpha_{;\eta}\alpha_{;\eta}\,\big)\,\eta^{\eta\eta}\,E^{-2}\,. \tag{13}$$

From (12) follows immediately, that in $n=1$ dimensions curvature vanishes identically, as one-dimensional geometric spaces are trivially always flat.

It also follows, that in $(n=2)$ dimensions curvature of the conformal space is proportional to the Laplacian, which in fact turns out to vanish identically,

$$R_{(2\mathrm{D})} \;=\; -6\,\Delta\alpha\,E^{-2} \;\equiv\; 0\,.$$

3 Variations

The variational derivative, varying a Lagrangian $\mathcal{L}$ over a function f, is given by

$$\begin{aligned} \delta_f\mathcal{L} \;:=\; \frac{\delta\mathcal{L}}{\delta f} \;&=\; \frac{\partial\mathcal{L}}{\partial f}-\left(\frac{\delta\mathcal{L}}{\delta f_{;a}}\right)_{;a} \qquad \text{(an infinite recursion)} \\ &=\; \frac{\partial\mathcal{L}}{\partial f}-\left(\frac{\partial\mathcal{L}}{\partial f_{;a}}\right)_{;a}+\left(\frac{\partial\mathcal{L}}{\partial f_{;ab}}\right)_{;ab}-\;\;\cdots\,, \end{aligned}$$

in such a way that the Euler-Lagrange equation reads

$$\delta_f\mathcal{L} \;\stackrel{!}{=}\; 0\,.$$

Lagrangians from Scalar Curvature

For the Lagrange density we choose the Ricci scalar,
together with a power of E^a as a scale factor,

$$\mathcal{L} \;=\; E^a R\,,$$

The totally scale-invariant Ricci scalar, expressed with covariant derivatives, is

$$R \;=\; (n-1)\,E^{-2}\left(-2\alpha_{;\eta\eta} + (n-2)\alpha_{;\eta}\alpha_{;\eta}\right)\eta^{\eta\eta}\,.$$

and so

$$E^a R \;=\; (n-1)\,E^{a-2}\left(-2\alpha_{;\eta\eta} + (n-2)\alpha_{;\eta}\alpha_{;\eta}\right)\eta^{\eta\eta}\,.$$

With

$$\begin{aligned}
\delta_\alpha E^a \;&=\; \frac{\partial}{\partial\alpha} e^{a\alpha} \;=\; a e^{a\alpha} \;=\; aE^a\,,\\
\delta_\alpha\left(\alpha_{;\mu\nu}\right) \;&=\; \left(\frac{\partial\alpha_{;\mu\nu}}{\partial\alpha_{;ab}}\right)_{;ab} \;=\; \left(\delta_\mu{}^a\delta_\nu{}^b\right)_{;ab} \;=\; 0\,,\\
\delta_\alpha\left(\alpha_{;\mu}\alpha_{;\nu}\right) \;&=\; -\left(\frac{\partial(\alpha_{;\mu}\alpha_{;\nu})}{\partial\alpha_{;a}}\right)_{;a} \;=\; -\left(\frac{\partial\alpha_{;\mu}}{\partial\alpha_{;a}}\alpha_{;\nu} + \alpha_{;\mu}\frac{\partial\alpha_{;\nu}}{\partial\alpha_{;a}}\right)_{;a} \;=\; -2\alpha_{;\mu\nu}\,,
\end{aligned}$$

we find in particular

$$\begin{aligned}
\delta_\alpha\left(E^{a-2}\alpha_{;\eta\eta}\eta^{\eta\eta}\right) \;&=\; E^{a-2}\,(a-2)\,\alpha_{;\eta\eta}\eta^{\eta\eta}\,,\\
\delta_\alpha\left(E^{a-2}\alpha_{;\eta}\alpha_{;\eta}\eta^{\eta\eta}\right) \;&=\; E^{a-2}\left(-2\,\alpha_{;\eta\eta} + (a-2)\,\alpha_{;\eta}\alpha_{;\eta}\right)\eta^{\eta\eta}\,,
\end{aligned}$$

and the total functional derivative,

$$\delta_\alpha\left(E^a R\right) \;=\; (n-1)E^{a-2}\Big(-2\big((a-2)+(n-2)\big)\,\alpha_{;\eta\eta} + (a-2)(n-2)\,\alpha_{;\eta}\alpha_{;\eta}\Big)\eta^{\eta\eta}\,,$$

which gives a generalized equation of motion,

$$\begin{aligned}
&\delta_\alpha\left(E^a R\right) \;\overset{!}{=}\; 0 \quad\Rightarrow\\
&\left(-2\big((a-2)+(n-2)\big)\,\alpha_{;\eta\eta} + (a-2)(n-2)\,\alpha_{;\eta}\alpha_{;\eta}\right)\eta^{\eta\eta} \;=\; 0\,.
\end{aligned}$$

4 Field Equations

By choosing $a = 2$, all product terms vanish, so the homogeneous d'Alembert operator on the logarithmic potential remains,

$$-2(n-2)\,E^{-2}\alpha_{;\eta\eta}\eta^{\eta\eta} \;=\; -2(n-2)\,\alpha_{;gg}g^{gg} \;=\; -2(n-2)\,\Box\,\alpha \;=\; 0\,,$$

which tells that the source density of the logarithmic potential vanishes, and in 4-dimensional spacetime gives a wave equation on the logarithmic potential,

$$\Box\,\alpha \;=\; \alpha_{;gg}g^{gg} \;=\; 0 \qquad \Rightarrow \qquad E^2\,\Box\,\alpha \;=\; \alpha_{;\eta\eta}\eta^{\eta\eta} \;=\; 0\,. \tag{14}$$

Effective Mean Curvature

The remaining part of the scale-invariant Ricci scalar, after vanishing of the source density of the conformal logarithm according to (14), is then

$$E^2R \;\stackrel{\star}{=}\; (n-1)(n-2)\,\alpha_{;\eta}\alpha_{;\eta}\,\eta^{\eta\eta}\,, \tag{15}$$

which reads: Mean curvature is proportional to the squared magnitude of the gradient of the conformal logarithm function. Thus mean curvature can be seen as an energy density of the logarithmic conformal field.

Only in 2D (and trivially in 1D), any conformal metric from a source-free conformal logarithm function gives a flat space; those are the well-known holomorphic and meromorphic functions on the space of complex numbers. The same does not hold true for higher dimensions, $n \geq 3$, and hence on 4D spacetime, where any conformal metric introduces a curvature.

In dimension higher than $n > 2$, any conformal function (which is not simply constant) curves space; there is no conformal metric which leaves spacetime flat except for a trivial rescaling of the whole universe.

5 Discussion

In the context of General Relativity, Hilbert chose for the Lagrange density in the 4D case

$$\mathcal{L} \;=\; \sqrt{g}R \;=\; E^nR\,,$$

to account for the volume element in some way, but this would not give a reasonable wave equation here. Instead, the wave equation arises only when the power E^2 exactly cancels out any dependence on size of the volume element, which means, the Lagrangian is 'scale-free' or 'scale invariant'.

So Hilbert's Lagrangian at least gives the source-free conformal wave equation in the case of $n = 2$ dimensions, but this does not describe spacetime as we need it.

For $n = 4$ dimensions, Hilbert's Lagrangian is $\mathcal{L} \ = \ E^4\, R$, which is not scale-invariant, that is, not independent on the absolute size of the volume element.

As an assumption of realism, each infinitesimal local volume element can not 'know' about its absolute size in the universal context, so in a 'general' relativistic theory, the action as well as the field equations should not depend on the absolute scale.

The Langrange density which is proposed instead is for n dimensions

$$\mathcal{L} \ = \ \sqrt[n]{g} R \ = \ E^2 R\,,$$

which in $n = 4$ dimensions effectively becomes

$$\mathcal{L} \ = \ \sqrt[4]{g} R \ = \ E^2 R\,,$$

The expressions $E^2 R$, $E^2 \,\Box\, \alpha$ are 'scale-free', which means, any infinitesimal volume element does not 'know' about its absolute size. In this sense this is a true near-field geometry (eine „wahre Nahegeometrie", as Hermann Weyl had asked for). Exactly in this situation the homogeneous wave equation arises.